Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 8, 1996, 327–333

REAL AND COMPLEX HOMOGENEOUS POLYNOMIAL ORDINARY DIFFERENTIAL EQUATIONS IN n-SPACE AND m-ARY REAL AND COMPLEX NON-ASSOCIATIVE ALGEBRAS IN n-SPACE

F. Brito — Daciberg L. Gonçalves

0. Introduction

There is a long-standing attempt to extend the theory of linear differential systems that are additively perturbed by higher-order terms to differential systems whose lowest degree terms are homogeneous forms of degree m with additive perturbations of degree greater than m. In order to do this, the first step must be the construction of a complete theory of differential systems whose rate functions are homogeneous of degree m (and no higher order perturbations). This will depend upon a full understanding of m-ary algebras over real or complex field, algebras which are commutative, but in general non-associative. The purpose of this work is to give some contributions to this problem. We will generalize results of C. Coleman [C1] and L. Markus [Ma]. Namely, the two results together say:

Theorem. *Let $A \cong \mathbb{R}^n$ be an m-ary real algebra. If $m = 2$ or n is odd, then A has at least one nilpotent or idempotent element. Moreover, the corresponding differential system has at least one line of critical points or a pair of opposite integral rays.*

1991 *Mathematics Subject Classification.* Primary 17A01, 34A34; Secondary 57R91.

See [Ma] and [C1]. More information about these algebras and their relations with differential systems can be found in [R].

We extend this result to the case where m is even without restriction on n. Namely we prove:

Theorem 3.1. *Let* $S = \{\dot{x}_i = \sum_{i_1,\dots,i_m=1}^{m} a^i_{i_1,\dots,i_m} x_{i_1} \dots x_{i_m} : i = 1,\dots,n\}$ *be a homogeneous differential system over* $\mathbb{R}$. *If* m *is even then* S *has either a line of critical points or two opposite rays which are non-critical integral rays.*

Corollary 3.2. *Let* m *be even. Then any real* m*-ary algebra over* $\mathbb{R}^n$ *has either a nilpotent or an idempotent element.*

In the remaining case where m is odd and n is even, we do not expect the result to be true without further hypotheses on the system. See Remark (2) of Section 3. Nevertheless we can show

Theorem 3.3. *If the function* $f(x) = (\dot{x}_1,\dots,\dot{x}_n)$ *misses one direction then* S *has a line of critical points. Otherwise, if the degree of* f *is different from* 1 *then* S *has two opposite rays which are non-critical integral rays.*

Although Theorem 3.1 and Corollary 3.2 are already known (see [BG] and [C2]), we obtain them easily from the results used to prove Theorem 3.4 and Corollary 3.5 stated below. Surprisingly, if we look at systems over $\mathbb{C}$, then we have similar results but basically without restrictions on m and n.

We prove:

Theorem 3.4. *Let* $S = \{\dot{x}_i = \sum_{i_1,\dots,i_m=1}^{m} a^i_{i_1,\dots,i_m} x_{i_1} \dots x_{i_m} : i = 1,\dots,n\}$ *be a homogeneous differential system over* $\mathbb{C}$. *Then there exists either a complex line of critical points or a complex line which is an invariant subset of the system. Moreover, if* $m \neq 1$ *and there is no complex line of critical points, then we can find* $\vec{\mu} \in \mathbb{C}^n$ *such that the complex line generated by* $\vec{\mu}$ *contains a pair of opposite integral rays of the system, namely* $\lambda\vec{\mu}$, $\lambda \in \mathbb{R}^+$, *and* $\lambda\vec{\mu}$, $\lambda \in \mathbb{R}^-$.

Corollary 3.5. *Let* A *be an* m*-ary algebra over* $\mathbb{C}^n$. *Then* A *has either a nilpotent or an idempotent element. In the case* $m \neq 1$, *if* A *has no nilpotent elements then we can find* $x \in A$ *such that* $\mu(x,\dots,x) = \lambda x$ *for some* $\lambda \in \mathbb{R}$.

This note is divided in three sections. In Section 1 we recall some relations between algebras and differential systems.

In Section 2 we study some geometrical problems. Namely, we consider maps $f : S^{2n+1} \to S^{2n+1}$ which are Z_m-equivariant, where Z_m acts freely in the first sphere and either freely or trivially in the second one.

Then we get some results about the existence of fixed points. We look at maps $f : S^{2n+1} \to S^{2n+1}$ which are S^1-equivariant and prove that they leave one orbit invariant.

In Section 3, we obtain results on differential systems over K and non-associative algebras over K, where K is the field of real or complex numbers. See Theorems 3.1, 3.3 and 3.4 and Corollaries 3.2 and 3.5. Finally, we comment on the case where the field is the quaternions.

We would like to thank Prof. E. Fadell for helpful conversations and suggestions which greatly improved an early version of this work.

1. Differential equations and non-associative algebras

We recall the relation between a special type of differential systems and non-associative algebras. This relation justifies the study of these algebras. For more details see [R].

Let

$$S=\left\{\dot{x}_i=\sum_{i_1,\ldots,i_m=1}^{m} a^i_{i_1,\ldots,i_m}x_{i_1}\ldots x_{i_m} : i=1,\ldots,n\right\}$$

be a homogeneous polynomial differential system in K^n, where K is the field of real or complex numbers. We can define an m-ary algebra A_S in K^n, where the m-ary multiplication μ_S is given, on basis elements, by

$$\mu_S(e_{i_1},\ldots,e_{i_m})=\sum_{i=1}^{m} a^i_{i_1,\ldots,i_m}e_i,$$

where $(e_1,\ldots,e_n)$ is the canonical basis of K^n.

Conversely, with every m-ary algebra A there is associated a system S_A (see [R]).

Definition 1.1. An element $x\in A_S$ is said to be *nilpotent* if $\mu_S(x,\ldots,x) = 0$, and *idempotent* if there exists $\lambda\in K$, $\lambda\neq 0$, such that $\mu_S(x,\ldots,x)=\lambda x$.

We now state two results relating the existence of nilpotent and idempotent elements in A_S to certain properties of the differential system S.

Proposition 1.2. *The algebra A_S has a nilpotent element x if and only if the line (K-line) generated by $x\in K^n$ is a line of critical points.*

Proposition 1.3. *An element $x\in A_S$ is idempotent if and only if the one-dimensional subspace generated by x is an invariant subspace of the system. Furthermore, there exists an idempotent element x such that $\mu_S(x,\ldots,x)=\lambda x$ with $\lambda\neq 0$ and λ real if and only if there exists a pair of opposite integral rays, namely $\{\alpha x:\alpha>0\}$ and $\{\alpha x:\alpha<0\}$ of the system S.*

The proofs of Propositions 1.2 and 1.3 are quite straightforward. See [C1], for example, for the real case.

We will generalize Theorem 10 of [C1], which says:

Theorem 1.4. *Let $K = \mathbb{R}$. If n is odd, then A_S has either a nilpotent or an idempotent element. Moreover, the system S has either a line of critical points or a pair of opposite integral rays.*

2. Vector fields and maps between spheres

We will consider maps which arise from differential systems. As before K is the field of real or complex numbers.

Let $\lambda \in S^1$ be a primitive mth root of unity, $m \neq 1$, and Z_m the cyclic group generated by λ. Since S^1 acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$, so does Z_m, by restriction.

Proposition 2.1. *Let $V : S^{2n+1} \to S^{2n+1}$ be a continuous map such that $V(\alpha x) = V(x)$ for all $x \in S^{2n+1}$ and $\alpha \in Z_m$. Then V has a fixed point.*

Proof. The map V factors through the lens space S^{2n+1}/Z_m and therefore the degree of V is divisible by m. The Lefschetz number of V is $1 - \deg(V) = 1 - km \neq 0$. Hence, V has a fixed point. □

Corollary 2.2. *Let $\vec{V}$ be a vector field over S^{2n+1} such that $\vec{V}(x) = \vec{V}(-x)$. Then $\vec{V}$ must have at least one singularity.*

Proof. The vector field $\vec{V}$ defines a function $V : S^{2n+1} \to S^{2n+1}$. By Proposition 2.1 for $m = 2$ the result follows. □

Proposition 2.3. *Let $V : S^{2n+1} \to S^{2n+1}$ be a continuous map such that $-V(x) = V(-x)$. Then V is surjective and if $\deg(V) \neq 1$ then V has a fixed point.*

Proof. Suppose V is not surjective. Take the equator $S^{2n} \subset S^{2n+1}$ which is perpendicular to the direction y which is not in the image of V. Set $\overline{V}(x) = PV(x)/\|PV(x)\|$ where PV is the orthogonal projection of $V(x)$ onto the subspace $\mathbb{R}^{2n+1}$ which contains S^{2n}. So we have a map $\overline{V} : S^{2n+1} \to S^{2n}$ which is Z_2-equivariant. This contradicts the Borsuk–Ulam theorem. See [D2]. So V must be surjective. The second part follows trivially from the Lefschetz fixed point theorem. □

Proposition 2.4. *Let $V : S^{2n+1} \to S^{2n+1}$ be an S^1-equivariant map, that is, $V(\lambda x) = \lambda V(x)$, $\lambda \in S^1$. Then there exist $x \in S^{2n+1}$ and $\lambda \in S^1$ such that $V(x) = \lambda x$.*

Proof. Let V_1 be the induced map in the quotient space $\mathbb{C}P^n = S^{2n+1}/S^1$, and consider the commutative diagram

$$\begin{array}{ccc} S^1_0 & \xrightarrow{\alpha} & S^1_1 \\ \downarrow & & \downarrow \\ S^{2n+1} & \xrightarrow{V} & S^{2n+1} \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \xrightarrow{V_1} & \mathbb{C}P^n \end{array}$$

where the induced map on the fibre S^1_0 is multiplication by $\alpha \in S^1$, for some α. It follows from the homotopy exact sequence that the induced map $V_{1\#} : \pi_2(\mathbb{C}P^n) \to \pi_2(\mathbb{C}P^n)$ is the identity and therefore, so is $V_1^* : H^2(\mathbb{C}P^n, \mathbb{Q}) \to H^2(\mathbb{C}P^n, \mathbb{Q})$. Here $\mathbb{Q}$ stands for the rational numbers. Hence, the Lefschetz number of V_1 is $n+1 \neq 0$ and so V_1 has a fixed point, which implies the result.□

3. Applications

In this section we obtain results on differential systems and consequently, on m-ary algebras, making use of the results of Section 2.

Theorem 3.1. *Let $S = \{\dot{x}_i = \sum_{i_1,\dots,i_m=1}^m a^i_{i_1,\dots,i_m} x_{i_1} \dots x_{i_m} : i = 1, \dots, n\}$ be a homogeneous differential system over $\mathbb{R}$. If m is even then S has either a line of critical points or two opposite rays which are non-critical integral rays.*

Proof. The case of n odd is already known (see [C1]). Hence, assume both m and n to be even.

Consider the map $f(x_1, \dots, x_n) = (\dot{x}_1, \dots, \dot{x}_n)$ restricted to S^{n-1}. Assuming the result does not hold, we have $f(x) \neq 0$ and $f(x) \neq \lambda x$ for all $x \in S^{n-1}$ and $\lambda \in \mathbb{R}$.

Let $\vec{V}(x)$ be the projection of $f(x)$ on the tangent space of S^{n-1} at the point x. So $\vec{V}(x)$ is a non-vanishing vector field which, with no loss of generality, can be assumed to have norm 1. But, since m is even, $f(x) = f(-x)$ and therefore $\vec{V}(x) = \vec{V}(-x)$, which is, by Corollary 2.2, a contradiction. □

Remarks. (1) The case $n = 2$ was already known (see Lemma 4 of [Ma]).

(2) If n is even and m is odd, the result does not hold. To see this consider the following example:

$$\dot{x}_i = \begin{cases} -(x_1^2 + \dots + x_n^2)^{(m-1)/2} x_{n-i+1}, & i = 1, \dots, n/2, \\ (x_1^2 + \dots + x_n^2)^{(m-1)/2} x_{n-i+1}, & i = n/2+1, \dots, n. \end{cases}$$

Corollary 3.2. *Let m be even. Then any real m-ary algebra over $\mathbb{R}^n$ has either a nilpotent or an idempotent element.*

For the purpose of the next results, let $V(x) = (\dot{x}_1, \dots, \dot{x}_n)/\|(\dot{x}_1, \dots, \dot{x}_n)\|$.

Theorem 3.3. *Let* $S = \{\dot{x}_i = \sum_{i_1,\dots,i_m=1}^m a^i_{i_1,\dots,i_m} x_{i_1} \dots x_{i_m} : i = 1,\dots,n\}$ *be a homogeneous differential system over* $\mathbb{R}$. *If* m *is odd and* n *is even, and* $f(x) = (\dot{x}_1,\dots,\dot{x}_n)$ *misses one direction, then* S *has a line of critical points. Otherwise, if* $\deg(V) \neq 1$ *then* S *has two opposite rays which are non-critical integral rays.*

Proof. Suppose that f has no singularity outside the origin. Then $V(x)$ is a well defined map from S^{2n+1} to S^{2n+1}. Since $V(x)$ is not surjective by Proposition 2.3, this is a contradiction. So S must have a singularity. Finally, if S has no singularity outside the origin then V is a well defined map, which by Proposition 2.3 implies the result. □

Theorem 3.4. *Let* $S = \{\dot{x}_i = \sum_{i_1,\dots,i_m=1}^m a^i_{i_1,\dots,i_m} x_{i_1} \dots x_{i_m} : i = 1,\dots,n\}$ *be a homogeneous differential system over* $\mathbb{C}$. *Then there exists either a complex line of critical points or a complex line which is an invariant subset of the system. Moreover, if* $m \neq 1$ *and there is no complex line of critical points, then we can find* $\vec{\mu} \in \mathbb{C}^n$ *such that the complex line generated by* $\vec{\mu}$ *contains a pair of opposite integral rays of the system, namely* $\lambda\vec{\mu}$, $\lambda \in \mathbb{R}^+$, *and* $\lambda\vec{\mu}$, $\lambda \in \mathbb{R}^-$.

Proof. Suppose S has no singularities outside the origin. Assume first $m \neq 1$. Consider $V(x_1,\dots,x_n) = (\dot{x}_1,\dots,\dot{x}_n)/\|(\dot{x}_1,\dots,\dot{x}_n)\|$ restricted to the sphere $S^{2n-1} \subset \mathbb{C}^n$. Then V satisfies the hypothesis of Proposition 2.1, and therefore there exists $x \in S^{2n-1}$ such that $V(x) = x$, which means $f(x) = \lambda x$ for some $\lambda \in \mathbb{R}^+$, where $f(x_1,\dots,x_n) = (\dot{x}_1,\dots,\dot{x}_n)$. The complex line generated by x proves the theorem.

Assume now $m = 1$. Proposition 2.4, applied to the map $V(x_1,\dots,x_n) = (\dot{x}_1,\dots\dot{x}_n)/\|(\dot{x}_1,\dots,\dot{x}_n)\|$, says that there exists $x \in S^{2n-1}$ such that $V(x) = \alpha x$ for some $\alpha \in S^1$. The complex line generated by x proves the result. □

Corollary 3.5. *Let* A *be an* m*-ary algebra over* $\mathbb{C}^n$. *Then* A *has either a nilpotent or an idempotent element. In the case* $m \neq 1$, *if* A *has no nilpotent elements then we can find* $x \in A$ *such that* $\mu(x,\dots,x) = \lambda x$ *for some* $\lambda \in \mathbb{R}$.

Remarks. (i) For the case $m = 1$, one cannot expect to get real rays which are integral rays of the system. Take, for example, $\dot{x} = ix$.

(ii) Results like Theorem 3.4 and Corollary 3.5 hold true for the field $\mathbb{H}$ of quaternions and follow directly from the complex case. In terms of algebras we get the following: every m-ary quaternionic algebra A over H^n has either a nilpotent or an idempotent element, i.e., there exists $x \in A$ such that $\mu(x,\dots,x) = \lambda x$ for some $\lambda \in \mathbb{C}$. In case $m \neq 1$, there exists $x \in A$ such that $\mu(x,\dots,x) = \lambda x$ for some $\lambda \in \mathbb{R}$.

REFERENCES

[BG] F. BRITO AND D. L. GONÇALVES, *Algebras nao associativas, sistemas diferenciais homogeneos e classes caracteristicas*, preprint, Dept. Math., University of São Paulo 8112 (1981), 1–6.

[C1] C. COLEMAN, *Growth and decay estimates near non-elementary stationary points*, Canad. J. Math. **22** (1970), 1156–1167.

[C2] ______, *Boundedness and unboundedness in polynomial systems*, Nonlinear Anal. **8** (1984), 1287-1294.

[D1] A. DOLD, *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[D2] ______, *Simple proofs of some Borsuk–Ulam results*, Contemp. Math., vol. 19, Amer. Math. Soc., 1983, pp. 65–69.

[Ma] L. MARKUS, *Quadratic differential equations and non-associative algebras*, Contributions to the Theory of Non-Linear Oscillations, vol. V, Princeton University Press, Princeton, N.J., 1960, pp. 185–212.

[R] H. RÖHRL, *Algebras and differential equations*, Nagoya Math. J. **68** (1977), 59–122.

Manuscript received March 10, 1996

FABIANO GUSTAVO BRAGA BRITO
Departamento de Matemática – IME-USP
Caixa Postal 66.281 – Ag. Cidade de São Paulo
05389-970 – São Paulo – SP, BRASIL

E-mail address: brito@ime.usp.br

DACIBERG LIMA GONÇALVES
Departamento de Matemática – IME-USP
Caixa Postal 66.281 – Ag. Cidade de São Paulo
05389-970 – São Paulo – SP, BRASIL

E-mail address: dlgoncal@ime.usp.br