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BASIC DEFINITIONS AND PROPERTIES OF TOPOLOGICAL BRANCHED COVERINGS

Artur Piękosz

1. Introduction

The aim of this paper is to examine topological branched coverings which were introduced in [5]. They appear naturally in algebraic and analytic geometry and they have been considered mostly in PL category (see for example [7], [2] and [4]). We introduce new notions which may be useful not only for examining topological branched coverings: a strong mapping at a point, a locally strong mapping and a spreading mapping. After proving that, under certain assumptions, the only branched coverings with the Absolute Covering Homotopy Property are unbranched coverings, we give two sufficient conditions for the Arc Lifting Property. We also characterize finite and locally finite nondegenerate graphs as branched coverings over the unit circle S^1 with one-point singular set.

2. Basic definitions

A continuous surjection $p : E \to B$ is called a (topological) *branched covering* if there exists a nowhere dense set $\Delta \subset B$ such that $p|p^{-1}(B\setminus\Delta) : p^{-1}(B\setminus\Delta) \to B\setminus\Delta$ is a covering mapping. The set $B\setminus\Delta$ is called a *regular set* of the branched covering p, whereas Δ its *singular set*. For a given branched covering $p : E \to B$ we define the *minimal singular set* $\Delta(p)$ consisting of all points $b \in B$ which have no evenly covered neighbourhood. This set is always closed.

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Assume that $p : E \to B$ is a branched covering with singular set Δ. We say that p is *without holes* when, for every $S \subset E$, the inclusion $\Delta \cap \overline{p(S)} \subset p(\overline{S})$ holds. On the other hand, if $p^{-1}(\Delta \cap \overline{p(S)}) \subset \overline{p^{-1}(p(S))}$ for every $S \subset E$ then we say that p is *without missing branches.*

EXAMPLE 1. Let $E = (([-1,0) \cup (0,1]) \times \{1\}) \cup ([-1,1] \times \{0\})$, $B = [-1,1]$ (we shall take the natural topologies in all cases unless otherwise stated) and $p(x,y) = x$. It is easy to see that $\Delta(p) = \{0\}$ and p has a "hole" at $(0,1)$.

EXAMPLE 2. Let $E = (\{0\} \times \{1\}) \cup ([-1,1] \times \{0\})$, $B = [-1,1]$ and $p(x,y) = x$. In this case p has "missing branches". To see this, take $S = (0,1] \times \{0\}$.

EXAMPLE 3. Every covering is a branched covering with $\Delta = \emptyset$. This is the reason why it is without holes and without missing branches.

A branched covering p is called *simple* if each $b \in \Delta$ has a neighbourhood U in B such that the punctured neighbourhood $U \setminus \{b\}$ is open and evenly covered. We define the *singular degree* of a branched covering as the maximal cardinality of singular fibres if it is finite, and ∞ otherwise. The *regular degree* of a branched covering is the same quantity for regular fibres. Moreover, we say that p has a *finite degree of branching* if for every $e \in p^{-1}(\Delta)$ there exist a natural number M and a neighbourhood U of e such that fibres of the mapping $p|U \setminus p^{-1}(\Delta) : U \setminus p^{-1}(\Delta) \to p\,(U) \setminus \Delta$ have at most M elements. Next, p is said to be *primitive* if its singular degree equals 1.

If every point $b \in B$ has a neighbourhood V such that each point $z \in p^{-1}(V)$ belongs to a certain set $Z \subset E$ which is homeomorphically mapped onto V by p, then p is said to be *decomposable into homeomorphisms* . Finally, let p be a simple branched covering with singular set Δ. We say that p is *with homeomorphisms on sheets* if for every $b \in \Delta$ and its open, evenly covered punctured neighbourhood $U \setminus \{b\}$ every homeomorphism $p|W_\alpha$ from a sheet W_α over $U \setminus \{b\}$ can be extended to a homeomorphism onto the neighbourhood U of b.

It is easy to see that every branched covering with homeomorphisms on sheets is decomposable into homeomorphisms.

3. Strong and locally strong mappings

Let $f : X \to Y$ be a mapping of topological spaces. We say that f is *strong at* $x \in X$ if for every neighbourhood U of x there exists a neighbourhood V of $f(x)$ such that $f^{-1}(V) \subset U$. If f is strong at each point of its domain then it is *strong.* We say that f is *locally strong at* $x \in X$ when there exists a neighbourhood U of x such that $f|U$ is strong at x. If f is locally strong at every $x \in X$ then f is called *locally strong.*

The following facts are obvious:

(1) A mapping $f : X \to Y$ is strong at $x \in X$ if and only if for every generalized sequence $\{x_\alpha\}_{\alpha\in A}$, where A is a directed set, the condition $f(x_\alpha) \to f(x)$ implies $x_\alpha \to x$.
(2) If f is strong at x and $f(x) \in \operatorname{int} f(X)$ (for example, if f is a surjection) then f is open at x. This means that for each neighbourhood U of x the set $f(U)$ is a vicinity (maybe not open) of $f(x)$.
(3) An open injection is strong.
(4) If X is a T_1-space and $f : X \to Y$ is strong at $x \in X$ then the fibre of x is a single point.
(5) A locally strong mapping may not be open (see Example 2).

PROPOSITION 1. *Let $p : E \to B$ be a branched covering with singular set Δ. Assume that p is strong at any point of $p^{-1}(\Delta)$. Then p is without holes. Moreover, if E is a T_1-space then p is primitive.*

PROOF. This follows from (1) and (4).

PROPOSITION 2. *If $p : E \to B$ is a branched covering with singular set Δ which is strong at points of $p^{-1}(\Delta)$, primitive*[1] *and simple then p is with homeomorphisms on sheets.*

PROOF. Apply the relevant definitions.

REMARK. The condition of strongness on $p^{-1}(\Delta)$ cannot be replaced by the condition of local strongness or openness.

EXAMPLE 4. Let $E = \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1],\ |y| = |x|\}$. We add the set $N = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1],\ y + x = 0\}$ to a basis of the topology on E, take $B = [-1, 1]$, $p(x, y) = x$ and $\Delta = \{0\}$. The mapping $p|N$ is strong at $(0, 0)$, so p is locally strong at $(0, 0)$ and open at $(0, 0)$. Nevertheless, for the sheet $W = \{(x, y) \in E \mid x = y,\ x \neq 0\}$ over $B \setminus \{0\}$, the homeomorphism $p|W : W \to B \setminus \{0\}$ cannot be extended to a homeomorphism onto B.

4. Spreading mappings

Let $f : X \to Y$ be a mapping of topological spaces. If the connected components of the preimages of all open sets in Y form a basis of the topology on X then we say that p is a *spreading mapping*. Likewise, for a point $x \in X$, if the family of connected components L of the preimages of neighbourhoods of $f(x)$ such that $x \in L$ forms a vicinity basis of x in X then p is called a *spreading mapping at x*.

[1] If E is a T_1-space then this assumption can be omitted.

The following facts are obvious:

(1) If $f : X \to Y$ is a spreading mapping at x then it is continuous at x.
(2) The domain of a spreading mapping is locally connected.
(3) $f : X \to Y$ is a spreading mapping if and only if it is a spreading mapping at every $e \in X$.
(4) If X is locally connected then any local homeomorphism $f : X \to Y$ is a spreading mapping.
(5) Let X be a locally connected space, $x \in X$, and Y be any topological space. If $f : X \to Y$ is continuous at x and strong at x then it is also a spreading mapping at x. The assumption that f is strong at x cannot be replaced by the assumption that f is locally strong at x (see the following example).

Example 5. Let $E = \bigcup_{i=1}^{\infty} C_i$, where C_i is the closed interval connecting the origin of $\mathbb{R}^2$ with the point $(1, 1/i)$. We add the set $N = \{(x, y) \in E \mid x \in C_j \Rightarrow x < 1/j\}$ to a basis of the topology induced on E from $\mathbb{R}^2$. Let $B = [0, 1]$, $p(x, y) = x$ and $\Delta = \{0\}$. The branched covering p with singular set Δ is locally strong and continuous. The preimages of neighbourhoods of $0 \in B$ are connected but they do not form a vicinity basis of $(0, 0)$.

Proposition 3. *Let E be a locally connected space and let $p : E \to B$ be a branched covering onto a topological space B with singular set Δ. If p is strong on $p^{-1}(\Delta)$ then it is a spreading mapping.*

Proof. This follows from (5) above.

If X and Y are locally connected T_1-spaces then any spreading mapping from X into Y is called a *spread* in the sense of R. H. Fox (see [3]). The language of spreads gives another possibility to define a branched covering. Every surjective branched covering in the sense of R. H. Fox is a branched covering in our sense.

5. The Absolute Covering Homotopy Property is not in general satisfied for branched coverings

One of the most important properties of unbranched coverings is the Absolute Covering Homotopy Property. A mapping $p : E \to B$ has this property when for every topological space X and every mapping $f : X \to E$, if $H : [0, 1] \times X \to B$ is a homotopy (i.e. it is continuous) for which $p \circ f = H(0, \cdot\,)$ then there exists a homotopy $\widetilde{H} : [0, 1] \times X \to E$ satisfying $p \circ \widetilde{H} = H$ and $\widetilde{H}(0, \cdot\,) = f$. The question whether branched coverings also have this property is natural. There is an example of a branched covering where only arcs of the base space are constant mappings (we say that B is totally pathwise disconnected in this case (see for instance [6, p. 31])). However, in "regular" cases branched coverings do not have the ACHP. This is shown in the theorem below.

Lemma 1. *Let $p : E \to B$ be a continuous mapping which has the ACHP. Assume that fibres of p are totally pathwise disconnected. If x and y belong to the same fibre and there exists an arc u such that u joins x to y and $p \circ u$ is a contractible loop then $x = y$.*

Proof. The proof is standard and left to the reader.

Theorem 1. *Let E be a locally arcwise connected space and B be a semilocally simply connected space. If $p : E \to B$ is an open and simple branched covering with discrete fibres and the ACHP then p is a covering* (*i.e.* $\Delta(p) = \emptyset$).

Proof. Suppose $\Delta(p) \neq \emptyset$ and fix $b \in \Delta(p)$. Notice that B is locally arcwise connected. There exists an arcwise connected neighbourhood U of b such that $U \setminus \{b\}$ is open, evenly covered and the mapping $\pi_1(U, b) \to \pi_1(B, b)$, induced by the inclusion, is trivial. Let $U \setminus \{b\} = \bigcup_{\alpha \in A} S_\alpha$ be the decomposition into connected components. The sets S_α are closed in $U \setminus \{b\}$, open and arcwise connected. Fix $c^\alpha \in S_\alpha$ for each $\alpha \in A$. If $p^{-1}(U \setminus \{b\}) = \bigcup_{i \in I} \widetilde{W}_i$ is a decomposition into sheets then, for a given $\alpha_0 \in A$, we obtain:

(1) There exists an arc $v : [0, 1] \to U$ such that $v(0) = c^{\alpha_0}$, $v(1) = b$.
(2) For each $i \in I$ the set $p^{-1}(c^{\alpha_0}) \cap \widetilde{W}_i$ consists of one point $c_i^{\alpha_0}$.
(3) By the ACHP, for every $i \in I$ there exists exactly one lifting v_i of an arc v such that $v_i(0) = c_i^{\alpha_0}$.

Let us denote $v_i(1)$ by e_i. We can define the following mapping:

$$\mathcal{E}_{\alpha_0} : I \ni i \mapsto e_i \in p^{-1}(b).$$

This definition does not depend on the choice of $c^{\alpha_0} \in S_{\alpha_0}$ and v. Indeed, take another $\widehat{c^{\alpha_0}} \in S_{\alpha_0}$ and an arc $\widehat{v} : [0, 1] \to U$ such that $\widehat{v}(0) = \widehat{c^{\alpha_0}}$, $\widehat{v}(1) = b$. For every $i \in I$ there exist a unique point $\widehat{c}_i^{\alpha_0} \in p^{-1}(\widehat{c^{\alpha_0}}) \cap \widetilde{W}_i$ and a unique lifting $\widehat{v}_i$ of $\widehat{v}$ for which $\widehat{v}_i(0) = \widehat{c_i^{\alpha_0}}$. Let $\widehat{e}_i = \widehat{v}_i(1)$. Notice that $p|p^{-1}(S_{\alpha_0}) \cap \widetilde{W}_i : p^{-1}(S_{\alpha_0}) \cap \widetilde{W}_i \to S_{\alpha_0}$ is a homeomorphism. Hence there exists an arc $z : [0, 1] \to p^{-1}(S_{\alpha_0}) \cap \widetilde{W}_i$ such that $z(0) = c^{\alpha_0}$, $z(1) = \widehat{c^{\alpha_0}}$. The arc $v_i^{-1} \cdot z \cdot \widehat{v}_i : [0, 1] \to p^{-1}(U)$ joins e_i to $\widehat{e}_i$. The mapping $p \circ (v_i^{-1} \cdot z \cdot \widehat{v}_i)$ is a loop which is contractible in B. By Lemma 1, $e_i = \widehat{e}_i$.

Injectivity of $\mathcal{E}_{\alpha_0}$. For given $i, j \in I$ such that $\mathcal{E}_{\alpha_0}(i) = \mathcal{E}_{\alpha_0}(j)$ we know that $v_i(1) = v_j(1)$, where v_i is the lifting of v which begins at $c_i^{\alpha_0}$, and v_j is the lifting of v which begins at $c_j^{\alpha_0}$. Hence $p \circ (v_i \cdot v_j^{-1})$ is a loop at c^{α_0} with the image in U. It is contractible because $p \circ (v_i \cdot v_j^{-1}) = v \cdot v^{-1}$. By Lemma 1, $c_i^{\alpha_0} = (v_i \cdot v_j^{-1})(0) = (v_i \cdot v_j^{-1})(1) = c_j^{\alpha_0}$, so $i = j$.

Surjectivity of $\mathcal{E}_{\alpha_0}$. Let $f \in p^{-1}(b)$. By the ACHP, there exists a lifting v_f of v which ends at f. There is a unique $i_0 \in I$ such that $c_{i_0}^{\alpha_0} = v_f(0) \in p^{-1}(c^{\alpha_0})$.

Thus $f = v_f(1) = e_{i_0}$. Similar bijections $\mathcal{E}_\alpha$ exist for all $\alpha \in A$. Let us define, for every $\alpha \in A$, the bijection

$$S_\alpha = \mathcal{E}_\alpha^{-1} \circ \mathcal{E}_{\alpha_0} : I \to I \quad (\alpha \in A)$$

and sets

$$W_i^\alpha = p^{-1}(S_\alpha) \cap \widetilde{W}_{S_\alpha(i)}, \quad W_i = \bigcup_{\alpha \in A} W_i^\alpha.$$

We shall prove that U is evenly covered. We have

$$p^{-1}(U) = \bigcup_{i \in I} (W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}),$$

$$(W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}) \cap (W_j \cup \{\mathcal{E}_{\alpha_0}(j)\}) = \emptyset \quad \text{for } i \neq j.$$

The sets $W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}$ $(i \in I)$ are pairwise disjoint and p maps each of them bijectively onto U. It suffices to prove that they are all open. All W_i $(i \in I)$ are open. We shall prove that the set $W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}$ is a neighbourhood of $\mathcal{E}_{\alpha_0}(i)$ for every $i \in I$. Let V be an arcwise connected neighbourhood of $\mathcal{E}_{\alpha_0}(i)$ contained in $\{\mathcal{E}_{\alpha_0}(i)\} \cup p^{-1}(U \setminus \{b\})$. Suppose that V is not contained in $W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}$. Then there exist $j \in I \setminus \{i\}$ and $d \in V \cap W_j$ which can be joined by an arc $l : [0,1] \to V$ to $\mathcal{E}_{\alpha_0}(i)$. Let $l' : [0,1] \to p^{-1}(V)$ be the lifting of $(p \circ l)^{-1}$ which begins at $l(1) = d$. The point $p(d)$ belongs to a certain S_{α_1}, so $d \in p^{-1}(S_{\alpha_1}) \cap \widetilde{W}_{S_{\alpha_1}(j)}$. Notice that l' defines the bijection $\mathcal{E}_{\alpha_1}$, that is,

$$l'(1) = \mathcal{E}_{\alpha_1}(S_{\alpha_1}(j)) = \mathcal{E}_{\alpha_0}(j).$$

The arc $l \cdot l'$ joins $\mathcal{E}_{\alpha_0}(i)$ to $\mathcal{E}_{\alpha_0}(j)$ and $p \circ (l \cdot l') = (p \circ l) \cdot (p \circ l)^{-1}$ is a contractible loop. By Lemma 1, $\mathcal{E}_{\alpha_0}(i) = \mathcal{E}_{\alpha_0}(j)$ and $i = j$. The contradiction proves that $V \subset W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}$. Thus all sets $W_i \cup \{\mathcal{E}_{\alpha_0}(i)\}$, for $i \in I$, are open and the proof is complete.

6. The Arc Lifting Property

Since branched coverings in "regular" cases do not have the ACHP in general, we shall seek sufficient conditions for the existence of a lifting of an arc. Clearly, we cannot expect the uniqueness of this lifting.

THEOREM 2. *Let $p : E \to B$ be a branched covering with singular set Δ. If p is decomposable into homeomorphisms then it has the Arc Lifting Property: for every arc $v : [0,1] \to B$ and $e \in E$ there exists an arc $\widetilde{v} : [0,1] \to E$ such that $\widetilde{v}(0) = e$ and $p \circ \widetilde{v} = v$.*

PROOF. Let an arc $\eta : [0,1] \to B$ and a point $x \in p^{-1}(\eta(0))$ be given. For every $t \in [0,1]$ we can find a neighbourhood U_t of $\eta(t)$ such that if $z \in p^{-1}(U_t)$ then there exists a set $Z \subset E$ such that $z \in Z$ and $p|Z : Z \to U_t$ is a homeomorphism.

The family $\{\eta^{-1}(U_t) : t \in [0,1]\}$ is open and covers $[0,1]$. By the Lebesgue lemma, there exists $\varepsilon > 0$ such that for all $t \in [0,1]$ we can find some t' for which $(t-\varepsilon, t+\varepsilon) \subset \eta^{-1}(U_{t'})$. We can put $\varepsilon = 1/N$, where N is an integer. Then $[j/N, (j+1)/N] \subset \eta^{-1}(U_{t_j})$ with some $t_j \in [0,1]$ and we can define the lifting $\widetilde{\eta}$ of η step by step.

Let Z_1 be chosen for $t = t_1$ and $z = x \in p^{-1}(U_{t_1})$. Put $\widetilde{\eta}_1 : [0, 1/N] \ni t \mapsto ((p|Z_1)^{-1} \circ \eta)(t) \in E$. Then $\widetilde{\eta}_1(0) = x$.

If the mapping $\widetilde{\eta}_{n-1} : [(n-2)/N, (n-1)/N] \to E$ is defined then we can choose Z_n for $t = t_n, z = \widetilde{\eta}_{n-1}((n-1)/N)$. The function $\widetilde{\eta}_N : [(n-1)/N, n/N] \ni t \mapsto ((p|Z_n)^{-1} \circ \eta)(t) \in E$ is continuous. By the bijectivity of $p|Z_n$, we have $\widetilde{\eta}_{n-1}((n-1)/N) = \widetilde{\eta}_n((n-1)/N)$. Then $\widetilde{\eta} = \bigcup_{i=1}^{N} \widetilde{\eta}_i$ is continuous and satisfies $\widetilde{\eta}(0) = \widetilde{\eta}_1(0) = x$ and $p \circ \widetilde{\eta}(t) = \eta(t)$ for every $t \in [0,1]$.

THEOREM 3. *Let $p : E \to B$ be a primitive branched covering which is strong at all points of $p^{-1}(\Delta)$. If Δ is discrete then p has the ALP.*

PROOF. We can assume that $\Delta = \Delta(p)$. Take an arc $\eta : [0,1] \to B$ and a point $x \in p^{-1}(\eta(0))$. For every $b \in \Delta$ there exists a neighbourhood U_b such that $U_b \cap \Delta = \{b\}$. The open family $\{\eta^{-1}(B \setminus \Delta)\} \cup \{\eta^{-1}(U_b) : b \in \Delta\}$ covers $[0,1]$. By the Lebesgue lemma, there exists a natural number N such that for every $n \in \{1, \ldots, N\}$ either (A) $\eta([(n-1)/N, n/N]) \cap \Delta = \emptyset$ or (B) $\eta([(n-1)/N, n/N]) \cap \Delta = \{b\}$, for some $b \in \Delta$. If (A) holds, then, by the ACHP for coverings, the lifting of $\eta|[(n-1)/N, n/N]$ exists. If (B) holds, then $p^{-1}(b_n) = \{e_n\}$ and the lifting of η on $[(n-1)/N, n/N]$ which begins at $x_n = p^{-1}(\eta((n-1)/N))$ is defined as follows:

In every connected component S_j ($j \in J$) of $\eta^{-1}(B \setminus \Delta) \cap [(n-1)/N, n/N]$ we choose a point d_j. If $(n-1)/N \notin S_j$, then d_j is any point, and if $(n-1)/N \in S_j$ then $d_j = (n-1)/N$. Further, we choose $e_j \in p^{-1}(\eta(d_j))$. If $d_j = (n-1)/N$ then $e_j = x_n$, if $d_j \neq (n-1)/N$ then we choose any e_j. On every S_j there exists a lifting $\xi_{nj} : S_j \to E$ of the arc $\eta|S_j$ by the ACHP for coverings and the fact that S_j is a locally finite union of closed intervals. The lifting $\widetilde{\eta}_n : [(n-1)/N, n/N] \to E$ is defined by

$$\widetilde{\eta}_n(t) = \begin{cases} e_n & \text{if } \eta(t) = b_n, \\ \xi_{nj}(t) & \text{if } t \in S_j. \end{cases}$$

By definition, we have $\widetilde{\eta}_n((n-1)/N) = x_n$, and $p \circ \widetilde{\eta}_n = \eta|[(n-1)/N, n/N]$. We get the lifting $\widetilde{\eta}$ of η taking the union of all $\widetilde{\eta}_n$ ($n = 1, \ldots, N$) as in the preceding theorem. It is continuous due to the assumption that p is strong at points of $p^{-1}(\Delta)$.

7. Graphs as branched coverings of S^1

Let us recall the definition of a graph. A topological pair (X, X^0) is called a *graph* if:

(A) X is a Hausdorff space (T_2),
(B) X^0 (the set of vertices) is a closed and discrete subspace of X,
(C) $X \setminus X^0$ is the disjoint union of a family $\mathcal{K} = \{e_i \mid i \in I\}$ of sets (open edges) which are homeomorphic to the open interval $(0, 1)$,
(D) for every $i \in I$, the set $\overline{e}_i$ (closed edge) is homeomorphic to S^1 or $[0, 1]$ and the set $\overline{e}_i \setminus e_i$ has one or two points,
(E) any subset $A \subset X$ is closed in X if and only if $A \cap \overline{e}_i$ is closed in $\overline{e}_i$ for every $i \in I$ (weak topology on X).

A graph X is *nondegenerate* if and only if the set of edges is not empty. It is *finite* if the number of vertices and edges is finite, and it is *locally finite* if for every $x \in X^0$ the number of edges e_i such that $x \in \overline{e}_i$ is finite.

Proposition 4. *Every nondegenerate graph (X, X^0) is a branched covering of the circle S^1 with one-point singular set $\{s\} = p(X^0)$.*

Proof. The proof is obvious.

Remark. The above mentioned branched covering is not necessarily without holes. To see this, take an infinite sequence of points in different open edges whose images converge to the singular point.

Lemma 2. *Assume that X is a Hausdorff space and $p : X \to S^1$ is a branched covering with singular set $\{s\}$ and discrete fibres. If the closure of any connected component S of $p^{-1}(S^1 \setminus \{s\})$ is compact then the following facts hold:*

(1) *∂S is a finite set contained in $X^0 = p^{-1}(s)$.*
(2) *For every $x \in \partial S$ there exists a generalized sequence $\{t_\alpha\}_{\alpha \in A}$ (A is a directed set) in $[0, 1]$ with all accumulation points in $\{0, 1\}$ and such that $h^{-1}(t_\alpha) \to x$.*
(3) *There exist points $0_S \in \overline{h^{-1}((0, 1/4])} \setminus h^{-1}((0, 1/4])$, $1_S \in \overline{h^{-1}([3/4, 1))} \setminus h^{-1}([3/4, 1))$.*
(4) *For every generalized sequence $\{x_\alpha\}_{\alpha \in A}$ in S the following implications hold: $h(x_\alpha) \to 0 \Rightarrow x_\alpha \to 0_S$ and $h(x_\alpha) \to 1 \Rightarrow x_\alpha \to 1_S$.*
(5) *For every neighbourhood U of 0_S (resp. 1_S) w X there exists $t \in (0, 1)$ such that $h^{-1}((0, t)) \subset U$ (resp. $h^{-1}((t, 1)) \subset U$).*
(6) *∂S has one or two points and $\overline{S}$ is homeomorphic to S^1 or to $[0, 1]$, where $h = (f|(0, 1))^{-1} \circ (p|S)$, and $f : [0, 1] \mapsto e^{2\pi it} \in S^1$ is a parametrization of the unit circle.*

Proof. The set $S^1 \setminus \{s\}$ is simply connected and locally arcwise connected. Thus the covering $p|X \setminus X^0 : X \setminus X^0 \to S^1 \setminus \{s\}$ is trivial and connected components of $p^{-1}(S^1 \setminus \{s\})$ are sheets over $S^1 \setminus \{s\}$. Consequently, S is open and its boundary is included in X^0.

(1) The boundary of S is a compact subset of X^0.

(2) Since h is a homeomorphism, no point of $(0,1)$ can be an accumulation point of $\{t_\alpha\}$.

(3) Follows from compactness of $\overline{S}$.

(4) We prove that $h(x_\alpha) \to 0 \Rightarrow x_\alpha \to 0_S$. We know that $\{x_\alpha\}_{\alpha\in A}$ has an accumulation point f_1 in ∂S. Assume that $f_1 \neq 0_S$. By (2) and (3), there exists a generalized sequence $\{t_\beta\}_{\beta\in B}$ in $(0,1/4]$ for which $h^{-1}(t_\beta) \to 0_S$. Only 0 is an accumulation point of $\{t_\beta\}_{\beta\in B}$. Therefore $t_\beta \to 0$. Let $\partial S = \{0_S, f_1, \ldots, f_m\}$. There exist disjoint neighbourhoods $U_0, U_1, \ldots, U_m$ of $0_S, f_1, \ldots, f_m$, respectively. The generalized sequences $\{t_\beta\}_{\beta\in B}$ and $\{h(x_\alpha)\}_{\alpha\in A}$ are convergent to 0. Hence there exist sequences of indices $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ such that $x_{\alpha_n} \in U_1$, $h^{-1}(t_{\beta_n}) \in U_0$, $\alpha_{n+1} \geq \alpha_n$, $\beta_{n+1} \geq \beta_n$ and $16x_{\alpha_{n+1}} < 4h^{-1}(t_{\beta_n}) < x_{\alpha_n} < 1/4$ for every natural n. In each closed interval $[t_{\beta_n}, h(x_{\alpha_n})]$, there exists t_n for which $h^{-1}(t_n) \in \partial U_0$. The sequence $\{h^{-1}(t_n)\}$ has an accumulation point in $\partial S \cap \partial U_0$ which does not belong to $\{0_S, f_1, \ldots, f_m\}$. The contradiction proves that $0_S = f_1$.

(5) Let us fix a neighbourhood U of 0_S. We assume that there exists a sequence $\{t_n\}$ in $[0,1]$ for which $t_n \to 0$ but $h^{-1}(t_n) \notin U$ for every natural n. The sequence $\{h^{-1}(t_n)\}$ is not convergent to 0_S. This contradicts (4).

(6) Let $y \in \partial S$. There exists a generalized sequence $\{t_\alpha\}_{\alpha\in A}$ in $[0,1]$ such that $h^{-1}(t_\alpha) \to y$ and all accumulation points of $\{t_\alpha\}_{\alpha\in A}$ belong to $\{0,1\}$. There are three cases:

(i) $t_\alpha \to 0$. Then, by (4) and (A), $y = 0_S$.
(ii) $t_\alpha \to 1$. Similarly, $y = 1_S$.
(iii) $\{0,1\}$ is the set of accumulation points of $\{t_\alpha\}_{\alpha\in A}$. There are subsequences convergent to 0 and 1, respectively. We get $y = 0_S = 1_S$.

If $0_S = 1_S$ then $p|\overline{S} : \overline{S} \to S^1$ is a homeomorphism. If $0_S \neq 1_S$ then we define the mapping $\widetilde{h} : \overline{S} \to [0,1]$ by

$$\widetilde{h}(x) = \begin{cases} 0 & \text{if } x = 0_S, \\ 1 & \text{if } x = 1_S, \\ h(x) & \text{if } x \in S. \end{cases}$$

The mapping $\widetilde{h}$ is continuous, bijective and defined on a compact set. So, it is a homeomorphism.

THEOREM 4. *For a Hausdorff space X, the following conditions are equivalent:*

I) *There exists a branched covering $p : X \to S^1$ with singular set $\{s\}$ which is proper and has finite fibres.*

II) *There exists a subspace X^0 of X such that (X, X^0) is a finite nondegenerate graph.*

Moreover, every nondegenerate finite graph is a branched covering of S^1 which is without holes and with one-point singular set $\{s\}$.

Proof. I) $\Rightarrow$ II). Take $X^0 = p^{-1}(s)$. Since the covering $p|X \setminus X^0$ is trivial, it is easy to see that axioms (A), (B), (C), (D) and (E) of a graph are satisfied (we take connected components of $X \setminus X^0$ as open edges). Clearly, this graph is finite and nondegenerate.

II) $\Rightarrow$ I). This follows from Proposition 4. The set X is compact, so p is proper.

If any of conditions I), II) is satisfied then p is closed, so, $p : E \to B$ is a branched covering without holes.

Theorem 5. *For a Hausdorff space X, the following conditions are equivalent:*

I) *X is locally connected and there exists a branched covering $p : X \to S^1$ with singular set $\{s\}$, finite degree of branching and discrete fibres, in which the closure of every connected component of $p^{-1}(S^1 \setminus \{s\})$ is compact.*

II) *There exists a subset X^0 of X for which (X, X^0) is a locally finite nondegenerate graph.*

Proof. I) $\Rightarrow$ II). Let $X^0 = p^{-1}(s)$. Axioms (A), (B), (C) and (D) are satisfied. Take $A \subset X$. If for every connected component L of $p^{-1}(S^1 \setminus \{s\})$ the set $A \cap \overline{L}$ is closed in $\overline{L}$ then this property also holds in X. We shall prove that the union

$$A = (A \cap X^0) \cup \bigcup\{\overline{L} \cap A \mid L \text{ is a connected component of } p^{-1}(S^1 \setminus \{s\})\}$$

is locally finite. It suffices to check this at points $x \in X^0$. The connected neighbourhood V of x which is contained in the neighbourhood $\{x\} \cup (X \setminus X^0)$ is a subset of $W = \{x\} \cup \bigcup\{L \text{ is a connected component of } p^{-1}(S^1 \setminus \{s\})\} \mid x \in \overline{L}\}$. Hence W is a neighbourhood of x. By assumption, there exists a neighbourhood U of x and a natural number M for which $p|U \setminus p^{-1}(\Delta)$ has fibres of not more than M elements each. If L is a connected component of $p^{-1}(S^1 \setminus \{s\})$ such that $x \in \overline{L}$, then $x = 0_L$ or $x = 1_L$. The number of L such that $x = 0_L$ is not greater than M. A similar property holds for $x = 1_L$. Hence there are not more than $2M$ connected components L such that $x \in \overline{L}$. Therefore A is closed in X. The reverse implication is trivial.

So, (X, X^0) is a nondegenerate graph. Actually, its local finiteness is already proved.

II) $\Rightarrow$ I). It is obvious that graphs are locally connected. We construct a branched covering in a standard way, so its fibres are discrete. Local finiteness of the graph implies the finite degree of branching of the branched covering. Since connected components of $p^{-1}(S^1 \setminus \{s\})\}$ are open edges, their closures are compact.

8. A topological proof of Bertini's theorem

Topological branched coverings may be applied in some theorems in analytic geometry. We give below the sketch of proof of a topological version of a theorem from Abhyankar's book [1] where the statement becomes more general and the proof is simpler. For the details see [5] .

Theorem 6 (cf. [1, (39.7)]). *Let Z be a connected topological manifold (without boundary) modelled on a real normed space E of dimension at least 2 and let Y be a simply connected and locally simply connected topological space. Suppose that V is a closed subset of $Y \times Z$ and $\pi : Y \times Z \to Y$ denotes the natural projection. Assume that $\pi_V : V \to Y$ is a branched covering whose regular fibres are finite and whose singular set $\Delta = \Delta(\pi_V)$ does not disconnect Y at any of its points.*[2] *Put $X = (Y \times Z) \setminus V$ and $L = \{p\} \times Z$, where $p \in Y \setminus \Delta$. If there exists a continuous mapping $h : Y \to Z$ whose graph is contained in X, then the inclusion $i : L \setminus V \hookrightarrow X$ induces an epimorphism $i_* : \pi_1(L \setminus V) \to \pi_1(X)$.*

Sketch of proof. To prove that $j_* : \pi_1(L\setminus V) \to \pi_1(X \setminus (\Delta \times Z))$, induced by inclusion, is an epimorphism take a loop $u = (f, g) : [0, 1] \to X \setminus (\Delta \times Z))$ at $(p, h(p))$. Define a new loop $w : [0, 1] \to X \setminus (\Delta \times Z))$ by the formula

$$w(t) = \begin{cases} u(2t) & \text{for } 0 \le t \le 1/2, \\ (f(2-2t), h(f(2-2t))) & \text{for } 1/2 < t \le 1. \end{cases}$$

Then $[u] = [w]$ and the loop w can be continuously transformed to a loop whose image is in $L \setminus V$ thanks to a generalization of (39.2) in [1] for topological manifolds. To prove that $k_* : \pi_1(X \setminus (\Delta \times Z)) \to \pi_1(X)$, induced by inclusion, is an epimorphism take a loop $u = (f, g) : [0, 1] \to X$ at $(p, h(p))$. Then the loop f at p is equivalent in Y to a loop v whose image does not intersect Δ (it suffices to take a union of local homotopies). We can choose v such that the loop (v, g), whose image is in $X \setminus (\Delta \times Z))$, is equivalent in X to the loop u.

[2] For every $y \in Y$ and its connected neighbourhood U there exists a neighbourhood $W \subset U$ such that $W \setminus \Delta$ is connected.

REFERENCES

[1] S. S. ABHYANKAR, *Local Analytic Geometry*, Academic Press, New York and London, 1964.

[2] I. BERSTEIN AND A. L. EDMONDS, *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979), 87–124.

[3] R. H. FOX, *Covering spaces with singularities*, A Symposium in Honour of S. Lefschetz, Princeton University Press, Princeton, N.J., 1957, pp. 243–257.

[4] H. M. HILDEN, *Every closed orientable 3-manifold is a 3-fold branched covering space of S^3*, Bull. Amer. Math. Soc. **80** (1974), 1243–1244.

[5] A. PIĘKOSZ, *A topological version of Bertini's theorem*, Ann. Polon. Math. **61** (1995), 89–93.

[6] L. A. STEEN, *Counterexamples in Topology*, Springer-Verlag, New York, 1978.

[7] A. W. TUCKER, *Branched and folded coverings*, Bull. Amer. Math. Soc. **42** (1936), 859–862.

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ARTUR PIĘKOSZ
Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, POLAND

E-mail address: pupiekos@cyf-kr.edu.pl