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EXISTENCE AND LOCALIZATION OF SOLUTIONS OF SECOND ORDER ELLIPTIC PROBLEMS USING LOWER AND UPPER SOLUTIONS IN THE REVERSED ORDER

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1. Introduction

Consider the semilinear elliptic problem

$$\mathcal{L}u = f(x,u) \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\mathcal{L}$ is a linear second order elliptic operator for which the maximum principle holds, $\mathcal{B}$ is a linear first order boundary operator and f is a nonlinear Carathéodory function.

We are concerned with the solvability of (1.1) in presence of lower and upper solutions. A classical basic result in this context says that if α is a lower solution and β is an upper solution satisfying

$$\alpha(x) \le \beta(x) \quad \text{for all } x \in \Omega, \tag{1.2}$$

then problem (1.1) has at least one solution u such that

$$\alpha(x) \le u(x) \le \beta(x) \quad \text{for all } x \in \Omega. \tag{1.3}$$

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In this paper, we want to discuss the situation where α and β satisfy the opposite ordering condition

$$\alpha(x) > \beta(x) \quad \text{for all } x \in \Omega. \tag{1.4}$$

The problem of the solvability of (1.1) in this frame was explicitly raised in the early seventies in [31] and became the subject of some works in the last two decades [4], [30], [20], [21], [29]. We point out that there are some concrete motivations to study this question. Indeed, it was observed in [27] that condition (1.2) turns out to be quite restrictive in some cases. For instance, if we denote by λ_1 the principal eigenvalue of the problem

$$\mathcal{L}u = \lambda u \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega,$$

and we assume that

$$\operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} \frac{\partial f}{\partial s}(x,s) > \lambda_1, \tag{1.5}$$

then, for any pair of lower and upper solutions α, β satisfying (1.2), we find that they must already be solutions and $\alpha = \beta$ (see also [10]). Hence, such upper and lower solutions seem inappropriate to deal with the case where f lies in some sense to the right of the first eigenvalue λ_1. However, in this situation, lower and upper solutions α, β satisfying the opposite ordering condition (1.4) arise naturally. As a very simple example, one can show that condition (1.5) implies the existence of such a pair of lower and upper solutions. We refer to Section 2 for a detailed discussion of the circumstances which give rise to lower and upper solutions satisfying (1.4).

However, it was pointed out by an example in [3] that, in general, the mere existence of a lower solution and an upper solution for which (1.4) holds is not sufficient to guarantee the solvability of (1.1). The example is essentially of the type

$$-\Delta u = \lambda_m u + \varphi_m \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where λ_m is an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ greater than λ_1 and φ_m is a corresponding nonzero eigenfunction. It is easily seen that (1.6) has no solution, although one can construct a lower solution α and an upper solution β, as multiples of the first eigenfunction φ_1, which satisfy

$$\beta(x) < 0 < \alpha(x) \quad \text{for all } x \in \Omega.$$

These considerations suggest that, in order to achieve the solvability, one should prevent the interference of f with the higher part of the spectrum. This can be expressed in various ways. In [4], where the first important contribution to this problem was given, it was proved that the existence of a lower solution

α and of an upper solution β (not necessarily satisfying any ordering condition) implies the solvability of (1.1), provided that

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} |f(x,s)-\lambda_1 s| < \infty.$$

More recently, when $\mathcal{L}$ is selfadjoint and $\mathcal{B}$ is either the Dirichlet or the Neumann boundary operator, the above mentioned result was generalized in [21] to unbounded perturbations of $\lambda_1 s$, satisfying

$$\liminf_{|s|\to\infty} f(x,s)/s \geq \lambda_1 \quad \text{uniformly a.e. in } \Omega, \tag{1.7}$$

and

$$\limsup_{|s|\to\infty} f(x,s)/s \leq \gamma(x) \quad \text{uniformly a.e. in } \Omega, \tag{1.8}$$

with $\gamma(x) \leq \lambda_2$ for a.e. $x \in \Omega$ and $\gamma(x) < \lambda_2$ on a subset of Ω of positive measure. A slight improvement of the main result in [21] has been obtained in [29]. It amounts to replacing condition (1.8) with

$$\limsup_{s\to\infty} f(x,s)/s \leq \gamma_+(x), \quad \limsup_{s\to-\infty} f(x,s)/s \leq \gamma_-(x)$$

uniformly a.e. in Ω, with $\gamma_+(x), \gamma_-(x) \leq \lambda_2$ for a.e. $x \in \Omega$ and $\gamma_+(x)$ or $\gamma_-(x) < \lambda_2$ on a subset of Ω of positive measure.

The proof of the result in [21] (as well as of that in [29]) exploits an idea introduced in [4], combined with some delicate estimates which are based on the maximum principle and a bootstrap technique. The method essentially consists of reducing the problem, through a Lyapunov–Schmidt decomposition together with degree and connectedness arguments, to the situation of well ordered lower and upper solutions, i.e. satisfying (1.2). However, this technique does not provide any information on the location of the solution, even in the case where α and β satisfy (1.4). Also, its definite linear character prevents the consideration of nonlinearities which exhibit an asymmetric behaviour at infinity; it imposes the rather unnatural restriction (1.7).

The aim of this paper is to provide, when condition (1.4) is assumed, a different approach to the problem, based on a direct use of the Leray–Schauder continuation method. The main novelty lies in the introduction of a suitable homotopy which eventually allows us to prove that the degree of the given problem is nonzero on an open bounded set conveniently defined in the function space. The advantages of this method are manifold. It gives some information about where the solution is situated. Indeed, when (1.4) holds, the solution u satisfies

$$\beta(x_0) \leq u(x_0) \leq \alpha(x_0) \quad \text{for some } x_0 \in \overline{\Omega},$$

instead of (1.3); further, we get information on the normal derivative of u in the case of Dirichlet boundary conditions. This simple fact is sometimes sufficient to prove the existence of multiple solutions. Moreover, we can deal with nonlinearities f, having linear growth, whose limits

$$\limsup_{s\to\infty} f(x,s)/s \quad \text{and} \quad \limsup_{s\to-\infty} f(x,s)/s$$

may be different. So that, in some situations, we are able to relate the asymptotic behaviour of f with the Dancer–Fučík spectrum of the homogeneous problem

$$\mathcal{L}u = \lambda_+ u^+ - \lambda_- u^- \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial\Omega.$$

Finally, since no restriction like (1.7) is required, it turns out that the existence of a lower solution α and an upper solution β satisfying (1.4) appears as a (rather weak) control with respect to the first eigenvalue λ_1, which yields the solvability of problem (1.1) when it is coupled with an additional noninterference condition with the rest of the (possibly generalized) spectrum.

However, we point out again that our approach requires that α and β satisfy (1.4), a condition which was not needed at all in [4] and [21].

2. Preliminaries, statements and remarks

Throughout this paper Ω will denote a bounded domain in $\mathbb{R}^N$, having a boundary $\partial\Omega$ of class $C^{1,1}$. Let $\mathcal{L}$ be the real second order strongly uniformly elliptic operator given by

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i} + a_0 u,$$

where $a_{ij} = a_{ji} \in C^{0,1}(\overline{\Omega})$ for $i,j = 1,\ldots,N$, $a_i \in L^\infty(\Omega)$ for $i = 0,\ldots,N$, with $a_0(x) \geq 0$ for a.e. $x \in \Omega$ and $a_0(x) > 0$ on a subset of Ω of positive measure. We also suppose that the boundary $\partial\Omega$ is the disjoint union of two closed subsets Γ_0 and Γ_1 each of which is an oriented $(N-1)$-dimensional submanifold of $\mathbb{R}^N$. Let us denote by $\nu \in C^1(\partial\Omega, \mathbb{R}^N)$ the unit outer normal to $\partial\Omega$ and by $\eta \in C^1(\Gamma_1, \mathbb{R}^N)$ a vector field satisfying $(\eta \mid \nu) > 0$ on Γ_1. Let $b_0 \in C^1(\Gamma_1)$ be a nonnegative function and define the boundary operator

$$\mathcal{B}u = \begin{cases} u & \text{on } \Gamma_0, \\ \partial u/\partial\eta + b_0 u & \text{on } \Gamma_1. \end{cases}$$

Let $p \in \,]N,\infty[$ be fixed. In this case, any function in $W^{2,p}(\Omega)$ is twice differentiable a.e. in Ω and belongs to $C^1(\overline{\Omega})$. Then we set

$$W^{2,p}_{\mathcal{B}}(\Omega) = \{u \in W^{2,p}(\Omega) \mid \mathcal{B}u = 0\},$$

where $\mathcal{B}$ is defined in the classical sense. Let $L : W^{2,p}_{\mathcal{B}}(\Omega) \to L^p(\Omega)$ be the operator defined by $Lu = \mathcal{L}u$ a.e. in Ω. It follows from the maximum principle and the regularity theory for elliptic equations that L is a linear homeomorphism (cf. e.g. [4], [8], [23]). We denote by λ_1 the principal eigenvalue of L; λ_1 is positive, simple and there exists a corresponding eigenfunction φ_1 such that $\varphi_1(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $(\partial\varphi_1/\partial\nu)(x) < 0$ for $x \in \Gamma_0$. Moreover, λ_1 is an eigenvalue of the adjoint operator L^*, with an associated eigenfunction φ_1^* satisfying $\varphi_1^*(x) > 0$ for $x \in \Omega$ (see [3]).

Let us consider the semilinear elliptic problem

$$\mathcal{L}u = f(x,u) \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Throughout the paper we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the L^p-Carathéodory conditions. This precisely means that $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $x \in \Omega$, $f(\cdot,s) : \Omega \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$, and for every $R > 0$, there is a function $\gamma_R \in L^p(\Omega)$ such that $|f(x,s)| \le \gamma_R(x)$ for a.e. $x \in \Omega$ and every $s \in [-R,R]$.

By a *lower solution* of (2.1) we mean a function $\alpha \in W^{2,p}(\Omega)$ such that

$$\mathcal{L}\alpha(x) \le f(x,\alpha(x)) \quad \text{for a.e. } x \in \Omega, \qquad \mathcal{B}\alpha(x) \le 0 \quad \text{for } x \in \partial\Omega.$$

An upper solution β is defined similarly by reversing the signs in the above inequalities. A solution of (2.1) can be thought of as a function u which is simultaneously a lower and an upper solution.

Generally we also suppose that f grows at most linearly, i.e.

(f_0) there exists a function $\gamma \in L^p(\Omega)$ such that

$$|f(x,s)| \le \gamma(x)(1+|s|) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}.$$

In the sequel, we will, however, need a more precise control on the linear growth of f. To express it, we first introduce the following condition:

(f_1) there exist functions $a,b,c,d \in L^p(\Omega)$ such that

$$\begin{aligned} a(x) &\le \liminf_{s\to\infty} f(x,s)/s \le \limsup_{s\to\infty} f(x,s)/s \le b(x), \\ c(x) &\le \liminf_{s\to-\infty} f(x,s)/s \le \limsup_{s\to-\infty} f(x,s)/s \le d(x), \end{aligned}$$

uniformly a.e. in Ω.

This means that for every $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that

$$\begin{aligned} a(x) - \varepsilon \le f(x,s)/s \le b(x) + \varepsilon \quad &\text{for a.e. } x \in \Omega \text{ and all } s \ge s_\varepsilon, \\ c(x) - \varepsilon \le f(x,s)/s \le d(x) + \varepsilon \quad &\text{for a.e. } x \in \Omega \text{ and all } s \le -s_\varepsilon. \end{aligned}$$

Next we give the following definition, where we set as usual $u^+ = \max\{u,0\}$ and $u^- = -\min\{u,0\}$.

Definition 2.1. Let $a, b, c, d \in L^p(\Omega)$ be functions satisfying $a(x) \le b(x)$ and $c(x) \le d(x)$ for a.e. $x \in \Omega$. We say that the box

$$[a,b] \times [c,d] = \{(q_+, q_-) \mid q_+, q_- \in L^p(\Omega),\ a(x) \le q_+(x) \le b(x),\\ c(x) \le q_-(x) \le d(x) \text{ for a.e. } x \in \Omega\}$$

is *admissible* if the following conditions hold:

(A) for every $(q_+, q_-) \in [a,b] \times [c,d]$, any nontrivial solution u of

$$\mathcal{L}u = q_+ u^+ - q_- u^- \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega,$$

is such that either $v = u$ or $v = -u$ satisfies

$$v(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) < 0 \text{ for } x \in \Gamma_0;$$

(B) there exist two continuous functions $q_+^\bullet$ (resp. $q_-^\bullet$) : $[0,1] \to L^p(\Omega)$, $\mu \mapsto q_+^\mu$ (resp. q_-^μ), such that

(i) $(q_+^1, q_-^1) \in [a,b] \times [c,d]$ and $q_+^1(x), q_-^1(x) \ge \lambda_1$ for a.e. $x \in \Omega$,
(ii) $q_+^0 = q_-^0$,
(iii) for every $\mu \in [0,1]$ the problem

$$\mathcal{L}u = q_+^\mu u^+ - q_-^\mu u^- \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial\Omega,$$

has only the trivial solution.

Roughly speaking, condition (A) expresses the fact that the box $[a,b] \times [c,d]$ does not interfere with any eigenvalue of L except λ_1, while condition (B) is an assumption of nondegeneracy on the degree. Now we are in a position to state the main result of this paper.

Theorem 2.2. *Assume condition* (f_1) *holds and the box* $[a,b] \times [c,d]$ *is admissible. Suppose that there exist a lower solution* α *and an upper solution* β *satisfying*

$$\beta(x) < \alpha(x) \quad \textit{for all } x \in \Omega. \tag{2.2}$$

Then problem (2.1) *has at least one solution* $u \in \overline{\mathcal{S}}$, *where*

$$\mathcal{S} = \{u \in C^1(\overline{\Omega}) \mid \mathcal{B}u = 0 \textit{ and } \beta(x_0) < u(x_0) < \alpha(x_0) \textit{ for some } x_0 \in \Omega\}.$$

Moreover, there exist a minimal solution v *and a maximal solution* w *with* $v, w \in \overline{\mathcal{S}}$, *i.e. there is no solution* $u \in \overline{\mathcal{S}}$ *such that*

$$u \ge v,\ u \ne v \quad \textit{or} \quad w \ge u,\ w \ne u.$$

There are three main aspects of this result we wish to comment: the construction of admissible boxes $[a,b] \times [c,d]$, the construction of lower and upper solutions satisfying (2.2) and the localization of the solution by means of the set $\mathcal{S}$. The rest of this section, which is subdivided in three parts, is devoted to this discussion.

2.1. Remarks on the construction of admissible boxes. The first criterion of admissibility we present holds for a general nonselfadjoint operator L as considered above. It is adapted from Lemma 2.2 of [2].

Proposition 2.3. *Let $a, c \in L^\infty(\Omega)$, and $b \in L^p(\Omega)$ be such that*

(i) *$a(x), c(x) \leq \lambda_1$ for a.e. $x \in \Omega$,*
(ii) *$b(x) \geq \lambda_1$ for a.e. $x \in \Omega$ and $b(x) > \lambda_1$ on a subset of Ω of positive measure.*

Then there exists $d \in \mathbb{R}$, with $d > \lambda_1$, such that the box $[a, b] \times [c, d]$ is admissible.

Theorem 2.2, together with Proposition 2.3, can be compared to Theorem 3.1 of [4], where the operators $\mathcal{L}$ and $\mathcal{B}$ were as general as here, but the nonlinearity f was supposed to be a bounded perturbation of $\lambda_1 s$, and no information on the location of the solution nor the existence of extremal solutions was obtained. However, it should be noted that in [4] condition (2.2) was not needed and f could depend also on ∇u.

The second criterion is confined to the case where the operator L is selfadjoint. It is related to Lemma 6.2 of [12]. More precisely, we suppose that $\mathcal{L}$ is formally selfadjoint, i.e.

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u,$$

with $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, N$, and, in the definition of $\mathcal{B}$, η is the conormal vector field. This includes in particular the case of Dirichlet or Neumann boundary conditions. In this frame, we can state a more precise result than Proposition 2.3, which makes use of the Dancer–Fučík spectrum for the operator L, i.e. the set $\mathcal{F}$ of those $(\lambda_+, \lambda_-) \in \mathbb{R}^2$ such that the problem

$$\mathcal{L}u = \lambda_+ u^+ - \lambda_- u^- \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega,$$

has a nontrivial solution. It has recently been proven in [12] (cf. [9] for some preliminary results in this direction) that $\mathcal{F}$ contains a curve $\mathcal{F}_2$ which passes through (λ_2, λ_2), where λ_2 is the second eigenvalue of L. The domain bounded by the lines $[\lambda_1, \infty[\times\{\lambda_1\}$, $\{\lambda_1\}\times[\lambda_1, \infty[$ and $\mathcal{F}_2$ does not intersect $\mathcal{F}$. The curve $\mathcal{F}_2$ is continuous, strictly decreasing, symmetric with respect to the diagonal. It is asymptotic to the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$, except when $N = 1$ and $\mathcal{B}$ is the Neumann boundary operator, i.e. $\Gamma_0 = \emptyset$ and $b_0 = 0$. We also recall that in the one-dimensional case it is not difficult (see e.g. [18]) to derive in some situations an explicit formula describing $\mathcal{F}_2$. For instance, if $\mathcal{L}u = -u'' + a_0 u$ with a_0 a positive constant and $\Omega =]0,\pi[$, then

$$\mathcal{F}_2 = \{(\lambda_+,\lambda_-) \mid \lambda_+,\lambda_- > 0 \text{ and } (\lambda_+ - a_0)^{-1/2} + (\lambda_- - a_0)^{-1/2} = 1\}$$

for $\mathcal{B}u = (u(0), u(\pi))$ (Dirichlet boundary conditions), and

$$\mathcal{F}_2 = \{(\lambda_+,\lambda_-) \mid \lambda_+,\lambda_- > 0 \text{ and } (\lambda_+ - a_0)^{-1/2} + (\lambda_- - a_0)^{-1/2} = 2\}$$

for $\mathcal{B}u = (u'(0), u'(\pi))$ (Neumann boundary conditions).

Proposition 2.4. *Assume that the operator L is selfadjoint. Let $(\lambda_+,\lambda_-) \in \mathcal{F}_2$ and let $a,b,c,d \in L^\infty(\Omega)$ satisfy $a(x) \le b(x)$, $c(x) \le d(x)$ and $\lambda_1 \le b(x) \le \lambda_+$, $\lambda_1 \le d(x) \le \lambda_-$ for a.e. $x \in \Omega$. Moreover, suppose that $\lambda_1 < b(x), \lambda_1 < d(x)$ on subsets of Ω of positive measure and $b(x) < \lambda_+, d(x) < \lambda_-$ on a common subset of Ω of positive measure. Then the box $[a,b] \times [c,d]$ is admissible.*

Using Proposition 2.4, we obtain the following result where the asymptotic behaviour of f is related to the second branch $\mathcal{F}_2$ of the Dancer–Fučík spectrum of L.

Theorem 2.5. *Suppose that the operator L is selfadjoint, let $(\lambda_+,\lambda_-) \in \mathcal{F}_2$ and assume that (f_0) holds with $\gamma \in L^\infty(\Omega)$. Moreover, suppose that*

- (f_2) *$\limsup_{s\to\infty} f(x,s)/s \le \lambda_+$ and $\limsup_{s\to-\infty} f(x,s)/s \le \lambda_-$ uniformly a.e. in Ω, with strict inequalities on a common subset of Ω of positive measure.*

Further, assume that there exist a lower solution α and an upper solution β satisfying (2.2). *Then the conclusion of Theorem* 2.2 *holds.*

An immediate corollary of Theorem 2.5 can be stated in the case where f can be written as

- (f_3) $f(x,s) = g(s) - h(x)$ with $g : \mathbb{R} \to \mathbb{R}$ continuous and $h \in L^\infty(\Omega)$.

The main feature of this result is that no growth restriction from below is imposed on g; yet, we lose the information on the localization of the solution.

Corollary 2.6. *Suppose that the operator L is selfadjoint. Let $(\lambda_+,\lambda_-) \in \mathcal{F}_2$ and assume that* (f_3) *holds together with*

- (f_2') *$\limsup_{s\to\infty} g(s)/s \le \lambda_+$ and $\limsup_{s\to-\infty} g(s)/s \le \lambda_-$, and at least one of these inequalities is strict.*

Further, suppose that there exist a lower solution α and an upper solution β satisfying (2.2). *Then problem* (2.1) *has at least one solution.*

Theorem 2.5 and Corollary 2.6 can be compared with the main result in [21] (see also [29]), where it is assumed that $\lambda_+ = \lambda_- = \lambda_2$ together with the additional restriction $\liminf_{|s|\to\infty} f(x,s)/s \ge \lambda_1$ uniformly a.e. in Ω. Variants of this result can be obtained where condition (f_2') is suitably weakened, introducing some restrictions on the potential $G(s) = \int_0^s g(t)\,dt$, as in [14], [22] and [24].

2.2. Remarks on the construction of lower and upper solutions. We describe some situations where lower and upper solutions satisfying (2.2) arise in a natural way. In this context, it is convenient to rewrite problem (2.1) in the equivalent form

$$\mathcal{L}u = \lambda_1 u + g(x,u) - h(x) \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the L^p-Carathéodory conditions and $h \in L^p(\Omega)$.

The first three results we present have been obtained in [21] and we refer to this paper for the proofs. These propositions are basically similar and will be deduced in a unified way (analogous results can be found in [27], when g lies to the left of λ_1).

We start by considering the case where a Dolph-type condition is assumed to the right of λ_1 (cf. [16]).

Proposition 2.7. *Assume that*

$$\liminf_{s\to\infty} g(x,s)/s \geq 0,$$

uniformly a.e. in Ω, with strict inequality on a subset of Ω of positive measure. Then, for any given h and for any given $t > 0$, there exists a lower solution α of (2.3) with $\alpha(x) \geq t\varphi_1(x)$ for $x \in \Omega$.

A similar condition on g assumed at $-\infty$ yields the existence of an upper solution β with $\beta(x) \leq -t\varphi_1(x)$ for $x \in \Omega$.

Now we consider the case where a Landesman–Lazer condition is satisfied to the right of λ_1 (cf. [28], [2]).

Proposition 2.8. *Assume that there exists a function $k \in L^p(\Omega)$ such that*

$$g(x,s) \geq k(x) \quad \textit{for a.e. } x \in \Omega \textit{ and every } s \geq 0.$$

Then, for any given h satisfying

$$\int_\Omega (\liminf_{s\to\infty} g(x,s))\varphi_1^*(x)\,dx > \int_\Omega h(x)\varphi_1^*(x)\,dx,$$

and for any given $t > 0$, there exists a lower solution α of (2.3) with $\alpha(x) \geq t\varphi_1(x)$ for $x \in \Omega$.

Similar conditions assumed for $s \leq 0$ yield the existence of an upper solution β with $\beta(x) \leq -t\varphi_1(x)$ for $x \in \Omega$.

Next we consider the case where a sign and orthogonality condition introduced by De Figueiredo and Ni is satisfied to the right of λ_1 (cf. [13], [25]).

Proposition 2.9. *Assume that*

$$g(x,s) \geq 0 \quad \textit{for a.e. } x \in \Omega \textit{ and every } s \geq 0.$$

Then, for any given h *satisfying*

$$\int_\Omega h(x)\varphi_1^*(x)\,dx = 0$$

and any given $t > 0$*, there exists a lower solution* α *of* (2.3) *with* $\alpha(x) \geq t\varphi_1(x)$ *for* $x \in \Omega$.

Again a similar condition assumed for $s \leq 0$ yields the existence of an upper solution β with $\beta(x) \leq -t\varphi_1(x)$ for $x \in \Omega$.

Remark 2.10. In the case where $\Gamma_0 = \emptyset$, so that $\mathcal{B}u = \partial u/\partial\eta + b_0 u$, it is sufficient to assume in Proposition 2.9 that the condition $g(x,s) \geq 0$ be satisfied only for s large, i.e. there exists $s_0 \geq 0$ such that $g(x,s) \geq 0$ for a.e. $x \in \Omega$ and all $s \geq s_0$. This is due to the fact that in this case $\min_{x\in\overline{\Omega}} \varphi_1(x) > 0$.

Further, if the first eigenfunction φ_1 of L is constant (which happens if and only if $\Gamma_0 = \emptyset$, a_0 is constant and $b_0 = 0$), then it is not difficult to see that the following criterion holds.

Proposition 2.11. *Assume that* φ_1 *is constant. Suppose that there are sequences* $(b_n)_n$ *and* $(c_n)_n$ *of nonnegative numbers, with* $b_n \to \infty$ *and* $c_n \to \infty$*, such that*

$$g(x,s) \geq 0 \quad \textit{for a.e. } x \in \Omega \textit{ and every } s \in [b_n, b_n + c_n].$$

Then, for any given h *satisfying*

$$\int_\Omega h(x)\,dx = 0,$$

there exists a sequence $(\alpha_n)_n$ *of lower solutions of* (2.3) *with* $\alpha_n(x) \in [b_n, b_n + c_n]$ *for* n *large and* $x \in \Omega$.

A similar condition assumed for $s \leq 0$ yields the existence of a sequence $(\beta_n)_n$ of upper solutions with $\beta_n(x) \in [-b_n - c_n, -b_n]$ for $x \in \Omega$.

Remark 2.12. It is obvious that Proposition 2.11 holds even if $\int_\Omega h(x)\,dx \neq 0$, provided that one assumes

$$g(x,s) \geq \frac{1}{|\Omega|}\int_\Omega h(x)\,dx \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [b_n, b_n + c_n].$$

Actually, a stronger result than Proposition 2.11 can be obtained, allowing the length c_n to go to zero but not too quickly.

Proposition 2.13. *Assume that φ_1 is constant. Let $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ be sequences of nonnegative numbers with $a_n \to \infty$, $b_n \to \infty$ and, for some $q \in]N/2, p[$ ($q = 1$ if $N = 1$), $c_n a_n^{p/q-1} \to \infty$. Suppose that*

$$g(x,s) \geq a_n \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [b_n, b_n + c_n].$$

Then, for any given h, there exists a sequence $(\alpha_n)_n$ of lower solutions of (2.3) *with $\alpha_n(x) \in [b_n, b_n + c_n]$ for n large and $x \in \Omega$.*

Remark 2.14. If $p = \infty$, it is easy to build lower solutions, taking $c_n = 0$ and assuming $\limsup_{s\to\infty} g(x,s) = \infty$ uniformly a.e. in Ω (cf. [20]).

A simple example of a class of functions having linear growth and satisfying the conditions of Proposition 2.13 is given by $g(s) = s|\sin(s^r)|$ with $0 \leq r < 2p/N$.

2.3. Remarks on the localization of the solution. In addition to the existence of a solution u, Theorem 2.2 also provides some information about where u is located; namely, we know that $u \in \overline{\mathcal{S}}$. We show in the proof of Theorem 2.2 that this precisely means that either there exists $x_0 \in \Omega \cup \Gamma_1$ such that $\beta(x_0) \leq u(x_0) \leq \alpha(x_0)$, or $u(x) > \alpha(x)$ for all $x \in \Omega \cup \Gamma_1$ and there exists $x_0 \in \Gamma_0$ such that $(\partial u/\partial\nu)(x_0) = (\partial\alpha/\partial\nu)(x_0)$, or $u(x) < \beta(x)$ for all $x \in \Omega\cup\Gamma_1$ and there exists $x_0 \in \Gamma_0$ such that $(\partial u/\partial\nu)(x_0) = (\partial\beta/\partial\nu)(x_0)$.

It should be observed that in this context one cannot generally guarantee that the solution u satisfies $\beta(x) \leq u(x) \leq \alpha(x)$ for all $x \in \Omega$. This is shown by the following simple example.

Example 2.15. Consider the linear Dirichlet problem

$$-u'' = 3u + \sin 2x \quad \text{in }]0,\pi[, \qquad u(0) = u(\pi) = 0.$$

The assumptions of Theorem 2.2 are satisfied, since $\alpha(x) = \sin x$ and $\beta(x) = -\sin x$ are respectively lower and upper solutions, with $\beta(x) < \alpha(x)$ for $x \in]0,\pi[$. Notice also that $u(x) = \sin 2x$ is the (unique) solution and is such that both $u-\alpha$ and $u - \beta$ change sign in $]0,\pi[$.

However, the information given by Theorem 2.2 is sometimes sufficient to prove the existence of multiple solutions. A typical result in this direction is the following.

Theorem 2.16. *Assume that there exist two lower solutions α_1, α_2 and two upper solutions β_1, β_2 satisfying*

$$\alpha_1(x) \leq \beta_1(x) < \beta_2(x) < \alpha_2(x) \quad \text{for } x \in \Omega\cup\Gamma_1$$

and

$$\frac{\partial\beta_1}{\partial\nu}(x) > \frac{\partial\beta_2}{\partial\nu}(x) \quad \text{for } x \in \Gamma_0.$$

Moreover, suppose that

(f$_4$) *there are functions $a, b \in L^p(\Omega)$ such that*

$$a(x) \leq \liminf_{s \to \infty} f(x,s)/s \leq \limsup_{s \to \infty} f(x,s)/s \leq b(x)$$

uniformly a.e. in Ω.

Then problem (2.1) *has at least two distinct solutions u_1 and u_2 with $u_1(x) \leq u_2(x)$ for all $x \in \Omega$. Moreover, u_1 satisfies*

$$\alpha_1(x) \leq u_1(x) \leq \beta_1(x) \quad \textit{for all } x \in \Omega \cup \Gamma_1$$

and

$$\frac{\partial u_1}{\partial \nu}(x) \geq \frac{\partial \beta_1}{\partial \nu}(x) \quad \textit{for all } x \in \Gamma_0,$$

and u_2 is such that either

$$u_2(x_0) \geq \beta_2(x_0) \quad \textit{for some } x_0 \in \Omega \cup \Gamma_1,$$

or

$$\frac{\partial u_2}{\partial \nu}(x_0) = \frac{\partial \beta_2}{\partial \nu}(x_0) \quad \textit{for some } x_0 \in \Gamma_0.$$

A result similar to Theorem 2.16 holds assuming the existence of two lower solutions α_1, α_2 and of two upper solutions β_1, β_2 such that $\beta_1(x) < \alpha_1(x) < \alpha_2(x) \leq \beta_2(x)$ for $x \in \Omega \cup \Gamma_1$, and $(\partial\alpha_1/\partial\nu)(x) > (\partial\alpha_2/\partial\nu)(x)$ for $x \in \Gamma_0$.

It is worth noticing that in Theorem 2.16 the assumption of the existence of two upper solutions β_1, β_2 can be replaced by the existence of a strict upper solution β, according to the following definition (a strict lower solution α can be defined similarly).

Definition 2.17. An upper solution β is called *strict* if there exists a constant $\delta > 0$ such that either

$$\mathcal{L}\beta(x) \geq f(x, \beta(x) + s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, \delta\varphi_1(x)],$$

or

$$\mathcal{L}\beta(x) \geq f(x, \beta(x) - s) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in [0, \delta\varphi_1(x)].$$

Hence, we obtain the following corollary of Theorem 2.16.

Corollary 2.18. *Assume* (f$_4$) *and suppose that there exist two lower solutions α_1, α_2 and a strict upper solution β satisfying*

$$\alpha_1(x) < \beta(x) < \alpha_2(x) \quad \textit{for all } x \in \Omega \cup \Gamma_1.$$

Then problem (2.1) *has at least two distinct solutions.*

These results naturally apply to the study of the following Ambrosetti–Prodi problem:

$$\mathcal{L}u = \lambda_1 u + g(x,u) - t\varphi_1 \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where t is a real parameter.

Proposition 2.19. *Assume that* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is continuous and*

(g_1) $\lim_{|s|\to\infty} g(x,s) = \infty$ *uniformly a.e. in* Ω*;*

(g_2) *there is a constant* γ *such that* $|g(x,s)| \le \gamma(s+1)$ *for* $x \in \Omega$ *and* $s \ge 0$*;*

(g_3) *there are constants* $K, s_0 > 0$ *such that either for each* $x \in \Omega$*, the function* $g(x,s) + Ks$ *is nondecreasing for* $|s| \le s_0$*, or for each* $x \in \Omega$*,* $g(x,s) - Ks$ *is nonincreasing for* $|s| \le s_0$*.*

Then there exists a real number t_0 *such that problem* (2.4) *has no solution for* $t < t_0$ *and has at least two solutions for* $t > t_0$*.*

Remark 2.20. It is not clear to us if the assumptions of Proposition 2.19 imply, in dimension $N > 1$, the existence of an a priori bound on the solutions of (2.4), when t varies in a compact set. This fact prevents us from establishing the existence of a solution for $t = t_0$. However, with respect to the classical results in this context, we are able to discuss the existence of multiple solutions of (2.4) under the very natural assumption (g_1), which replaces the usual (stronger) condition: there exists a constant $\varepsilon > 0$ such that

$$\liminf_{|s|\to\infty} g(x,s)/s \ge \varepsilon \quad \text{uniformly in } \Omega.$$

From this point of view Proposition 2.19 can be seen as a partial extension to the higher dimensional case, as well as to more general differential operators and boundary conditions, of some results obtained in [7] and [26]. We also point out that condition (g_3) was already used in [6] and [7], and is for instance satisfied if g is of class C^1 with respect to s near 0.

Finally, if we suppose that $\Gamma_0 = \emptyset$, a_0 is constant and $b_0 = 0$, so that one can take $\varphi_1 = 1$, then the following variant of Proposition 2.19 can be proved, where condition (g_1) is considerably relaxed and (g_3) is omitted.

Proposition 2.20. *Assume that* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies*

(g_1') $\limsup_{|s|\to\infty} g(x,s) = \infty$ *uniformly a.e. in* Ω

and (g_2)*. Then there exists* t_0*, with possibly* $t_0 = -\infty$*, such that the Neumann problem*

$$\mathcal{L}u = \lambda_1 u + g(x,u) - t \quad \text{in } \Omega, \qquad \partial u/\partial\eta = 0 \quad \text{on } \partial\Omega,$$

has no solution for $t < t_0$ *and at least two solutions for* $t > t_0$*.*

3. Proofs

Lemma 3.1. *Let* $v \in C^1(\overline{\Omega})$ *be a function such that:*

(i) $v(x) > 0$ *for all* $x \in \Omega \cup \Gamma_1$*,*

(ii) $v(x) = 0$ *and* $(\partial v/\partial\nu)(x) < 0$ *for all* $x \in \Gamma_0$*.*

Then the following holds:

(j) *for any sequence* $(v_n)_n \subset C^1(\overline{\Omega})$ *with* $v_n(x) = 0$ *for all* $x \in \Gamma_0$ *and* $v_n \to v$ *in* $C^1(\overline{\Omega})$, *and for any constant* $c \in]0,1[$, *one has*

$$v_n(x) \geq cv(x) \quad \textit{for all } x \in \Omega \textit{ and } n \textit{ large enough;}$$

(jj) *for any* $w \in C^1(\overline{\Omega})$ *with* $w(x) \leq 0$ *for all* $x \in \Gamma_0$, *there is a constant* $c > 0$ *such that*

$$w(x) \leq cv(x) \quad \textit{for all } x \in \Omega.$$

Proof. First, we observe that there exists a constant $\delta_0 > 0$ such that, for any $\delta \in]0, \delta_0]$, one can find a compact tubular neighbourhood N_δ of Γ_0 in $\overline{\Omega}$ such that the map $(y,t) \mapsto y - t\nu(y)$ is a C^1-homeomorphism from $\Gamma_0 \times [0,\delta]$ onto N_δ (cf. [17]). It is also convenient, for any function $v \in C^1(\overline{\Omega})$, to extend $\partial v/\partial\nu$ by continuity to the whole of N_δ writing

$$\frac{\partial v}{\partial \nu}(x) = (\nabla v(y - t\nu(y)) \mid \nu(y)) \quad \text{if } x = y - t\nu(y).$$

Now we are in a position to prove (j). We know that, taking $\delta \in]0, \delta_0]$ small enough, $\max_{x \in N_\delta}(\partial v/\partial\nu)(x) = -M < 0$. Hence, for any given $c \in]0,1[$ and for all n large enough,

$$\frac{\partial v_n}{\partial \nu}(x) < \frac{\partial v}{\partial \nu}(x) + (1-c)M \leq c\frac{\partial v}{\partial \nu}(x) \quad \text{for all } x \in N_\delta,$$

and then, writing $x = y - t\nu(y)$,

$$v_n(x) - cv(x) = -\frac{\partial(v_n - cv)}{\partial \nu}(y - \tau\nu(y))t \geq 0$$

for some $\tau \in]0, t[$. Hence, we get

$$v_n(x) \geq cv(x) \quad \text{for all } x \in N_\delta.$$

On the other hand, we have $\inf_{x \in \overline{\Omega} \setminus N_\delta} v(x) = m > 0$. Then, for all n large enough,

$$v_n(x) \geq v(x) - (1-c)m \geq cv(x) \quad \text{for all } x \in \overline{\Omega} \setminus N_\delta,$$

and (j) follows.

Next, to prove (jj), we take N_δ in such a way that $\max_{x \in N_\delta}(\partial v/\partial\nu)(x) = -M < 0$ and a constant $c_1 > 0$ such that

$$-c_1 M \leq \min_{x \in N_\delta} \frac{\partial w}{\partial \nu}(x).$$

Let $x = y - t\nu(y) \in N_\delta$. We can write

$$c_1 v(x) - w(x) = c_1 v(y) - w(y) - t\frac{\partial(c_1 v - w)}{\partial \nu}(y - \tau\nu(y)) \geq 0$$

for some $\tau \in]0,t[$, and then $c_1 v(x) \geq w(x)$ for all $x \in N_\delta$. On the other hand, we take a constant $c_2 > 0$ such that

$$c_2 \inf_{x\in\overline{\Omega}\setminus N_\delta} v(x) \geq \sup_{x\in\overline{\Omega}\setminus N_\delta} w(x).$$

Hence, taking $c = \max\{c_1, c_2\}$, we get the conclusion. □

Lemma 3.2 (Strong maximum principle). *Let $\mathcal{L}$, $\mathcal{B}$ and Ω be as in Section 2 and let $v \in W^{2,p}(\Omega)$ be given. Then the following holds.*

(i) (Interior form) *Let $x_0 \in \Omega$ and let B be an open ball centered at x_0 and contained in Ω. If $\mathcal{L}v \geq 0$ in B, $v(x) \geq v(x_0)$ for all $x \in B$ and $v(x_0) \leq 0$, then $v(x) = v(x_0)$ for all $x \in B$.*

(ii) (Boundary form) *Let $x_0 \in \partial\Omega$ and let B be an open ball contained in Ω with $x_0 \in \partial B$. If $\mathcal{L}v \geq 0$ in B, $v(x) > v(x_0)$ for all $x \in B$ and $v(x_0) \leq 0$, then $(\partial v/\partial\zeta)(x_0) < 0$ for each ζ satisfying $(\zeta \mid \nu) > 0$.*

(iii) (Global form) *Let $k \geq 0$ be a constant. If $\mathcal{L}v + kv \geq 0$ in Ω and $\mathcal{B}v \geq 0$ on $\partial\Omega$, then either $v = 0$ in Ω, or $v(x) > 0$ for all $x \in \Omega \cup \Gamma_1$ and $(\partial v/\partial\nu)(x) < 0$ for all $x \in \Gamma_0$.*

Proof. This follows from [19], p. 188 and p. 33, and from [11], p. 49. See also [3], p. 634. □

Proof of Theorem 2.2. This proof is divided in several steps. Throughout we assume that α and β are lower and upper solutions which are not already solutions of problem (2.1).

Step 1. *Definition of a homotopy.* Set

$$\gamma(x,s;\mu) = s - \max\{0, \mu - |s - \alpha(x) - \mu|\} + \max\{0, \mu - |s - \beta(x) + \mu|\}$$

and

$$f(x,s;\mu) = f(x, \gamma(x,s;\mu))$$

for a.e. $x \in \Omega$, every $s \in \mathbb{R}$ and $\mu \in [0,1]$. Define

$$k(x,s) = \begin{cases} f(x,\alpha(x)) + q^1_+(x)(s-\alpha(x)) & \text{if } s \geq \alpha(x), \\ f(x,s) & \text{if } \alpha(x) > s > \beta(x), \\ f(x,\beta(x)) + q^1_-(x)(s-\beta(x)) & \text{if } \beta(x) \geq s, \end{cases}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Consider the homotopy

$$\mathcal{L}u = (1-\mu)f(x,u;\mu) + \mu k(x,u) \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

with $\mu \in [0,1]$.

We recall from Section 2 that the differential operator $\mathcal{L}$ and the boundary operator $\mathcal{B}$ induce a linear homeomorphism $L : W^{2,p}_{\mathcal{B}}(\Omega) \to L^p(\Omega)$, with inverse K. The compact embedding of $W^{2,p}(\Omega)$ into $C^1(\overline{\Omega})$ implies that K :

$L^p(\Omega) \to C^1(\overline{\Omega})$ is compact. Moreover, for each $\mu \in [0,1]$, consider the operator $N_\mu : C^1(\overline{\Omega}) \to L^p(\Omega)$ defined by $N_\mu u = (1-\mu) f(\cdot, u; \mu) + \mu k(\cdot, u)$; N_μ is continuous and maps bounded sets into bounded sets. Hence, for each $\mu \in [0,1]$, the operator $KN_\mu : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is completely continuous and its fixed points are precisely the solutions of (3.1).

Step 2. *Construction of an open set.*

Claim 1. *There exists a constant $C > 0$ such that if u is a solution of* (3.1), *for some $\mu \in [0,1]$, which satisfies $\beta(x_0) \le u(x_0) \le \alpha(x_0)$ for some $x_0 \in \Omega$, then*

$$\|u\|_{C^1} < C.$$

Proof. Assume by contradiction that, for each n, there exists a solution $u = u_n$ of (3.1) for some $\mu = \mu_n \in [0,1]$ such that

$$\beta(x_n) \le u_n(x_n) \le \alpha(x_n) \quad \text{for some } x_n \in \Omega, \tag{3.2}$$

and

$$\|u_n\|_{C^1} \ge n.$$

Define

$$g_+(x,s) = \begin{cases} \max\{a(x), \min\{f(x,s)/s, b(x)\}\} & \text{if } s \ge 1, \\ g_+(x,1) & \text{if } s < 1, \end{cases}$$
$$g_-(x,s) = \begin{cases} \max\{c(x), \min\{f(x,s)/s, d(x)\}\} & \text{if } s \le -1, \\ g_-(x,-1) & \text{if } s > -1 \end{cases}$$

and

$$h(x,s;\mu) = f(x,s;\mu) - g_+(x,s)s^+ + g_-(x,s)s^-$$

for a.e. $x \in \Omega$, every $s \in \mathbb{R}$ and $\mu \in [0,1]$. Notice that

$$g_+(x,s) \in [a(x), b(x)], \quad g_-(x,s) \in [c(x), d(x)],$$

and for any $\varepsilon > 0$, there exists $\gamma_\varepsilon \in L^p(\Omega)$ such that

$$|h(x,s;\mu)| \le \varepsilon |s| + \gamma_\varepsilon(x)$$

for a.e. $x \in \Omega$, every $s \in \mathbb{R}$ and $\mu \in [0,1]$ (see e.g. [24]). In a similar way, we can write

$$k(x,s) = q^1_+(x)s^+ - q^1_-(x)s^- + r(x,s)$$

with, for some $\delta \in L^p(\Omega)$,

$$|r(x,s)| \le \delta(x) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}.$$

Next, setting $v_n = u_n / \|u_n\|_{C^1}$, we have

$$\mathcal{L} v_n = \widehat{q}^{\,n}_+ v_n^+ - \widehat{q}^{\,n}_- v_n^- + \widehat{r}_n \quad \text{in } \Omega, \qquad \mathcal{B} v_n = 0 \quad \text{on } \partial\Omega, \tag{3.3}$$

where, for each n, we put

$$\widehat{q}^{\,n}_{+}(x) = (1-\mu_n)g_+(x,u_n(x)) + \mu_n q^1_+(x) \in [a(x),b(x)],$$
$$\widehat{q}^{\,n}_{-}(x) = (1-\mu_n)g_-(x,u_n(x)) + \mu_n q^1_-(x) \in [c(x),d(x)],$$

and

$$\widehat{r}_n(x) = (1-\mu_n)\frac{h(x,u_n(x);\mu_n)}{\|u_n\|_{C^1}} + \mu_n\frac{r(x,u_n(x))}{\|u_n\|_{C^1}}$$

for a.e. $x \in \Omega$. The functions $\widehat{r}_n$ are such that, for any $\varepsilon > 0$, there exists a function $\vartheta_\varepsilon \in L^p(\Omega)$ (independent of n) such that

$$|\widehat{r}_n(x)| \le \varepsilon + \frac{\vartheta_\varepsilon(x)}{\|u_n\|_{C^1}} \quad \text{for a.e. } x \in \Omega.$$

Rewrite (3.3) in the form

$$Lv_n = \widehat{q}^{\,n}_{+}v_n^+ - \widehat{q}^{\,n}_{-}v_n^- + \widehat{r}_n \tag{3.4}$$

and recall that $L : W^{2,p}_{\mathcal{B}}(\Omega) \to L^p(\Omega)$ is a linear homeomorphism. Since the right hand side of (3.4) is bounded in $L^p(\Omega)$, we see that $(v_n)_n$ is bounded in $W^{2,p}(\Omega)$ and hence, going to a subsequence, $v_n \to v$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$, with $\|v\|_{C^1} = 1$. We can also suppose that $\widehat{q}^{\,n}_{+} \to \widehat{q}_+$ and $\widehat{q}^{\,n}_{-} \to \widehat{q}_-$ weakly in $L^p(\Omega)$, with $\widehat{q}_+(x) \in [a(x),b(x)]$ and $\widehat{q}_-(x) \in [c(x),d(x)]$ for a.e. $x \in \Omega$ (as convex sets are weakly closed in $L^p(\Omega)$). Further, the weak continuity of L implies that v is a nontrivial solution of

$$\mathcal{L}v = \widehat{q}_+v^+ - \widehat{q}_-v^- \quad \text{in } \Omega, \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega.$$

From the admissibility condition (Definition 2.1, part (A)) of $[a,b]\times[c,d]$, we derive that $u = v$ or $u = -v$ satisfies $u(x) > 0$ for $x \in \Omega\cup\Gamma_1$ and $(\partial u/\partial\nu)(x) < 0$ for $x \in \Gamma_0$. Assume for instance that v is positive. By Lemma 3.1 there exist constants $c_1, c_2 > 0$ such that, for n large,

$$v_n(x) \ge c_1 v(x), \quad c_2 v(x) \ge \alpha(x)$$

and

$$u_n(x) = \|u_n\|_{C^1} v_n(x) \ge c_1\|u_n\|_{C^1} v(x) \ge c_2 v(x) \ge \alpha(x)$$

for $x \in \overline{\Omega}$. This contradicts (3.2) and concludes the proof of Claim 1.

Now, we define in the Banach space

$$C^1_{\mathcal{B}}(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) \mid \mathcal{B}u = 0\},$$

endowed with the C^1-norm, the following open bounded set:

$$\mathcal{O} = \{u \in C^1_{\mathcal{B}}(\overline{\Omega}) \mid \|u\|_{C^1} < C \text{ and } \beta(x_0) < u(x_0) < \alpha(x_0) \text{ for some } x_0 \in \Omega\},$$

where C is the constant given in Claim 1.

Claim 2. *Every solution* $u \in \overline{\mathcal{O}}$ *of problem* (3.1), *for* $\mu \in \,]0,1]$, *is such that* $\beta(x_0) < u(x_0) < \alpha(x_0)$ *for some* $x_0 \in \Omega$.

Proof. Let $u \in \overline{\mathcal{O}}$ be a solution of (3.1) for some $\mu \in \,]0,1]$, and suppose by contradiction that for instance

$$u(x) \geq \alpha(x) \quad \text{for all } x \in \Omega. \tag{3.5}$$

Hence, $u \in \partial\mathcal{O}$ and three different cases may occur:

(i) there exists $x_0 \in \Omega$ such that $u(x_0) = \alpha(x_0)$;

(ii) $u(x) > \alpha(x)$ for all $x \in \Omega$ and there exists $x_0 \in \Gamma_1$ such that $u(x_0) = \alpha(x_0)$;

(iii) $u(x) > \alpha(x)$ for all $x \in \Omega \cup \Gamma_1$ and there exists $x_0 \in \Gamma_0$ such that $(\partial u/\partial\nu)(x_0) = (\partial\alpha/\partial\nu)(x_0)$.

Indeed, as $u \in \partial\mathcal{O}$, there is a sequence $(u_n)_n \subset \mathcal{O}$ such that $u_n \to u$ in $C^1(\overline{\Omega})$. Setting $v_n = u_n - \alpha$ and $v = u - \alpha$, we have $v_n \to v$ in $C^1(\overline{\Omega})$, with $v_n(x) = 0$ for $x \in \Gamma_0$, $v(x) \geq 0$ for $x \in \overline{\Omega}$ and $v(x) = 0$ for $x \in \Gamma_0$. Moreover, for each n, there exists $x_n \in \Omega$ such that $v_n(x_n) < 0$. We can assume that, going to a subsequence, $x_n \to x_0 \in \overline{\Omega}$. Clearly, we have $v(x_0) = 0$. Hence, if $x_0 \in \Omega \cup \Gamma_1$, (i) or (ii) follows. Finally, if $v(x) > 0$ for $x \in \Omega \cup \Gamma_1$, it is obvious that $(\partial v/\partial\nu)(x) \leq 0$ for $x \in \Gamma_0$. On the other hand, we cannot have $(\partial v/\partial\nu)(x) < 0$ for all $x \in \Gamma_0$. In fact, in such a case, by Lemma 3.1, if n is large enough, we obtain $v_n(x) \geq \frac{1}{2}v(x) > 0$ for $x \in \Omega$: a contradiction.

Now, using (3.5), we get

$$\begin{aligned} \mathcal{L}(u-\alpha) &\geq (1-\mu)(f(x,u;\mu) - f(x,\alpha)) + \mu(k(x,u) - f(x,\alpha)) \\ &= (1-\mu)(f(x,u;\mu) - f(x,\alpha)) + \mu q_+^1(x)(u-\alpha) \end{aligned}$$

for a.e. $x \in \Omega$. Observe that $u \neq \alpha$. Indeed, otherwise we would have

$$\mathcal{L}\alpha = (1-\mu)f(x,\alpha;\mu) + \mu k(x,\alpha) = f(x,\alpha) \quad \text{in } \Omega, \qquad \mathcal{B}\alpha = 0 \quad \text{on } \partial\Omega,$$

that is, α would be a solution of (2.1).

In case (i), we can find an open ball B centered at x_0, with $B \subset \Omega$, such that

$$\mathcal{L}(u-\alpha)(x) \geq \mu q_+^1(x)(u-\alpha(x)) \geq 0 \quad \text{for a.e. } x \in B. \tag{3.6}$$

Moreover, we can also suppose that there is a point $x_1 \in B$ where $u(x_1) > \alpha(x_1)$. Since $u(x) \geq \alpha(x)$ for $x \in B$ and $u(x_0) = \alpha(x_0)$ the interior form of the strong maximum principle (cf. Lemma 3.2) implies that $u = \alpha$ in B, which is a contradiction.

In case (ii), we can find an open ball B, with $B \subset \Omega$ and $x_0 \in \partial B$, such that (3.6) holds. Since $u(x) > \alpha(x)$ for $x \in B$ and $u(x_0) = \alpha(x_0)$, the boundary form of the strong maximum principle (cf. Lemma 3.2) implies that $(\partial u/\partial\zeta)(x_0) < (\partial\alpha/\partial\zeta)(x_0)$ for any vector ζ satisfying $(\zeta \mid \nu) > 0$. This in particular yields

$$\begin{aligned} 0 = \mathcal{B}u(x_0) &= \frac{\partial u}{\partial \eta}(x_0) + b_0(x_0)u(x_0) \\ &< \frac{\partial \alpha}{\partial \eta}(x_0) + b_0(x_0)\alpha(x_0) = \mathcal{B}\alpha(x_0) \le 0, \end{aligned}$$

which is a contradiction.

In case (iii), we find, arguing as in case (ii), that $(\partial u/\partial\nu)(x_0) < (\partial\alpha/\partial\nu)(x_0)$: a contradiction. This concludes the proof of Claim 2.

Similarly, we can treat the case where, instead of (3.5), one has $u(x) \le \beta(x)$ for all $x \in \Omega$.

Step 3. *Computation of the degree.* Suppose that, for $\mu = 0$, problem (3.1) has no solution on $\partial\mathcal{O}$. From Step 2, it follows that the degree

$$d(I - KN_\mu, \mathcal{O}, 0)$$

is defined and independent of $\mu \in [0,1]$. For $\mu = 1$, problem (3.1) reduces to

$$\mathcal{L}u = k(x,u) \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega. \tag{3.7}$$

Claim 1. *There is no solution u of* (3.7) *such that $u(x) \ge \alpha(x)$ for all $x \in \Omega$.*

Proof. If $u(x) \ge \alpha(x)$ for all $x \in \Omega$, then u satisfies, for a.e. $x \in \Omega$,

$$\mathcal{L}u = f(x,\alpha) + q_+^1(x)(u - \alpha).$$

Hence, we deduce that $u \ne \alpha$ because otherwise α would be a solution of (2.1). Moreover, let $\widehat{\alpha}$ be the solution of

$$\mathcal{L}\widehat{\alpha} = \mathcal{L}\alpha \quad \text{in } \Omega, \qquad \mathcal{B}\widehat{\alpha} = 0 \quad \text{on } \partial\Omega.$$

Since

$$\begin{cases} \mathcal{L}(\widehat{\alpha} - \alpha) = 0 & \text{in } \Omega, \\ \mathcal{B}(\widehat{\alpha} - \alpha) \ge 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}(u - \alpha) \ge 0 & \text{in } \Omega, \\ \mathcal{B}(u - \alpha) \ge 0 & \text{on } \partial\Omega, \end{cases}$$

the global form of the strong maximum principle (cf. Lemma 3.2) implies that $\widehat{\alpha}(x) \ge \alpha(x)$ and $u(x) > \alpha(x)$ for all $x \in \Omega$. Therefore, we have, for a.e. $x \in \Omega$,

$$\begin{aligned} \mathcal{L}(u - \widehat{\alpha}) - \lambda_1(u - \widehat{\alpha}) &= \mathcal{L}(u - \alpha) - \lambda_1(u - \alpha) + \lambda_1(\widehat{\alpha} - \alpha) \\ &\ge \mathcal{L}u - f(x,\alpha) - q_+^1(x)(u - \alpha) + (q_+^1(x) - \lambda_1)(u - \alpha) \\ &= (q_+^1(x) - \lambda_1)(u - \alpha). \end{aligned}$$

Since $(\mathcal{L}-\lambda_1)(u-\widehat{\alpha}) \in \operatorname{Range}(L-\lambda_1 I)$ and $\operatorname{Range}(L-\lambda_1 I) = \operatorname{Ker}(L^*-\lambda_1 I)^\perp$ (as $L-\lambda_1 I$ is a Fredholm operator of index zero), we get

$$0 = \int_\Omega (\mathcal{L}-\lambda_1)(u-\widehat{\alpha})\varphi_1^* \, dx \geq \int_\Omega (q_+^1(x)-\lambda_1)(u-\alpha)\varphi_1^* \, dx \geq 0,$$

which implies $q_+^1 = \lambda_1$. This contradicts condition (iii) in Definition 2.1, part (B), and concludes the proof of Claim 1.

In a similar way we can prove the following.

Claim 2. *There is no solution u of* (3.7) *such that $u(x) \leq \beta(x)$ for $x \in \Omega$.*

As a consequence of the above claims we have

$$d(I-KN_1, \mathcal{O}, 0) = d(I-KN_1, B_R, 0),$$

where B_R is any ball in $C^1_{\mathcal{B}}(\overline{\Omega})$ of center 0 and radius R such that $\mathcal{O} \subset B_R$. Taking into account the admissibility conditions (Definition 2.1, part (B)), we now define the function

$$k(x,s;\mu) = \begin{cases} f(x,\alpha(x)) + q_+^\mu(x)(s-\alpha(x)) & \text{if } s \geq \alpha(x), \\ f(x,s) & \text{if } \alpha(x) > s > \beta(x), \\ f(x,\beta(x)) + q_-^\mu(x)(s-\beta(x)) & \text{if } \beta(x) \geq s, \end{cases}$$

for a.e. $x \in \Omega$, every $s \in \mathbb{R}$ and $\mu \in [0,1]$. Moreover, for each $\mu \in [0,1]$, we define the operator $\widehat{N}_\mu : C^1(\overline{\Omega}) \to L^p(\Omega)$ by $\widehat{N}_\mu(u) = k(\cdot,u;\mu)$. Of course, $\widehat{N}_\mu$ is continuous and maps bounded sets into bounded sets, and $\widehat{N}_1 = N_1$. Next, we consider the homotopy

$$\begin{aligned} &\mathcal{L}u - q_+^\mu u^+ + q_-^\mu u^- = \mu(k(x,u;\mu) - q_+^\mu u^+ + q_-^\mu u^-) \quad \text{in } \Omega, \\ &\mathcal{B}u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.8}$$

for $\mu \in [0,1]$. It is clear from the definition of $k(x,s;\mu)$ that the right hand side of (3.8) is bounded in $L^p(\Omega)$.

Claim 3. *There exists a constant $D > 0$ such that for any $\mu \in [0,1]$ and every solution u of* (3.8), *one has $\|u\|_{C^1} < D$.*

Proof. Assume by contradiction that, for each n, there exist some $\mu = \mu_n \in [0,1]$ and a solution $u = u_n$ of (3.8) such that $\|u_n\|_{C^1} \geq n$. Setting $v_n = u_n/\|u_n\|_{C^1}$, we get

$$\begin{aligned} &\mathcal{L}v_n - q_+^{\mu_n} v_n^+ + q_-^{\mu_n} v_n^- = \mu_n(k(x,u_n;\mu_n) - q_+^{\mu_n} u_n^+ + q_-^{\mu_n} u_n^-)/\|u_n\|_{C^1} \quad \text{in } \Omega, \\ &\mathcal{B}v_n = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

As the right hand side of the equation remains bounded in $L^p(\Omega)$, we deduce that $(v_n)_n$ is bounded in $W^{2,p}(\Omega)$ and therefore, going to a subsequence, $v_n \to v$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$, with $\|v\|_{C^1} = 1$. We can also assume that $\mu_n \to \mu \in [0,1]$ and hence $q_+^{\mu_n} \to q_+^{\mu}$, $q_-^{\mu_n} \to q_-^{\mu}$ in $L^p(\Omega)$ (by Definition 2.1, part (B)). Moreover, v satisfies

$$\mathcal{L}v = q_+^{\mu} v^+ - q_-^{\mu} v^- \quad \text{in } \Omega, \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega.$$

Since by the admissibility conditions this problem has only the trivial solution, we get a contradiction. This concludes the proof of Claim 3.

Now, take any constant $R > D$. We have

$$d(I - K\widehat{N}_0, B_R, 0) = d(I - K\widehat{N}_1, B_R, 0) = d(I - KN_1, B_R, 0).$$

Further, as $I - K\widehat{N}_0$ is linear, bounded and one-to-one,

$$|d(I - K\widehat{N}_0, B_R, 0)| = 1,$$

which implies

$$|d(I - KN_1, B_R, 0)| = 1$$

and therefore

$$|d(I - KN_0, \mathcal{O}, 0)| = 1.$$

Step 4. *Existence of extremal solutions.* The proof of the existence of a minimal solution v and a maximal solution w, with $v, w \in \overline{\mathcal{S}}$, is based on an application of Zorn's Lemma. Define the set

$$\Sigma = \{u \in \overline{\mathcal{S}} \mid u \text{ solves } (2.1)\},$$

which can be (partially) ordered by the usual ordering: $u_1 \le u_2$ if and only if $u_1(x) \le u_2(x)$ for $x \in \overline{\Omega}$. We will show that Σ possesses a maximal element. One should proceed similarly for proving the existence of a minimal element. Let $\mathcal{U} = \{u_i \mid i \in I\}$ be a totally ordered subset of Σ. We want to prove that $\mathcal{U}$ has an upper bound in Σ. Set

$$u(x) = \sup_{i \in I} u_i(x) \quad \text{for } x \in \overline{\Omega}.$$

Note that, by Claim 1 in Step 2, $\mathcal{U}$ is uniformly bounded and equicontinuous. Hence, the function u is well-defined and continuous on $\overline{\Omega}$. Let $\mathcal{D} = \{x_m \mid m \in \mathbb{N}\}$ be a countable dense subset of $\overline{\Omega}$ and define a sequence in $\mathcal{U}$ as follows:

- for $n = 1$, take $u_1 \in \mathcal{U}$ such that $u_1(x_1) \ge u(x_1) - 1$,
- for $n = 2$, take $u_2 \in \mathcal{U}$, with $u_2 \ge u_1$, such that

$$u_2(x_2) \ge u(x_2) - 1/2, \quad u_2(x_1) \ge u(x_1) - 1/2,$$

and so on. In this way, we construct a sequence $(u_n)_n \subset \mathcal{U}$, with $u_1 \le u_2 \le \ldots \le u_n \le u_{n+1} \le \ldots$, such that

$$u_n(x_k) \ge u(x_k) - 1/n \quad \text{for } 1 \le k \le n.$$

It is clear that $(u_n(x_m))_n$ converges to $u(x_m)$ for each $x_m \in \mathcal{D}$, i.e. $(u_n)_n$ converges to u pointwise on $\mathcal{D}$. On the other hand, as $(u_n)_n$ is uniformly bounded and equicontinuous, there exists a subsequence converging uniformly in $\overline{\Omega}$ to some function v. Actually, by monotonicity, the whole sequence $(u_n)_n$ converges to v. Hence $v = u$ on $\mathcal{D}$ and, by continuity, $v = u$ on $\overline{\Omega}$. Moreover, as each u_n is a solution, it follows that $(u_n)_n$ is bounded in $W^{2,p}(\Omega)$ and then there is a subsequence of $(u_n)_n$ which converges weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to some function $w \in W^{2,p}(\Omega)$. Hence, w is a solution of (2.1), with $w \in \overline{\mathcal{S}}$ and $w = u$ in $\overline{\Omega}$. This shows that $\mathcal{U}$ has an upper bound in Σ. Zorn's Lemma yields the conclusion.

This concludes the proof of Theorem 2.2. $\square$

Proof of Proposition 2.3. *Verification of* (A). Let u be a nontrivial solution of

$$\mathcal{L}u = q_+u^+ - q_-u^- \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega.$$

First, we observe that if $v = u$ or $v = -u$ satisfies $v(x) \geq 0$ for all $x \in \Omega$, then the conclusion follows from the global form of the maximum principle (cf. Lemma 3.2). Indeed, we consider the problem

$$\mathcal{L}v + kv = (q + k)v \quad \text{in } \Omega, \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega,$$

where we set $q(x) = q_+(x)$ if $v = u$ or $q(x) = q_-(x)$ if $v = -u$ and k is a constant satisfying

$$k \geq \max\{\|a\|_\infty, \|c\|_\infty\}.$$

Since the right hand side of the equation is nonnegative we get the conclusion.

Therefore, assume by contradiction that, for any given n, there exist $q_+^n, q_-^n \in L^p(\Omega)$ satisfying

$$q_+^n(x) \in [a(x), b(x)], \quad q_-^n(x) \in [c(x), \lambda_1 + 1/n], \quad \text{for a.e. } x \in \Omega,$$

and a solution u_n of the problem

$$\mathcal{L}u_n = q_+^n u_n^+ - q_-^n u_n^- \quad \text{in } \Omega, \qquad \mathcal{B}u_n = 0 \quad \text{on } \partial\Omega,$$

which changes sign in Ω. It is not restrictive also to suppose that $\|u_n\|_{C_1} = 1$. Hence we can assume that, passing to a subsequence, $u_n \to u$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$, with $\|u\|_{C^1} = 1$. Moreover, as each u_n changes sign, we see by Lemma 3.1 that either there exists $x_0 \in \Omega \cup \Gamma_1$ such that $u(x_0) = 0$ or there exists $x_0 \in \Gamma_0$ such that $(\partial u/\partial\nu)(x_0) = 0$. Since $(q_+^n)_n$ and $(q_-^n)_n$ are bounded in $L^p(\Omega)$, we can also suppose that $q_+^n \to q_+$ and $q_-^n \to q_-$ weakly in $L^p(\Omega)$, with

$$q_+(x) \in [a(x), b(x)], \quad q_-(x) \in [c(x), \lambda_1], \quad \text{for a.e. } x \in \Omega.$$

Further, we deduce that u is a nontrivial solution of

$$\mathcal{L}u = q_+u^+ - q_-u^- \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega.$$

If $v = u$ or $v = -u$ satisfies $v(x) \geq 0$ for all $x \in \Omega$, we get immediately a contradiction, by using the maximum principle as before. Therefore, suppose that u changes sign in Ω. By Lemma 3.1 we know that there are constants $t > 0$ such that

$$t\varphi_1(x) + u(x) \geq 0 \quad \text{for } x \in \overline{\Omega}.$$

Clearly, the set of all these constants t admits a minimum, say $\bar{t} > 0$. Define $v = \bar{t}\varphi_1 + u$. Of course, $v(x) \geq 0$ for $x \in \Omega$ and $v \neq 0$. Take a constant k with $k \geq \|a\|_\infty$, and compute

$$\begin{aligned}\mathcal{L}v + kv &= \lambda_1\bar{t}\varphi_1 + q_+u^+ - q_-u^- + k\bar{t}\varphi_1 + ku^+ - ku^- \\ &= \lambda_1(\bar{t}\varphi_1 - u^-) + (\lambda_1 - q_-)u^- + (q_+ + k)u^+ + k(\bar{t}\varphi_1 - u^-).\end{aligned}$$

As the right hand side of this equation is nonnegative, we see that v satisfies

$$\mathcal{L}v + kv \geq 0 \quad \text{in } \Omega, \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega.$$

Then the maximum principle implies that $v(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $(\partial v/\partial\nu)(x) < 0$ for $x \in \Gamma_0$. Lemma 3.1 yields finally the existence of a constant $\varepsilon > 0$ such that $v(x) \geq \varepsilon\varphi_1(x)$ for $x \in \Omega$, thus contradicting the definition of $\bar{t}$.

Verification of (B). Set

$$q(x) = \min\{b(x), d\} \quad \text{for a.e. } x \in \Omega.$$

Clearly, $q \in L^p(\Omega)$ and satisfies $q(x) \geq \lambda_1$ for a.e. $x \in \Omega$, with strict inequality on a subset of Ω of positive measure, and $(q, q) \in [a, b] \times [c, d]$. Then we define $q_+^\mu = q_-^\mu = q$ for all $\mu \in [0, 1]$. In order to prove that condition (iii) holds, we observe that any nontrivial solution u of

$$\mathcal{L}u = qu \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega,$$

is such that $v = u$ or $v = -u$ satisfies $v(x) > 0$ for all $x \in \Omega$. Now, multiply the equation by φ_1^*. Integrating on Ω and recalling $\mathrm{Range}(L - \lambda_1 I) = \mathrm{Ker}(L^* - \lambda_1 I)^\perp$ yields

$$\int_\Omega (q - \lambda_1)u\varphi_1^*\,dx = 0,$$

which is impossible. □

Proof of Proposition 2.4. *Verification of* (A). Let u be a nontrivial solution of

$$\mathcal{L}u = q_+u^+ - q_-u^- \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

As previously observed, it is sufficient to prove that $v = u$ or $v = -u$ satisfies $v(x) \geq 0$ for all $x \in \Omega$. Therefore, suppose by contradiction that u changes sign in Ω. Set $r = (\lambda_- - \lambda_1)/(\lambda_+ - \lambda_1)$ and define $w = u^+ - su^-$ with

$$s = \frac{1}{r}\frac{\int_\Omega u^+\varphi_1\,dx}{\int_\Omega u^-\varphi_1\,dx} > 0.$$

Clearly, we have $w^+ = u^+$, $w^- = su^-$ and

$$\int_\Omega (w^+ - rw^-)\varphi_1\,dx = \int_\Omega (u^+ - rsu^-)\varphi_1\,dx = 0.$$

Hence, by Corollary 2.2 of [12], we get

$$\begin{aligned}\int_\Omega \mathcal{L}u^+u^+\,dx + s^2\int_\Omega \mathcal{L}u^-u^-\,dx &= \int_\Omega \mathcal{L}w^+w^+\,dx + \int_\Omega \mathcal{L}w^-w^-\,dx\\ &\geq \int_\Omega \lambda_+(w^+)^2\,dx + \int_\Omega \lambda_-(w^-)^2\,dx\\ &= \int_\Omega \lambda_+(u^+)^2\,dx + \int_\Omega \lambda_- s^2(u^-)^2\,dx.\end{aligned}$$

Here and in the sequel, it is understood that

$$\int_\Omega \mathcal{L}v^\pm v^\pm\,dx = \sum_{i,j=1}^N \int_\Omega a_{ij}\frac{\partial v^\pm}{\partial x_i}\frac{\partial v^\pm}{\partial x_j}\,dx + \int_\Omega a_0\,(v^\pm)^2\,dx$$

for any $v \in H^1(\Omega)$. On the other hand, from the equation in (3.9), we have

$$\int_\Omega \mathcal{L}u^+u^+\,dx = \int_\Omega q_+(u^+)^2\,dx \quad\text{and}\quad \int_\Omega \mathcal{L}u^-u^-\,dx = \int_\Omega q_-(u^-)^2\,dx.$$

Hence, we obtain

$$\int_\Omega (\lambda_+ - q_+)(u^+)^2\,dx + s^2\int_\Omega (\lambda_- - q_-)(u^-)^2\,dx = 0,$$

which is impossible, because $q_+(x) \leq \lambda_+$, $q_-(x) \leq \lambda_-$ for a.e. $x \in \Omega$, and $q_+(x) < \lambda_+$, $q_-(x) < \lambda_-$ on a common subset of Ω of positive measure.

Verification of (B). Fix a number ϑ such that $\lambda_1 < \vartheta < \min\{\lambda_+, \lambda_-\}$ and set $q_+^\mu = \mu b + (1-\mu)\vartheta$ and $q_-^\mu = \mu d + (1-\mu)\vartheta$ for $\mu \in [0,1]$. Clearly, conditions (i) and (ii) are fulfilled. In order to verify (iii), we suppose by contradiction that the problem

$$\mathcal{L}u = q_+^\mu u^+ - q_-^\mu u^- \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega,$$

has, for some $\mu \in [0,1]$, a nontrivial solution u. Using the argument in part A with $\overline{b}(x) = \max\{b(x),\vartheta\}$ and $\overline{d}(x) = \max\{d(x),\vartheta\}$ for a.e. $x \in \Omega$, we find that $v = u$ or $v = -u$ satisfies $v(x) > 0$ for all $x \in \Omega$. Assume for instance that u is positive. Hence it satisfies

$$\mathcal{L}u - \lambda_1 u = (q_+^\mu - \lambda_1)u \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega.$$

Multiplying the equation by φ_1 and integrating on Ω, we obtain

$$\int_\Omega (q_+^\mu - \lambda_1) u \varphi_1 \, dx = 0,$$

which is impossible since $q_+^\mu(x) \geq \lambda_1$ for a.e. $x \in \Omega$, with strict inequality on a subset of Ω of positive measure. $\square$

Proof of Theorem 2.5. By assumption (f_0) we can find functions $a, b, c, d \in L^\infty(\Omega)$ such that

$$\begin{aligned} a(x) &\leq \liminf_{s\to\infty} f(x,s)/s \leq \limsup_{s\to\infty} f(x,s)/s \leq b(x), \\ c(x) &\leq \liminf_{s\to-\infty} f(x,s)/s \leq \limsup_{s\to-\infty} f(x,s)/s \leq d(x), \end{aligned}$$

uniformly a.e. in Ω, which satisfy $\lambda_1 \leq b(x)$ and $\lambda_1 \leq d(x)$ for a.e. $x \in \Omega$, with strict inequality on subsets of Ω of positive measure. By assumption (f_2), they can be chosen to satisfy

$$b(x) \leq \lambda_+, \quad d(x) \leq \lambda_-$$

for a.e. $x \in \Omega$, and

$$b(x) < \lambda_+, \quad d(x) < \lambda_-$$

on a common subset of Ω of positive measure. Accordingly, by Proposition 2.4 the box $[a, b] \times [c, d]$ is admissible. Then the conclusion follows from Theorem 2.2. $\square$

Proof of Corollary 2.6. By the properties of $\mathcal{F}_2$, we can find another point $(\overline{\lambda}_+, \overline{\lambda}_-) \in \mathcal{F}_2$ and constants $b > \lambda_1$ and $d > \lambda_1$ such that

$$\limsup_{s\to\infty} g(s)/s = b < \overline{\lambda}_+ \quad \text{and} \quad \limsup_{s\to-\infty} g(s)/s = d < \overline{\lambda}_-.$$

Moreover, observe that if $\liminf_{s\to\infty} g(s) = -\infty$, then there exists a constant $B > 0$, with $B \geq \max_{\overline{\Omega}} \alpha$, such that $g(B) \leq h(x)$ for a.e. $x \in \Omega$. Hence, one easily sees that B is an upper solution satisfying $B \geq \alpha(x)$ for $x \in \Omega$. Similarly, if $\limsup_{s\to-\infty} g(s) = \infty$, then there exists a constant $A < 0$, with $A \leq \min_{\overline{\Omega}} \beta$, such that $g(A) \geq h(x)$ for a.e. $x \in \Omega$. Hence, A is a lower solution satisfying $A \leq \beta(x)$ for $x \in \Omega$. In either case, we can conclude the existence of a solution by classical results (see e.g. [1], [15]). Accordingly, we can also suppose that

$$\liminf_{s\to\pm\infty} g(s)/s \geq 0.$$

The conclusion then follows from Theorem 2.5, taking into account that the box $[0, b] \times [0, d]$ is admissible. $\square$

Proof of Proposition 2.13. For each n, define the following excess function:

$$h_n(x) = \begin{cases} h(x) - a'_n & \text{if } h(x) > a'_n, \\ 0 & \text{if } |h(x)| \le a'_n, \\ h(x) + a'_n & \text{if } h(x) < -a'_n, \end{cases}$$

where we set $a'_n = a_n/2$. Clearly, we have $\|h - h_n\|_\infty \le a'_n$. Also, we compute

$$\begin{aligned} \bigg(\int_\Omega |h_n|^q\,dx\bigg)^{1/q} &= \bigg(\int_{\Omega_n} |h - a'_n|^q\,dx\bigg)^{1/q} \\ &\le \bigg(\int_{\Omega_n} |h|^q\,dx\bigg)^{1/q} + \bigg(\int_{\Omega_n} (a'_n)^q\,dx\bigg)^{1/q}, \end{aligned}$$

where $\Omega_n = \{x \in \Omega \mid |h(x)| \ge a'_n\}$. If $p > q$, we can write

$$\bigg(\int_{\Omega_n} |h|^q\,dx\bigg)^{1/q} \le \bigg(\int_{\Omega_n} |h|^q |h/a'_n|^{p-q}\,dx\bigg)^{1/q} \le \|h\|_{L^p}^{p/q}/(a'_n)^{p/q-1}$$

and

$$\bigg(\int_{\Omega_n} (a'_n)^q\,dx\bigg)^{1/q} \le a'_n \bigg(\int_{\Omega_n} |h/a'_n|^p\,dx\bigg)^{1/q} \le \|h\|_{L^p}^{p/q}/(a'_n)^{p/q-1},$$

which finally yields

$$\|h_n\|_q \le 2\|h\|_p^{p/q}/(a'_n)^{p/q-1}.$$

Now, we are in a position to build lower solutions of the type $\alpha_n = w_n + b_n + c_n/2$, where w_n is the solution of

$$\mathcal{L}w_n = -h_n \quad \text{in } \Omega, \qquad \mathcal{B}w_n = 0 \quad \text{on } \partial\Omega.$$

Since $p > N$ and $q \in \,]N/2, p[$ ($q = 1$ if $N = 1$), we obtain

$$\|w_n\|_\infty \le c\|h_n\|_q \le 2\,c\,\|h\|_p^{p/q}/(a'_n)^{p/q-1},$$

where c is a constant independent of n. Recall that as φ_1 is constant, we have $a_0 = \lambda_1$ and $\mathcal{B}u = \partial u/\partial\eta$. Accordingly, for all n large enough, we have $\alpha_n(x) \in [b_n, b_n + c_n]$ for $x \in \Omega$,

$$\begin{aligned} \mathcal{L}\alpha_n &= \mathcal{L}w_n + a_0\,(b_n + c_n/2) \\ &\le -h_n - (h - h_n) + \|h - h_n\|_\infty + \lambda_1\alpha_n + \lambda_1\|w_n\|_\infty + g(x, \alpha_n) - a'_n - a'_n \\ &\le \lambda_1\alpha_n + g(x, \alpha_n) - h \quad \text{in } \Omega \end{aligned}$$

and $\mathcal{B}\alpha_n = 0$ on $\partial\Omega$. □

Proof of Theorem 2.16. The existence of a solution u_1 such that

$$\alpha_1(x) \le u_1(x) \le \beta_1(x) \quad \text{for } x \in \Omega$$

can be established by standard arguments (cf. [1], [15]). It is obvious that the condition $(\partial u_1/\partial\nu)(x) \ge (\partial\beta_1/\partial\nu)(x)$ for $x \in \Gamma_0$ is fulfilled as well.

In order to prove the existence of a second solution u_2, we apply Theorem 2.2 to a modified problem. Define

$$\widetilde{f}(x,s) = \begin{cases} f(x,s) & \text{if } s > u_1(x), \\ f(x,u_1(x)) & \text{if } s \le u_1(x), \end{cases}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. Clearly, $\widetilde{f}$ satisfies

$$\lim_{s\to-\infty} \widetilde{f}(x,s)/s = 0$$

and

$$a(x) \le \liminf_{s\to\infty} \widetilde{f}(x,s)/s \le \limsup_{s\to\infty} \widetilde{f}(x,s)/s \le b(x)$$

uniformly a.e. in Ω. Set $c(x) = 0$ for $x \in \Omega$ and suppose, without loss of generality, that $a(x) \le \lambda_1$ and $b(x) \ge \lambda_1$ for a.e. $x \in \Omega$, and $b(x) > \lambda_1$ on a subset of Ω of positive measure. Hence Proposition 2.3 yields the existence of a constant $d > \lambda_1$ such that the box $[a,b] \times [c,d]$ is admissible. Theorem 2.2 then guarantees the existence of a solution u_2 of the problem

$$\mathcal{L}u_2 = \widetilde{f}(x,u_2) \quad \text{in } \Omega, \qquad \mathcal{B}u_2 = 0 \quad \text{on } \partial\Omega,$$

If we prove that $u_2(x) \ge u_1(x)$ for all $x \in \Omega$, then u_2 will be a solution of problem (2.1) too. Assume by contradiction that $\min_{\overline{\Omega}}(u_2 - u_1) < 0$. We start observing that $\max_{\overline{\Omega}}(u_2 - u_1) > 0$. Indeed, otherwise we would have

$$\mathcal{L}(u_2 - u_1) \ge \widetilde{f}(x,u_2) - \widetilde{f}(x,u_1) = 0 \quad \text{in } \Omega, \qquad \mathcal{B}(u_2 - u_1) \ge 0 \quad \text{on } \partial\Omega,$$

and the global form of the strong maximum principle (cf. Lemma 3.2) would imply $u_2 = u_1$ in Ω. Moreover, it is obvious that $\min_{\overline{\Omega}}(u_2 - u_1) < 0$ cannot be attained on Γ_0. Accordingly, two cases may occur:

(i) there exist a point $x_0 \in \Omega$ and an open ball B, centered at x_0 and contained in Ω, such that $\min_{\overline{\Omega}}(u_2 - u_1) = u_2(x_0) - u_1(x_0) \le u_2(x) - u_1(x) < 0$ for $x \in B$, $u_2 - u_1$ is not constant in B and $\mathcal{L}(u_2 - u_1) \ge 0$ in B, or

(ii) there exist a point $x_0 \in \Gamma_1$ and an open ball B contained in Ω, with $x_0 \in \partial B$, such that $\min_{\overline{\Omega}}(u_2 - u_1) = u_2(x_0) - u_1(x_0) < u_2(x) - u_1(x) < 0$ for $x \in B$, and $\mathcal{L}(u_2 - u_1) \ge 0$ in B.

In case (i), the interior form of the strong maximum principle (cf. Lemma 3.2) implies that $u_2 - u_1$ is constant in B: a contradiction. In case (ii), the boundary form of the strong maximum principle (cf. Lemma 3.2) implies that $(\partial u_2/\partial\zeta)(x_0) < (\partial u_1/\partial\zeta)(x_0)$ for each ζ satisfying $(\zeta \mid \nu) > 0$, and therefore

$$\begin{aligned} 0 = (\mathcal{B}u_2)(x_0) &= \frac{\partial u_2}{\partial\eta}(x_0) + b_0(x_0)\, u_2(x_0) \\ &< \frac{\partial u_1}{\partial\eta}(x_0) + b_0(x_0) u_1(x_0) = (\mathcal{B}u_1)(x_0) \le 0; \end{aligned}$$

a contradiction.

Moreover, according to Theorem 2.2, u_2 is such that either $u_2(x_0) \geq \beta_2(x_0)$ for some $x_0 \in \Omega \cup \Gamma_1$, or $(\partial u_2/\partial\nu)(x_0) = (\partial\beta_2/\partial\nu)(x_0)$ for some $x_0 \in \Gamma_0$. So that we can conclude that u_1 differs from u_2, because either there exists $x_0 \in \Omega \cup \Gamma_1$ such that

$$u_1(x_0) \leq \beta_1(x_0) < \beta_2(x_0) \leq u_2(x_0),$$

or there exists $x_0 \in \Gamma_0$ such that

$$\frac{\partial u_1}{\partial\nu}(x_0) \geq \frac{\partial\beta_1}{\partial\nu}(x_0) > \frac{\partial\beta_2}{\partial\nu}(x_0) = \frac{\partial u_2}{\partial\nu}(x_0). \qquad \square$$

Proof of Corollary 2.18. In order to apply Theorem 2.16 we have to prove that there exist two upper solutions β_1, β_2 satisfying $\alpha_1(x) \leq \beta_1(x) < \beta_2(x) < \alpha_2(x)$ for $x \in \Omega \cup \Gamma_1$ and $(\partial\beta_1/\partial\nu)(x) > (\partial\beta_2/\partial\nu)(x)$ for $x \in \Gamma_0$. Assume that in Definition 2.17 the first alternative holds, i.e.

$$\mathcal{L}\beta(x) \geq f(x, \beta(x) + s) \quad \text{for a.e. } x \in \Omega \text{ and } s \in [0, \delta\varphi_1(x)].$$

One should proceed similarly in the other case. Set $\beta_1 = \beta$ and $\beta_2 = \beta + \varepsilon\varphi_1$, for some $\varepsilon \in]0, \delta]$ to be determined later. It is clear that β_2 is an upper solution and, by the properties of φ_1, that $\beta_1(x) < \beta_2(x)$ for $x \in \Omega \cup \Gamma_1$ and $(\partial\beta_1/\partial\nu)(x) > (\partial\beta_2/\partial\nu)(x)$ for $x \in \Gamma_0$. Hence, we only have to verify that, for some $\varepsilon \in]0, \delta]$,

$$\beta(x) + \varepsilon\varphi_1(x) = \beta_2(x) < \alpha_2(x) \quad \text{for } x \in \Omega \cup \Gamma_1.$$

In order to prove this, observe that $\alpha_2(x) - \beta(x) = 0$ for $x \in \Gamma_0$ and $\alpha_2(x) - \beta(x) > 0$ for $x \in \Omega \cup \Gamma_1$. If we show that

$$\frac{\partial\alpha_2}{\partial\nu}(x) - \frac{\partial\beta}{\partial\nu}(x) < 0 \quad \text{for } x \in \Gamma_0,$$

Lemma 3.1 will imply that

$$\alpha_2(x) - \beta(x) \geq \varepsilon\varphi_1(x) \quad \text{for } x \in \Omega,$$

provided that $\varepsilon > 0$ is chosen small enough. Assume by contradiction that there exists $x_0 \in \Gamma_0$ such that

$$\frac{\partial\alpha_2}{\partial\nu}(x_0) - \frac{\partial\beta}{\partial\nu}(x_0) = 0$$

and set for convenience $w = \alpha_2 - \beta - \delta\varphi_1$, where $\delta > 0$ comes from Definition 2.17. Since $(\partial w/\partial\nu)(x_0) > 0$, we can find an open ball B centered at x_0 such that $(\partial w/\partial\nu)(x) > 0$ for all $x \in B \cap \overline{\Omega}$, where as usual the normal derivative of w has been extended to a tubular neighbourhood of Γ_0 in $\overline{\Omega}$. As $w(x) = 0$ for $x \in \Gamma_0$, we can conclude that $w(x) < 0$ for $x \in B \cap \Omega$, that is, $0 < \alpha_2(x) - \beta(x) < \delta\varphi_1(x)$ for $x \in B \cap \Omega$. Hence, by Definition 2.17, we get

$$\mathcal{L}(\beta - \alpha_2)(x) \geq f(x, \beta(x) + (\alpha_2(x) - \beta(x))) - f(x, \alpha_2(x)) = 0$$

for a.e. $x \in B \cap \Omega$. The boundary form of the strong maximum principle (cf. Lemma 3.2) then implies that $(\partial\beta/\partial\nu)(x_0) < (\partial\alpha_2/\partial\nu)(x_0)$, which is a contradiction. □

PROOF OF PROPOSITION 2.19. First we observe that, by condition (g_1) and Proposition 2.8, for every t and for every $r > 0$ there exists a lower solution $\alpha_2(x) \geq r\varphi_1(x)$ for $x \in \overline{\Omega}$. From a similar argument (see also [27]), we obtain a lower solution $\alpha_1(x) \leq -r\varphi_1(x)$ for $x \in \overline{\Omega}$.

Next, we prove that there exists t^* such that problem (2.4) has an upper solution β for $t = t^*$. Set

$$M = \max\{|\lambda_1 s + g(x,s)| \mid x \in \overline{\Omega} \text{ and } s \in [-1,1]\}.$$

Let C be a compact subset of Ω, to be specified later, and define the function

$$h(x) = \begin{cases} M & \text{for } x \in \Omega \setminus C, \\ 0 & \text{for } x \in C. \end{cases}$$

We will show that the solution β of the problem

$$\mathcal{L}\beta = h \quad \text{in } \Omega, \qquad \mathcal{B}\beta = 0 \quad \text{on } \partial\Omega,$$

is the desired upper solution provided that C is suitably chosen. Indeed, taking $q > N/2$, we have, for some constant $c > 0$,

$$\|\beta\|_\infty \leq c\|h\|_q = cM|\Omega \setminus C|^{1/q}.$$

Hence, if C is large enough in Ω, we get $\beta(x) \in [-1,1]$ for $x \in \Omega$. Then we choose $t^* > 0$ such that $t^*\varphi_1(x) \geq M$ for $x \in C$. Accordingly, we have

$$\begin{aligned} \mathcal{L}\beta(x) &\geq \lambda_1\beta(x) + g(x,\beta(x)) - M + h(x) \\ &\geq \lambda_1\beta(x) + g(x,\beta(x)) - t^*\varphi_1(x) \end{aligned}$$

for $x \in \Omega$, that is, β is an upper solution of (2.4) for $t = t^*$.

Next we show that if $\overline{\beta}$ is an upper solution of (2.4) for some $\overline{t}$, with $\overline{\beta}(x) = 0$ for $x \in \Gamma_0$, then $\overline{\beta}$ is a strict upper solution for each $t > \overline{t}$. We must prove that one of the conditions of Definition 2.17 is fulfilled. To this end we suppose that the second alternative of condition (g_3) holds. In the other case we proceed similarly. Fix $t > \overline{t}$ and compute

$$\begin{aligned} \mathcal{L}\overline{\beta}(x) &\geq \lambda_1\overline{\beta}(x) + g(x,\overline{\beta}(x)) - \overline{t}\varphi_1(x) \\ &= \lambda_1(\overline{\beta}(x) + s) + g(x,\overline{\beta}(x) + s) - t\varphi_1(x) \\ &\quad + g(x,\overline{\beta}(x)) - g(x,\overline{\beta}(x) + s) - \lambda_1 s + (t - \overline{t})\varphi_1(x) \end{aligned}$$

for a.e. $x \in \Omega$ and $s \in \mathbb{R}$. Now take a tubular neighbourhood N of Γ_0 in $\overline{\Omega}$ such that $|\overline{\beta}(x)| \le s_0/2$ for $x \in N$. Then we can find a number $\delta > 0$ so small that, by the properties of φ_1, we have

$$\begin{aligned} &g(x,\overline{\beta}(x)) - g(x,\overline{\beta}(x)+s) - \lambda_1 s + (t-\overline{t})\varphi_1(x) \\ &\qquad \ge (-K\delta - \lambda_1\delta + t - \overline{t})\varphi_1(x) \ge 0 \quad \text{for } x \in N \text{ and } s \in [0,\delta\varphi_1(x)]. \end{aligned}$$

On the other hand, using the continuity of g and the properties of φ_1, we get

$$g(x,\overline{\beta}(x)) - g(x,\overline{\beta}(x)+s) - \lambda_1 s + (t-\overline{t})\varphi_1(x) \ge 0$$

for $x \in \Omega \setminus N$ and $s \in [0,\delta\varphi_1(x)]$. Hence we derive that $\overline{\beta}$ is a strict upper solution according to Definition 2.17.

In order to conclude the proof we set

$$t_0 = \inf\{t \mid \text{problem (2.4) has a solution}\}.$$

Note that t_0 is a real number. Indeed, it follows from the above discussion that (2.4) has at least one solution for each $t \ge t^*$. Hence $t_0 < \infty$. On the other hand, if (2.4) has a solution u for some t, multiplying (2.4) by φ_1^* and integrating we obtain

$$t\int_\Omega \varphi_1\varphi_1^*\,dx = \int_\Omega g(x,u)\varphi_1^*\,dx.$$

Since by (g_1) there exist a constant m such that $g(x,s) \ge m$ for every $x \in \Omega$ and $s \in \mathbb{R}$, we get

$$t \ge m\frac{\int_\Omega \varphi_1^*\,dx}{\int_\Omega \varphi_1\varphi_1^*\,dx},$$

which implies $t_0 > -\infty$.

Finally, for proving that, for every $t > t_0$, problem (2.4) has at least two solutions, we observe that if (2.4) has a solution $\widetilde{u}$ for some $\widetilde{t}$, then $\widetilde{u}$ is a strict upper solution for each $t > \widetilde{t}$ and, for the same t, there exist two lower solutions α_1, α_2 satisfying $\alpha_1(x) < \widetilde{u}(x) < \alpha_2(x)$ for $x \in \Omega \cup \Gamma_1$. These facts, together with assumption (g_2), allow us to apply Corollary 2.18 in order to get the conclusion. □

Proof of Proposition 2.20. It is similar to the proof of Proposition 2.19, taking into account Proposition 2.13 and Remark 2.14. □

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