

ON SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH “MAXIMA”

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The method of *a priori* estimates based on the Leray–Schauder topological degree theory is developed to establish the existence of solutions to general boundary value problems for differential equations with “maxima”.

1. Introduction

In the theory of automatic control of various technical systems it often occurs that the law of regulation depends on maximum values of some regulated state parameters over certain time intervals [8]. This is especially the case for stabilization systems, where the regulated quantity usually represents the deviation of some state parameters from the given value. The mathematical models for such systems naturally include boundary value problems for differential equations with “maxima”. As a typical example, consider the following rather general boundary value problem on a finite time interval:

$$(1) \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}(t), \max_{\tau \in S(t)} \mathbf{x}(\tau)), & t \in [a, b], \\ \mathbf{x}(t) = 0, & t \notin [a, b], \\ \phi(\mathbf{x}) = 0, \end{cases}$$

where $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$ is an unknown vector function, $S(t) \subset \mathbb{R}$, $\max_{\tau \in S(t)} \mathbf{x}(\tau)$ stands for the vector with the components $\max_{\tau \in S(t)} x_i(\tau)$, $i = 1, \dots, n$, $\mathbf{f} :$

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$[a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and ϕ is some vector functional over a space of vector functions on $[a, b]$ to be specified in the sequel, which represents the boundary conditions. We emphasize that the additional assumption $\mathbf{x}(t) = 0$ for $t \notin [a, b]$ is unnecessary if $S(t) \subset [a, b]$ for all $t \in [a, b]$. In the opposite case, however, one can expect different types of such additional assumption depending on applications. For example, instead of $\mathbf{x} = 0$ it can be required that $\mathbf{x}(t) = \boldsymbol{\varphi}(t)$ outside the interval $[a, b]$, where $\boldsymbol{\varphi}(\cdot) \in \mathbb{R}^n$ is some given vector function, or $\mathbf{x}(t) = \mathbf{x}(a)$ for $t \leq a$ and $\mathbf{x}(t) = \mathbf{x}(b)$ for $t \geq b$. Furthermore, the right side of the differential equation in (1) can also be more complex, e.g. there can occur the dependence on maximum values of different components of the state vector \mathbf{x} on different time intervals. It is also worth noting that in most real systems the law of regulation depends only on the past and present state, and thus $S(t) \subset [-\infty, t]$. In this case one observes an obvious analogy with retarded functional differential equations.

The questions of solvability of problems of the above type have recently been raised by V. G. Angelov, D. D. Bainov and S. G. Hristova [1, 4]. In particular, in [1] initial value problems for some differential equations with “maxima” were considered in a rather restricted space of continuously differentiable functions with bounded (on the whole real axis) derivatives, and Cauchy–Picard’s type existence and uniqueness results were obtained under suitable Lipschitz continuity assumptions on the right side. In [4] the periodic boundary value problem was studied by means of monotone iterative techniques with $S(t) := [t - h, t]$, $h > 0$, and under some monotonicity assumptions on the right side.

In this paper we continue the research in this field and develop the method of *a priori* estimates to get existence results for rather general boundary value problems of type (1). It will be shown that such problems fit well in the theory of abstract functional differential equations [3], and their solutions will be searched in spaces of absolutely continuous vector functions. Our results stated in Section 3 are essentially based on a special development of the Leray–Schauder topological degree theory, which will be presented in Section 2. In the general statements of Section 3 we do not assume that $S(t) \subset [-\infty, t]$, thus allowing the dependence of the law of regulation on the future. However, we show in a particular example how such a requirement can help in establishing existence of solutions via majorant techniques like those developed for general functional differential equations by V. P. Maksimov [3].

2. Abstract groundwork

Function spaces. Let $L^p(a, b)$ stand for the standard Lebesgue space of functions integrable on the interval (a, b) with exponent $1 \leq p < \infty$ (or essentially bounded when $p = \infty$). In the sequel we make an extensive use of the space $L^p((a, b); \mathbb{R}^n)$ (denoted by L_n^p for brevity) of functions with values in \mathbb{R}^n with

components in $L^p(a, b)$, equipped with its usual norm

$$\|\mathbf{x}\|_p := \begin{cases} (\int_a^b |\mathbf{x}(\tau)|^p d\tau)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{[a,b]} |\mathbf{x}(t)|, & p = \infty, \end{cases}$$

where $|\cdot|$ stands for the norm in \mathbb{R}^n . We denote by AC_n^p the space of vector functions with absolutely continuous components and with derivatives in L_n^p , equipped with the norm $\|\mathbf{x}\|_{AC_n^p} := |\mathbf{x}(a)| + \|\dot{\mathbf{x}}\|_p$. Also we denote by $C_n[a, b]$ (C_n , for short) the space of all \mathbb{R}^n -valued functions with continuous components, equipped with the usual maximum norm.

Abstract scheme. Suppose we have to solve a nonlinear boundary value problem

$$(2) \quad \begin{cases} Lx = F(x), \\ \phi(x) = 0, \end{cases}$$

where $x \in X$ is an unknown, $L : X \rightarrow Y$ is some linear operator, $F : X \rightarrow Y$ is a nonlinear operator, $\phi : X \rightarrow \mathbb{R}^n$ is a vector functional, generally speaking, nonlinear, and X and Y are given Banach spaces. Let X be isomorphic to the direct product of some Banach space E and the finite-dimensional space \mathbb{R}^n by an isomorphism

$$J : E \times \mathbb{R}^n \ni (u, \lambda) \mapsto \Lambda u + D\lambda \in X,$$

where $\Lambda : E \rightarrow X$ and $D : \mathbb{R}^n \rightarrow X$. Assuming that $Q = L\Lambda : E \rightarrow E$ is invertible, we can then reduce the problem (2) to the form

$$(3) \quad \begin{cases} u = \mathcal{F}(u, \lambda), \\ \mathbf{D}(u, \lambda) = 0, \end{cases}$$

where $u \in E$ and $\lambda \in \mathbb{R}^n$ are unknowns, $\mathcal{F} : E \times \mathbb{R}^n \rightarrow E$ is a nonlinear operator and $\mathbf{D} : E \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a nonlinear vector functional. To study the solvability of the latter we apply the topological degree theory.

Leray–Schauder degree theory. For the purpose of this paper the Leray–Schauder topological degree theory will be applied to mappings of the form

$$(4) \quad \Psi(u, \lambda) = \begin{Bmatrix} u - \mathcal{F}(u, \lambda) \\ \mathbf{D}(u, \lambda) \end{Bmatrix}.$$

Introduce regions (open bounded subsets) $\Omega_1 \subset E$ and $\Omega_2 \subset \mathbb{R}^n$ and denote their boundaries by $\partial\Omega_1$ and $\partial\Omega_2$ respectively. Also, let $\Omega = \Omega_1 \times \Omega_2$.

To develop the Leray–Schauder degree theory one needs the compactness of Ψ , which is provided by the following assumption:

- (C) \mathcal{F} is a compact and continuous operator, and \mathbf{D} is a continuous vector functional which maps bounded sets into bounded sets.

In fact, (C) clearly implies that the mapping Ψ is a compact perturbation of the identity in $E \times \mathbb{R}^n$. Then the additional requirement of nondegeneracy of the mapping Ψ on $\partial\Omega$ is enough to define correctly the topological degree $\deg(\Psi, \Omega, 0)$, having all the ordinary properties (see [7]). We say that the mapping Ψ (the system (3)) is *topologically nontrivial* on the regions $\Omega_1 \subset E$ and $\Omega_2 \subset \mathbb{R}^n$ (is $V(\Omega_1, \Omega_2)$, for short) whenever $\deg(\Psi, \Omega, 0) \neq 0$. Thus, if the system (3) is $V(\Omega_1, \Omega_2)$, then it admits at least one solution $(u, \lambda) \in \Omega$.

To calculate the topological degree of Ψ we will use the following result extending the analogous statements by J. Cronin [7] and S. A. Vavilov [10]. Consider an auxiliary finite-dimensional continuous vector field $\mathbf{D}_0(\lambda) : \mathbb{R}^n \rightarrow \mathbb{R}^n$. In the sequel we will always take $\mathbf{D}_0(\lambda) := \mathbf{D}(0, \lambda)$.

THEOREM 1. *Suppose $\Omega_1 \subset E$ is convex, \mathcal{F} is a compact and continuous operator, \mathbf{D} is a continuous vector functional which maps bounded sets into bounded sets, and:*

- (i) $\mathcal{F}(\partial\Omega_1, \text{cl}\Omega_2) \subset \Omega_1$;
- (ii) $\forall u \in \text{cl}\Omega_1, \forall \lambda \in \partial\Omega_2, 0 \leq |\mathbf{D}(u, \lambda) - \mathbf{D}_0(\lambda)| < |\mathbf{D}_0(\lambda)|$.

Then $\deg(\Psi, \Omega, 0) = \deg(\mathbf{D}_0, \Omega_2, 0)$. In particular, when the latter is not zero, then the system (3) has at least one solution $(u', \lambda') \in \Omega$. The set of such solutions can be approximated by the Galerkin numerical scheme applied to (3).

REMARK. Suppose $0 \in \Omega_1$. The statement of the theorem remains valid if (i) is replaced by

$$\lambda \in \text{cl}\Omega_2, t \in [0, 1], u = t\mathcal{F}(u, \lambda) \Rightarrow u \notin \partial\Omega_1.$$

If, in particular, Ω_1 is an open ball with center zero, then the latter condition follows from the existence of a uniform *a priori* estimate for $u \in E$ from the first equation of (3).

PROOF OF THEOREM 1. Choose some $u_0 \in \Omega_1$ and consider the homotopy

$$\Psi_t(u, \lambda) = \left\{ \begin{array}{l} u - (1-t)u_0 - t\mathcal{F}(u, \lambda) \\ \mathbf{D}_0(\lambda) + t(\mathbf{D}(u, \lambda) - \mathbf{D}_0(\lambda)) \end{array} \right\}, \quad t \in [0, 1].$$

Since $\partial\Omega = \partial\Omega_1 \times \text{cl}\Omega_2 \cup \text{cl}\Omega_1 \times \partial\Omega_2$, we observe that $\Psi_t \neq 0$ on $\partial\Omega$ for all $t \in [0, 1]$. Furthermore, clearly $\deg(\Psi_0, \Omega, 0) = \deg(\mathbf{D}_0, \Omega_2, 0)$ by the properties of the degree [6]. Thus, noting the compactness of the above homotopy and applying the homotopy invariance of the degree, we conclude the proof. \square

To calculate the topological degree of Ψ we will also use the result stated in [9] and connected with the analysis of the “iterated” mapping

$$(5) \quad \Psi^{(1)}(u, \lambda) = \left\{ \begin{array}{l} u - \mathcal{F}(u, \lambda) \\ \mathbf{D}(\mathcal{F}(u, \lambda), \lambda) \end{array} \right\}.$$

THEOREM 2. *If both Ψ and $\Psi^{(1)}$ have the compactness property (C), and one of them is nondegenerate on $\partial\Omega$, then so is the other. Moreover, in this case*

$$\deg(\Psi, \Omega, 0) = \deg(\Psi^{(1)}, \Omega, 0).$$

PROOF. The first part of the statement is obvious, for the sets of zeros of Ψ and $\Psi^{(1)}$ coincide. To prove the second part, note that the “vector fields” Ψ and $\Psi^{(1)}$ can have opposite directions on $\partial\Omega$ only if $u = \mathcal{F}(u, \lambda)$. But in this case their finite-dimensional components are equal and nonzero due to nondegeneracy. Therefore, Ψ and $\Psi^{(1)}$ never have opposite directions, which implies the statement. \square

3. Existence results

Let the boundary conditions in (1) be represented by a nonlinear vector functional $\phi : AC_n^q \rightarrow \mathbb{R}^n$, $1 < q < \infty$. Applying the isomorphism between AC_n^q and $L_n^q \times \mathbb{R}^n$ given by the formula

$$J : L_n^q \times \mathbb{R}^n \ni (\mathbf{u}, \lambda) \mapsto \mathbf{x} = \int_a^t \mathbf{u}(\tau) d\tau + \lambda \in AC_n^q,$$

we reduce the original problem to the following system of type (3):

$$(6) \quad \begin{cases} \mathbf{u}(t) = \mathbf{f}\left(t, \lambda + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \lambda + \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right), \\ \phi\left(\lambda + \int_a^t \mathbf{u}(\tau) d\tau\right) = 0. \end{cases}$$

Here and in the sequel we write for brevity $\tilde{S}(t) := S(t) \cap [a, b]$ and

$$0_S(t) = \begin{cases} 0, & S(t) \neq \tilde{S}(t), \\ -\infty, & S(t) = \tilde{S}(t). \end{cases}$$

Assume, further, that the set function $S(\cdot)$ takes closed values and is measurable in the sense that for any open $V \subset \mathbb{R}$ the set

$$S^{-1}(V) := \{t \mid S(t) \cap V \neq \emptyset\}$$

is measurable. Note that there are various measurability criteria for set functions (see, for instance, Chapter III of [5]). In particular, our assumptions hold when $S(t) := [r(t), s(t)]$, where $r(\cdot)$ and $s(\cdot)$ are measurable (not necessarily almost everywhere finite) functions. Finally, define

$$\gamma := \left(\frac{\sup \bigcup_{t \in [a, b]} \tilde{S}(t) - a}{b - a} \right)^{(q-1)/q}.$$

To get a solvability result for the original problem, we can now use the topological degree theory by applying Theorems 1 and 2.

We now pass to important particular examples.

Consider a general nonlinear two-point boundary value problem for a differential equation with “maxima”:

$$(7) \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}(t), \max_{\tau \in S(t)} \mathbf{x}(\tau)), & t \in [a, b], \\ \mathbf{x}(t) = 0, & t \notin [a, b], \\ \mathbf{h}(\mathbf{x}(a), \mathbf{x}(b)) = 0, \end{cases}$$

where $\mathbf{h} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$. After the above reduction the system (6) assumes the form

$$(8) \quad \begin{cases} \mathbf{u}(t) = \mathbf{f}\left(t, \boldsymbol{\lambda} + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \boldsymbol{\lambda} + \max_{\tau \in \bar{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right), \\ \mathbf{h}\left(\boldsymbol{\lambda}, \boldsymbol{\lambda} + \int_a^b \mathbf{u}(\tau) d\tau\right) = 0. \end{cases}$$

THEOREM 3. *Assume that the following conditions hold:*

- (i) $\mathbf{f}(t, \mathbf{x}, \mathbf{y})$ is a Carathéodory vector function (i.e. continuous in $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$ for a.e. $t \in [a, b]$ and measurable in t for each (\mathbf{x}, \mathbf{y})), while for a.e. $t \in [a, b]$,

$$\begin{aligned} |\mathbf{x}| \leq U, |\mathbf{y}| \leq V &\Rightarrow |\mathbf{f}(t, \mathbf{x}, \mathbf{y})| \leq \alpha(t, U, V), \quad \|\alpha(\cdot, U, V)\|_q \leq \Phi(U, V), \\ \left. \begin{array}{l} |\mathbf{x}_1| \leq U, |\mathbf{x}_2| \leq U \\ |\mathbf{y}_1| \leq V, |\mathbf{y}_2| \leq V \end{array} \right\} &\Rightarrow \left| \int_a^b (\mathbf{f}(t, \mathbf{x}_1, \mathbf{y}_1) - \mathbf{f}(t, \mathbf{x}_2, \mathbf{y}_2)) dt \right| \leq \Phi_1(U, V). \end{aligned}$$

- (ii) The vector function $\mathbf{h}(\mathbf{x}, \mathbf{y})$ is continuous, and

$$|\mathbf{x}| \leq U, |\mathbf{y}_1| \leq V, |\mathbf{y}_2| \leq V \Rightarrow |\mathbf{h}(\mathbf{x}, \mathbf{y}_1) - \mathbf{h}(\mathbf{x}, \mathbf{y}_2)| \leq \delta(U, V)|\mathbf{y}_1 - \mathbf{y}_2|.$$

- (iii) $|\mathbf{D}_0(\boldsymbol{\lambda})| \geq \beta(\varrho_2) > 0$ if $|\boldsymbol{\lambda}| = \varrho_2$, while $\deg(\mathbf{D}_0, |\boldsymbol{\lambda}| < \varrho_2, 0) \neq 0$, where

$$\mathbf{D}_0(\boldsymbol{\lambda}) = \mathbf{h}\left(\boldsymbol{\lambda}, \boldsymbol{\lambda} + \int_a^b \mathbf{f}(\tau, \boldsymbol{\lambda}, \max(0_S(\tau), \boldsymbol{\lambda})) d\tau\right).$$

- (iv) We have

$$(9) \quad \begin{cases} \Phi(\varrho_2 + \varrho_1, \varrho_2 + \gamma\varrho_1) < \varrho_1/(b-a)^{(q-1)/q}, \\ \delta(\varrho_2, \varrho_2 + \Phi(\varrho_2 + \varrho_1, \varrho_2 + \gamma\varrho_1))\Phi_1(\varrho_2 + \varrho_1, \varrho_2 + \gamma\varrho_1) < \beta(\varrho_2). \end{cases}$$

Then the two-point boundary value problem (7) has at least one solution $\mathbf{x} \in AC_n^q$, $1 < q < \infty$, satisfying $\|\dot{\mathbf{x}}\|_q < \varrho_1/(b-a)^{(q-1)/q}$ and $|\mathbf{x}(a)| < \varrho_2$.

REMARKS. 1. In (i) one may choose $\Phi_1(U, V) := 2(b-a)^{(q-1)/q}\Phi(U, V)$. However, in applications one can often get sharper estimates.

2. The theorem also opens the way to numerical treatment of the original problem (7). Namely, the set of pairs $(\mathbf{u}, \boldsymbol{\lambda})$ in appropriate regions (see the proof), which corresponds to the solutions of (7) found in the theorem, can be approximated by the Galerkin numerical scheme applied to (8). The same refers to all the statements below.

PROOF OF THEOREM 3. Substituting the first equation of (8) into the second, we obtain the system

$$(10) \quad \begin{cases} \mathbf{u}(t) = \mathbf{f}\left(t, \boldsymbol{\lambda} + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \boldsymbol{\lambda} + \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right), \\ \mathbf{h}\left(\boldsymbol{\lambda}, \boldsymbol{\lambda} + \int_a^b \mathbf{f}\left(t, \boldsymbol{\lambda} + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \boldsymbol{\lambda} + \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right) dt\right) = 0. \end{cases}$$

Obviously to any solution $(\mathbf{u}, \boldsymbol{\lambda}) \in L_n^q \times \mathbb{R}^n$ of the latter there corresponds a solution $\mathbf{x} \in AC_n^q$ of the original problem (7). Consider the open balls $B_1 \subset L_n^q$ and $B_2 \subset \mathbb{R}^n$,

$$B_1 := \{\mathbf{u} \mid \|\mathbf{u}\|_q < \varrho_1 / (b - a)^{(q-1)/q}\}, \quad B_2 := \{\boldsymbol{\lambda} \mid |\boldsymbol{\lambda}| < \varrho_2\}.$$

The statement will be proven if we show that the system (10) is $V(B_1, B_2)$. For this purpose we apply Theorem 1.

To verify the compactness assumption (C), note that according to (i) the Nemytskiĭ operator

$$N : L_n^\infty \times L_n^\infty \ni (\mathbf{v}_1, \mathbf{v}_2) \mapsto \mathbf{f}(\cdot, \mathbf{v}_1(\cdot), \mathbf{v}_2(\cdot)) \in L_n^q$$

is continuous and maps bounded sets into bounded sets. Now define formally an operator M on the space C_n by

$$(M\mathbf{v})(t) := \max_{\tau \in \tilde{S}(t)} \mathbf{v}(t).$$

For any $\mathbf{v} \in C_n$ clearly $(M\mathbf{v})(t)$ is measurable due to the Krasnosel’skiĭ–Ladyzhenskii lemma (see Lemma III.39 of [5] or Theorem 6.2 of [2]) and bounded. Thus $M : C_n \rightarrow L_n^\infty$ and obviously it maps bounded sets into bounded sets. The continuity of M follows from the fact that if a sequence of continuous functions converges uniformly on $[a, b]$, then their maxima on any compact subset of $[a, b]$ converge to the maximum of the limit function on the same subset, the rate of the latter convergence being independent of the choice of the subset. Hence one easily shows that the compactness of the nonlinear operator on the right side of the first equation of (10) is ensured by the compactness of the imbedding $AC_n^q \subset C_n$, while the continuity of the vector functional corresponding to the second equation is provided by (ii). To conclude the proof it remains to observe that conditions (i) and (ii) of Theorem 1 are ensured by (9). \square

As an important particular case consider the boundary value problem

$$(11) \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}(t), \max_{\tau \in S(t)} \mathbf{x}(\tau)), & t \in [a, b], \\ \mathbf{x}(t) = 0, & t \notin [a, b], \\ \mathbf{x}(a) = \mathbf{x}(b) + \xi \mathbf{h}_1(\mathbf{x}(a), \mathbf{x}(b)), \end{cases}$$

where $\mathbf{h}_1 : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ and $\xi \in \mathbb{R}$ is a perturbation parameter. For $\xi = 0$ this is a periodic-type boundary value problem (however, one should not speak of periodic solutions unless $S(t) \subset [-\infty, t]$). One is therefore interested both in existence of solutions for $\xi = 0$ and in whether they do not disappear for small values of ξ .

THEOREM 4. *Assume that the following conditions hold:*

(i) $\mathbf{f}(t, \mathbf{x}, \mathbf{y})$ is a Carathéodory vector function, and for a.e. $t \in [a, b]$,

$$\begin{aligned} |\mathbf{x}| \leq U, |\mathbf{y}| \leq V &\Rightarrow |\mathbf{f}(t, \mathbf{x}, \mathbf{y})| \leq \alpha(t, U, V), \quad \|\alpha(\cdot, U, V)\|_q \leq \Phi(U, V), \\ \left. \begin{array}{l} |\mathbf{x}_1| \leq U, |\mathbf{x}_2| \leq U \\ |\mathbf{y}_1| \leq V, |\mathbf{y}_2| \leq V \end{array} \right\} &\Rightarrow \left| \int_a^b (\mathbf{f}(t, \mathbf{x}_1, \mathbf{y}_1) - \mathbf{f}(t, \mathbf{x}_2, \mathbf{y}_2)) dt \right| \leq \Phi_1(U, V). \end{aligned}$$

(ii) The vector function $\mathbf{h}_1(\mathbf{x}, \mathbf{y})$ is continuous.

(iii) $|\mathbf{D}_0(\lambda)| \geq \beta(\varrho_2) > 0$ if $|\lambda| = \varrho_2$, while $\deg(\mathbf{D}_0, |\lambda| < \varrho_2, 0) \neq 0$, where

$$\mathbf{D}_0(\lambda) = \int_a^b \mathbf{f}(\tau, \lambda, \max(0_S(\tau), \lambda)) d\tau.$$

(iv) We have

$$(12) \quad \begin{cases} \Phi(\varrho_2 + \varrho_1, \varrho_2 + \gamma\varrho_1) < \varrho_1/(b-a)^{(q-1)/q}, \\ \Phi_1(\varrho_2 + \varrho_1, \varrho_2 + \gamma\varrho_1) < \beta(\varrho_2). \end{cases}$$

Then there is $\xi^* > 0$ such that for each ξ with $|\xi| \leq \xi^*$ the boundary value problem (11) has at least one solution $\mathbf{x}_\xi \in AC_n^q$, $1 < q < \infty$, satisfying $\|\dot{\mathbf{x}}_\xi\|_q < \varrho_1/(b-a)^{(q-1)/q}$ and $|\mathbf{x}_\xi(a)| < \varrho_2$.

REMARK. As will be clear from the proof, one can in fact also assert that as $\xi \rightarrow 0$, the sets of solutions to (11) found in the theorem are uniformly “attracted” to the set of solutions to the unperturbed problem with $\xi = 0$ (see [6]).

PROOF. Write out the system of type (3) corresponding to the original problem:

$$(13) \quad \begin{cases} \mathbf{u}(t) = \mathbf{f}\left(t, \lambda + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \lambda + \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right), \\ \int_a^b \mathbf{u}(\tau) d\tau + \xi \mathbf{h}_1\left(\lambda, \lambda + \int_a^b \mathbf{u}(\tau) d\tau\right) = 0, \end{cases}$$

and consider the iteration, as introduced by (5), of the system obtained by setting $\xi = 0$:

$$(14) \quad \begin{cases} \mathbf{u}(t) = \mathbf{f}\left(t, \lambda + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \lambda + \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right), \\ \int_a^b \mathbf{f}\left(t, \lambda + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \lambda + \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right) dt = 0. \end{cases}$$

Let B_1 and B_2 be the balls introduced in the proof of Theorem 3. The assumptions imply that (14) is $V(B_1, B_2)$ according to Theorem 1. Applying the iteration theorem 2, one observes that the system (13) for $\xi = 0$ is also $V(B_1, B_2)$. The desired conclusion follows now from the stability of topological degree with respect to small perturbations of the mapping [6]. \square

Another classical application is the Cauchy problem

$$(15) \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}(t), \max_{\tau \in S(t)} \mathbf{x}(\tau)), & t \in [a, b], \\ \mathbf{x}(t) = 0, & t \notin [a, b], \\ \mathbf{x}(a) = \boldsymbol{\lambda}_0, \end{cases}$$

where $\boldsymbol{\lambda}_0 \in \mathbb{R}^n$. It is worth noting that the system (8) in this case is reduced to a single equation:

$$(16) \quad \mathbf{u}(t) = \mathbf{f}\left(t, \boldsymbol{\lambda}_0 + \int_a^t \mathbf{u}(\tau) d\tau, \max\left(0_S, \boldsymbol{\lambda}_0 + \max_{\tau \in S(t)} \int_a^\tau \mathbf{u}(s) ds\right)\right).$$

The following statement can be obtained either as a simple corollary of Theorem 3 or independently by applying the Schauder fixed point principle to (16).

THEOREM 5. *Assume that the following conditions hold:*

(i) $\mathbf{f}(t, \mathbf{x}, \mathbf{y})$ is a Carathéodory vector function, and for a.e. $t \in [a, b]$,

$$|\mathbf{x}| \leq U, |\mathbf{y}| \leq V \Rightarrow |\mathbf{f}(t, \mathbf{x}, \mathbf{y})| \leq \alpha(t, U, V), \quad \|\alpha(\cdot, U, V)\|_q \leq \Phi(U, V).$$

(ii) *We have*

$$(17) \quad \Phi(|\boldsymbol{\lambda}_0| + \varrho_1, |\boldsymbol{\lambda}_0| + \gamma\varrho_1) < \varrho_1 / (b - a)^{(q-1)/q}.$$

Then the Cauchy problem (15) admits at least one solution $\mathbf{x} \in AC_n^q$, $1 < q < \infty$, satisfying $\|\dot{\mathbf{x}}\|_q < \varrho_1 / (b - a)^{(q-1)/q}$.

It should be noted that usually one can get significantly stronger results on solvability of boundary value problems of the above types provided that the assumption $S(t) \subset [-\infty, t]$ holds. In fact, the operator M introduced in the proof of Theorem 3 is a Volterra operator. This simplifies the situation and enables one to use various majorant techniques described in [3] (in particular, Cauchy majorant problems) to get the necessary *a priori* estimates. To show this we make the above-mentioned assumption in the following simple example which rather frequently appears in applications (see, for instance, the model of stabilization system for a direct current generator [4]):

$$(18) \quad \begin{cases} \dot{\mathbf{x}} = \mathcal{F}(t) + \mathcal{B}(t) \max_{\tau \in S(t)} \mathbf{x}(\tau), & t \in [a, b], \\ \mathbf{x}(t) = \mathbf{x}(a) = \mathbf{x}(b), & t \leq a, \end{cases}$$

where $\mathcal{B}(\cdot)$ is an $n \times n$ matrix function, and $\mathcal{F}(\cdot)$ is a vector function. For brevity set $B := \int_a^b \mathcal{B}(t) dt$, $F := \int_a^b \mathcal{F}(t) dt$, $\beta_1 := \int_a^b \|\mathcal{B}(t)\| dt$, $f(t) := |\mathcal{F}(t)|$,

$b(t) := \|\mathcal{B}(t)\|$, where $\|\cdot\|$ stands for the matrix norm compatible with the norm in \mathbb{R}^n . Also, let

$$\mu(t) := b(t) \exp \left(\int_a^t b(\tau) d\tau \right).$$

THEOREM 6. *Let $f \in L_1^q$, $b \in L_1^q$ and $\theta := \inf_{|\lambda|=1} |B\lambda| > 0$. Then the problem (18) admits at least one solution $\mathbf{x} \in AC_n^q$, $1 < q < \infty$, provided that*

$$(b - a)^{(q-1)/q} \|\mu\|_q \beta_1 < \theta.$$

PROOF. Write out the system of equations of type (3) corresponding to the problem (18):

$$(19) \quad \begin{cases} \mathbf{u}(t) = \mathcal{F}(t) + \mathcal{B}(t) \left(\boldsymbol{\lambda} + \max \left(0_S, \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds \right) \right), \\ F + B\boldsymbol{\lambda} + \int_a^b \mathcal{B}(t) \max \left(0_S, \max_{\tau \in \tilde{S}(t)} \int_a^\tau \mathbf{u}(s) ds \right) dt = 0. \end{cases}$$

We will prove that for some $B_1 \subset L_n^q$ and $B_2 \subset \mathbb{R}^n$ this system is $V(B_1, B_2)$ according to Theorem 1, and thus show the statement. In fact, let B_1 be as in the proof of Theorem 3 and $B_2 \subset \mathbb{R}^n$ be the ball $|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*| < \varrho_2$, $\boldsymbol{\lambda}^* := -B^{-1}F$. Consider the auxiliary vector field

$$\mathbf{D}_0(\boldsymbol{\lambda}) := F + B\boldsymbol{\lambda}.$$

It is clear that $\text{deg}(\mathbf{D}_0, B_2, 0) \neq 0$ and $|\mathbf{D}_0(\boldsymbol{\lambda})| \geq \theta \varrho_2$ when $|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*| = \varrho_2$. Condition (ii) of Theorem 1 then holds provided that

$$(20) \quad \beta_1 \varrho_1 < \theta \varrho_2.$$

Now turn to the first equation of (19). To abbreviate the notation, let $y(t) := |\mathbf{u}(t)|$ and $y_0 := |\boldsymbol{\lambda}|$. Obviously this equation implies

$$y(t) \leq f(t) + b(t) \left(y_0 + \int_a^t y(\tau) d\tau \right),$$

and constructing the Cauchy majorant problem and using a Chaplygin type result on differential inequalities [3] one easily concludes that

$$y(t) \leq f(t) + \mu(t) \left(y_0 + \int_a^t (f(\tau)/\mu(\tau)) d\tau \right).$$

Hence if $\boldsymbol{\lambda} \in \text{cl } B_2$, then $\|\mathbf{u}\|_q \leq c_1 + \|\mu\|_q (\varrho_2 + c_2)$, where c_1, c_2 are some positive constants. Condition (i) of Theorem 1 will then hold provided that

$$(21) \quad c_1 + \|\mu\|_q (\varrho_2 + c_2) < \varrho_1 / (b - a)^{(q-1)/q}.$$

It remains to note that under the conditions of the theorem being proved the relations (20) and (21) are valid simultaneously for sufficiently large $\varrho_2 > 0$. \square

Analogous results can be easily provided for more general problems

$$(22) \quad \begin{cases} \dot{\mathbf{x}} = \mathcal{F}(t) + \mathcal{A}(t)\mathbf{x}(t) + \mathcal{B}(t) \max_{\tau \in S(t)} \mathbf{x}(\tau), & t \in [a, b], \\ \mathbf{x}(t) = \mathbf{x}(a) = \mathbf{x}(b), & t \leq a, \end{cases}$$

where $\mathcal{A}(\cdot)$ is an $n \times n$ matrix function. In this case seeking solutions in the form

$$\mathbf{x}(t) = X(t)\boldsymbol{\lambda} + X(t) \int_a^t X^{-1}(\tau)\mathbf{u}(\tau) d\tau,$$

where $X(t)$ is the fundamental matrix of the system $\dot{\mathbf{x}} - \mathcal{A}(t)\mathbf{x} = 0$, we obtain an auxiliary system of type (19) to be analyzed,

$$\begin{cases} \mathbf{u}(t) = \mathcal{F}(t) + \mathcal{B}(t) \max \left(\boldsymbol{\lambda}, X(t)\boldsymbol{\lambda} + \max_{\tau \in \tilde{S}(t)} X(\tau) \int_a^\tau X^{-1}(s)\mathbf{u}(s) ds \right), \\ \tilde{F} + (X(b) - I_n)\boldsymbol{\lambda} \\ + X(b) \int_a^b \mathcal{B}(t) \max \left(\boldsymbol{\lambda}, X(t)\boldsymbol{\lambda} + \max_{\tau \in \tilde{S}(t)} X(\tau) \int_a^\tau X^{-1}(s)\mathbf{u}(s) ds \right) dt = 0, \end{cases}$$

where $\tilde{F} = X(b) \int_a^b X^{-1}(t)\mathcal{F}(t) dt$ and I_n is the $n \times n$ identity matrix.

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