

REMARKS TO THE VIETORIS THEOREM

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1. The Vietoris theorem

In 1927 L. Vietoris [32] proved his famous theorem stating that a continuous surjective map with fibres acyclic with respect to the homology theory constructed in [32] (based on the notion of true cycles and applicable to metric spaces) induces an isomorphism of the homology groups. Vietoris's homology theory developed into the Čech homology groups of compact spaces and Čech cohomology groups for arbitrary spaces. The Vietoris theorem evolved as well and the following remarkable result was established (see [2], [3]).

THEOREM 1.1 (Begle, 1950). *Let X, Y be paracompact spaces, G be an abelian group and let $f : X \rightarrow Y$ be a continuous closed surjection. If there is an integer $N \geq 0$ such that, for each $0 \leq k < N$ and $y \in Y$, $\check{H}^k(f^{-1}(y); G) = \check{H}^k(*; G)$, then the homomorphism $f^* = \check{H}^k(f) : \check{H}^k(Y; G) \rightarrow \check{H}^k(X; G)$, induced by f , is*

- (i) *an isomorphism if $0 \leq k \leq N - 1$;*
- (ii) *a monomorphism if $0 \leq k \leq N$.*

Above $\check{H}^*(\cdot; G)$ stands for the Čech cohomology theory with coefficients in the group G and $*$ is a one-point space.

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Perhaps the most elementary proof of this result was given by J. D. Lawson [19] in 1973. He showed that it is a simple consequence of his main theorem stating that any two taut cohomology theories on a paracompact space coinciding on points are isomorphic.

In 1964 Sklyarenko [27] obtained a significant generalization of the Vietoris–Begle theorem. Let again $f : X \rightarrow Y$ be a closed surjection and let \mathcal{A} be a sheaf of abelian groups over Y . For an integer $k \geq 0$, define

$$\begin{aligned} s^0(f; \mathcal{A}) &:= \{y \in Y \mid H^0(f^{-1}(y); \mathcal{A}^*) \neq \mathcal{A}_y\}, \\ s^k(f; \mathcal{A}) &:= \{y \in Y \mid H^k(f^{-1}(y); \mathcal{A}^*) \neq 0\}^1 \end{aligned}$$

($H^*(\cdot; \mathcal{A})$ denotes the cohomology theory with coefficients in the sheaf \mathcal{A}) and, for an integer $N \geq 1$, let

$$i^N(f; \mathcal{A}) := \inf\{n \geq 0 \mid \max_{0 \leq k \leq N-1} \{\text{rd}_Y(s^k(f; \mathcal{A})) + k\} + 1 < n\}.$$
²

If there is no $n \geq 0$ such that $\text{rd}_Y(s^k(f; \mathcal{A})) + k + 1 < n$ for $0 \leq k \leq N - 1$, then we put $i^N(f; \mathcal{A}) = \infty$. Additionally, let

$$i^0(f; \mathcal{A}) = -\infty, \quad i(f; \mathcal{A}) = \sup_{N \geq 0} i^N(f; \mathcal{A}).$$

THEOREM 1.2 (Sklyarenko, 1964). *If there is $N \geq 0$ such that $i^N(f; \mathcal{A}) \leq N$, then, for $q \geq 0$,*

$$f^* = \check{H}^q(f) : H^q(Y; \mathcal{A}) \rightarrow H^q(X; \mathcal{A}^*)$$

is an epimorphism if $q = i^N(f; \mathcal{A}) - 1$, an isomorphism if $i^N(f; \mathcal{A}) \leq q \leq N - 1$ and a monomorphism if $q = N$. If $i(f; \mathcal{A}) < \infty$, then f^ is an epimorphism for $q = i(f; \mathcal{A}) - 1$ and an isomorphism for $q \geq i(f; \mathcal{A})$.*

REMARK 1.3. (i) In particular, suppose that B is a closed subspace of Y , $A = f^{-1}(B)$ and G is an abelian group; if we let the sheaf \mathcal{A} be constant and equal to G over $Y \setminus B$ and 0 over other points of Y , then $H^q(X; \mathcal{A}^*) = \check{H}^q(X, A; G)$ and $H^q(Y; \mathcal{A}) = \check{H}^q(Y, B; G)$ for any $q \geq 0$. Thus if, for some $N \geq 0$, $m = \inf\{n \mid \max_{0 \leq k \leq N-1} \{\text{rd}_Y(s) + k\} + 1 < n\} \leq N$, where $s = \{y \in Y \setminus B \mid \check{H}^k(f^{-1}(y); G) \neq \check{H}^k(*; G)\}$, then $f^* : \check{H}^q(Y, B; G) \rightarrow \check{H}^q(X, A; G)$ is an epimorphism for $m - 1 \leq q \leq N - 1$ and a monomorphism for $m \leq q \leq N$. If, for $0 \leq k \leq N - 1$, the sets $\{y \in Y \setminus B \mid \check{H}^k(f^{-1}(y); G) \neq \check{H}^k(*; G)\}$ are empty, then above $m = 0$ and we obtain a relative version of the Vietoris–Begle theorem.

¹ \mathcal{A}_y denotes the fibre of \mathcal{A} over $y \in Y$. A sheaf over the whole space and those induced by it over subspaces are denoted by the same letter. By \mathcal{A}^* we denote the inverse image of \mathcal{A} under f .

²For $A \subset Y$, $\text{rd}_Y(A) := \sup\{\dim C \mid C \text{ is closed in } Y, C \subset A\}$. Moreover, we let $\text{rd}_Y(\emptyset) = -\infty$.

(ii) Observe that the standard way of obtaining a relative version of the Vietoris–Begle theorem for groups $\check{H}^*(\cdot, \cdot; G)$ via exact cohomological sequences and the five-lemma would give a worse result under stronger assumptions. However, recalling that

$$\check{H}^*(X, A; G) = \check{H}^*(X/A, a_0; G), \quad \check{H}^*(Y, B; G) = \check{H}^*(Y/B, b_0; G)$$

and $f^* = (f_A : (X/A, a_0) \rightarrow (Y/B, b_0))^*$, we shall also get the result stated in (i) directly applying Theorem 1.2 and taking a sheaf over Y/B constant and equal to G .

(iii) A. Białynicki-Birula [4] generalized the Vietoris–Begle theorem in another direction. He considers three spaces X , Y and T and closed surjections $f : X \rightarrow Y$, $g : Y \rightarrow T$ and $h = g \circ f$ and shows that the assertion of this theorem holds provided, for each $t \in T$, f induces an isomorphism $\check{H}^q(g^{-1}(t); G) \rightarrow \check{H}^q(h^{-1}(t); G)$ for $0 \leq q \leq N - 1$ and a monomorphism for $q = N$. Sklyarenko [27] extends this result in a similar manner as before.

Let us now collect several properties of the above defined number $i^N(f; \mathcal{A})$ where $N \geq 0$ and \mathcal{A} is an arbitrary sheaf.

PROPOSITION 1.4.

- (i) *The sequence $N \mapsto i^N(f; \mathcal{A})$ is nondecreasing. If $\dim < \infty$, then for each $N \geq \dim X + 1$,*

$$i^N(f; \mathcal{A}) = i^{\dim X + 1}(f; \mathcal{A}) = i(f; \mathcal{A}).$$

If $\dim X, \dim Y < \infty$, then for each $N \geq 0$, $i^N(f; \mathcal{A}) \leq \dim Y + \min\{\dim X, N - 1\} + 2$.

- (ii) *If $f_B = f|_A : A \rightarrow B$ (recall that $A = f^{-1}(B)$), then $i^N(f_B; \mathcal{A}) \leq i^N(f; \mathcal{A})$ for any $N \geq 0$.*
- (iii) *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow Z$ be closed surjections. If, for some $N \geq 1$, $i^N(f_1; \mathcal{A}) = 0$, then $i^N(f_2 \circ f_1; \mathcal{A}) = i^N(f_2; \mathcal{A})$.*
- (iv) *Let \mathcal{A}_B be the sheaf induced by \mathcal{A} equal to \mathcal{A} over $Y \setminus B$ and 0 elsewhere. Then $i^N(f; \mathcal{A}_B) \leq i^N(f; \mathcal{A})$.*

In the sequel if the sheaf \mathcal{A} is constant and equal to G (resp. $G = \mathbb{Z}$), then in the above notation we write $\check{H}^*(\cdot; G), s^*(\cdot; G), i^*(\cdot; G)$ (resp. $\check{H}^*(\cdot), s^*(\cdot)$ and $i^*(\cdot)$) unless it leads to an ambiguity.

Let us derive a simple corollary from the Sklyarenko theorem. As before by \mathbb{Z}_B we denote the sheaf constantly equal to \mathbb{Z} over an open set $Y \setminus B$ and 0 on B .

COROLLARY 1.5. *Let $f : X \rightarrow Y$ be a perfect³ surjection such that $A = f^{-1}(B)$, where $B \subset Y$ is closed, and suppose that, for some $N \geq 0$, $i^N(f; \mathbb{Z}_B)$*

³That is, closed with compact fibres.

$< N$. If the space Y is compact or the group G is finitely generated, then for each $q \geq 0$, the induced homomorphism

$$f^* : \check{H}^q(Y, B; G) \rightarrow \check{H}^q(X, A; G)$$

is an epimorphism for $q = i^N(f; \mathbb{Z}_B) - 1$, an isomorphism for $i^N(f; \mathbb{Z}_B) \leq q \leq N - 2$ and a monomorphism for $q = N - 1$. If $i(f; \mathbb{Z}_B) < \infty$, then f^* is an epimorphism for $q = i(f; \mathbb{Z}_B) - 1$ and an isomorphism for $q \geq i(f; \mathbb{Z}_B)$.

PROOF. The assertion holds for $G = \mathbb{Z}$. Since the universal coefficient sequence for the Čech cohomology is exact whenever Y is compact (hence X is also compact because f is proper) or the group G is finitely generated (see [30]), the proof is complete. \square

2. The Vietoris theorem and spectra

Since it is well-known that the Čech cohomology $\check{H}^*(\cdot; G)$ corresponds to the spectrum $\mathbf{K}(G) = \{K(G, n)\}_{n=-\infty}^\infty$ ⁴ (that is, $\check{H}^n(X; G) = \mathbf{K}(G)^n(X)$), the following question arises:

QUESTION 2.1. *Is there a version of the Vietoris–Begle theorem for spectral cohomologies?*

Recall that if $\mathbf{E} = \{E_n, e_n\}_{n=-\infty}^\infty$ is a CW-spectrum (i.e. for each $n \in \mathbb{Z}$, E_n and $e_n : SE_n \rightarrow E_{n+1}$ belong to the category of pointed CW-complexes), then for a pointed CW-complex X and an integer n , one defines an abelian group $\mathbf{E}^n(X)$, a suspension isomorphism $\sigma_n : \mathbf{E}^{n+1}(SX) \rightarrow \mathbf{E}^n(X)$ ⁵ and shows that $\mathbf{E}^*(\cdot) = \{\mathbf{E}^n(\cdot), \sigma^n(\cdot)\}_{n=-\infty}^\infty$ is an (extraordinary reduced) cohomology theory on the category of pointed CW-complexes (see [31]). As usual the cofunctor $\mathbf{E}^*(\cdot)$ may be extended to the (reduced) cohomology cofunctor $\mathbf{E}^*(\cdot)$ on the category of all CW-complexes.

Moreover, the groups $\mathbf{E}^n(X)$, X being any (pointed) topological space, are defined as $\varinjlim_{\alpha \in \Lambda} \{\mathbf{E}^n(X_\alpha), p_{\alpha\beta}^*, \Lambda\}$, where $\{X_\alpha, p_{\alpha\beta}, \alpha \in \Lambda\}$ is the Čech system of the space X (see [25]), i.e. one considers the Čech extension of the cofunctor $\mathbf{E}^*(\cdot)$ from the category of (pointed) CW-complexes to the category of all (pointed) topological spaces.

A positive answer to Question 2.1 is given by the following result (see [11]).

THEOREM 2.2 (Dydak–Kozłowski, 1991). *Let \mathbf{E} be an arbitrary spectrum and let $f : X \rightarrow Y$ be a closed surjection of paracompact spaces. If the Brouwer–Čech dimension $\text{Ind } Y = N_Y < \infty$ and, for some integer m ,*

$$(1) \quad f^* : \mathbf{E}^m(\{y\}) \rightarrow \mathbf{E}^m(f^{-1}(y))$$

⁴For $n \geq 1$, $K(G, n)$ is an Eilenberg–MacLane space of (G, n) -type and, for $n \leq 0$, $K(G, n) = \{*\}$.

⁵For $k \geq 1$, $S^k X$ denotes the k th (reduced) suspension of X .

is an isomorphism for each $m \leq n \leq m + N_Y$ and $y \in Y$, then

$$f^* : \mathbf{E}^q(Y) \rightarrow \mathbf{E}^q(X)$$

is an isomorphism for $q = m + N_Y$ and a monomorphism for $q = m + N_Y + 1$.

In [11] an example is provided showing that if instead of (1) one assumes that

$$(2) \quad \mathbf{E}^n(f^{-1}(y)) \cong \mathbf{E}^n(\{y\}),$$

then the assertion does not hold.

However, if $\mathbf{E} = \mathbf{K}(G)$, then clearly assumptions (1) and (2) are equivalent. Hence in this case the Dydak–Kozłowski theorem constitutes a version of the classical Vietoris–Begle theorem. Indeed, suppose that $m \geq 1$ and put $N = m + N_Y + 1$. Clearly, if $0 \leq k \leq m - 1$, then $\text{rd}_Y(s^k(f; G)) \leq N_Y$ and, for $m \leq k \leq N - 1$, $s^k(f; G) = \emptyset$. Hence $i^N(f; G) \leq N$. The Sklyarenko theorem states that $f^* : \check{H}^q(Y; G) \rightarrow \check{H}^q(X; G)$ is (at least) an epimorphism for $q = N - 1$ and a monomorphism for $q = N$. The Dydak–Kozłowski result gives additionally that for $q = N - 1$ it is also an isomorphism. If $m \leq 0$, then again putting $N = m + N_Y + 1$, we see that $i^N(f; G) = 0$. Looking carefully at the Dydak–Kozłowski result, we observe that in this case its assertion gives the same information as the Sklyarenko theorem or the Vietoris–Begle theorem:

$f^* : \check{H}^q(Y; G) \rightarrow \check{H}^q(X; G)$ is an isomorphism for any $0 \leq q \leq N - 1$ and a monomorphism for $q = N$.

Let \mathbf{E} be an Ω -spectrum. This means that the map $e'_n : E_n \rightarrow \Omega E_{n+1}$ dual to e_n (i.e. given by the formula $e'_n(x)(s) = e_n[s, x]$ for $x \in E_n$ and $s \in S^1$) is a weak homotopy equivalence. Hence, in particular, for any $k \geq 0$ and $n \in \mathbb{Z}$, we have $\pi_k(E_n) \cong \pi_k(\Omega E_{n+1}) \cong \pi_{k+1}(E_{n+1})$. For instance, $\mathbf{K}(G)$ is an Ω -spectrum. A basic result is that in this case $\mathbf{E}^n(X) \cong [X; E_n]$. In this case Theorem 2.2 has a particularly nice form: it states that f induces a bijective correspondence

$$f^\# : [Y; E_n] \rightarrow [X; E_n], \quad n = m + N_Y,$$

between the respective sets of homotopy classes. It therefore might be viewed as a *homotopy* version of the Vietoris theorem.

However, in the next sections we shall be interested in homotopy properties of f stated in terms of the behaviour of $f^\# : [Y; S^n] \rightarrow [X; S^n]$. Unfortunately, the spherical spectrum $\mathbf{S} = \{S^n\}_{n=-\infty}^\infty$ ⁶ is not an Ω -spectrum. Such spectra give rise to a type of stable cohomology theories; hence the result of Dydak–Kozłowski should be viewed rather as a result of a *stable* character.

⁶For $n < 0$, we put $S^n = \{*\}$.

To see that better, let $\mathbf{E} = \mathbf{S}$. Clearly, for any paracompact space X and $n \in \mathbb{Z}$,

$$\mathbf{S}^n(X) = \varinjlim_{k \geq 0} [S^k X; S^{n+k}]$$

is, by definition, the n th stable cohomotopy group $\pi_s^n(X)$ of X .

If, for instance, X and Y are paracompact spaces, $\dim Y \leq \text{Ind } Y = N_Y < \infty$, $\dim X \leq N_X \leq \infty$ and $f : X \rightarrow Y$ is a closed surjection such that, for an integer m ,

$$\forall m \leq q \leq m + N_Y \quad \forall y \in Y \quad \pi_s^q(f^{-1}(y)) = \pi_s^q(\{y\}),$$

then $f^* : \pi_s^n(Y) \cong \pi_s^n(X)$ for $n = m + N_Y$. By the suspension theorem (see [30, Theorem 8.5.11]), $\pi_s^n(X) \cong [S^k X; S^{n+k}]$ and $\pi_s^n(Y) \cong [S^l Y; S^{n+l}]$ where k, l are such that $2n + k \geq N_X + 2$ and $2n + l \geq N_Y + 2$. In particular, if $n \geq (\max\{N_X, N_Y\} - k)/2 + 1$ for some $k \geq 0$, then $(S^k f)^\# : [S^k Y; S^{n+k}] \rightarrow [S^k X; S^{n+k}]$ ($n = m + N_Y$) is a bijection. This is perhaps the best we can achieve with regard to the homotopy behaviour of f under the above assumptions.

A natural question arises:

QUESTION 2.3. *What assumptions give more information on the transformation $f^\# : [Y; S^n] \rightarrow [X; S^n]$, induced by f ; when n is in the unstable area, i.e. $n < \max\{N_X, N_Y\}/2 + 1$ or, more generally, on $(S^k f)^\# : [S^k Y; S^{n+k}] \rightarrow [S^k X; S^{n+k}]$ when $\max\{N_X, N_Y\} > 2n + k - 2$?*

Some light onto these questions is shed by the following result, which is a generalization of a theorem proved by the author [17].

THEOREM 2.4. *Let X, Y be paracompact spaces, $\dim X = N_X$, $\dim Y = N_Y$, and let P be a compact (metric) k -connected ANR, $k \geq 1$. If $f : X \rightarrow Y$ is a closed surjection such that, for $N \geq N_X + N_Y + 2$, $i^N(f) < N$, then the transformation*

$$f^\# : [Y; P] \rightarrow [X; P]$$

induced by f is a surjection if $i^N(f) - 2 \leq k$ and a bijection if $i^N(f) - 1 \leq k$.

The proof of a further extension of this result will be given in the next section.

Let us note the following

COROLLARY 2.5. *Assume that $f : X \rightarrow Y$ is as above and let $n \geq 2$. For any $k \geq 0$, the transformation*

$$(S^k f)^\# : [S^k Y; S^{n+k}] \rightarrow [S^k X; S^{n+k}]$$

is a surjection if $i^N(f) - 1 \leq n$ and a bijection if $i^N(f) \leq n$.

PROOF. If $k = 0$, then it is enough to observe that S^n is $(n - 1)$ -connected and invoke Theorem 2.3 putting $P = S^n$. The case $k \geq 1$ will be treated in the next section. □

3. (Co)homotopy version of the Vietoris–Sklyarenko theorem

As above all *spaces* are supposed to be paracompact and a *pair* (X, A) consists of a space X and its *closed* subset A ; a pair (X, A) is a (*complete*) ANR-*pair* if X and A are (complete) metric absolute neighbourhood retracts.

DEFINITION 3.1. We say that a topological space X is of (n, m) -*type*, $1 \leq n, m \leq \infty$, if it is path connected and $\pi_i(X) = 0$ for each $i \leq n - 1$ or $i \geq m + 1$. More generally, a pair (X, A) is of (n, m) -*type* if it is 1-connected and $\pi_i(X, A) = 0$ if $2 \leq i \leq n - 1$ or $i \geq m + 1$.

Clearly, from the definition it follows that if X (resp. (X, A)) is of (n, m) -type and is not ∞ -connected, then $m \geq n$ and $n < \infty$. Obviously X (resp. (X, A)) is of (n, ∞) -type if and only if X (resp. (X, A)) is $(n - 1)$ -connected.

In the rest of this section we assume that pairs (X, X') , (Y, Y') and a *perfect surjection*

$$f : (X, X') \rightarrow (Y, Y') \text{ such that } X' = f^{-1}(Y')$$

are given.

Moreover, we make

ASSUMPTION 3.2.

- (i) (P, P') is a paracompact pair having the homotopy type of a complete ANR-pair;
- (ii) P is of (n, m) -type, $1 \leq n, m \leq \infty$;
- (iii) (P, P') is of $(n + 1, m + 1)$ -type;
- (iv) P (resp. (P, P')) is homotopically simple, that is, i -simple for any $i \geq 1$ (resp. $i \geq 2$);
- (v) if Y (resp. Y') is not compact, we assume that $\pi_i(P)$ (resp. $\pi_i(P')$) is finitely generated for any $i \geq 1$.

REMARK 3.3. (i) Condition 3.2(i) holds for instance if (P, P') is a CW-pair (each CW-pair has the homotopy type of a polyhedral pair which is homotopy equivalent to itself endowed with the metric topology (see [20]); a simplicial pair with the metric topology is homotopy equivalent to its telescope which is a complete ANR-pair (see [10]); or an ANR-pair (homotopy dominated by a CW-pair, hence having the homotopy type of a CW-pair (see [20])).

(ii) Observe that if conditions 3.2(i)–(iii) hold, and the space P (resp. P') is simply connected and has the homotopy type of a compact ANR whenever Y (resp. Y') is not compact, then all the above assumptions are satisfied. Indeed, if Y (resp. Y') is not compact, then P (resp. P') has the homotopy type of a simply connected compact polyhedron (see [34]). Hence, by the generalized Hurewicz theorem [30, Chapter 9.6, Corollary 16], $\pi_i(P)$ (resp. $\pi_i(P')$) is a finitely generated abelian group for any $i \geq 2$.

Let us now state the main results of this section.

THEOREM 3.4. *Suppose that, for some $N \geq 1$, we have $i^N(f) < N$.*

- (i) (Case $m = \infty$) *If $N \geq \dim X + \dim Y + 2$, then the transformations $f^\# : [Y, Y'; P, P'] \rightarrow [X, X'; P, P']$ and $f^\# : [Y; P] \rightarrow [X; P]$ induced by f are:*
1. *surjective if $n \geq i^N(f) - 1$;*
 2. *bijection if $n \geq i^N(f)$ (see also Remark 3.7 below).*
- (ii) (Case $m < \infty$) *The transformations $f^\# : [Y, Y'; P, P'] \rightarrow [X, X'; P, P']$ and $f^\# : [Y, P] \rightarrow [X, P]$ are:*
1. *surjective if $i^N(f) - 1 \leq n$ and $m \leq N - 2$;*
 2. *injective if $i^N(f) \leq n$ and $m \leq N - 1$.*

Consequently, $f^\#$ is a bijection if $i^N(f) \leq n$ and $m \leq N - 2$.

THEOREM 3.5. *Suppose that there is $N \geq 1$ such that $i^N(f; \mathcal{A}) < N$, where $\mathcal{A} = \mathbb{Z}_Y$,⁷ and let p be an arbitrary point in P .*

- (i) (Case $m = \infty$) *If $N \geq \dim X + \dim Y + 2$, then $f^\# : [Y, Y'; P, p] \rightarrow [X, X'; P, p]$ is:*
1. *a surjection for $n \geq i^N(f) - 1$;*
 2. *a bijection for $n \geq i^N(f)$.*
- (ii) (Case $m < \infty$) *The transformation $f^\#$ is:*
1. *a surjection if $i^N(f) - 1 \leq n$ and $m \leq N - 2$;*
 2. *an injection if $i^N(f) \leq n$ and $m \leq N - 1$.*

Consequently, if $i^N(f) \leq n$ and $m \leq N - 2$, then $f^\#$ is a bijection⁸.

The reader will easily see the analogies of Theorems 3.4, 3.5 and Corollary 1.5. On the other hand, if either Y is compact or the abelian group G is finitely generated, $n \geq 1$ and P is the Eilenberg–MacLane complex $K(G, n)$ (which is a space of (n, n) -type and—as a CW-complex—homotopy equivalent to some complete ANR), then $\check{H}^n(X, X'; G) = [X, X'; P, *]$, $\check{H}^n(Y, Y'; G) = [Y, Y'; P, *]$ where $* \in P$, and, by 3.5, $f^* : \check{H}^n(Y, Y'; G) \rightarrow \check{H}^n(X, X'; G)$ is an epimorphism for $i^N(f) - 1 \leq n \leq N - 2$ and a monomorphism for $i^N(f) \leq n \leq N - 1$. Therefore Theorems 3.4 and 3.5 constitute sort of generalizations of Corollary 1.5. These results correspond to an unpublished result of Kozłowski (see [33, Appendix B, p. 117]) and to results stated in another paper by Dydak and Kozłowski (see [10, Corollary 1]); cf. also [16].

Before we give the proofs of Theorems 3.4 and 3.5, let us consider the following examples showing the nature of Assumptions 3.2 and those of Theorems 3.4 or 3.5.

⁷Recall that \mathbb{Z}_Y denotes the sheaf constantly equal to \mathbb{Z} over $Y \setminus Y'$ and trivial over Y' .

⁸Observe that 3.5 does not follow directly from 3.4. If we put $P' = \{p\}$, then (P, P') is not of $(n + 1, m + 1)$ -type.

EXAMPLE 3.6. (i) Kahn [15] gives an example of an acyclic compact metric space X , $\dim X = \infty$ with admitting an essential map $g : X \rightarrow P = S^3$. Taking $f : X \rightarrow Y = *$ we see that $f^\# : [Y; P] \rightarrow [X; P]$ is not surjective.

(ii) Dranishnikov [6] gives an example of a compact metric space Y' such that $\dim Y' = \infty$ but the (integral) *cohomological dimension* $c\text{-dim } Y'$ is 3. By the theorem of Edwards (see [33, Section 6, p. 113]), there is a compact metric space X' with $\dim X' = 3$ and a *cell-like* map⁹ $p' : X' \rightarrow Y'$. Clearly, for any $N \geq 1$, $i^N(f') = 0$. There is a closed subspace $Y \subset Y'$ with $\dim Y = \infty$ and $[Y; S^4] \neq 0$ but $[X; S^4] = 0$ where $X = f'^{-1}(Y)$. Hence $f^\#$, where $f = f'|_X : X \rightarrow Y$, is not injective (recall that, by Proposition 1.4, $i(f) = 0$).

(iii) Let Σ be the Alexander horned sphere in S^3 . By the Alexander duality (see e.g. [30]), $S^3 \setminus \Sigma$ has two components A, B each with (singular) homology of a point. Take the component A having a nontrivial fundamental group and let $X = \Sigma \cup A$. Then $\check{H}^q(X) = H_{3-q}(S^3, B)$ for any $q \geq 0$; hence X is acyclic. Since Σ is a neighbourhood retract of S^3 , one verifies easily that $P = X$ is an ANR. Now, let $f : X \rightarrow Y = *$. Clearly $[X; P] \neq 0$; hence $f^\#$ is not surjective (observe that $\pi_1(P)$ is not abelian, so P is not homotopically simple).

(iv) Let $f : S^3 \rightarrow S^2$ be the Hopf fibration. Evidently, for any $N \geq 1$, $i^N(f) = 5$, but for $P = S^3$ neither assertion of Theorem 3.4 holds.

Our approach to the proof of the above results is direct, classical and essentially based on obstruction theory (see e.g. [14, Chapter VI]). In order to proceed further we recall some rather well-known notions and introduce the necessary notation.

The family of all locally finite open coverings of a space X is denoted by $\Omega(X)$. For a pair (X, A) and a covering $\mathfrak{A} \in \Omega(X)$, let $(X_{\mathfrak{A}}, A_{\mathfrak{A}})$ be a polyhedral pair where $X_{\mathfrak{A}}$ (resp. $A_{\mathfrak{A}}$) is the space of the nerve of \mathfrak{A} (resp. of the covering $\mathfrak{A}|_A = \{U \cap A \mid U \in \mathfrak{A}\}$) endowed with the weak topology. (Observe that the family $\{(\mathfrak{A}, \mathfrak{A}|_A) \mid \mathfrak{A} \in \Omega(X)\}$ is cofinal in the family of all open coverings of the pair (X, A) directed by the usual relation “ \preceq ” of refinement).

Let $f : (X, A) \rightarrow (Y, B)$ (resp. $g : A \rightarrow Y$). After [14, Chapter II, Ex. B] we say that a covering $\mathfrak{A} \in \Omega(X)$ is a *bridge* for f (resp. for g) if there is a *bridge map* $f_{\mathfrak{A}} : (X_{\mathfrak{A}}, A_{\mathfrak{A}}) \rightarrow (Y, B)$ (resp. $g_{\mathfrak{A}} : A_{\mathfrak{A}} \rightarrow Y$) such that, for any canonical map $p_{\mathfrak{A}} : (X, A) \rightarrow (X_{\mathfrak{A}}, A_{\mathfrak{A}})$, $f_{\mathfrak{A}} \circ p_{\mathfrak{A}} \simeq f$ (resp. $g_{\mathfrak{A}} \circ (p_{\mathfrak{A}}|_A) \simeq g$).

It is well-known that if (Y, B) is an ANR-pair, then:

- (i) there exists a bridge $\mathfrak{A} \in \Omega(X)$ of f (resp. of g);
- (ii) a refinement of a bridge is again a bridge.

PROOF OF THEOREM 3.4. Clearly if $n = \infty$ or $n > m$, then P and (P, P') are ∞ -connected and the assertions follow trivially. Therefore in the sequel we

⁹That is, having fibres with trivial shape.

assume that $n < \infty$ and $n \leq m$. Moreover, we may assume that actually (P, P') is a complete ANR-pair. Recall also that any complete ANR is an absolute neighbourhood extensor for the class of paracompact spaces; hence each paracompact pair has the homotopy extension property (HEP) with respect to it.

Let Z (resp. Z') be the cylinder of $f : X \rightarrow Y$ (resp. of $f' = f|_{X'} : X' \rightarrow Y'$). Therefore X', X and Z' are closed subspaces of Z , $X' \subset Z'$ and $\dim Z \leq \dim X + \dim Y + 1$ (obviously Z is paracompact). There is a (strong) deformation retraction $r : (Z, Z') \rightarrow (Y, Y')$ and $f = r \circ i$ where $i : (X, X') \rightarrow (Z, Z')$ is the inclusion. Evidently:

(1) $r^\# : [Y, Y'; P, P'] \rightarrow [Z, Z'; P, P']$ is a bijection.

(2) Let G be an abelian group which is finitely generated whenever Y is not compact. For any $q \geq 0$, $\check{H}^q(Z; G) = \check{H}^q(Y; G)$, $\check{H}^q(Z'; G) = \check{H}^q(Y'; G)$ and hence, by Corollary 1.5, for any $i^N(f) \leq q \leq N - 1$, $\check{H}^q(Z, X; G) = 0 = \check{H}^q(Z', X'; G)$.

Let $i_1 : (X, X') \rightarrow (Z, Z')$ be the inclusion.

(3) To prove surjectivity of $i_1^\#$ and $i^\# : [Z; P] \rightarrow [X; P]$ let $i^N(f) - 1 \leq n$ (and $m \leq N - 2$ in case (ii)) and let $g : (X, X') \rightarrow (P, P')$. We claim that there is an extension $\bar{g} : Z \rightarrow P$ of g .

Indeed, there is a bridge $\mathfrak{A} \in \Omega(Z)$ of g with a bridge map $g_{\mathfrak{A}} : X_{\mathfrak{A}} \rightarrow P$. Since P is $(n - 1)$ -connected, $g_{\mathfrak{A}}$ has an extension $\bar{g}_{\mathfrak{A}} : Z_{\mathfrak{A}}^n \cup X_{\mathfrak{A}} \rightarrow P$ (here and below $Z_{\mathfrak{A}}^n$ denotes the n -dimensional skeleton of $Z_{\mathfrak{A}}$). Assume that $g_{\mathfrak{A}}$ has an extension $\bar{g}_{\mathfrak{A}} : Z_{\mathfrak{A}}^k \cup X_{\mathfrak{A}} \rightarrow P$ for some $n \leq k$. Observe that under our assumptions, $\pi_k(P)$ is an abelian group which is finitely generated when Y is not compact. The $(k + 1)$ -dimensional obstruction set $\mathcal{O}^{k+1}(g_{\mathfrak{A}}) \subset \check{H}^{k+1}(Z_{\mathfrak{A}}, X_{\mathfrak{A}}; \pi_k(P))$ is nonempty. Let $c \in \mathcal{O}^{k+1}(g_{\mathfrak{A}})$. Since $i^N(f) \leq k + 1$, $\check{H}^{k+1}(Z, X; \pi_k(P)) = 0$, and there is a bridge $\mathfrak{B} \in \Omega(Z)$, $\mathfrak{A} \preceq \mathfrak{B}$, of g such that, for any canonical projection $p_{\mathfrak{A}\mathfrak{B}} : (Z_{\mathfrak{B}}, X_{\mathfrak{B}}) \rightarrow (Z_{\mathfrak{A}}, X_{\mathfrak{A}})$, $0 = p_{\mathfrak{A}\mathfrak{B}}^*(c) \in \mathcal{O}^{k+1}(g_{\mathfrak{A}} \circ (p_{\mathfrak{A}\mathfrak{B}}|_{X_{\mathfrak{B}}})) \subset \check{H}^{k+1}(Z_{\mathfrak{B}}, X_{\mathfrak{B}}; \pi_k(P))$. Hence the map $g_{\mathfrak{B}} = g_{\mathfrak{A}} \circ (p_{\mathfrak{A}\mathfrak{B}}|_{X_{\mathfrak{B}}})$, being a \mathfrak{B} -bridge map for g , has an extension $\bar{g}_{\mathfrak{B}} : Z_{\mathfrak{B}}^{k+1} \cup X_{\mathfrak{B}} \rightarrow P$.

After a finite number of steps, we get a bridge $\mathfrak{D} \in \Omega(Z)$ of g with a bridge map $g_{\mathfrak{D}} : X_{\mathfrak{D}} \rightarrow P$ having an extension $\bar{g}_{\mathfrak{D}} : Z_{\mathfrak{D}}^l \cup X_{\mathfrak{D}} \rightarrow P$ where $l = \min\{N - 1, \dim Z\}$. If $\dim Z \leq \dim X + \dim Y + 1 \leq N - 1$, then $Z_{\mathfrak{D}}^l = Z_{\mathfrak{D}}$. Otherwise, if $m \leq N - 2$, then, since $\pi_i(P) = 0$ for $i \geq m + 1$, we may further extend $\bar{g}_{\mathfrak{D}}$ to get a map $Z_{\mathfrak{D}} \rightarrow P$ denoted, as before, by $\bar{g}_{\mathfrak{D}}$.

By HEP of (Z, X) with respect to P , g has the desired extension onto Z . Therefore $i_1^\# : [Z, X'; P, P'] \rightarrow [X, X'; P, P']$ and $i^\# : [Z; P] \rightarrow [X; P]$ are surjections.

(4) To prove the injectivity of $i_1^\#$ and $i^\#$, let $i^N(f) \leq n$ (and $m \leq N - 1$ in case (ii)) and consider maps $g_j : (Z, X') \rightarrow (P, P')$, $j = 0, 1$, such that

$h : g_0 \circ i_1 \simeq g_1 \circ i_1$. Define $\bar{h} : Z \times \{0, 1\} \cup X \times I \rightarrow P$ by the formula

$$\bar{h}(z, t) = \begin{cases} g_t(z) & \text{if } t = 0, 1, z \in Z, \\ h(z, t) & \text{if } t \in I, z \in X. \end{cases}$$

Since $\check{H}^q(Z \times I, Z \times \{0, 1\} \cup X \times I; \pi_{q-1}(P)) = \check{H}^{q-1}(Z, X; \pi_{q-1}(P)) = 0$ for $i^N(f) + 1 \leq q \leq N$, arguing as in (3), we get an extension $H : Z \times I \rightarrow P$ of \bar{h} , with $H(X' \times I) \subset P'$, and thus a homotopy $H : g_0 \simeq g_1$.

This already completes the proof of the part concerning $f^\# : [Y; P] \rightarrow [X; P]$.

In order to proceed with the relative case consider the inclusions

$$(Z', X') \xrightarrow{i_2} (Z, X') \xrightarrow{j} (Z, Z').$$

(5) Assume that $i^N(f) - 1 \leq n$ (and $m \leq N - 2$ in case (ii)). First we shall prove that, given a map $g : (Z', X') \rightarrow (P, P')$, there is a map $\bar{g} : (Z', X') \rightarrow (P, P')$ such that $\bar{g}(Z') \subset P'$ and $\bar{g} \simeq g \text{ rel } X'$.

There is a bridge $\mathfrak{A} \in \Omega(Z')$ of g with a bridge map $g_{\mathfrak{A}} : (Z'_{\mathfrak{A}}, X'_{\mathfrak{A}}) \rightarrow (P, P')$.

Since (P, P') is n -connected (and, at least, 1-connected) there exists a map $\bar{g}_{\mathfrak{A}} : (Z'_{\mathfrak{A}}, X'_{\mathfrak{A}}) \rightarrow (P, P')$ such that $\bar{g}_{\mathfrak{A}} \simeq g_{\mathfrak{A}}$ and $\bar{g}_{\mathfrak{A}}(Z'^k_{\mathfrak{A}}) \subset P'$ for some $k \geq \max\{n, 1\}$.

Reasoning similarly to (3) but applying the theory of obstructions to the deformation we get a bridge $\mathfrak{B} \in \Omega(Z')$ and a bridge map $g_{\mathfrak{B}} : (Z'_{\mathfrak{B}}, X'_{\mathfrak{B}}) \rightarrow (P, P')$ such that $g_{\mathfrak{B}}(Z') \subset P'$. Let $g' = g_{\mathfrak{B}} \circ p_{\mathfrak{B}}$ where $p_{\mathfrak{B}} : (Z', X') \rightarrow (Z'_{\mathfrak{B}}, X'_{\mathfrak{B}})$ is the canonical map. Then $g' \simeq g$ and $g'(Z') \subset P'$. Since (Z', X') (resp. $(Z' \times I, Z' \times \{0, 1\} \cup X' \times I)$) has HEP with respect to P' (resp. P), we get the required \bar{g} .

(6) Let $g : (Z, X') \rightarrow (P, P')$. By (5), there is a map $\bar{g} : (Z', X') \rightarrow (P, P')$ such that $\bar{g}(Z') \subset P'$ and $h : \bar{g} \simeq g \circ i_2 \text{ rel } X'$. Clearly h can be extended to a homotopy $H : Z \times I \rightarrow P$ such that $H(\cdot, 1) = g$ and $H(\cdot, 0)|_{Z'} = \bar{g}$. Therefore $H(Z' \times \{0\}) \subset P'$ and $H(\cdot, 0) \circ j \simeq g$. Hence $j^\#[H(\cdot, 0)] = [g]$ and $j^\# : [Z, Z'; P, P'] \rightarrow [Z, X'; P, P']$ is surjective.

(7) To prove injectivity of $j^\#$ suppose that $i^N(f) \leq n$ (and $m \leq N - 1$ in case (ii)), $g_i : (Z, Z') \rightarrow (P, P')$, $i = 0, 1$, and $h : g_0 \circ j \simeq g_1 \circ j$. Evidently $h(Z' \times \{0, 1\} \cup X' \times I) \subset P'$. Since $\check{H}^q(Z' \times I, Z' \times \{0, 1\} \cup X' \times I; \pi_q(P, P')) = \check{H}^{q-1}(Z', X'; \pi_q(P, P')) = 0$ for $i^N(f) + 1 \leq k + 1 \leq q \leq N$, arguing as in (5), there is $\bar{h} : (Z' \times I, Z' \times \{0, 1\} \cup X' \times I) \rightarrow (P, P')$ such that $\bar{h}(Z' \times I) \subset P'$ and $H : \bar{h} \simeq (h|_{Z' \times I}) \text{ rel } Z' \times \{0, 1\} \cup X' \times I$. Let $\bar{H} : (Z \times \{0, 1\} \cup Z' \times I, Z' \times \{0, 1\} \cup X' \times I) \times I \rightarrow (P, P')$ be given by the formula

$$\bar{H}(z, t, \lambda) = \begin{cases} g_t(z) & \text{if } t = 0, 1, \lambda \in I, z \in Z, \\ H(z, t, \lambda) & \text{if } t, \lambda \in I, z \in Z'. \end{cases}$$

Since $\overline{H}(\cdot, \cdot, 1) = h|_{Z \times \{0, 1\} \cup Z' \times I}$ has the extension h onto $Z \times I$, the map $\overline{H}(\cdot, \cdot, 0)$ has an extension $G : (Z \times I, Z' \times I) \rightarrow (P, P')$ and $G : g_0 \simeq g_1$.

Summing up: $i^\# : [Z, Z'; P, P'] \rightarrow [X, X'; P, P']$ is the composition $i_1^\# \circ j^\#$; hence, in case (i) it is a surjection for $n \geq i^N(f) - 1$, an injection if $i^N(f) \leq n$. In case (ii) it is a surjection if $i^N(f) - 1 \leq n \leq m \leq N - 2$ and an injection for $i^N(f) \leq n \leq m \leq N - 1$. Since $f^\# = i^\# \circ r^\#$, in view of (1), we complete the proof. \square

REMARKS 3.7. (i) Observe that it is enough to assume that $N \geq \dim Z + 1$. Moreover, note that $i^N(f) < N$ for $N \geq \dim Z + 1$ if and only if $i(f) < N$ since, for such N , $i^N(f) = i^{\dim X + 1}(f) = i(f)$ (see Proposition 1.4). Recall that if $\dim X, \dim Y < \infty$, then $i(f) < \infty$. Therefore $f^\#$ is surjective (bijective) whenever $n \geq i(f) - 1$ ($n \geq i(f)$).

(ii) The main tool of the above proof is $\check{H}^q(Z, X; G) = 0 = \check{H}^q(Z', X'; G)$ if G is an abelian group (finitely generated whenever Y is not compact) and $i^N(f) \leq q \leq N - 1$.

PROOF OF THEOREM 3.5. Let $\overline{X} = X/X', \overline{Y} = Y/Y'$ and let $\varphi_X : X \rightarrow \overline{X}$, $\varphi_Y : Y \rightarrow \overline{Y}$ be the canonical projections. If $\overline{x} = \varphi_X(X')$ and $\overline{y} = \varphi_Y(Y')$, then $\varphi_X^\# : [\overline{X}, \overline{x}; P, p] \rightarrow [X, X'; P, p]$ and $\varphi_Y^\# : [\overline{Y}, \overline{y}; P, p] \rightarrow [Y, Y'; P, p]$ are bijections. Define $\overline{f} : (\overline{X}, \overline{x}) \rightarrow (\overline{Y}, \overline{y})$ by the formula $\overline{f} \circ \varphi_X = \varphi_Y \circ f$. Clearly \overline{f} is a continuous closed surjection and the spaces $\overline{X}, \overline{Y}$ are paracompact.

Observe now that $i^N(\overline{f}) = i^N(f; \mathcal{A})$. Since $\overline{f}^{-1}(\overline{y}) = \{\overline{x}\}$ we may argue as in the proof of Theorem 3.4 to get the desired result. \square

If in Theorem 3.4, $m = \infty$, we can also get a different result, but still one needs some dimension restrictions. We have seen that assumptions concerning the cohomological dimension will not do (recall Example 3.6(i)). However, the deformation dimension [25], [24] seems to be a right choice.

We give here an absolute version of the result, leaving the relative case to the reader.

PROPOSITION 3.8. *Let $f : X \rightarrow Y$ be a closed surjection between paracompact spaces such that, for some $N \geq 2$, the deformation dimensions $\text{def dim } X, \text{def dim } Y \leq N - 2$ and $i^N(f) < N$. If P satisfies Assumptions 3.2(i), (iv) and (v) and is $(n - 1)$ -connected, then $f^\# : [Y; P] \rightarrow [X; P]$ is a surjection for $i^N(f) - 1 \leq n$ and an injection for $i^N(f) \leq n$.*

PROOF. First observe that we may replace P by a homotopy equivalent CW-complex (still denoted by P). By attaching k -cells, $k \geq N$, to P , we obtain a CW-complex Q such that $\pi_i(Q) = 0$ for each $i \geq N - 1$. Hence Q is of $(n, N - 2)$ -type. Since the $(N - 1)$ -dimensional skeletons of P and Q coincide and $\text{def dim } X, \text{def dim } Y \leq N - 2$, the inclusion $i : P \rightarrow Q$ induces bijections $i_\# : [X; P] \rightarrow [X; Q]$ and $i_\# : [Y; P] \rightarrow [Y; Q]$. On the other hand, since Q has

the homotopy type of an absolute neighbourhood retract for paracompact spaces, in view of 3.4, we see that $f^\# : [Y; Q] \rightarrow [X; Q]$ is a surjection for $i^N(f) - 1 \leq n$ and an injection for $i^N(f) \leq n$. This completes the proof. \square

Since evidently $\text{def dim } X \leq \dim X$ and $\text{def dim } Y \leq \dim Y$ we get an obvious augmentation of 3.4. However, 3.8 is not an extension of 3.4. Indeed, if $\max\{\dim X, \dim Y\} + 2 = M$ and $\max\{\text{def dim } X, \text{def dim } Y\} + 2 = N$, then clearly $M \geq N$ and $i^M(f) \geq i^N(f)$, so there is no relation between the numbers $N - i^N(f)$ and $M - i^M(f)$.

Let us note the following corollary:

COROLLARY 3.9. *Let X and Y be homotopically simple complete ANRs such that $\pi_i(X)$ and $\pi_i(Y)$ are finitely generated groups for any $i \geq 1$. If X is of (n, m) -type, Y is of $(n, m + 1)$ -type, $1 \leq n, m \leq \infty$, $f : X \rightarrow Y$ is a closed surjection such that, for some $N \geq 0$, $i^N(f) < N$, $i^N(f) \leq n$ and either*

- (i) $m \leq N - 2$; or
- (ii) $\dim X + \dim Y + 2 \leq N$; or
- (iii) $\max\{\text{def dim } X, \text{def dim } Y\} + 2 \leq N$,

then f is a homotopy equivalence.

PROOF. We easily find that in each case $f^\# : [Y; X] \rightarrow [X; X]$ is a surjection; hence there is $g : Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$. For a finitely generated abelian group G , $g^* : \check{H}^q(X; G) \rightarrow \check{H}^q(Y; G)$ is a monomorphism for any $q \geq 0$ and an isomorphism for $i^N(f) \leq q \leq N - 1$. Therefore $\check{H}^q(Z_g, Y; G) = 0$ (Z_g is the cylinder of g) if $i^N(f) + 1 \leq q \leq N$. Arguing as in the proof of Theorem 3.4, we deduce that $g^\# : [X; Y] \rightarrow [Y; Y]$ is a surjection; hence there is $f' : X \rightarrow Y$ such that $f' \circ g \simeq \text{id}_Y$. We see therefore that $g_\# : \pi_i(Y) \rightarrow \pi_i(X)$ and $f'_\# : \pi_i(X) \rightarrow \pi_i(Y)$ are isomorphisms for any $i \geq 1$. This completes the proof. \square

4. Vietoris theorem and cohomotopy groups

S. Smale [28] was perhaps the first to observe that a Vietoris type theorem holds in terms of homotopy groups. Let X be an arbitrary topological space, let Y be paracompact and locally n -connected ($n \geq 1$); if instead of acyclicity one assumes that each fibre of a given perfect (or merely closed) surjection $f : X \rightarrow Y$ is n -connected and locally $(n - 1)$ -connected, then $f_\# : \pi_q(X, x) \rightarrow \pi_q(Y, f(x))$ is an isomorphism for any $0 \leq q \leq n$. If additionally Y is dominated by a polyhedron (i.e. has the homotopy type of a CW-complex), then $f_\#$ is an epimorphism for $q = n + 1$ as well. It was shown independently by many authors that the assumption concerning the fibres of f may be still relaxed. Namely (see e.g. [7], [1], [18]), one can suppose that, for each $y \in Y$ and each neighbourhood

U of $f^{-1}(y)$ there is a neighbourhood $V \subset U$ of $f^{-1}(y)$ such that any singular k -sphere in V is inessential in U ($0 \leq k \leq n$). In other words, the fibres of f should have the UV^n -property. Compare also the papers [9] and [26].

As a simple corollary we get a Vietoris type result stated in terms of cohomotopy groups. However, again our hypotheses are not “categorical”: the assumption concerning the fibres is stated in the language of cohomology.

For in Theorem 3.4 one may take for instance $(P, P') = (B^n, S^{n-1})$ (where B^n is the n -dimensional closed unit ball), $n \geq 2$. If $(P, P') = (B^1, S^0)$, then, although the assumptions are not satisfied, the assertion also holds if $i^N(f) = 0$.

Taking $(P, P') = (S^n, s_0)$, where s_0 is a base point, $n \geq 1$, we see that a perfect surjection $f : (X, X') \rightarrow (Y, Y')$ induces a map

$$f^\# : [Y, Y'; S^n, s_0] \rightarrow [X, X'; S^n, s_0]$$

which is a surjection if $n = i(f) - 1$ and a bijection if $i(f) \leq n$ provided $\dim X, \dim Y < \infty$.

This assertion also holds when $i(f) = 0$ and $(P, P') = (S^0, s_0)$.

In [21], it is proved that given a compact pair (X, A) with $\dim X < \infty$ and $\check{H}^q(X, A) = 0$ for $q \geq 2m - 1$, $m \geq 1$, the set $\pi^n(X, A) := [X, A; S^n, s_0]$, $n \geq m$, admits the structure of an abelian group by the usual Borsuk method (see [29], [14]; cf. also [22]).

Essentially by the same methods one can introduce a group structure in the set $\pi^n(X, A)$, $n \geq m$, where (X, A) is a pair with $\dim X < \infty$ and $\check{H}^q(X, A) = 0$ for $q \geq 2m - 1$ ¹⁰. Moreover, if $f : (X, A) \rightarrow (Y, B)$, $\dim Y < \infty$ and $\check{H}^q(Y, B) = 0$ for $n \geq 2m - 1$, then $f^\# : \pi^n(Y, B) \rightarrow \pi^n(X, A)$, $n \geq m$, is a homomorphism.

In view of Theorem 3.5 we have

THEOREM 4.1. *Let $m \geq 1$ and let $f : (X, X') \rightarrow (Y, Y')$ be a perfect surjection of finite-dimensional paracompact spaces. Moreover, let $\check{H}^q(Y, Y') = 0$ for $q \geq 2m - 1$.*

- (i) *If $n = \max\{m, i(f) - 1\}$, then $f^\# : \pi^n(Y, Y') \rightarrow \pi^n(X, X')$ is an epimorphism.*
- (ii) *If $\max\{m, i(f)\} \leq n$, then $f^\#$ is an isomorphism.*

Below, we denote by $\Sigma^k X$ ($S^k X$) the k th unreduced (reduced) suspension of a (pointed) space X .

COROLLARY 4.2. *If $f : X \rightarrow Y$ is a perfect surjection of finite-dimensional (pointed) paracompact spaces, then for each $k \geq 0$, $(\Sigma^k f)^\# : \pi^{n+k}(\Sigma^k Y) \rightarrow \pi^{n+k}(\Sigma^k X)$ (resp. $(S^k f)^\# : \pi^{n+k}(S^k Y) \rightarrow \pi^{n+k}(S^k X)$) is a surjection for $n =$*

¹⁰If $\dim X \leq 2m - 1$, then this follows from the suspension theorem.

$i(f) - 1$ and a bijection for $n \geq i(f)$. If $k \geq 2$, then $(S^k f)^\#$ is an epimorphism and an isomorphism, respectively.

One argues as in the proof of 3.4 but instead of the cylinder of $\Sigma^k f$ (resp. $S^k f$) one can consider the space $\Sigma^k Z$ (resp. $S^k Z$) where Z is the cylinder of f .

Moreover, in view of [25, (4.2)] and Proposition 3.8 we have

COROLLARY 4.3. *If $f : X \rightarrow Y$ is a perfect surjection such that, for some $m \geq 1$, $i^N(f) < N$ where $\max\{\text{def dim } X, \text{def dim } Y\} < \max\{N - 1, 2m - 1\}$, then $\pi^n(X)$ and $\pi^n(Y)$ admit the structures of abelian groups for $n \geq m$ and $f^\# : \pi^n(Y) \rightarrow \pi^n(X)$ is an epimorphism for $n = \max\{i^N(f) - 1, m\}$ and an isomorphism for $n \geq \max\{i^N(f), m\}$.*

The above facts have straightforward implications in coincidence (fixed-point) theory.

Let $\dim X < \infty$, $f : (X, X') \rightarrow (B^{m+1}, S^m)$ be a perfect surjection and let $g : X \rightarrow \mathbb{R}^{m+1}$ be a map such that $(f - g)(X) \subset \mathbb{R}^{n+1}$, $m \geq n \geq i(f)$ (that means that $f = (f_1, f_2)$, $g = (g_1, g_2)$ where $f_1, g_1 : X \rightarrow \mathbb{R}^{n+1}$ and $f_2 = g_2 : X \rightarrow \mathbb{R}^{m-n}$). Suppose that $(f - g)(X') \subset \mathbb{R}^{n+1} \setminus \{0\}$. Therefore, without loss of generality we may assume that $f - g : (X, X') \rightarrow (B^{n+1}, S^n)$.

PROPOSITION 4.4. *If the element $(f^\#)^{-1}([f - g]) \in [B^{m+1}, S^m; B^{n+1}, S^n] \cong \pi^n(S^m)$ is nontrivial, then f and g have a coincidence, i.e. there is $x_0 \in X$ such that $f(x_0) = g(x_0)$.*

PROOF. Assume to the contrary that $(f - g)(X) \subset B^{n+1} \setminus \{0\}$. Hence there is $F : (X, X') \rightarrow (B^{n+1}, S^n)$ such that $f - g \simeq F \text{ rel } X'$ and $F(X) \subset S^n$. There is $F' : B^{m+1} \rightarrow S^n$ such that $F' \circ f \simeq F : (X, X') \rightarrow (B^{n+1}, S^n)$. Since F' is inessential and $[f - g] = f^\#[F']$ we get a contradiction. \square

In particular, we obtain

COROLLARY 4.5. *Let $f : (X, X') \rightarrow (B^{n+1}, S^n)$, $n \geq 0$ be a perfect surjection and let $\dim X < \infty$. If $g : X \rightarrow \mathbb{R}^{n+1}$ is such that*

$$(3) \quad \|g(x)\| \leq \|f(x)\| \quad \text{for } x \in X'$$

and $i(f) \leq n$, then f and g have a coincidence.

PROOF. Assumption (3) yields immediately $f - g \simeq f$, i.e. $(f^\#)^{-1}([f - g])$ is nontrivial. \square

REMARK 4.6. (i) Clearly assumption (3) (called the Rothe type condition) is necessary to show that $f - g \simeq f$. Any other condition of this type (like Altman's or Krasnosel'skii's) would do (see [8, II.(5.1)]) for the fixed point theoretical context.

(ii) One easily sees that the element $(f^\#)^{-1}([f - g]) \in \pi^n(S^m)$ in Proposition 4.4 constitutes a natural generalization of the *fixed-point index*. For if $(X, X') = (B^{m+1}, S^m)$, $m = n$ and $f = \text{id}$, then $(f^\#)^{-1}([f - g]) = \text{ind } g$ (see [5]).

(iii) The results stated above are valid when the domain X of the perfect surjection f is a finite-dimensional paracompact space. However, these results still hold true with obvious modifications if we assume that $\text{def dim } X \leq N - 2$ and $i^N(f) < N$ for some $N \geq 2$.

An idea underlying Proposition 4.4 suggests a way of extending Theorem 4.1. Namely, dimension restrictions should be replaced by restrictions on the “admissible” category of maps.

Let $(E, \|\cdot\|)$ be a Banach space, $\dim E \leq \infty$, and consider the following generalized Leray–Schauder category $\mathcal{LS}(E)$ (cf. [12], [13], [23]):

- Objects of this category are pairs (X, φ) where X is a paracompact space and $\varphi : X \rightarrow E$ is a *proper*¹¹ map such that $\varphi^{-1}(L)$ is a compact finite-dimensional subset of X for each finite-dimensional (linear) subspace $L \subset E$ ¹².
- Morphisms between given objects (X, φ) and (Y, ψ) are maps $f : X \rightarrow Y$ such that $\psi \circ f = \varphi$.

We say that an object (X, φ) is *regular* if there exists a positive integer $m_0 \leq \dim E$ such that, for all linear subspaces L of E with $\dim L \geq m_0$ and $q \geq 2 \dim L - 3$, $\check{H}^q(\varphi^{-1}(L)) = 0$.

EXAMPLE 4.7. (i) The simplest example of a regular object in $\mathcal{LS}(E)$ with $\dim E \geq 4$ is as follows: take a closed bounded $X \subset E$ and $\varphi = i : X \hookrightarrow E$.

(ii) More generally: if E' is a Banach space, $\Phi : E' \rightarrow E$ is a Fredholm operator of positive index k ¹³, $X \subset E'$ is closed bounded and $\varphi = \Phi|_X$, then (X, φ) is a regular object provided $\dim E \geq 4 + k$.

Let $(X, \varphi) \in \mathcal{LS}(E)$. By a φ -field we understand a *compact*¹⁴ map $g : X \rightarrow E$ such that $\varphi(x) \neq g(x)$ for all $x \in X$. We say that two φ -fields $g_i : X \rightarrow E$, $i = 0, 1$, are φ -homotopic (written $g_0 \simeq_\varphi g_1$) if there is a compact map $h : X \times [0, 1] \rightarrow X$, called a φ -homotopy, such that $h(\cdot, i) = g_i$, $i = 0, 1$, and $\varphi(x) \neq h(x, t)$ for all $x \in X$ and $t \in [0, 1]$.

Clearly “ \simeq_φ ” is an equivalence relation; its equivalence classes are denoted by $[g]_\varphi$ where g is a φ -field; the set of all φ -homotopy classes is denoted by $\pi^E(X, \varphi)$.

¹¹That is, $\varphi^{-1}(K)$ is compact for each compact $K \subset E$ (e.g. a perfect map is proper).

¹²The reader will see that it is enough here to assume that $\varphi^{-1}(L)$ is a compactum of finite deformation dimension.

¹³That is, a bounded linear operator with finite-dimensional null-space $\text{Ker}(\Phi)$ and closed range $\text{R}(\Phi)$ of codimension $\dim \text{Ker}(\Phi) - k$.

¹⁴That is, $\text{cl } g(X)$ is compact in E .

It is clear that π^E is an h -cofunctor (the notion of a homotopy of morphisms in $\mathcal{L}S(E)$ is obvious) from $\mathcal{L}S(E)$ to the category of sets and, given a morphism $f : (X, \varphi) \rightarrow (Y, \psi)$, the morphism $\pi^E(f) : \pi^E(Y, \psi) \rightarrow \pi^E(X, \varphi)$ is given by $\pi^E(f)([g]_\psi) = [g \circ f]_\varphi$ for any ψ -field $g : Y \rightarrow E$.

THEOREM 4.8.

- (i) Given $(X, \varphi), (Y, \psi) \in \mathcal{L}S(E)$ and a morphism $f : (X, \varphi) \rightarrow (Y, \psi)$, if $f : X \rightarrow Y$ is a perfect surjection with $i(f) < \infty$, then

$$\pi^E(f) : \pi^E(Y, \psi) \rightarrow \pi^E(X, \varphi)$$

is a bijection.

- (ii) For any regular object $(X, \varphi) \in \mathcal{L}(E)$, the set $\pi^E(X, \varphi)$ has the structure of an abelian group.
- (iii) If (Y, ψ) is a regular object, then so is (X, φ) and $\pi^E(f)$ is an isomorphism.

Before we proceed with the proof, let us introduce an *orientation* in E (see [13]) and recall that if L, N are finite-dimensional linear subspaces in E , $L \subset N$ and $\dim L + 1 = \dim N$, then this orientation determines two closed subspaces N_+ and N_- of N such that $N_+ \cap N_- = L$ and $N_+ \cup N_- = N$.

PROOF OF THEOREM 4.8. (i) Take $[g]_\varphi \in \pi^E(X, \varphi)$ where $g : X \rightarrow E$ is a φ -field. Since g is compact the set $K = \text{cl } g(X)$ is compact. For any $\varepsilon > 0$, there is a finite-dimensional linear subspace E_ε of E and a Schauder projection (see e.g. [8]) $p_\varepsilon : K \rightarrow E_\varepsilon$ such that $\|p_\varepsilon(x) - x\| < \varepsilon$ for $x \in K$.

CLAIM. *There is a unique (up to ψ -homotopy) ψ -field $q : Y \rightarrow E$ such that $\pi^E([q]_\psi) = [g]_\varphi$.*

(1) *Existence.* Since φ is proper, there exists $\varepsilon > 0$ such that, for all $x \in X$, $\|\varphi(x) - p \circ g(x)\| \geq 2\varepsilon$ where we have put $p := p_\varepsilon$. Let $L := E_\varepsilon$ and assume, without loss of generality, that $m + 1 := \dim L \geq i(f) + 1$.

In order to simplify the notation, we let $X_L = \varphi^{-1}(L)$, $\varphi_L = \varphi|_{X_L} : X_L \rightarrow L$, $Y_L = \psi^{-1}(L)$, $\psi_L = \psi|_{Y_L} : Y_L \rightarrow L$, $K_L = \text{cl conv } p(K)$ and $g_L = p \circ g|_{X_L} : X_L \rightarrow L$. It is obvious that $g_L(X_L) \subset K_L$ and $f_L = f|_{X_L}$ may be regarded as a perfect surjection $X_L \rightarrow Y_L$ and $i(f_L) \leq i(f)$ (see Proposition 1.4(ii)).

Evidently $\varphi_L - g_L : X_L \rightarrow L \setminus \{0\}$. In view of Theorem 3.4, there is a unique (up to homotopy) map $Q : Y_L \rightarrow L \setminus \{0\}$ such that $H_L : Q \circ f_L \simeq \varphi_L - g_L : X_L \rightarrow L \setminus \{0\}$.

(*) Let q be a compact extension onto Y of the map $Y_L \ni y \mapsto \psi_L(y) - Q(y) \in L$. Clearly $\psi(y) \neq q(y)$ for $y \in Y$, thus q is a ψ -field. Moreover, if $\bar{Q} \simeq Q : Y_L \rightarrow L \setminus \{0\}$ and a ψ -field \bar{q} is a compact extension onto Y of the map $Y_L \ni y \mapsto \psi_L(y) - \bar{Q}(y) \in L$, then $q \simeq_\psi \bar{q}$. Conversely, given a ψ -field

$\bar{q} : Y \rightarrow L$ such that there is a ψ -homotopy $h : Y \times [0, 1] \rightarrow L$ joining q to \bar{q} , we have $Q \simeq \bar{Q} = (\psi - \bar{q})|_{Y_L} : Y_L \rightarrow L \setminus \{0\}$.

Let $h_1 : X \times [0, 1] \rightarrow L \subset E$ be a compact extension of the map

$$X \times \{0, 1\} \cup X_L \times [0, 1] \ni (x, t) \mapsto \begin{cases} q \circ f(x) & \text{for } t = 0, x \in X, \\ p \circ g(x) & \text{for } t = 1, x \in X, \\ \varphi_L(x) - H_L(x, t) & \text{for } t \in [0, 1], x \in X_L. \end{cases}$$

It is easy to see that h_1 is a φ -homotopy joining $q \circ f$ to $p \circ g$. Now $p \circ g \simeq_\varphi g$ through the linear homotopy $X \times [0, 1] \ni (x, t) \mapsto h_2(x, t) = (1 - t)p \circ g(x) + tg(x) \in E$. Composing h_1 and h_2 and taking into account the choice of ε , we have established the existence part of the Claim.

(2) *Uniqueness.* In order to show the uniqueness of $[q]_\psi$ suppose that $q' : Y \rightarrow E$ is another ψ -field such that there is a compact φ -homotopy $h : q' \circ f \simeq_\varphi g : X \rightarrow E$. In view of the choice of ε , we may actually assume that $h : q' \circ f \simeq_\varphi p \circ g$.

I. First suppose that h is a finite-dimensional map, i.e. $h(X \times [0, 1]) \subset N$ (in particular $q'(Y) \subset N$) where N is a linear subspace of E , $\dim N < \infty$, and $L \subset N$.

(j) If $N = L$, then defining $Q' : Y_L \rightarrow L$ by $Q'(y) = \psi_L(y) - q'(y)$, $y \in Y_L$, we see that $Q' \circ f_L \simeq \varphi_L - g_L$ and, by the uniqueness of $[Q]$, $Q' \simeq Q$; therefore $q' \simeq_\psi q$ by (*) above.

(jj) Suppose that $L \subsetneq N$. As above let $X_N = \varphi^{-1}(N)$, $\varphi_N := \varphi|_{X_N}$, $Y_N = \psi^{-1}(N)$, $\psi_N := \psi|_{Y_N}$ and $g_N := p \circ g|_{X_N} : X_N \rightarrow L \subset N$. Moreover $f_N = f|_{X_N}$ may be regarded as a perfect surjection $X_N \rightarrow Y_N$ and $i(f_N) \leq i(f)$. If $Q' = \psi_N - q'|_{Y_N}$, then $Q' \circ f_N \simeq \varphi_N - g_N : X_N \rightarrow N \setminus \{0\}$. Take any subspace N_1 with $L \subset N_1 \subset N$ such that $N = N_1 \oplus Z$ (direct sum) where $\dim Z = 1$.

We shall show that there are a ψ -field $q_1 : Y \rightarrow N_1$ such that $q_1 \simeq_\psi q'$ and a φ -homotopy $h_1 : q_1 \circ f \simeq p \circ g$ such that $h_1(X \times [0, 1]) \subset N_1$.

Let $X_1 := \varphi^{-1}(N_1)$, $\varphi_1 := \varphi|_{X_1}$, $Y_1 := \psi^{-1}(N_1)$, $\psi_1 := \psi|_{Y_1}$, $f_1 := f|_{X_1} : X_1 \rightarrow Y_1$ and $g_1 := p \circ g|_{X_1} : X_1 \rightarrow L \subset N_1$. Evidently there is a compact map $Q_1 : Y_1 \rightarrow N_1 \setminus \{0\}$ such that $H_1 : Q_1 \circ f_1 \simeq \varphi_1 - g_1 : X_1 \rightarrow N_1 \setminus \{0\}$.

Define $q_1 : Y \rightarrow N_1$ to be a compact extension of the map $Y_1 \ni y \mapsto \psi_1(y) - Q_1(y)$; q_1 is clearly a ψ -field.

Let N^\pm be the *open* half-spaces determined by N_1 and Z and let $X_N^\pm := \varphi^{-1}(N^\pm)$, $Y_N^\pm := \psi^{-1}(N^\pm)$. Then $X_N = X_N^+ \cup X_1 \cup X_N^-$, $Y_N = Y_N^+ \cup Y_1 \cup Y_N^-$. If $Q_N = \psi_N - q_1$, then $Q_N(Y_N^\pm) \subset N^\pm$ and $Q_N(Y_N) \subset N \setminus \{0\}$. Consider a map $D : X_N \times \{0, 1\} \cup X_1 \times [0, 1] \rightarrow N$ given by

$$D(x, t) = \begin{cases} Q_N \circ f_N(x) & \text{for } x \in X_N, t = 0, \\ \varphi_N(x) - g_N(x) & \text{for } x \in X_N, t = 1, \\ H_1(x, t) & \text{for } x \in X_1, t \in [0, 1]. \end{cases}$$

Since $Q_N \circ f_N|_{X_1} = Q_1 \circ f_1$ and $(\varphi_N - g_N)|_{X_1} = \varphi_1 - g_1$, we see that D is well-defined and compact. Clearly $Q_N \circ f_N(x), \varphi_N(x) - g_N(x) \in N^\pm$ for any $x \in X_N^\pm$. We now take a compact extension H_N of D over $X_N \times [0, 1]$ such that $H_N(X_N^\pm \times [0, 1]) \subset N^\pm$ showing that $H_N : Q_N \circ f_N \simeq \varphi_N - g_N : X_N \rightarrow N \setminus \{0\}$. By the uniqueness of $[Q']$, we have $Q_N \simeq Q' : Y_N \rightarrow N \setminus \{0\}$. Therefore $q_1 \simeq_\psi q'$ and a map $h_1 : X \times [0, 1] \rightarrow N_1$ which is a compact extension of the map

$$X \times \{0, 1\} \cup X_1 \times [0, 1] \ni (x, t) \mapsto \begin{cases} q_1 \circ f(x) & \text{for } t = 0, x \in X, \\ p \circ g(x) & \text{for } t = 1, x \in X, \\ \varphi_1(x) - H_1(x, t) & \text{for } t \in [0, 1], x \in X_1, \end{cases}$$

provides a φ -homotopy joining $q_1 \circ f$ to $p \circ g$.

After at most $\dim N - \dim L$ steps we get a ψ -field $\bar{q} : X \rightarrow L$ such that $\bar{q} \simeq_\psi q'$ and a φ -homotopy $h_1 : \bar{q} \circ f \simeq_\varphi p \circ g : X \rightarrow L$. By (j) above, $\bar{q} \simeq_\psi q$; hence $q \simeq_\psi q'$.

II. If the original h is not finite-dimensional, then take $\varepsilon' > 0, \varepsilon' < \varepsilon$, such that $\|p' \circ h(x, t) - \varphi(x)\| \geq \varepsilon'$ for $x \in X, t \in [0, 1]$ where $p' : \text{cl } h(X \times [0, 1]) \rightarrow N \supset L, \dim N < \infty$, is a Schauder projection with $\|p'(x) - x\| < \varepsilon'$ for $x \in \text{cl } h(X \times [0, 1])$. Now $p' \circ h : p' \circ q' \circ f \simeq_\varphi p' \circ p \circ g$ and $p' \circ p \circ g \simeq_\varphi p \circ g$. Clearly $q' \simeq_\psi p' \circ q'$, and by part I, $p' \circ q' \simeq_\psi q$. This completes the proof of part (i) of the theorem.

(ii) For any linear subspace $L \subset E$ with $\dim L = m + 1$, we have $\Sigma^L(\varphi^{-1}(L)) = [\varphi^{-1}(L), L \setminus \{0\}] \cong [\varphi^{-1}(L), S^m] = \pi^m(\varphi^{-1}(L))$ because $L \setminus \{0\}$ is homotopy equivalent to S^m .

Assume that (X, φ) is a regular object, i.e. if $\dim L = m + 1 \geq m_0$, then $\check{H}^q(\varphi^{-1}(L)) = 0$ for $q \geq 2 \dim L - 3 = 2m - 1$. Hence the set $\pi^m(\varphi^{-1}(L))$ and thus $\Sigma^L(\varphi^{-1}(L))$ has the structure of an abelian group provided $\dim L = m + 1 \geq m_0$.

We shall denote by Λ the family of all linear subspaces $L \subset E$ with $\dim L \geq m_0$ directed by inclusion.

Let $L, N \in \Lambda, L \subset N$ and $\dim L = m + 1 = \dim N - 1$. In this case L cuts N into two closed half-spaces denoted by N_+ and N_- . If $X_N := \varphi^{-1}(N), X_L := \varphi^{-1}(L)$ and $X_\pm^N := \varphi^{-1}(N_\pm)$, then $X_N = X_+^N \cup X_-^N$ and $X_L = X_+^N \cap X_-^N$. As usual, one defines the coboundary homomorphism Δ_{LN} of the triad $(X_N; X_+^N, X_-^N)$ putting $\Delta_{LN} = j^\# \circ (k^\#)^{-1} \circ \delta : \pi^m(X_L) \rightarrow \pi^{m+1}(X_N)$ where $\delta : \pi^m(X_L) \rightarrow \pi^{m+1}(X_+^N, X_L)$ is the coboundary homomorphism of the pair (X_+^N, X_L) and $k : (X_+^N, X_L) \rightarrow (X_N, X_-^N), j : X_N \rightarrow (X_N, X_-^N)$ are the inclusions, $k^\#$ being the excision isomorphism. Clearly Δ_{LN} induces a homomorphism (denoted by the same symbol) $\Delta_{LN} : \Sigma^L(X_L) \rightarrow \Sigma^N(X_N)$.

If $L \in \Lambda$ and $L \subset N \in \Lambda$, then there is a chain of subspaces $L = L_0 \subset L_1 \subset \dots \subset L_{n+1} = N$ such that $\dim L_{i+1} = \dim L_i + 1$ for $i = 0, \dots, n$. Hence we may

define a homomorphism $\Delta_{LN} = \Delta_{L_n L_{n+1}} \circ \dots \circ \Delta_{L_0 L_1} : \Sigma^L(X_L) \rightarrow \Sigma^N(X_N)$. One shows easily that Δ_{LN} is well-defined, i.e. it does not depend on the choice of the above chain of subspaces.

Given a third subspace $M \in \Lambda$, $M \supset N$, we see that $\Delta_{LM} = \Delta_{NM} \circ \Delta_{LN}$. Additionally we put $\Delta_{LL} = \text{id}$.

(**) Let us note the following simple property. Assume that $L \in \Lambda$ and $G : X_L \rightarrow L \setminus \{0\}$ and consider a compact extension $g : X \rightarrow L$ onto X of the map $X_L \ni x \mapsto \varphi(x) - G(x) \in L$. Evidently g is a φ -field. Let $L \subset N \in \Lambda$ and let $Q = (\varphi - g)|_{X_N}$. Then $\Delta_{LN}([G]) = [Q]$.

Indeed, suppose first that $\dim N = \dim L + 1$. Observe that $Q|_{X_L} = G$; moreover, $Q(X_{\pm}^N) \subset N_{\pm}$. Then, by the very definition of the homomorphism Δ_{LN} , $[Q] = \Delta_{LN}([G])$. If $\dim N > \dim L + 1$, then in order to get the assertion one can iterate the above argument.

Hence we may define a direct system $\Sigma = \{\Sigma^L(X_L), \Delta_{LN} \mid L, N \in \Lambda, L \subset N\}$ of abelian groups. Let

$$\Sigma^E(X, \varphi) := \varinjlim_{L \in \Lambda} \Sigma^L(X_L)$$

and let $\sigma^L : \Sigma^L(X_L) \rightarrow \Sigma^E(X, \varphi)$, $L \in \Lambda$, be the canonical homomorphism.

We shall show that there is a 1-1 (set) correspondence between $\Sigma^E(X, \varphi)$ and $\pi^E(X, \varphi)$.

To this end, for each $L \in \Lambda$, consider a transformation $\xi^L : \Sigma^L(X_L) \rightarrow \pi^E(X, \varphi)$ which assigns to the homotopy class $[G]$ of a map $G : X_L \rightarrow L \setminus \{0\}$ the homotopy class $[g]_{\varphi}$ of the φ -field $g : X \rightarrow L$ which is an arbitrary compact extension onto X of the map $X_L \ni x \mapsto G^L(x) = \varphi(x) - G(x)$. We have already seen (recall (*) above) that ξ^L is well-defined (and even injective if we restrict “admissible” φ -homotopies to those mapping $X \times [0, 1]$ into L). It is a matter of simple calculation to check that the family $\{\xi^L \mid L \in \Lambda\}$ of set-transformations is compatible with Σ (treated as a direct system in the category of sets), i.e. for any $L, N \in \Lambda$ with $L \subset N$, $\xi_L \circ \Delta_{LN} = \xi_N$. Hence there is a unique (limit) set-transformation $\xi : \Sigma^E(X, \varphi) \rightarrow \pi^E(X, \varphi)$ such that $\xi \circ \sigma^L = \xi^L$.

The transformation ξ is bijective.

Indeed, we have already shown that given a φ -field $g : X \rightarrow E$ there is a finite-dimensional subspace L (without loss of generality we may assume that $L \in \Lambda$) and a φ -field $g_L : X \rightarrow L$ which is φ -homotopic to g ; hence $[g]_{\varphi} = \xi^L([G])$ where $X_L \ni x \mapsto G(x) = \varphi(x) - g(x) \in L \setminus \{0\}$. In other words, $\pi^E(X, \varphi) = \bigcup_{L \in \Lambda} \xi^L(\Sigma^L(X_L))$.

Suppose now that there are subspaces $L, N \in \Lambda$ and maps $G_0 : X_L \rightarrow L \setminus \{0\}$, $G_1 : X_N \rightarrow N \setminus \{0\}$ such that $\xi^L([G_0]) = \xi^N([G_1])$; thus if a φ -field $g_i : X \rightarrow E$ represents $\xi^L([G_i])$, $i = 0, 1$, then there is a φ -homotopy $h : g_0 \simeq_{\varphi} g_1 : X \rightarrow E$. If h itself is finite-dimensional, say $h(X \times [0, 1]) \subset M \in \Lambda$ (clearly $L, N \subset M$),

and $X_M \ni x \mapsto Q_i(x) = \varphi(x) - g_i(x) \in M \setminus \{0\}$, $i = 0, 1$, then $Q_0 \simeq Q_1 : X_M \rightarrow M \setminus \{0\}$ and, in view of (**) above, $\Delta_{LM}([G_0]) = [Q_0] = [Q_1] = \Delta_{NM}([G_1])$.

If h is not finite-dimensional, then arguing as in part (2) II of the proof of (i), we obtain the same assertion. This completes the proof of (ii).

(iii) If the object (Y, ψ) is regular, then in view the Sklyarenko theorem, so is (X, φ) . Now assertion (iii) follows from (i) and (ii) because Δ_{LN} behaves well with respect to induced homomorphisms, in other words Δ_{LN} is a natural transformation of the cofunctor Σ^L to Σ^N . \square

EXAMPLE 4.9. For some instances, the group $\pi^E(X, \varphi)$ may be easily computed. If E' is a Banach space, $\dim E = \infty$, $\Phi : E' \rightarrow E$ is a Fredholm operator of index $k \geq 0$, $X = S$ is the unit sphere in E' and $\varphi = \Phi|_S$, then (S, φ) is a regular object and $\pi^E(S, \varphi) = \pi_s^k(S^0)$ is the k th stable cohomotopy group of spheres. To see that, let Λ' be a subfamily in Λ consisting of linear subspaces of sufficiently large dimension and in “general position” with respect to the range $R(\Phi)$, that is, $L \in \Lambda'$ if and only if $L = L' \oplus Z$ where $Z \oplus R(\Phi) = E$ and $L' \subset R(\Phi)$. Clearly Λ' is cofinal in Λ and thus $\pi^E(S, \varphi) = \varinjlim_{L \in \Lambda'} \pi^{\dim L - 1}(S^{\dim L - 1 + k})$ because $\varphi^{-1}(L) = S \cap \Phi^{-1}(L')$ and $\dim \Phi^{-1}(L') = \dim L' + \dim \text{Ker}(\Phi) = \dim L + k$. If $\dim L$ is sufficiently large, then by the suspension theorem, $\pi^{\dim L - 1}(S^{\dim L - 1 + k}) \cong \pi_s^k(S^0)$.

The introduced cofunctor π^E admits a generalization in analogy to the *infinite-dimensional stable cohomotopy* theory of Geřba (see [12]).

Namely assume that $\dim E = \infty$ and we are given a filtration $\{E^n\}_{n=0}^\infty$ of linear subspaces such that $\dim E^n = n$, $E^n \subset E^{n+1}$, and a family of complementing closed subspaces $\{E_n\}$, i.e. $E^n \oplus E_n = E$ and $E_n \subset E_{n-1}$ for each $n \geq 1$. Additionally let Z be a straight line (with a fixed orientation) lying in $\bigcap_{n=0}^\infty E_n$.

For each object (X, φ) in $\mathcal{LS}(E)$, a closed $A \subset X$ and $n \geq 0$ let $\pi^{\infty-n}(X, A; \varphi)$ be the set of all φ -homotopy classes of compact maps $g : X \rightarrow E$ such that $(\varphi - g)(X) \subset E_n \setminus \{0\}$ and $(\varphi - g)(A) \subset E_n \setminus Z_-$ where Z_- is the negative (open) half-line (φ -homotopies are compact maps $h : X \times [0, 1] \rightarrow E$ such that $(\varphi - h)(\cdot, t) : (X, A) \rightarrow (E_n \setminus \{0\}, E_n \setminus Z_-)$).

It is easy to see that if $A = \emptyset$, then $\pi^{\infty-n}(X, \varphi) = \pi^{E_n}(X, \varphi_n)$ where $\varphi_n = p_n \circ \varphi$ and $p_n : E \rightarrow E_n$ is the linear projection parallel to E^n . Moreover, for a linear subspace L in “general position” with respect to E_n (i.e. $L = E^n \oplus L'$ where $L' \subset E_n$) we have $\varphi^{-1}(L) = \varphi_n^{-1}(L')$. The family of such subspaces is cofinal in the family of all subspaces.

Evidently if (X, φ) is a regular object, then so is (X, φ_n) . Therefore in this case one can pull back the group structure from $\pi^{E_n}(X, \varphi_n)$ onto $\pi^{\infty-n}(X, \varphi)$, $n \geq 0$. Reasoning similarly one shows that the set $\pi^{\infty-n}(X, A; \varphi)$ also admits such a structure.

Arguing as in [12] (with necessary modifications suggested by the proof above) one shows that the family $\{\pi^{\infty-*}\}_{n=0}^{\infty}$ of cofunctors gives rise to an (extraordinary) cohomology theory (i.e. satisfies the Eilenberg–Steenrod axioms save the dimension axiom)—the so-called *infinite-dimensional stable cohomotopy theory*.

It is also easy to show that if $f : (X, \varphi) \rightarrow (Y, \psi)$ in $\mathcal{LS}(E)$ is a perfect surjection, $A \subset X$, $B \subset Y$ are closed, $f^{-1}(B) = A$ and $i(f) < \infty$, then $\pi^{\infty-n}(f) : \pi^{\infty-n}(Y, B; \psi) \rightarrow \pi^{\infty-n}(X, A; \varphi)$ is a bijection (resp. an isomorphism provided the object (Y, ψ) is regular) for any $n \geq 0$.

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