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# THE FOLD COMPLEMENTARITY PROBLEM AND THE ORDER COMPLEMENTARITY PROBLEM

G. Isac

## Introduction

We consider the Fold Complementarity Problem, which is one of the recent subjects in complementarity theory. It is a mathematical model used in economics in the study of distributive problems (cf. [25], [5]). A particular case is the $k$-Fold Complementarity Problem, studied using a variant of the notion of $Z$-function by A. Villar [25]. In this way, Villar obtained the solution of this problem as a solution of a minimization problem. We will study the Fold Complementarity Problem by a topological method. We will show that this method is also applicable to systems of Fold Complementarity Problems. We work towards two aims. First, we show that the Fold Complementarity Problem is exactly equivalent to an Order Complementarity Problem. This is important, since in this way the Fold Complementarity Problem is transformed into a nonlinear equation, or a fixed point problem, and hence we can use the theory of Order Complementarity Problems [9]–[15]. Second, we show that the Order Complementarity Problem associated with the Fold Complementarity Problem is naturally prepared to apply the topological index defined by Opoĭtsev [20] for continuous admissible mappings, defined on solid convex cones. We remark that to define this topological index it is not necessary to introduce a complicated

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mathematical structure, but just to use a special homotopy. Hence, this index is relatively simple to define and use. Certainly, it can also be used to other complementarity problems. It is interesting to compare this theory with the classical topological degree used recently in the study of the Linear Complementarity Problem (cf. [1], [7], [8], [13]). In the last part we consider systems of Fold Complementarity Problems and we finish with some comments about numerical methods for solving the Fold Complementarity Problem. This subject can be considered, on the other hand, as an interesting application of nonlinear analysis in economics.

## Preliminaries

We consider in this paper only the Euclidean space $(\mathbb{R}^m, \| \ \|)$. It is well known that $\mathbb{R}^m_+$ is a pointed closed convex cone and with respect to the ordering $x \le y \Leftrightarrow y - x \in \mathbb{R}^m_+$, $\mathbb{R}^m$ is a lattice ordered vector space. We write $x \vee y = \sup(x,y)$ and $x \wedge y = \inf(x,y)$. The cone $\mathbb{R}^m_+$ is *normal*, that is, there exists $\delta \ge 1$ such that, for every pair $x, y \in \mathbb{R}^m_+$, $\|x\| \le \delta \|x+y\|$, it is *solid*, that is, its topological interior is nonempty, and it is *regular*, that is, every order bounded increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^m_+$ is convergent. For every $r > 0$ we define

$$S_+(r) = \{x \in \mathbb{R}^m_+ \mid \|x\| = r\} \quad \text{and} \quad B_+(r) = \{x \in \mathbb{R}^m_+ \mid \|x\| \le r\}.$$

All the mappings considered in this paper are assumed to be continuous.

If $K \subset \mathbb{R}^m$ is a closed convex cone, we denote by $K^*$ the dual of $K$, that is,

$$K^* = \{y \in \mathbb{R}^m \mid \langle x, y\rangle \ge 0 \text{ for all } x \in K\}.$$

We say that $K$ is *selfadjoint* if and only if $K = K^*$.

We denote the boundary of $K$ by $\partial K$. If $A \subset \mathbb{R}^m_+$ we denote by $CA$ the set $\{x \in \mathbb{R}^m \mid x \notin A\}$.

## The Fold Complementarity Problem

An important chapter in economics is the study of distributive problems (cf. [25]). To facilitate understanding we consider the particular case where only goods are being distributed.

Consider a distributive problem involving $n$ agents (consumers) and, for the $j$th agent, $k_j$ goods ($k_j \ge 2,\ j = 1, \dots, n$). $\mathbb{R}^{k_j}_+$ stands for the *consumption set* of the $j$th consumer ($j = 1, \dots, n$).

Let $N = \sum_{j=1}^n k_j$. A point $x \in \mathbb{R}^N_+$ denotes an *allocation*, which can be written as $x = (x_1, \dots, x_j, \dots, x_n)$, where $x_j = (x_{j1}, \dots, x_{jk_j})$ for all $j = 1, \dots, n$.

*Given a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ of utility values the problem is to find the amount of goods and their corresponding distribution so that these utility* *levels are actually reached, and if some agent ends up with utility greater than his component $v_j$ then he should receive no goods.*

If $u_j : \mathbb{R}_+^N \to \mathbb{R}$ denotes the $j$th agent's payoff function, we have the $n$-vector function $u(x) = [u_1(x), \ldots, u_n(x)]$ of agents' payoffs for all $x \in \mathbb{R}_+^N$. In order to allow for the presence of consumption externalities, agents' preferences are defined over the entire $\mathbb{R}_+^N$. The mathematical model of our problem is

$$(1) \qquad \begin{cases} \text{find } x \in \mathbb{R}_+^N \text{ such that} \\ \text{(i) } u(x) \geq v, \\ \text{(ii) } u_j(x) > v_j \Rightarrow x_j = 0. \end{cases}$$

If $0_n$ (resp. $0_{k_j}$) $(j = 1, \ldots, n)$ denotes the origin of $\mathbb{R}^n$ (resp. $\mathbb{R}^{k_j}$) and $F(x) := u(x) - v$ for all $x \in \mathbb{R}^N$, then problem (1) has the following form named the *Fold Complementarity Problem*:

$$\text{(FCP)} \qquad \begin{cases} \text{find } x \in \mathbb{R}_+^N \text{ such that} \\ \text{(i) } F(x) \geq 0_n, \\ \text{(ii) } F_j(x) > 0 \Rightarrow x_j = 0_{k_j}. \end{cases}$$

If in this model $k$ is given and $k_j = k$ for every $j = 1, \ldots, n$ we obtain the Fold Complementarity Problem studied in [25]. Another interesting distributive problem seems to be the following.

Under the same assumptions as in problem (1), *we may consider the problem to find the amount of goods and their corresponding distribution so that these utility levels are actually reached, and if some agent ends up with utility greater than his component $v_j$, then he should receive no goods or receive strictly less than $k_j$ goods.*

We name this problem the *Special Fold Complementarity Problem* and its mathematical model is

$$\text{(SFCP)} \qquad \begin{cases} \text{find } x \in \mathbb{R}_+^N \text{ such that} \\ \text{(i) } F(x) \geq 0_n, \\ \text{(ii) } F_j(x) > 0 \Rightarrow x_j = 0_{k_j} \\ \text{or some components of } x_j \text{ are zero.} \end{cases}$$

## The Order Complementarity Problem

We consider the Euclidean space $\mathbb{R}^N$ (where $N = \sum_{j=1}^n k_j$) endowed with the ordering "$\leq$" defined by the cone $\mathbb{R}_+^N$. The ordered vector space $(\mathbb{R}^N, \mathbb{R}_+^N)$ is a vector lattice and $\mathbb{R}_+^N$ is a normal, solid and regular cone. Moreover, $\mathbb{R}_+^N$ is a selfadjoint cone and $(\mathbb{R}^N, \mathbb{R}_+^N)$ is a Hilbert lattice, that is, the following conditions are satisfied:

(h$_1$) $\||x|\| = \|x\|$ for all $x \in \mathbb{R}^N$ (i.e. the norm is absolute),
(h$_2$) $0 \le x \le y$ implies $\|x\| \le \|y\|$ for all $x, y \in \mathbb{R}^N_+$.

Given $r$ mappings $T_1, \dots, T_r : \mathbb{R}^N_+ \to \mathbb{R}^N$, the *General Order Complementarity Problem* associated with the family $\{T_i\}_{i=1}^r$ and the cone $\mathbb{R}^N_+$ is

$$\mathrm{GOCP}(\{T_i\}_{i=1}^r, \mathbb{R}^N_+) \qquad \begin{cases} \text{find } x_0 \in \mathbb{R}^N_+ \text{ such that} \\ \bigwedge(T_1(x_0), \dots, T_r(x_0)) = 0. \end{cases}$$

This problem has many interesting applications and it has been studied in [9]–[15].

If in the problem $\mathrm{GOCP}(\{T_i\}_{i=1}^r, \mathbb{R}^N_+)$, $r = 2$ and $T_1(x) = \Im(x)$ (the identity mapping on $\mathbb{R}^N_+$) then we have the (classical) Order Complementarity Problem, which is equivalent (since $(\mathbb{R}^N, \mathbb{R}^N_+)$ is a Hilbert lattice) to the classical complementarity problem

$$\mathrm{CP}(T_2, \mathbb{R}^N) \qquad \begin{cases} \text{find } x_0 \in \mathbb{R}^N_+ \text{ such that} \\ T_2(x_0) \in \mathbb{R}^N_+ \text{ and } \langle x_0, T_2(x_0)\rangle = 0. \end{cases}$$

Also, the problem $\mathrm{GOCP}(\{T_i\}_{i=1}^2, \mathbb{R}^N_+)$ is known as the *Implicit Order Complementarity Problem* and it is equivalent to the (classical) Implicit Complementarity Problem,

$$\mathrm{ICP}(T_1, T_2, \mathbb{R}^N_+) \qquad \begin{cases} \text{find } x_0 \in \mathbb{R}^N_+ \text{ such that} \\ T_1(x_0), T_2(x_0) \in \mathbb{R}^N_+ \text{ and } \langle T_1(x_0), T_2(x_0)\rangle = 0. \end{cases}$$

Now, we show that each Fold Complementarity Problem is equivalent to an Order Complementarity Problem. Thus, we consider the problems (FCP) and (SFCP), as defined above. Since $k_j \ge 2$ (for every $j = 1, \dots, n$) we can define the immersion $\Psi : \mathbb{R}^n \to \mathbb{R}^N$ by

$$\Psi(x_1, \dots, x_j, \dots, x_n) = (\underbrace{x_1, \dots, x_1}_{k_1 \text{ times}}, \dots, \underbrace{x_j, \dots, x_j}_{k_j \text{ times}}, \dots, \underbrace{x_n, \dots, x_n}_{k_n \text{ times}})$$

for all $x = (x_1, \dots, x_j, \dots, x_n) \in \mathbb{R}^n$, and the mapping $G : \mathbb{R}^N_+ \to \mathbb{R}^n$ by

$$G(x_1, \dots, x_N) = \left( \bigwedge(x_1, \dots, x_{k_1}), \bigwedge(x_{k_1+1}, \dots, x_{k_1+k_2}), \dots, \bigwedge(x_{\alpha+1}, \dots, x_N) \right)$$

where $\alpha = \sum_{j=1}^{n-1} k_j$ and $x = (x_1, \dots, x_N) \in \mathbb{R}^N_+$.

With the problem (FCP) we associate the mappings $T_1(x) = \Im(x)$ and $T_2(x) = \Psi(F(x))$. Similarly, with the problem (SFCP) we associate the mappings $S_1(x) = \Psi(G(x))$ and $S_2(x) = \Psi(F(x))$. Evidently $T_1, T_2, S_1, S_2 : \mathbb{R}^N_+ \to \mathbb{R}^N$. We have the following results.

Proposition 1. *The problem* (FCP) *is equivalent to* $\mathrm{GOCP}(\{T_i\}_{i=1}^2, \mathbb{R}_+^N)$.

Proof. Suppose that $x^* = (x_1^*, \dots, x_N^*) \in \mathbb{R}_+^N$ is a solution of (FCP). Let $x_r^*$ be an arbitrary component of $x^*$. There exists an index $j$ $(1 \le j \le n)$ such that $x_r^*$ is a component of $x_j$, for example, we may suppose that $x_r^* = x_{j_s}^*$ $(1 \le s \le k_j)$. Because $F(x^*) > 0_n$, we have $F_j(x^*) = 0$, or $F_j(x^*) > 0$. If $F_j(x^*) = 0$, then $x_r^* \wedge F_j(x^*) = 0$. If $F_j(x^*) > 0$, then $x_j = 0_{k_j}$, which implies in particular that $x_r^* = 0$, and hence again $x_r^* \wedge F_j(x^*) = 0$. In this way and using the definitions of $T_1$ and $T_2$ we deduce that $x^*$ is a solution of $\mathrm{GOCP}(\{T_i\}_{i=1}^2, \mathbb{R}^N)$.

Conversely, suppose that $x^* = (x_1^*, \dots, x_N^*) \in \mathbb{R}_+^N$ is a solution of the problem $\mathrm{GOCP}(\{T_i\}_{i=1}^2, \mathbb{R}_+^N)$. We have $\Psi(F(x^*)) \ge 0_N$, which implies that $F(x^*) \ge 0_N$. If $F_j(x^*) > 0$ then obviously, the definition of the mapping $T_2$ implies that $x_j^* = 0_{k_j}$ and the proposition is proved. □

Proposition 2. *The problem* (SFCP) *is equivalent to* $\mathrm{GOCP}(\{S_i^2\}_{i=1}, \mathbb{R}_+^N)$.

Proof. The proof is similar to the proof of Proposition 1. □

### A topological index with respect to a cone

We present in this section the topological index introduced by V. I. Opoĭtsev [20]. This index is defined using homotopy only and it is strongly based on the rotation of a vector field, as defined and used by Krasnosel'skiĭ (cf. [18]).

Let $N = \sum_{j=1}^n k_j$, where $k_j \ge 2$ for every $j = 1, \dots, n$.

Definition 1. We say that a continuous mapping $f : \mathbb{R}_+^N \to \mathbb{R}^N$ is $\mathbb{R}_+^N$-*admissible* if and only if $f(\partial \mathbb{R}_+^N) \subset \mathrm{Cint}(\mathbb{R}_+^N)$.

Remark. In Definition 1 we use the fact that $\mathbb{R}_+^N$ is a solid cone.

Definition 2. Two mappings $f_0, f_1 : \mathbb{R}_+^N \to \mathbb{R}^N$ are said to be $\mathbb{R}_+^N$-*homotopic* on the set $D \subset \mathbb{R}_+^N$ if there exists a mapping $H : D \times [0,1] \to \mathbb{R}^N$ such that

(1) $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in D$,
(2) $H(x,t)$ is $\mathbb{R}_+^N$-admissible for any fixed $t \in [0,1]$,
(3) $H(x,t) \ne 0$ for all $x \in D$ and $t \in [0,1]$ (i.e. $H$ is nonsingular).

We denote again by $\Im$ the identity mapping in $\mathbb{R}_+^N$. Given a positive number $r$ and an $h_0 \in \mathbb{R}_+^N$ such that $\|h_0\| > 1$, we define $H_r(x; h_0) = x - rh_0$ for every $x \in \mathbb{R}_+^N$.

Definition 3. Let $f : \mathbb{R}_+^N \to \mathbb{R}^N$ be an $\mathbb{R}_+^N$-admissible mapping. We say that $f$ has *index zero* at the distance $r > 0$ from the origin, and write $\mathrm{ind}(f, r) = 0$, if $f$ is $\mathbb{R}_+^N$-homotopic on $S_+(r)$ to the mapping $H_r(x; h_0)$, and we say that $f$ has *index* 1 at the distance $r > 0$ from the origin, and write $\mathrm{ind}(f, r) = 1$, if $f$ is $\mathbb{R}_+^N$-homotopic on $S_+(r)$ to the mapping $\Im$.

Remarks. 1. In Definition 3 the number $r > 0$ can be very small or very large.

2. We remark that all the mappings $H_\alpha(x) = x - \alpha h_0$ $(\alpha \geq r)$ on $S_+(r)$ are $\mathbb{R}^N_+$-homotopic to $H_r(x; h_0)$.

3. When we use this index we work only with the set $S_+(r)$ but not with the boundary of the set $B_+(r) = \{x \in \mathbb{R}^N_+ \mid \|x\| \leq r\}$.

This topological index seems to be suitable for Fold Complementarity Problems, since in the definition of these problems the mapping $F$ is defined only on the cone $\mathbb{R}^N_+$.

**The topological index and the Fold Complementarity Problem**

In this section we use the topological index introduced above to solve Fold Complementarity Problems. We associate with the problem (FCP) the mapping $\mathcal{E}(x) = \Im(x) \wedge T_2(x)$ and with the problem (SFCP) the mapping $\mathcal{E}_S(x) = S_1(x) \wedge S_2(x)$. The mappings $\mathcal{E}$ and $\mathcal{E}_S$ are from $\mathbb{R}^N_+$ into $\mathbb{R}^N$ and they are continuous since we suppose all the time that $F$ is continuous.

We suppose $N = \sum_{j=1}^n k_j$. We recall that the mapping $f(x) = (f_1(x), \ldots, f_N(x))$ from $\mathbb{R}^N_+$ into $\mathbb{R}^N$ is said to be *off-diagonal negative* if and only if for every $i = 1, \ldots, N$ we have $f_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) \leq 0$ if $x_j \geq 0$ for $j \neq i$.

All the results presented in this section are based on the following lemma:

Lemma (Opoĭtsev [20]). *If an $\mathbb{R}^N_+$-admissible mapping $\Phi : \mathbb{R}^N_+ \to \mathbb{R}^N$ coincides with the identity mapping $\Im$ on the set $S_+(\varrho)$ (with $\varrho > 0$) then the equation $\Phi(x) = 0$ has a solution $x_* \in \mathbb{R}^N_+$ such that $\|x_*\| \leq \varrho$.*

Proof. We can show that the rotation of the vector field $\Phi(x)$ at the boundary of $B_+(\varrho)$ is nonzero. □

Proposition 3. *The mappings $\mathcal{E}$ and $\mathcal{E}_S$ are $\mathbb{R}^N_+$-admissible and off-diagonal negative.*

Proof. Let $x \in \partial\mathbb{R}^N_+$ be an arbitrary point. Since $x$ has one or more components equal to zero from the definition of $\mathcal{E}(x)$ (respectively, $\mathcal{E}_S(x)$) we see that $\mathcal{E}(x)$ (respectively, $\mathcal{E}_S(x)$) has one or more components zero or negative. Hence $\mathcal{E}(x) \notin \operatorname{int}(\mathbb{R}^N_+)$ (respectively, $\mathcal{E}_S(x) \notin \operatorname{int}(\mathbb{R}^N_+)$), which means that $\mathcal{E}(x)$ (respectively $\mathcal{E}_S(x)$) is $\mathbb{R}^N_+$-admissible.

If now $x = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) \in \mathbb{R}^N_+$ then, using again the definition of $\mathcal{E}(x)$ (respectively, $\mathcal{E}_S(x)$) we see that $\mathcal{E}(x)$ (respectively, $\mathcal{E}_S(x)$) has the $i$th component negative or zero, which implies that $\mathcal{E}$ (respectively, $\mathcal{E}_S$) is off-diagonal negative. □

THEOREM 4. *If, for a positive number $r$, we have* $\operatorname{ind}(\mathcal{E}, r) = 1$ (*respectively,* $\operatorname{ind}(\mathcal{E}_S, r) = 1$)*, then the problem* (FCP) (*respectively,* (SFCP)) *has at least one solution.*

PROOF. Since the problem (FCP) (respectively, (SFCP)) is equivalent to $\mathrm{GOCP}(\{T_i\}_{i=1}^2, \mathbb{R}_+^N)$ (respectively, $\mathrm{GOCP}(\{S_i\}_{i=1}^2, \mathbb{R}_+^N)$), we must show that the equation $\mathcal{E}(x) = 0$ (respectively, $\mathcal{E}_S(x) = 0$) has a solution in $\mathbb{R}_+^N$. Because $\operatorname{ind}(\mathcal{E}, r) = 1$ we know that $\mathcal{E}(x)$ and $\Im(x)$ are $\mathbb{R}_+^N$-homotopic on $S_+(r)$. Let $H$ be this homotopy: $H(x, 0) = \Im(x)$ and $H(x, 1) = \mathcal{E}(x)$. We now consider the mapping $\mathcal{E}^*$ defined on $\mathbb{R}_+^N$ by

$$\mathcal{E}^*(x) = \begin{cases} \mathcal{E}(x) & \text{if } \|x\| \le r, \\ \dfrac{\|x\|}{r} H\left(\dfrac{rx}{\|x\|}, 2 - \dfrac{\|x\|}{r}\right) & \text{if } r \le \|x\| \le 2r, \\ \Im(x) & \text{if } \|x\| \ge 2r. \end{cases}$$

We apply the Lemma to the mapping $\mathcal{E}^*$ and the set $S_+(2r)$ (i.e. with $\varrho = 2r$) and we obtain a solution $x^*$ of the equation $\mathcal{E}^*(x) = 0$ such that $\|x^*\| \le 2r$. Since the homotopy $H(x, t)$ is nonsingular, we must have $\|x^*\| \le r$, which implies that $\mathcal{E}(x^*) = 0$. For the equation $\mathcal{E}_S(x) = 0$ the proof is similar. □

For practical problems it is important to know if a Fold Complementarity Problem has a nonzero solution. The next results are in this sense.

THEOREM 5. *If for the mapping $\mathcal{E}$* (*respectively, $\mathcal{E}_S$*) *there exist* $0 < r_1 < r_2$ *such that* $\operatorname{ind}(\mathcal{E}, r_1) = 0$ *and* $\operatorname{ind}(\mathcal{E}, r_2) = 1$ (*respectively,* $\operatorname{ind}(\mathcal{E}_S, r_1) = 0$ *and* $\operatorname{ind}(\mathcal{E}_S, r_2) = 1$)*, then the problem* (FCP) (*respectively,* (SFCP)) *has at least one nonzero solution $x^*$.*

PROOF. Since $\operatorname{ind}(\mathcal{E}, r_1) = 0$, there exists an $\mathbb{R}_+^N$-homotopy $H(x, t)$ on $S_+(r_1)$ such that $H(x, 0) = \mathcal{E}(x)$ and $H(x, 1) = H_{r_1}(x; h_0)$. We define on $\mathbb{R}_+^N$ the mapping $\mathcal{E}^0$ by

$$\mathcal{E}^0(x) = \begin{cases} (x - r_1 h_0)/2 & \text{if } \|x\| \le r_1/2, \\ \dfrac{\|x\|}{r_1} H\left(\dfrac{r_1 x}{\|x\|}, \dfrac{2\|x\|}{r_1} - 1\right) & \text{if } r_1/2 \le \|x\| \le r_1, \\ \mathcal{E}(x) & \text{if } \|x\| \ge r_1. \end{cases}$$

Since $r_2 > r_1$ we have $\mathcal{E}^0(x) = \mathcal{E}(x)$ for every $x \in \mathbb{R}_+^N$ such that $\|x\| > r_1$ and hence $\operatorname{ind}(\mathcal{E}^0, r_2) = 1$. Applying Theorem 4 we obtain an element $x^* \in \mathbb{R}_+^N$ such that $\mathcal{E}^0(x^*) = 0$. Since $H(x, t)$ is a nonsingular $\mathbb{R}_+^N$-homotopy we must have $\|x^*\| \ge r_1$, which implies that $\mathcal{E}(x^*) = 0$. For $\mathcal{E}_S$ the proof is similar. □

THEOREM 6. *If for the mapping $\mathcal{E}$* (*respectively, $\mathcal{E}_S$*) *there exists $r > 0$* (*possibly very small*) *with $r^2 < 1$ such that* $\operatorname{ind}(\mathcal{E}, r) = 1$ *and* $\operatorname{ind}(\mathcal{E}, 1/r) = 0$ (*respectively,* $\operatorname{ind}(\mathcal{E}_S, r) = 1$ *and* $\operatorname{ind}(\mathcal{E}_S, 1/r) = 0$) *then the problem* (FCP) (*respectively,* (SFCP)) *has at least one nonzero solution.*

Proof. In this case we reduce the problem to the situation studied in Theorem 5, considering the mapping

$$\Re(x) = \begin{cases} \|x\|^2\mathcal{E}(x/\|x\|^2) & \text{if } \|x\| \geq r, \\ r^2\mathcal{E}(x/r^2) & \text{if } \|x\| < r, \end{cases}$$

defined for all $x \in \mathbb{R}^N_+$. The mapping $\Re$ is continuous and $\mathbb{R}^N_+$-admissible.

If $H^1(x,t)$ is an $\mathbb{R}^N_+$-homotopy on $S_+(r)$ from $\mathcal{E}(x)$ to $\Im(x)$ and $H^2(x,t)$ an $\mathbb{R}^N_+$-homotopy on $S_+(1/r)$ from $\mathcal{E}(x)$ to $H_{1/r}(x;h_0)$, then $\|x\|H^1(x/\|x\|^2,t)$ is an $\mathbb{R}^N_+$-homotopy on $S_+(1/r)$ from $\Re(x)$ to $\Im(x)$ and $\|x\|^2H^2(x/\|x\|^2,t)$ an $\mathbb{R}^N_+$-homotopy on $S_+(r)$ from $\Re(x)$ to $H_r(x;h_0)$ (since $H_{1/r}(x;h_0)$ and $H_r(x;h_0)$ are $\mathbb{R}^N_+$-homotopic because $1/r > r$). Hence $\operatorname{ind}(\Re, r) = 0$ and $\operatorname{ind}(\Re, 1/r) = 1$.

By Theorem 5 the equation $\Re(x) = 0$ has a solution $x^*$ such that $\|x^*\| \geq r$. Evidently, $x^0 = x^*/\|x^*\|$ is a nonzero solution of the equation $\mathcal{E}(x) = 0$. The same proof works for the problem (SFCP) using the mapping $\mathcal{E}_S$ and the equation $\mathcal{E}_S(x) = 0$. □

Remark. In the proofs of Theorems 4 and 5 we esentially used Opoĭtsev's ideas. Theorem 6 is a new result.

For the next result we introduce the following notation.

If $x, y \in \mathbb{R}^N$ we write $x \nleq y$ if there exists at least one $i$ $(1 \leq i \leq N)$ such that $x_i > y_i$.

Theorem 7. *If the following assumptions are satisfied:*

(1) *for $r_1 > 0$ (possibly, sufficiently small) we have $0 \nleq \mathcal{E}(x)$ for all $x \in S_+(r_1)$,*
(2) *for $r_2 > 0$ (possibly, sufficiently large) and such that $0 < r_1 < r_2$ we have $\mathcal{E}(x) \nleq 0$ for all $x \in S_+(r_2)$,*

*then the problem* (FCP) *has a nonzero solution.*

Proof. Taking a real number $\delta > 0$, possibly sufficiently large, we consider the linear homotopy

$$H(x,t) = t\mathcal{E}(x) + (1-t)(x - \delta h_0), \tag{2}$$

with $h_0 \in \mathbb{R}^N_+$ such that $\|h_0\| > 1$. We show that there exists $\delta$ such that $H(x,t)$ is an $\mathbb{R}^N_+$-homotopy on $S_+(r_1)$. Indeed, we remark that $H(x,t)$ is an $\mathbb{R}^N_+$-admissible mapping for any fixed $t \in [0,1]$. To prove that there exists $\delta \geq r_1$ such that $H(x,t)$ is a nonsingular homotopy on $S_+(r_1)$, we assume the contrary. Hence, we can suppose that there exists a sequence $\{\delta_n\}$ such that $\{\delta_n\} \to \infty$ as $n \to \infty$ and two sequences, $\{x^n\} \subset S_+(r_1)$ and $\{t_n\} \subset [0,1]$ such that

$$t_n\mathcal{E}(x^n) + (1-t_n)x^n = (1-t_n)\delta_n h_0 \quad \text{for every } n \in \mathbb{N}. \tag{3}$$

Using the compactness and (3) we find (considering a subsequence if necessary) that $\{x^n\} \to x^*$ and $\{t_n\} \to t^*$. But from (3) we see that $\{(1-t_n)\delta_n\} \to \alpha \geq 0$, and since $\{\delta_n\} \to \infty$ we must have $\{t_n\} \to 1$.

Computing the limit in (3) we deduce that $\mathcal{E}(x^*) = \alpha h_0 \geq 0$, which is a contradiction with assumption (1). Hence, there exists $\delta \geq r_1$ such that $H(x,t)$ is an $\mathbb{R}^N_+$-homotopy from $\mathcal{E}(x)$ to $H_{r_1}(x; h_0)$ (replacing $h_0$ by $\delta h_0/r_1$ if necessary), which implies that $\operatorname{ind}(\mathcal{E}, r_1) = 0$. On the other hand, from assumption (2) and the fact that $\mathcal{E}(x)$ is off-diagonal negative we see that for $r_2 > 0$ possibly sufficiently large and such that $r_1 < r_2$, $\mathcal{E}(x)$ and $\Im(x)$ are $\mathbb{R}^N_+$-homotopic on $S_+(r_2)$ by the linear homotopy $H(x,t) = tx + (1-t)\mathcal{E}(x)$, which is nonsingular for every $t \in [0,1]$ and every $x \in S_+(r_2)$. Thus, $\operatorname{ind}(\mathcal{E}, r_2) = 1$. Applying now Theorem 5 we conclude that the problem (FCP) has a nonzero solution. □

Theorem 8. *If the following assumptions are satisfied:*

(1) *for* $r > 0$ (*possibly, sufficiently small*) *such that* $r^2 < 1$, *we have* $\mathcal{E}(x) \nleq 0$ *for all* $x \in S_+(r)$,

(2) $0 \nleq \mathcal{E}(x)$ *for all* $x \in S_+(1/r)$,

*then the problem* (FCP) *has a nonzero solution.*

Proof. The proof is similar to the proof of Theorem 7 but using Theorem 6. □

Remarks. (1) Theorems 7 and 8 are also valid for the problem (SFCP). Certainly in this case we use the mapping $\mathcal{E}_S(x)$.

(2) The conditions used in Theorems 7 and 8 are very natural in the case of Fold Complementarity Problems, since in this case the mapping $F(x)$ has the form $F(x) = u(x) - v$ and conditions (i) and (ii) are consequences of the order relation between the utility function $u(x)$ and the utility value $\nu$.

We denote by $[x]^+$ the vector $\sup\{0, x\}$ in $\mathbb{R}^N$ ordered by $\mathbb{R}^N_+$.

Theorem 9. *If for* $r > 0$ *sufficiently large we have*

$$\|[\Im(x) - \Psi(F(x))]^+\| < \|\Im(x) \wedge \Psi(F(x))\| \quad \text{for all } x \in S_+(r) \tag{4}$$

*then the problem* (FCP) *has at least one solution.*

Proof. We remark that the linear homotopy $H(x,t) = (1-t)\Im(x) + t\mathcal{E}(x)$, $t \in [0,1]$, is $\mathbb{R}^N_+$-admissible and since we have

$$\begin{aligned}\|\Im(x) - \mathcal{E}(x)\| &= \|\Im(x) - (-(\Im(x) \vee (-\Psi(F(x)))))\| \\ &= \|0 \vee (\Im(x) - \Psi(F(x)))\| = \|[\Im(x) - \Psi(F(x))]^+\| \\ &< \|\Im(x) \wedge \Psi(F(x))\| = \|\mathcal{E}(x)\|\end{aligned}$$

for all $x \in S_+(r)$, we can apply the Poincaré–Bohl Theorem [18] to deduce that $H(x,t)$ is an $\mathbb{R}^N_+$-homotopy from $\mathcal{E}(x)$ to $\Im(x)$ on $S_+(r)$. The theorem is a consequence of Theorem 4. $\square$

Remark. Theorem 9 is valid also for the problem (SFCP), but in this case condition (4) must be replaced by

$$\|(\Im(x) - S_1(x)) \vee (\Im(x) - S_2(x))\| \le \|S_1(x) \wedge S_2(x)\| \tag{5}$$

for all $x \in S_+(r)$.

### The index at infinity and the Hyers–Ulam stability

In this section we establish an interesting relation between the computation of the index for $r$ sufficiently large and the Hyers–Ulam stability of mappings. Our aim is not to introduce the reader in the theory of Hyers–Ulam stability, but to show how we can use some results obtained recently in this theory to compute the index.

Definition 5. We say that a mapping $f : \mathbb{R}_+ \to \mathbb{R}^N$ is *$\psi$-additive* if and only if there exists $\theta \ge 0$ and a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \psi(t)/t = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\psi(\|x\|) + \psi(\|y\|))$$

for all $x, y \in \mathbb{R}^N$.

The following result is a particular case of Theorem 1 proved in [16].

Theorem 10. *If $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous $\psi$-additive mapping and the function $\psi$ satisfies the following assumptions:*

(1) *$\psi(ts) \le \psi(t)\psi(s)$ for all $t, s \in \mathbb{R}_+$,*
(2) *$\psi(t) < t$ for all $t > 1$,*

*then there exists a unique linear mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - \psi(2)}\psi(\|x\|)$$

*for all $x \in \mathbb{R}^N$. Moreover,*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for all } x \in \mathbb{R}^N.$$

Remarks. (1) Since $\mathbb{R}^N$ is a finite-dimensional vector space, $T$ is continuous and since $\lim_{\|x\|\to\infty} \|f(x) - T(x)\|/\|x\| = 0$, we see that $T$ is the *asymptotic derivative* of $f$.

(2) As we showed in [17], if $f = af_1 + bf_2$, where $a, b \in \mathbb{R}$, $f_1$, $f_2$ are continuous, $f_1$ is $\psi_1$-additive, $f_2$ is $\psi_2$-additive and $\psi_1$, $\psi_2$ satisfy conditions (1)

and (2) of Theorem 10 then the asymptotic derivative of $af_1 + bf_2$ is also the linear operator $T$ defined by

$$T(x) = \lim_{n\to\infty} \frac{af_1(2^n c) + bf_2(2^n x)}{2^n} \quad \text{for all } x \in \mathbb{R}^N.$$

We say that two mappings $f, g : \mathbb{R}^N_+ \to \mathbb{R}^N$ are $\mathbb{R}^N_+$-*asymptotically equivalent* if

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|f(x) - g(x)\|}{\|x\|} = 0.$$

Theorem 11. *If the mapping $\mathcal{E}$ is $\mathbb{R}^N_+$-asymptotically equivalent to a $\psi$-additive mapping $H : \mathbb{R}^N \to \mathbb{R}^N$ and the following assumptions are satisfied:*

(1) *$\psi$ satisfies assumptions* (1) *and* (2) *of Theorem* 10*,*
(2) *$H$ is off-diagonal negative with respect to $\mathbb{R}^N_+$,*
(3) *the operator $T(x) = \lim_{n\to\infty} H(2^n x)/2^n$ is nonsingular on $\mathbb{R}^N$ (i.e. $T(x) \neq 0$ whenever $x \neq 0$),*
(4) *$T$ does not have in $\mathbb{R}^N_+$ eigenvectors corresponding to real eigenvalues $\lambda < 0$,*

*then* $\operatorname{ind}(\mathcal{E}, r) = 1$ *for $r$ sufficiently large.*

Proof. By Theorem 10, the linear operator $T$ is well defined and it is the asymptotic derivative of $H$. Moreover, since $H$ is off-diagonal negative, so is $T$. The operator $T$ is also the asymptotic derivative of $\mathcal{E}$ along the cone $\mathbb{R}^N_+$. Indeed, we have

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|\mathcal{E}(x) - T(x)\|}{\|x\|} \le \lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|\mathcal{E}(x) - H(x)\|}{\|x\|} + \lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|H(x) - T(x)\|}{\|x\|} = 0.$$

We now show that $\operatorname{ind}(\mathcal{E}, r) = \operatorname{ind}(T, r)$ for $r$ sufficiently large. Indeed, since $T$ is nonsingular on $\mathbb{R}^N_+$, we have $\inf\{\|T(x)\| \mid x \in \mathbb{R}^N_+,\ \|x\| = 1\} = \alpha > 0$ and because

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|\mathcal{E}(x) - T(x)\|}{\|x\|} = 0,$$

we deduce that for $r$ sufficiently large,

$$\frac{\|\mathcal{E}(x) - T(x)\|}{\|x\|} < \frac{\|T(x)\|}{\|x\|}$$

for all $x \in S_+(r)$, which implies $\|\mathcal{E}(x) - T(x)\| < \|T(x)\|$ for all $x \in S_+(r)$.

Finally, we deduce that $\mathcal{E}$ and $T$ are $\mathbb{R}^N_+$-homotopic on $S_+(r)$, using the linear homotopy and the Poincaré–Bohl theorem, for $r$ sufficiently large, which implies that $\operatorname{ind}(\mathcal{E}, r) = \operatorname{ind}(T, r)$. Because $T$ does not have in $\mathbb{R}^N_+$ eigenvectors corresponding to a real eigenvalue $\lambda < 0$, we find, using linear homotopy and the fact that $T$ is off-diagonal negative, that $\operatorname{ind}(T, r) = 1$ and the theorem is proved. $\square$

Remark. There is an analogue of Theorem 11 for the mapping $\mathcal{E}_S$.

Another possibility to use $\psi$-additive mappings in the computation of the index at infinity is given by the following theorem. We consider again the problem (FCP) and the mapping $\mathcal{E}(x) = x \wedge \Psi(F(x))$ from $\mathbb{R}^N_+$ into $\mathbb{R}^N$.

Theorem 12. *Let $F$ be a mapping such that $\Psi(F(x))$ is $\mathbb{R}^N_+$-asymptotically equivalent to a $\psi$-additive mapping $H : \mathbb{R}^N \to \mathbb{R}^N$, with $\psi$ satisfying assumptions* (1) *and* (2) *of Theorem* 10. *Let $T(x) = \lim_{n\to\infty} H(2^n x)/2^n$ be the linear mapping associated with $H$ by Theorem* 10. *If the mapping $x \to x \wedge T(x)$ is nonsingular on $\mathbb{R}^N_+$ and it does not have in $\mathbb{R}^N_+$ eigenvectors corresponding to real eigenvalues $\lambda < 0$, then $\operatorname{ind}(\mathcal{E}, r) = 1$ for $r$ sufficiently large.*

Proof. The proof is similar to the proof of Theorem 11, but using the following facts. First, the mapping $x \to x \wedge T(x)$ is off-diagonal negative without the assumption that $\Psi(F(x))$ is off-diagonal negative. Second, even if the mapping $x \to x \wedge T(x)$ is nonlinear we have

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|\mathcal{E}(x) - x \wedge T(x)\|}{\|x\|} = 0.$$

Indeed, we have

$$\begin{aligned}
|\mathcal{E}(x) - x \wedge T(x)| &= |x \wedge \Psi(F(x)) - x \wedge T(x)| \\
&= |(-x) \vee (-T(x)) - (-x) \vee (-\Psi(F(x)))| \\
&= |(-x) + [x - T(x)]^+ - ((-x) + [x - \Psi(F(x))]^+)| \\
&\leq |\Psi(f(x)) - T(x)|.
\end{aligned}$$

Since $\mathbb{R}^N$ is a Hilbert lattice we have $\|\mathcal{E}(x) - x \wedge T(x)\| \leq \|\Psi(F(x)) - T(x)\|$, which implies $\|\mathcal{E}(x) - x \wedge T(x)\| \leq \|\Psi(F(x)) - H(x)\| + \|H(x) - T(x)\|$, and hence

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{R}^N_+}} \frac{\|\mathcal{E}(x) - x \wedge T(x)\|}{\|x\|} = 0.$$

Third, $\inf\{\|x \wedge T(x)\| \mid x \in \mathbb{R}^N_+,\ \|x\| = 1\} = \alpha > 0$. This relation is true since $S_+(1)$ is compact and the mapping $x \to \|x \wedge T(x)\|$ is continuous and strictly positive on $S_+(1)$. Now, for $r$ sufficiently large, we have $\|\mathcal{E}(x) - x \wedge T(x)\| < \|x \wedge T(x)\|$ for all $x \in S_+(r)$ and as in the proof of Theorem 11 we can show that $\operatorname{ind}(\mathcal{E}, r) = \operatorname{ind}(\Im \wedge T, r) = 1$. $\square$

Remarks. 1. A theorem similar to Theorem 12 holds for the mapping $\mathcal{E}_S$.

2. Theorems 11 and 12 are valid if the mapping $H$ is a linear combination of $\psi$-additive mappings.

**Systems of Fold Complementarity Problems**

(A) We now consider a *planned distributive problem*, that is, we consider a distributive problem involving a period of time divided in $m$ subperiods $t_1, \dots, t_m$ and $n$ agents (consumers). For the $j$th agent we consider $k_j$ goods ($k_j \geq 2$, $j = 1, \dots, n$) and we set $N = \sum_{j=1}^{n} k_j$. $\mathbb{R}_+^{k_j}$ denotes the *allocation* set for the $j$th agent and $x = (x_1, \dots, x_j, \dots, x_n)$ for all $j = 1, \dots, n$. Let $u_j^{t_r} : \mathbb{R}_+^N \to \mathbb{R}$ be the $j$th agent's payoff function corresponding to the period $t_r$.

We have $m$ functions from $\mathbb{R}_+^N$ to $\mathbb{R}^n$,

$$u^r(x) = [u_1^{t_r}(x), \dots, u_n^{t_r}(x)], \quad r = 1, \dots, m,$$

and we define the functions

$$F^r(x) := u^r(x) - v \quad \text{for all } x \in \mathbb{R}_+^N,\ r = 1, \dots, m.$$

*Given a vector of utility values $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ the problem is to find the amount of goods and their corresponding distribution so that these utility levels are actually reached and if some agent ends up with utility greater than his component $v_j$ in at least one period $t_r$ ($r = 1, \dots, m$), then he should receive no goods.*

If $\Psi$ is the function used in the study of the problem (FCP), then an element $x_* \in \mathbb{R}_+^N$ is a solution of the above problem if $x_*$ satisfies the system

$$\Im(x) \wedge \Psi(F^r(x)) = 0, \quad r = 1, \dots, m. \tag{6}$$

(B) Another problem similar to the problem described in (A) is the following. Consider a firm producing through $n$ divisions, in a highly competitive environment (think of a chain store, for instance). The incumbent firm is willing to avoid new firms pouring into the market, and tries to implement a deterrence policy by making zero profits. A point $x_j \in \mathbb{R}$ denotes the vector of inputs to be allotted centrally to the $j$th division. Assuming that prices are given, let $c_j^r$, $\Re_j^r$ describe the $j$th division's total cost and revenue in period $r$. These are functions of $x = (x_1, \dots, x_n)$ (since some divisions may well serve overlapping markets), and may change from one period to another, due to the $j$th division's promotion policy or production technology. Define now $u_j^r(x)$ for $x \in \mathbb{R}^N$, as follows: $u_j^r(x) = c_j^r - \Re_j^r(x)$; $j = 1, \dots, n$, $r = 1, \dots, m$. A solution to system (6) (with $v = 0_n$), that is, satisfying

(i) $u_j^r(x) \geq 0$ for all $j$ and $r$,
(ii) $x_j = 0$ if $u_j^r(x) > 0$ for some $r$,

gives us an allocation of inputs such that:

(1) No division will exhibit positive profits (that corresponds to the entry deterrence policy).

(2) If a division is going to make losses in a single period, then it will be closed down (i.e. $x_j = 0$).

The following result gives us the possibility to study the system (6) also by means of the $\mathbb{R}^N_+$-index. We define the mappings $\mathcal{E}^r$ $(r = 1, \ldots, m)$ by $\mathcal{E}^r(x) = \Im(x) \wedge \Psi(F^r(x))$ for all $x \in \mathbb{R}^N_+$.

Theorem 13. *The system* (6) *is equivalent to the following general order complementarity problem:*

$$\mathrm{GOCP}(T_1, T_2, \mathbb{R}^N_+) \qquad \begin{cases} \textit{find } x_* \in \mathbb{R}^N_+ \textit{ such that} \\ T_1(x_*) \wedge T_2(x_*) = 0, \end{cases}$$

*where* $T_1(x) = \bigwedge_{r=1}^m \mathcal{E}^r(x)$ *and* $T_2(x) = -\sum_{r=1}^m \mathcal{E}^r(x)$.

Proof. If $x_*$ is a solution of the system (6) then we see immediately that $x_*$ is a solution of $\mathrm{GOCP}(T_1, T_2, \mathbb{R}^N_+)$. Conversely, let $x_* \in \mathbb{R}^N_+$ be a solution of the problem $\mathrm{GOCP}(T_1, T_2, \mathbb{R}^N_+)$. We have

$$\bigwedge_{r=1}^m \mathcal{E}^r(x) \geq 0 \tag{7}$$

and

$$-\sum_{r=1}^m \mathcal{E}^r(x) \geq 0. \tag{8}$$

From (7) we have $\mathcal{E}^r(x_*) \geq 0$ for all $r = 1, \ldots, m$, which implies

$$\sum_{r=1}^m \mathcal{E}^r(x_*) \geq 0. \tag{9}$$

Hence, from (8) and (9) we deduce

$$\sum_{r=1}^m \mathcal{E}^r(x_*) = 0 \tag{10}$$

and since every $\mathcal{E}^r(x_*)$ is positive we find that $\mathcal{E}^r(x_*) = 0$ for all $r = 1, \ldots, m$, that is, $x_*$ is a solution of the system (6).

Remark. The mapping $\mathcal{E}^{**}(x) = \bigwedge(\bigwedge_{r=1}^m \mathcal{E}^r(x), -\sum_{r=1}^m \mathcal{E}^r(x))$ is $\mathbb{R}^N_+$-admissible and hence we can use the index defined in Definition 3.

## Comments

1. Because the Fold Complementarity Problem can be transformed into a fixed point problem and $\mathbb{R}^N_+$ is normal and regular, we can use the iterative methods developed in our papers [11], [12], [15] based on the $\wedge$-monotone increasing mappings and the coupled fixed points associated with heterotonic operators [19].

2. It is also interesting to study the Fold Complementarity Problem by some special fixed point theorems as for example Horn's fixed point theorem (Theorem 7 of [6]) or the fixed point theorem for quasimonotone mappings proved recently by J. Guillerme [3].

3. As numerical methods for solving the Fold Complementarity Problem we can use

(i) the iterative methods developed in [11], [12], [15],
(ii) the numerical methods based on $B$-differentiable mappings [21]–[24],
(iii) the global optimization as indicated in [12].

**Open problems.** 1. It is interesting to study the solvability of the Fold Complementarity Problem by the index theory when the function $F$ is a $Z$-function in the sense of A. Villar [25].

2. It is important to study the spectrum with respect to $\mathbb{R}^N_+$ of the nonlinear operator $\Im(x) \wedge T(x)$, when $T$ is the linear operator defined in Theorem 10 (using possibly the properties of the mapping $f$).

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G. Isac
Department of Mathematics and Computer Science
Royal Military College of Canada
Kingston, Ontario K7K 5L0, CANADA