Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 8, 1996, 295–314

EXISTENCE RESULTS FOR RESONANT PERTURBATIONS OF THE FUČIK SPECTRUM

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1. Introduction

1.1. Let Ω be a bounded smooth domain in $\mathbb{R}^m$, $m \geq 1$. The so-called *Fučik spectrum* of $-\Delta$ on Ω with Dirichlet boundary condition is the set $\Sigma_0 = \Sigma(H_0^1(\Omega))$ of all pairs $(\alpha,\lambda) \in \mathbb{R}^2$ for which there exists a nonzero solution of the problem

$$(\mathrm{P})_{(\alpha,\lambda)} \qquad -\Delta u = \lambda u_+ - \alpha u_-, \quad u \in H_0^1(\Omega),$$

where $u_+ = \max\{u, 0\}$ and $u_- = u_+ - u$ (see [Fu], where this notion of spectrum was introduced). In this way, when $(\alpha,\lambda) \in \Sigma_0$, problem $(\mathrm{P})_{(\alpha,\lambda)}$ will be called a *resonant problem* and it is clear that Σ_0 contains the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ as well as the points (λ_k, λ_k), where $0 < \lambda_1 < \lambda_2 < \ldots$ are the distinct eigenvalues of $-\Delta$ on $H_0^1(\Omega)$. Also, in the one-dimensional case $m = 1$ (say, with $\Omega = (0,T)$), it is easy to see that Σ_0 is the union of the two lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ and the curves

$$\Gamma_{n,p} = \{(\alpha,\lambda) \in \mathbb{R}_+^2 \mid n/\sqrt{\alpha} + p/\sqrt{\lambda} = T/\pi\},$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, $n, p \geq 1$ are integers with $|n - p| = 0$ or 1, and $n + p \geq 2$. The situation in the case of a general domain $\Omega \subset \mathbb{R}^m$ is much more

1991 *Mathematics Subject Classification.* 35J25, 35J50.

Research of the second author supported by the F.N.R.S. (Belgium) and the European Network ERBCHRXCT940555. The second author wishes to thank the Department of Mathematical Sciences, UNLV, for their hospitality during her stay in May 1994.

delicate and only a few general facts are known about Σ_0 (cf. [Ca, Da, dF-Go, Ga-Ka, Ma, Ru]). Among other results, [Da] shows that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in Σ_0, and [dF-Go] shows that

$$\Sigma_0 = \mathbb{R} \times \{\lambda_1\} \cup \{\lambda_1\} \times \mathbb{R} \cup C_2 \cup \Sigma_0^1$$

where $C_2 = \{(\alpha(r), \lambda(r)) \in \mathbb{R}^2 \mid r > 0\}$ is a continuous, strictly decreasing curve asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$, with $\alpha(r) > \lambda_1$ and $\lambda(r) > \lambda_1$ for all $r > 0$, and where the set Σ_0^1 is contained in the component of $\mathbb{R}^2 \setminus C_2$ to which the point (λ_1, λ_1) does not belong. Recently, we learned about the results of [Sc3] which, extending the methods of [Ca], provide the most complete description of the Fučik spectrum and complement all the previously known results. In particular, it is shown in [Sc3] that there exist decreasing continuous curves $C_{N,1}$ and $C_{N,2}$ passing through each eigenvalue λ_N (with $C_{N,2}$ *above* $C_{N,1}$, and possibly coincident) such that, in each square $\Lambda_N = [\lambda_{N-1}, \lambda_{N+1}] \times [\lambda_{N-1}, \lambda_{N+1}]$, there are no points in the Fučik spectrum inside Λ_N lying below $C_{N,1}$ or above $C_{N,2}$.

1.2. In this paper we consider perturbations

$$\text{(P)} \qquad -\Delta u = \lambda u_+ - \alpha u_- + g(x,u), \quad u \in H_0^1(\Omega),$$

of the resonant problem $(\text{P})_{(\alpha,\lambda)}$ with $(\alpha, \lambda) \in C_2$ which are also *resonant* in the sense that the primitive $G(x,s) = \int_0^s g(x,\sigma)\,d\sigma$ satisfies

$$\text{(R)} \qquad \lim_{|s|\to\infty} 2G(x,s)/s^2 = 0 \quad \text{uniformly for a.e. } x \in \Omega.$$

Keeping in mind the situation $\alpha = \lambda = \lambda_2$ with $g(x,s) = g(s) + h(x)$, $h \in L^\infty(\Omega)$ and $g(s)$ having finite limits $g(\pm\infty) = \lim_{s\to\pm\infty} g(s)$ which satisfy $\pm g(\pm\infty) > 0$, $g(-\infty) < g(s) < g(\infty)$ or $\pm g(\pm\infty) < 0$, $g(\infty) < g(s) < g(-\infty)$ (*Landesman–Lazer* [L-L] *situations*), it is clear that additional conditions are necessary for (P) to have a solution. Here, we assume that $G(x,s)$ is *nonquadratic at infinity* ([Co-Ma2]) in the sense that either $(\text{NQ})_+$ or $(\text{NQ})_-$ below holds:

$$(\text{NQ})_\pm \qquad \lim_{|s|\to\infty} [sg(x,s) - 2G(x,s)] = \pm\infty \quad \text{uniformly for a.e. } x \in \Omega.$$

We note that the Landesman–Lazer situation recalled above automatically satisfies $(\text{NQ})_-$ when $\pm g(\pm\infty) > 0$ holds [resp. satisfies $(\text{NQ})_+$ when $\pm g(\pm\infty) < 0$ holds] and $|h|_\infty < \min\{|g(+\infty)|, |g(-\infty)|\}$. We also note that $(\text{NQ})_\pm$ allow many situations where

$$0 = \lim_{|s|\to 0} g(x,s)/s < \limsup_{|s|\to\infty} g(x,s)/s = \infty,$$

for which one has *crossing* of infinitely many eigenvalues (see Section 4). In this respect, our result will complement and extend other results in the literature on *resonant* and *double-resonant* problems (e.g. [Co-Ol], [Co-Ma1,2], [Cu], [Cu-Go], [dF-Go], [dF-Mi], [dF-Ru], [Gn-Mi], [Mw-Wd-Wi], [Sc1,2], [Si1,2], [So]).

Our approach is variational and is motivated by the works [Co-Ma, Cu, Cu-Go]. Throughout this paper, the *perturbation* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be a Carathéodory function with *subcritical growth*, that is,

$$\text{(SG)} \qquad |g(x,s)| \le a_0|s|^{p-1} + b_0 \quad \text{for a.e. } x \in \Omega,\ s \in \mathbb{R},$$

for some $a_0, b_0 > 0$ and $1 < p < 2m/(m-2)$ if $m \ge 3$ [resp. $1 < p < \infty$ if $m = 1, 2$]. So, the weak solutions of (P) are precisely the critical points of the associated C^1-functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

$$(1) \qquad I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u_+^2 - \alpha u_-^2)\,dx - \int_\Omega G(x,u)\,dx = q(u) - N(u).$$

In Section 2 we prove

Theorem 1. *Let* $(\alpha,\lambda) \in C_2$. *Under conditions* (R) *and* $(\mathrm{NQ})_+$, *problem* (P) *has at least one weak solution* $u \in H_0^1(\Omega)$.

1.3. From a variational point of view, the geometry of the function I is such that it is *anticoercive* on the one-dimensional subspace $\langle\phi_1\rangle$ spanned by the first eigenfunction of $-\Delta$ on $H_0^1(\Omega)$. On the other hand, as a consequence of $(\mathrm{NQ})_+$ and of the variational characterization of C_2 given in [dF-Go], the functional I is *bounded from below* on a corresponding cone $M \subset H_0^1(\Omega)$ of codimension 1 which "separates" $t\phi_1$ and $-t\phi_1$ for all $t > 0$. Moreover, either nonquadraticity condition $(\mathrm{NQ})_+$ or $(\mathrm{NQ})_-$ implies (cf. [Co-Ma2]) a compactness condition of Palais–Smale type [Ce], so that a variant of the Saddle-Point Theorem of Rabinowitz [Ra] can be used to prove existence of a critical point for I.

Now, when the nonquadraticity condition $(\mathrm{NQ})_-$ is assumed instead of $(\mathrm{NQ})_+$, the functional I turns out to be anticoercive on any two-dimensional half-space $\langle\phi_1\rangle \oplus \mathbb{R}_+ v$ where v is a nonzero solution of $(\mathrm{P})_{(\alpha,\lambda)}$. The variational characterization of C_2 in [dF-Go] provides one such v, but there is no known way (cf. Section 4) of determining all these "generalized eigenfunctions" of $(\mathrm{P})_{(\alpha,\lambda)}$. This information is clearly needed for the geometry of the functional in order to possibly apply some *higher linking* variational result. Moreover, although the Fučik spectrum is explicitly known in the one-dimensional situation, no variational characterization of that spectrum is known under Dirichlet boundary conditions. When periodic boundary conditions are assumed instead of the Dirichlet condition, the corresponding Fučik spectrum has been characterized by [dF-Ru] using variational methods. Therefore, we will next consider the corresponding one-dimensional problem under periodic boundary conditions

$$(\mathrm{P})_1 \qquad \begin{cases} -u'' = \lambda u_+ - \alpha u_- + g(x,u), \\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi), \end{cases}$$

and prove in Section 3 the following:

Theorem 2. *Assume* $(\alpha,\lambda) \in C_N$, $N \geq 2$ *and that* g *satisfies* (R). *Then*

(a) *if* $(\mathrm{NQ})_+$ *holds, then problem* $(\mathrm{P})_1$ *has at least one solution;*

(b) *if* $(\mathrm{NQ})_-$ *holds and* $\alpha \geq \lambda_{N-1}$, $\lambda \geq \lambda_{N-1}$ *then* $(\mathrm{P})_1$ *has at least one solution.*

2. The Dirichlet problem

In this section we prove Theorem 1. The proof will be a consequence of Theorem 2.1 and Corollary 2.5 below.

2.1. We start by recalling a compactness condition of Palais–Smale type introduced by Cerami [Ce] (see also [Ba-Be-F], [Co-Ma2], [Gn-Mi] for applications). Let $(X, \|\cdot\|)$ be a real Banach space. A functional $I \in C^1(X,\mathbb{R})$ is said to satisfy *condition* (C) *at the level* $c \in \mathbb{R}$ if the following holds:

(C) Any sequence $(u_n) \subset X$ such that $I(u_n) \to c$ and $(1+\|u\|)\|I'(u_n)\| \to 0$ has a convergent subsequence.

If (C) holds for all $c \in \mathbb{R}$ we say that I *satisfies condition* (C).

In [Ba-Be-F, Theorem 1.3] it was shown that condition (C) suffices to get a deformation theorem. Then, by standard minimax arguments, the following critical point theorem holds true (see [Mw-Wi]).

Theorem 2.1 [Mw-Wi]. *Let* $I \in C^1(X,\mathbb{R})$ *satisfy condition* (C). *Assume that there exist a compact metric space* K, *a closed part* $K_0 \subset K$ *and a map* $h_0 \in C(K_0, K)$ *such that*

$$\max_{z\in K_0} I(h_0(z)) < \max_{z \in K} I(h(z)) \tag{2.1}$$

for any $h \in \Gamma = \{h \in C(K,X) \mid h|_{K_0} = h_0\}$. *Set*

$$c = \inf_{h\in\Gamma} \max_{u \in h(K)} I(u).$$

Then c *is a critical value of* I.

2.2. Let us now consider $X = H_0^1(\Omega)$ and the functional I defined in (1.1) associated with problem (P). We have the following two analogues of Lemmas 1.2 and 3.1 of [Co-Ma2].

Lemma 2.2. *Assume that g satisfies* (R) *and either* $(\mathrm{NQ})_+$ *or* $(\mathrm{NQ})_-$. *Then I satisfies condition* (C).

Proof. In view of the growth condition (SC) on g, it suffices to show that "(C)-sequences" are bounded. Assume that $(\mathrm{NQ})_+$ holds true (the proof under $(\mathrm{NQ})_-$ is analogous). By contradiction, suppose that I does not satisfy condition (C) for some $c \in \mathbb{R}$. Then there exists a sequence $u_n \in H_0^1(\Omega)$ such that $I(u_n) \to c$ and $\|I'(u_n)\| \cdot \|u_n\| \to 0$ but $\|u_n\| \to \infty$. Hence

$$\lim_{n\to\infty} \int_\Omega (g(x,u_n)u_n - 2G(x,u_n))\,dx = \lim_{n\to\infty} (2I(u_n) - (I'(u_n),u_n)) = 2c. \tag{2.2}$$

On the other hand, we claim that there exists some subset $\Omega_0 \subset \Omega$, with $|\Omega_0| > 0$, such that $|u_n(x)| \to \infty$ for a.e. $x \in \Omega_0$. To prove this claim, consider the sequence $\widehat{u}_n = u_n/\|u_n\|$ that converges to some $\widehat{u}$ in the weak sense in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. By Fatou's lemma and condition (R), we have

$$\limsup_{n\to\infty} \frac{1}{\|u_n\|^2} \int_\Omega G(x,u_n(x))\,dx \le 0$$

and, consequently,

$$0 = \lim_{n\to\infty} \frac{1}{\|u_n\|^2} I(u_n) \ge \frac{1}{2}(1 - \lambda\|\widehat{u}_+\|_2^2 - \alpha\|\widehat{u}_-\|_2^2).$$

Hence $\widehat{u} \ne 0$ and it suffices to choose $\Omega_0 = \{x \in \Omega \mid \widehat{u}(x) \ne 0\}$ to prove the claim.

Now, using hypothesis $(\mathrm{NQ})_+$ we obtain

$$\lim_{n\to\infty} [g(x,u_n(x))u_n(x) - 2G(x,u_n(x))] = \infty \quad \text{for a.e. } x \in \Omega_0.$$

Moreover, since $(\mathrm{NQ})_+$ and (SG) imply that

$$g(x,u_n)u_n(x) - 2G(x,u_n(x)) \ge -M \quad \text{for a.e. } x \in \Omega,$$

for some $M > 0$, we conclude that

$$\begin{aligned}\liminf_{n\to\infty} &\int_\Omega (g(x,u_n)u_n - 2G(x,u_n))\,dx \\ &\ge \liminf_{n\to\infty} \int_{\Omega_0} (g(x,u_n)u_n - 2G(x,u_n))\,dx - M|\Omega \setminus \Omega_0| = \infty,\end{aligned}$$

which contradicts (2.2). □

Lemma 2.3. *Assume that g satisfies* (R).

(a) *If* $(\mathrm{NQ})_+$ *holds then* $\lim_{|s|\to\infty} G(x,s) = -\infty$.
(b) *If* $(\mathrm{NQ})_-$ *holds then* $\lim_{|s|\to\infty} G(x,s) = \infty$.

Proof. (a) By $(\mathrm{NQ})_+$, given $M > 0$ there exists $s_M > 0$ such that

$$g(x,s)s - 2G(x,s) \geq M \quad \forall |s| \geq s_M, \text{ for a.e. } x \in \Omega. \tag{2.3}$$

Using (2.3) and integrating the identity

$$\frac{d}{ds}\left(\frac{G(x,s)}{s^2}\right) = \frac{g(x,s)s - 2G(x,s)}{s^3}$$

over $[s,\overline{s}] \subset [s_M,\infty]$ gives the inequality

$$\frac{G(x,\overline{s})}{\overline{s}^2} - \frac{G(x,s)}{s^2} \geq -\frac{M}{2}\left(\frac{1}{\overline{s}^2} - \frac{1}{s^2}\right)$$

so that, in view of (R), we obtain

$$G(x,s) \leq -M/2 \quad \forall s \geq s_M, \text{ for a.e. } x \in \Omega.$$

Analogously we prove that $G(x,s) \leq -M/2$ for all $s \leq -s_M$ and a.e. $x \in \Omega$. Since M is arbitrary, we conclude that (a) holds true. The proof of (b) is similar and we omit it. □

2.3. We shall use these two lemmas to study the geometry of the functional I in the next proposition. We recall that, according to our notation, ϕ_1 is the positive eigenfunction associated with the first eigenvalue λ_1 of $-\Delta$ in $H_0^1(\Omega)$ and satisfying $\|\phi_1\|_2 = 1$.

Proposition 2.4. *Assume* (R) *and* $(\mathrm{NQ})_+$. *Consider the subspace* $V = \langle\phi_1\rangle$ *and the cone*

$$M = \left\{u \in H_0^1(\Omega) \;\middle|\; \int_\Omega u_+\phi_1\,dx = \frac{\alpha - \lambda_1}{\lambda - \lambda_1}\int_\Omega u_-\phi_1\,dx\right\}.$$

Then

(i) $\lim_{\|u\|\to\infty, u\in V} I(u) = -\infty$;
(ii) $\inf_{u\in M} I(u) > -\infty$.

Proof. (i) Let $u = t\phi_1$, $t \geq 0$. We have

$$I(u) = \frac{1}{2}t^2(\lambda_1 - \lambda) - \int_\Omega G(x,t\phi_1(x))\,dx$$

Since $\lim_{|s|\to\infty} 2G(x,s)/s^2 = 0$ uniformly for a.e. $x \in \Omega$, it follows that, for any $\varepsilon > 0$, there exists $M > 0$ such that $2G(x,s) \geq -\varepsilon s^2 - M$ for all $s \in \mathbb{R}$. Hence

$$I(u) \leq \frac{1}{2}t^2(\lambda_1 - \lambda) + \frac{\varepsilon t^2}{2} + \frac{M}{2}|\Omega|.$$

Similarly, for $u = t\phi_1$, $t \leq 0$, we find

$$I(u) \leq \frac{1}{2}t^2(\lambda_1 - \alpha) + \frac{\varepsilon t^2}{2} + \frac{M}{2}|\Omega|.$$

Therefore, by choosing $0 < \varepsilon < \min\{\alpha - \lambda_1, \lambda - \lambda_1\}$ we obtain

$$\lim_{|t|\to\infty} I(t\phi_1) = -\infty.$$

(ii) We shall use the following property of the set M (cf. [dF-Go]):

$$\int_\Omega |\nabla u|^2\, dx \geq \lambda \int_\Omega u_+^2\, dx + \alpha \int_\Omega u_-^2\, dx \quad \forall u \in M. \tag{2.4}$$

Notice that (2.4) is equivalent to the estimate $q(u) \geq 0$ for all $u \in M$. On the other hand, since $\lim_{|s|\to\infty} G(x,s) = -\infty$ uniformly for a.e. $x \in \Omega$ (cf. Lemma 2.3(a)), there exists $M > 0$ such that $G(x,s) \leq M$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$, hence

$$N(u) \leq M|\Omega| \quad \forall u \in H_0^1(\Omega),$$

and we finally obtain the estimate

$$I(u) = q(u) - N(u) \geq -M|\Omega| \quad \forall u \in M,$$

which proves (ii). □

Corollary 2.5. *There exists $T > 0$ such that*

$$\max\{I(T\phi_1), I(-T\phi_1)\} < \inf_{h\in\Lambda} \max_{t\in[-1,1]} I(h(t)),$$

where $\Lambda = \{h \in C([-1,1], H_0^1(\Omega)) \mid h(\pm 1) = \pm T\phi_1\}$.

Remark 2.6. We apply Theorem 2.1 to the functional I with $K = [-1,1]$, $K_0 = \{\pm 1\}$ and $h_0(\pm 1) = \pm T\phi_1$ (i.e., the "mountain pass" result of Ambrosetti–Rabinowitz) to conclude that

$$c = \inf_{h\in\Lambda} \max_{t\in[-1,1]} I(h(t))$$

is a critical value of I.

Proof of Corollary 2.5. By Proposition 2.4(i) we can choose $T > 0$ such that

$$\max\{I(T\phi_1), I(-T\phi_1)\} < \inf_{u\in M} I(u).$$

Let $h \in C([-1,1],\, H_0^1(\Omega))$, $h(\pm 1) = \pm T\phi_1$. It is easy to check that

$$\biggl(\int_\Omega \biggl(h(1)_+ - \frac{\alpha - \lambda_1}{\lambda - \lambda_1} h(1)_-\biggr)\phi_1\, dx\biggr)\biggl(\int_\Omega \biggl(h(-1)_+ - \frac{\alpha - \lambda_1}{\lambda - \lambda_1} h(-1)_-\biggr)\phi_1\, dx\biggr) < 0.$$

Then, by continuity, there exists some $\bar{t} \in\,]-1,1[$ such that

$$\int_\Omega \biggl(h(\bar{t})_+ - \frac{\alpha - \lambda_1}{\lambda - \lambda_1} h(\bar{t})_-\biggr)\phi_1\, dx = 0,$$

i.e., $h(\bar{t}) \in M$. Therefore, we have

$$\max_{t\in[1,1]} I(h(t)) \geq I(h(\bar{t})) \geq \inf_M I > \max\{I(T\phi_1), I(-T\phi_1)\},$$

which concludes the proof. □

3. The periodic problem

3.1. In this section we consider the periodic problem

$$(\mathrm{P})_1 \qquad \begin{cases} -u''(t) = \lambda u_+(t) - \alpha u_-(t) + g(t,u), & t \in]0,2\pi[, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases}$$

where g satisfies the regularity and growth conditions of Section 1.

We shall assume here that $(\alpha,\lambda) \in \Sigma_{\mathrm{p}}$, the Fučik spectrum in the periodic case. This spectrum can be easily computed to be the union of the two lines $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ and the sequence of curves C_N given by

$$C_N = \{(\alpha,\lambda) \in \mathbb{R}^2_+ \mid 1/\sqrt{\alpha} + 1/\sqrt{\lambda} = 2/(N-1)\}, \quad N \geq 2.$$

We recall that $\lambda_N = (N-1)^2$, $N \in \mathbb{N}$, is the sequence of eigenvalues of $(-u'', [0,2\pi])$ with periodic boundary conditions. Notice that, according to the previous notation, the point (λ_N, λ_N) belongs to C_N for all $N \geq 2$.

Let us denote by $H^1_{2\pi}$ the subspace of $H^1(0,2\pi)$ consisting of the 2π-periodic functions. As in the Dirichlet case (see Section 2), we associate with problem $(\mathrm{P})_1$ the functional $I : H^1_{2\pi} \to \mathbb{R}$ defined by

$$I(u) = q(u) - N(u),$$

where $q = q_{(\alpha,\lambda)}$ is the quadratic form

$$q(u) = \tfrac{1}{2}(\|u'\|_2^2 - \lambda\|u_+\|_2^2 - \alpha\|u_-\|_2^2)$$

and N is the nonlinear functional

$$N(u) = \int_0^{2\pi} G(t,u(t))\,dt, \quad u \in H^1_{2\pi}.$$

3.2. It can be easily checked that Lemmas 2.2 and 2.3 also hold in this periodic setting, for any $(\alpha,\lambda) \in \Sigma_{\mathrm{p}}$. In order to describe the geometry of the functional I under hypothesis (a) or (b) of Theorem 2, we will make use of the results in [dF-Ru] concerning two variational characterizations (v.c. for short) of the set Σ_{p}. We briefly recall these results and introduce some necessary notation.

First v.c. Set $M = \{u \in H^1_{2\pi} \mid \|u\|_2 = 1\}$. Let $(\alpha,\lambda) \in C_N$, $N \geq 2$. Then

$$0 = \inf_{A\in\Gamma_N} \max_{u\in A} q_{(\alpha,\lambda)}(u), \tag{3.1}$$

where

$$\Gamma_N = \{A \subset M \mid A \text{ is compact, invariant and } \gamma_0(A) \geq N-1\}.$$

Here "invariant" means invariant under the group of translations T_θ, $\theta \in S^1$ $(= \mathbb{R}/2\pi\mathbb{Z})$, which is defined as follows:

$$T_\theta(u) = u(\theta + \cdot), \quad u \in H^1_{2\pi}.$$

The map γ_0 in the definition of Γ_N is the "relative S^1-index of Benci" (cf. [Bs-Ls-Mn-Ru]).

Second v.c. We denote by S_k the sphere of $\mathbb{R}\times\mathbb{C}^k$ with respect to the euclidean norm. That is,

$$S_k = \left\{ z = (x, z_1, \ldots, z_k) \in \mathbb{R} \times \mathbb{C}^k \;\middle|\; |z|^2 = x^2 + \sum_{i=1}^{k} |z_i|^2 = 1 \right\},$$

and we set

$$S_0 = \{-1, 1\}.$$

On $\mathbb{R} \times \mathbb{C}^k$ consider the following S^1-action:

$$S_\theta(x, z_1, \ldots, z_k) = (x, e^{i\theta} z_1, \ldots, e^{i\theta} z_k), \quad \theta \in S^1,$$

and observe that $S_\theta(S_k) = S_k$ for all $\theta \in S^1$.

From now on we will use the same notation S_θ for the action defined above regardless of the number of components k, $k \in \mathbb{N}$.

Let $(\alpha, \lambda) \in C_N$, $N \geq 2$, be given. Let $h_0 : S_{N-2} \to M$ be a continuous map satisfying[1]

$$\text{(3.2)} \qquad \begin{aligned} &\text{(i) } h_0 \circ S_\theta = T_\theta \circ h_0 && \forall \theta \in S^1, \\ &\text{(ii) } h_0(x, 0, \ldots, 0) = x && \forall x \in \mathbb{R} \cap S_{N-2}, \\ &\text{(iii) } \max_{u \in h_0(S_{N-2})} q_{(\alpha,\lambda)}(u) < 0. \end{aligned}$$

Then

$$\text{(3.3)} \qquad \inf_{h \in \Lambda_{N-1}} \max_{u \in h(S^+_{N-1})} q_{(\alpha,\lambda)}(u) = 0,$$

where $S^+_k = \{z = (x, z_1, \ldots, z_k) \in S_k \mid z_k \geq 0\}$ and

$$\Lambda_{N-1} = \{h \in C(S^+_{N-1}, M) \mid h|_{S_{N-2}} = h_0\}.$$

3.3. Let us start the proof of Theorem 2(a). It is a consequence of Corollary 3.2 below.

[1] The existence of such maps h_0 is also proved in [dF-Ru].

Proposition 3.1. *Assume that* $(\alpha,\lambda)\in C_N$ *and that* g *satisfies* (R) *and* $(\mathrm{NQ})_+$. *Let* $h_0 : S_{N-2}\to M$ *be any continuous map satisfying condition* (3.2). *Then*

(i) $\lim_{R\to\infty}\max_{u\in Rh_0(S_{N-2})} I(u) = -\infty$;
(ii) *If* $\Lambda_R = \{h\in C(S^+_{N-1}, H^1_{2\pi}) \mid h|_{S_{N-2}} = Rh_0\}$ *then, for* $R>0$ *sufficiently large, we have the strict inequality*

$$\max_{u\in Rh_0(S_{N-2})} I(u) < \inf_{h\in\Lambda_R}\max_{u\in h_0(S^+_{N-1})} I(u).$$

Proof. (i) Condition (R) implies that, for any $\varepsilon>0$, there exists $M>0$ such that

$$G(x,s) > -\varepsilon s^2 - M \quad \forall s\in\mathbb{R}.$$

For a given $\varepsilon>0$ (to be determined later), we have the estimate

$$I(u)\le q(u) + \varepsilon\|u\|_2^2 + 2\pi M. \tag{3.4}$$

Let $u\in Rh_0(S_{N-2})$. Then $\|u\|_2^2 = R^2$ and, by (3.4),

$$I(u)\le q(u)+\varepsilon R^2 + 2\pi M = R^2(q(u/\|u\|_2)+\varepsilon) + 2\pi M,$$

hence

$$\max_{u\in Rh_0(S_{N-2})} I(u)\le R^2(\max_{v\in h_0(S_{N-2})} q(v)+\varepsilon) + 2\pi M. \tag{3.5}$$

Since by (3.2) we have $\max_{v\in h_0(S_{N-2})} q(v) < 0$, we can choose $0<\varepsilon< -\max_{v\in h_0(S_{N-2})} q(v)$ in (3.4) to conclude, from (3.5), that

$$\lim_{R\to\infty}\max_{u\in Rh_0(S_{N-2})} I(u) = -\infty.$$

(ii) From Lemma 2.3(a) we deduce that there exists $M>0$ such that

$$G(t,s)\le M \quad \forall s\in\mathbb{R},\ \text{for a.e. } t\in[0,2\pi].$$

Therefore,

$$I(u)\ge q(u) - 2\pi M \quad \forall u\in H^1_{2\pi}. \tag{3.6}$$

Let us fix $R>0$ in part (i) so that

$$\max_{u\in Rh_0(S_{N-2})} I(u) < -2\pi M$$

and set Λ_R as in the proposition. Given $h\in\Lambda_R$, we distinguish two cases:

(a) $0\in h(S^+_{N-1})$. Then

$$\max_{u\in h(S^+_{N-1})} I(u)\ge I(0) = 0 > -2\pi M.$$

(b) $0 \notin h(S^+_{N-1})$. Consider the map $\overline{h} : S^+_{N-1} \to M$ defined by $\overline{h}(z) = h(z)/\|h(z)\|_2$. Obviously $\overline{h} \in \Lambda_{N-1}$ and then, by the second v.c.,

$$\max_{v\in\overline{h}(S^+_{N-1})} q(v) \geq \inf_{\Lambda_{N-1}} \max q = 0.$$

We now use (3.6) and the above inequality to obtain the estimate

$$\max_{u\in h(S^+_{N-1})} \frac{I(u)+2\pi M}{\|u\|_2^2} \geq \max_{u\in h(S^+_{N-1})} \frac{q(u)}{\|u\|_2^2} = \max_{v\in\overline{h}(S^+_{N-1})} q(v) \geq 0,$$

so that

$$\max_{u\in h(S^+_{N-1})} I(u) > -2\pi M.$$

In both cases we have

$$\max_{u\in h(S^+_{N-1})} I(u) > -2\pi M > \max_{u\in Rh_0(S_{N-2})} I(u),$$

which concludes the proof of (ii). □

Corollary 3.2. *Under the hypotheses of Theorem* 2(a)*, problem* (P)$_1$ *admits at least one solution* $u \in H^1_{2\pi}$.

3.4. Next we deal with case (b) of Theorem 2, where condition (NQ)$_-$ is assumed instead of (NQ)$_+$. Since, by Lemma 2.3, $\lim_{|s|\to\infty} G(t,s) = \infty$, the functional I is then anticoercive in any direction $v \in H^1_{2\pi} \setminus \{0\}$ with $q(v) < 0$ or $q(v) = 0$. Examples of functions v with $q(v) = 0$ are the solutions of problem $(P)_{(\alpha,\lambda)}$:

$$(P)_{(\alpha,\lambda)} \qquad \begin{cases} -u''(t) = \lambda u_+(t) - \alpha u_-(t), \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases}$$

Let us introduce some new notation. Given $\lambda_1,\dots,\lambda_{N-1}$, the $N-1$ first eigenvalues of $(-u'', H^1_{2\pi})$, and $(\lambda,\alpha) \in C_N$, denote by E_j, $j = 1,\dots,N-1$, the eigenspace associated with λ_j and by F the set of solutions of $(P)_{(\alpha,\lambda)}$. We write

$$\begin{cases} E_1 = \mathbb{R}, \\ E_j = \{r\phi_j(\theta+\cdot) \mid r \geq 0,\ \theta \in S^1\}, \quad j \geq 2, \\ F = \{rv(\theta+\cdot) \mid r \geq 0,\ \theta \in S^1\}, \end{cases}$$

for some (fixed) λ_j-eigenfunction ϕ_j with $\|\phi_j\|_2 = 1, \phi_1 = 1$ and some $v \in F$ with $\|v\|_2 = 1$. Set $E = \bigoplus_{j=1}^{N-1} E_j$ and assume the following *condition* $(*)$ on $(\alpha,\lambda) \in C_N$:

$$(*) \qquad \alpha \geq \lambda_{N-1}, \quad \lambda \geq \lambda_{N-1}.$$

Observe that since $(\lambda_{N-1},\lambda_{N-1}) \notin C_N$ we have $\min\{\alpha-\lambda_{N-1}, \lambda-\lambda_{N-1}\} > 0$.

This condition $(*)$ allows us to prove the following lemma.

Lemma 3.3. *Let $h_1 : S_{N-1} \to M$ be the map defined by*

$$h_1(x, r_1 e^{i\theta_1}, \dots, r_{N-1} e^{i\theta_{N-1}}) = \frac{x + \sum_{i=1}^{N-2} r_i \phi_{i+1}(\theta_i + \cdot) + r_{N-1} v(\theta_{N-1} + \cdot)}{\|x + \sum_{i=1}^{N-2} r_i \phi_{i+1}(\theta_i + \cdot) + r_{N-1} v(\theta_{N-1} + \cdot)\|_2}.$$

Then h_1 is a well defined continuous map which satisfies (3.2). *Moreover, there exists some $c > 0$ such that*

$$q(u) \le -c\|P(u)\|_2^2 \quad \forall u \in h_1(S_{N-1}), \tag{3.7}$$

where $P : E + F \to E$, $P(u_0 + w) = u_0$ for $u_0 \in E$ and $w \in F$.

Proof. Let us prove that h_1 is well defined. Continuity and condition (3.2) follow directly from the definition.

We write each $u \in h_1(S_{N-1})$ in the form

$$u = \frac{u_0 + w}{\|u_0 + w\|_2},$$

where $u_0 \in E$ and $w \in F$ with $\|u_0\|_2 \cdot \|w\|_2 \neq 0$. Then u is well defined if we show that $u_0 + w \neq 0$, i.e. $F \cap E = \{0\}$. This is obviously true when $\alpha = \lambda = \lambda_N$ because then $F = E_N$ and it is well known that $E_N \cap E = \{0\}$. In the case $\alpha \neq \lambda$, observe that the solutions $w \in F$ of problem $(P)_{(\alpha,\lambda)}$ are at most of class C^2 whereas the functions $u_0 \in E$ are C^∞. Thus again $E \cap F = \{0\}$. Also notice that $E \cap F = \{0\}$ implies that the map P in the lemma is well defined. It is easy to check that P is continuous.

Let us now prove (3.7). We write the elements $u \in h_1(S_{N-1})$ in the form

$$u = \varrho u_0 + w, \quad \varrho \in \mathbb{R},\ u_0 \in E \cap M,\ w \in F.$$

Fix $u_0 \in E \cap M$ and $w \in F$. Define the map $g : \mathbb{R} \to \mathbb{R}$ letting $g(\varrho) = q(\varrho u_0 + w)$. Then (3.7) is equivalent to showing that there exists a constant $c = c(\alpha, \lambda) > 0$ such that

$$g(\varrho) \le -c\varrho^2 \quad \forall \varrho \in \mathbb{R}. \tag{3.7$'$}$$

In order to prove (3.7)$'$ we first state some properties of the function g:

(a) $g(0) = q(w) = 0$, since $w \in F$;

(b) g is differentiable and

$$g'(\varrho) = 2\bigg(\int_0^{2\pi} (\varrho u_0 + w)' u_0' \, dt - \lambda \int_0^{2\pi} (\varrho u_0 + w)_+ u_0 \, dt + \alpha \int_0^{2\pi} (\varrho u_0 + w)_- u_0 \, dt\bigg);$$

(c) g' is differentiable and

$$g''(\varrho) = 2\bigg(\int_0^{2\pi} u_0'^2 \, dt - \lambda \int_{\{\varrho u_0 + w > 0\}} u_0^2 \, dt - \alpha \int_{\{\varrho u_0 + w < 0\}} u_0^2 \, dt\bigg).$$

As a matter of fact, the differentiability of g' follows from the property below (see [So] and [Cu] for a proof):

$$|\{t \mid (\varrho u_0 + w)(t) = 0\}| = 0 \quad \forall \varrho \in \mathbb{R},\ \forall u_0 \in E,\ \forall w \in F \text{ such that } \varrho u_0 + w \neq 0.$$

Now, since $u_0 \in E \cap M$, we have $\int_0^{2\pi} u_0'^2\,dt \le \lambda_{N-1} \int_0^{2\pi} u_0^2\,dt = \lambda_{N-1}$ and, using (c), we conclude that

$$g''(\varrho) \le 2\max\{\lambda_{N-1} - \lambda, \lambda_{N-1} - \alpha\}. \tag{3.8}$$

Set $c = \min\{\alpha - \lambda_{N-1}, \lambda - \lambda_{N-1}\}$. From condition $(*)$ we have $c > 0$. Finally, from (a), (b) and the Mean Value Theorem, we obtain

$$g(\varrho) = \tfrac{1}{2} g''(\tau\varrho)\varrho^2 \quad \text{for some } \tau \in \,]0,1[,$$

so that, using (3.8), we conclude the proof of claim $(3.7)'$. □

Lemma 3.4. *Assume that $(\alpha, \lambda) \in C_N$ satisfies condition $(*)$ and consider h_1 defined in Lemma 3.3. Let*

$$\Lambda = \{h \in C(S_N^+, M) \mid h|_{S_{N-1}} = h_1\}.$$

Then there exists some constant $d > 0$ such that

$$\inf_{h\in\Lambda} \max_{u \in h(S_{N-1}^+)} q(u) \ge d.$$

Proof. We take a point $(\alpha', \lambda') \in C_{N+1}$ such that $\alpha < \alpha'$ and $\lambda < \lambda'$. Then, letting $d = \min\{\lambda' - \lambda, \alpha' - \alpha\} > 0$, we have

$$q(u) = q_{(\alpha,\lambda)}(u) \ge q_{(\alpha',\lambda')}(u) + d\|u\|_2^2 \quad \forall u \in H^1_{2\pi} \tag{3.9}$$

and, therefore,

$$\inf_{\Lambda} \max q \ge \inf_{\Lambda} \max q_{(\alpha',\lambda')} + d.$$

On the other hand, the map h_1 satisfies condition (3.2) relative to (α', λ'). Indeed,

$$\max_{u\in h_1(S_{N-1})} q_{(\alpha',\lambda')}(u) \le \max_{u\in h_1(S_{N-1})} q(u) - d \le 0 - d < 0$$

since $\max_{u\in h_1(S_{N-1})} q(u) \le 0$ in view of (3.7). We can then apply the second v.c. to (α', λ') to conclude that

$$0 = \inf_{\Lambda} \max q_{(\alpha',\lambda')},$$

and hence, from (3.9), that

$$\inf_{\Lambda} \max q \ge d.$$ □

We are now ready to announce the following result on the geometry of the functional I in the case (b) of Theorem 2. The proof of statement (b) in Theorem 2 will follow from Corollary 3.6 of the following proposition.

Proposition 3.5. *Let* $(\alpha,\lambda)\in C_N$ *satisfy condition* $(*)$. *Assume that* g *satisfies conditions* (R) *and* $(\mathrm{NQ})_-$. *Then*

(i) $\lim_{R\to\infty}\max_{u\in Rh_1(S_{N-1})} I(u)=-\infty$;
(ii) *If* $\Lambda_R=\{h\in C(S_N^+,H^1_{2\pi})\mid h|_{S_{N-1}}=Rh_1\}$ *then, for* R *sufficiently large, we have*

$$\max_{u\in Rh_1(S_{N-1})} I(u)<\inf_{h\in\Lambda_R}\max_{u\in h(S_N^+)} I(u).$$

Proof. (i) We know from Lemma 2.3 that $\lim_{|s|\to\infty}G(t,s)=\infty$. Therefore, there exists some $M>0$ such that

$$G(t,s)\geq M\quad\forall s\in\mathbb{R},\ \text{for a.e. } t,$$

so that we obtain the following estimate from above for the functional I:

$$I(u)=q(u)-N(u)\leq q(u)-2\pi M. \tag{3.10}$$

Now, by contradiction, suppose that (i) does not hold. Then there exist $A>0$ and a sequence (m_k), with $m_k\to\infty$, such that

$$-A<\max_{u\in m_kh_1(S_{N-1})} I(u).$$

Choose a corresponding sequence of points $u_k\in h_1(S_{N-1})$ such that

$$-A<I(m_ku_k) \tag{3.11}$$

and write $m_ku_k=\alpha_ku_{0,k}+w_k$, $u_{0,k}\in E\cap M$, $w_k\in F,\alpha_k\geq 0$. We distinguish two cases:

Case 1: $\lim_{k\to\infty}\alpha_k=\infty$ (for some subsequence). Using Lemma 3.3 and (3.10), we obtain

$$I(m_ku_k)\leq -c\alpha_k^2-2\pi M\to-\infty\quad\text{as } k\to\infty,$$

which contradicts (3.11).

Case 2: (α_k) *is bounded.* Write

$$u_k=\frac{\alpha_k}{m_k}u_{0,k}+\frac{w_k}{m_k}\quad\text{and}\quad v_k=\frac{w_k}{m_k}\in F.$$

We then have

$$\|v_k\|_2=\left\|u_k-\frac{\alpha_k}{m_k}u_{0,k}\right\|_2\to 1, \tag{3.12}$$

and, noticing that $\|w'\|_2^2=\lambda\|w_+\|_2^2+\alpha\|w_-\|_2^2$ for all $w\in F$, we see from (3.12) that $(\|v_k\|)$ is bounded. Therefore, there exist a subsequence of (v_k) (still denoted by v_k) and some $v_0\in H^1_{2\pi}$ such that

$$v_k\rightharpoonup v_0\quad\text{in } H^1_{2\pi},\qquad v_k\to v_0\quad\text{in } L^2.$$

Since $\alpha_k/m_k \to 0$ as $k \to \infty$, we also have

$$u_k \rightharpoonup v_0 \quad \text{in } H^1_{2\pi}, \qquad u_k \to v_0 \quad \text{in } L^2.$$

In particular, $\|v_0\|_2 = \lim_{k\to\infty} \|u_k\|_2 = 1$, so that $v_0(t) \neq 0$ for a.e. t and, therefore,

$$m_k|u_k(t)| \to \infty \quad \text{as } k \to \infty, \text{ for a.e. } t. \tag{3.13}$$

Finally, using the fact that $\lim_{|s|\to\infty} G(x,s) = \infty$ and Fatou's lemma, it follows that

$$N(m_k u_k) = \int_0^{2\pi} G(t, m_k u_k(t))\, dt \to \infty \quad \text{as } k \to \infty$$

and, hence,

$$I(m_k u_k) \le 0 - N(m_k u_k) \to -\infty \quad \text{as } k \to \infty, \tag{3.14}$$

which contradicts (3.11). This concludes the proof of (i).

(ii) Let $0 < \varepsilon < d$ where d is the constant in Lemma 3.4. Condition (R) implies that there exists $a_\varepsilon \in L^1$ such that

$$G(t,s) \le \varepsilon s^2 + a_\varepsilon(t) \quad \forall s \in \mathbb{R}, \text{ for a.e. } t,$$

so that

$$I(u) \ge q(u) - \varepsilon \|u\|_2^2 - \|a_\varepsilon\|_1.$$

Fix $R > 0$ in (i) such that

$$\max_{u \in R h_1(S_{N-1})} I(u) < -\|a_\varepsilon\|_1 \tag{3.15}$$

and let $h \in \Lambda_R$. We distinguish 2 cases:

(a) $0 \in h(S_N^+)$. Then

$$\max_{u \in h(S_N^+)} I(u) \ge 0 \ge -\|a_\varepsilon\|_1. \tag{3.16}$$

(b) $0 \notin h(S_N^+)$. Then, defining $\overline{h} : S_N^+ \to M$ by $\overline{h}(z) = h(z)/\|h(z)\|_2$, it is clear that $h \in \Lambda$ and, therefore,

$$\max_{u \in h(S_N^+)} \frac{I(u) + \|a_\varepsilon\|_1}{\|u\|_2^2} \ge \max_{v \in \overline{h}(S_N^+)} q(v) - \varepsilon \ge \inf_{\Lambda} \max q - \varepsilon = d - \varepsilon > 0,$$

so that

$$\max_{u \in h(S_N^+)} I(u) > -\|a_\varepsilon\|_1. \tag{3.17}$$

The proof of (ii) readily follows from (3.15)–(3.17). □

4. Some examples and final comments

In this section, as stated in the introduction, we provide some nontrivial examples where both the *resonance* condition (R) and the *nonquadraticity at infinity* condition $(\mathrm{NQ})_\pm$ hold, and yet the nonlinearity *crosses infinitely many eigenvalues*. We will also try to illustrate the technical difficulties which may arise in higher dimensions, even in the particular case of a simple domain $\Omega \subset \mathbb{R}^2$ and with resonance (from above) at the second branch C_2 of the Fučik spectrum.

4.1. We start with some examples (cf. also [Co-Ma]).

Example 1. Let $\psi : [1,\infty] \to \mathbb{R}$ be any continuous function such that

$$\psi(s) \geq 0, \quad \int_1^\infty \psi(t)\,dt < \infty,$$

and define $H(s) = d + \int_1^s \psi(t)\,dt$, $s \geq 1$, where $d \in \mathbb{R}$ is such that $H(s) \to 1$ as $s \to \infty$ and H is supposed to be extended to the whole real line as an even function of class C^1 with $H(0) = 1$. Then it is easy to see that the function

$$G(s) = \tfrac{1}{2}(\lambda s_+^2 + \alpha s_-^2)[H(s) - 1]$$

satisfies conditions (R) and $(\mathrm{NQ})_+$. Moreover, one has *crossing of all the eigenvalues* in the sense that

$$0 = \lim_{|s|\to 0} g(s)/s < \limsup_{|s|\to\infty} g(s)/s = \infty.$$

Example 2. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (R) and $(\mathrm{NQ})_+$, with the quantity $sg(s) - 2G(s)$ tending to ∞ *at least linearly*, that is, such that

$$\delta(G) := \liminf_{|s|\to\infty} \frac{sg(s) - 2G(s)}{|s|} = \delta > 0.$$

In this case, we can think of $\delta(G)$ as a measure of *deviation from quadraticity at infinity* for the function G (clearly, $\delta(G) = \infty$ is possible). Now, consider the *resonant problem*

$$-\Delta u = \lambda u_+ - \alpha u_- + g(u) + h(x), \quad u \in H_0^1(\Omega), \tag{4.1}$$

where $h \in L^\infty(\Omega)$ is given. Then, as an immediate consequence of Theorem 1, we obtain the following:

Corollary. *Let $(\alpha, \lambda) \in C_2$. Under the above conditions, problem* (4.1) *has a weak solution $u \in H_0^1(\Omega)$ provided that $|h|_\infty < \delta$. In particular, if $\delta(G) = \infty$, then problem* (4.1) *has a solution for any given $h \in L^\infty$.*

We should note that the above result is in the spirit of the Landesman–Lazer results, as we briefly pointed out in the introduction. In fact, assume that the limits $g(\pm\infty) = \lim_{s\to\pm\infty} g(s)$ exist and are distinct, say $g(-\infty) <$

$g(\infty)$. Without loss of generality (by subtracting and adding the average of these numbers on the right hand side of (4.1)), we may assume that $g(-\infty) = -g(\infty) < 0$. Then, in this context, an analogue of the *Landesman–Lazer* situation might state that a weak solution exists provided that

$$\text{(LL)} \qquad |h_k|_\infty < g(\infty),$$

where h_k denotes the *projection* of h onto the set of "generalized (α, λ)-eigenfunctions".

Example 3. Although (as shown above) these situations of nonquadraticity at infinity partially extend the Landesman–Lazer situations, they do not easily compare with those of Ahmad–Lazer–Paul (cf. [A-L-P], [Ra2]). Indeed, in the context of problem (4.1) (with $h = 0$ for simplicity), an analogue of the *Ahmad–Lazer–Paul* situation might state that a solution exists provided that

$$\text{(ALP)} \qquad g(s) \text{ is } \textit{bounded} \text{ and } G(\pm\infty) = \lim_{|s|\to\infty} G(s) = \infty.$$

Consider the following examples of continuous functions $g : \mathbb{R} \to \mathbb{R}$:

(a) $g(s)$ is odd and such that $G(s) = s\log(s)$ for $s > 0$ large;
(b) $g(s)$ is odd, 4-periodic, and such that $g(s) > 0$ for $0 < s < 2$ and $g(s) < 0$ for $2 < s < 4$, with $\int_0^2 g = 4$, $\int_2^4 g = -2$, $g(1) = 1$ and $G(1) = 2$.

In case (a), it is easy to check that g satisfies both conditions (R) and $(\mathrm{NQ})_-$, but it does not satisfy (ALP) above, since $g(s) = 1 + \log(s)$ (for $s > 0$ large) is *not bounded*. On the other hand, in case (b), the function g clearly satisfies (ALP), while $(\mathrm{NQ})_\pm$ does not hold since a simple calculation shows that $(4j-3)g(4j-3) - 2G(4j-3) = -3$ for all $j \geq 1$.

Finally, we would like to make a few comments on these resonant Fučik problems, nonquadratic at infinity, which hopefully will illustrate their intrinsic complexity when trying to handle them by variational methods in the case of higher dimensional domains. As we saw in this paper, some specific knowledge of the Fučik spectrum and of variational properties of the "generalized eigenfunctions" was needed in order to determine the geometry of the functional I and be able to use min-max techniques. More precisely, when (α, λ) belonged to C_2 and $(\mathrm{NQ})_-$ was assumed, we would have resonance from above at (α, λ) (since $\lim_{|s|\to\infty} G(x, s) = \infty$ by Lemma 2.3(b)) and, as a result, the functional I was anticoercive on any two-dimensional half-space $\langle\phi_1\rangle \oplus \mathbb{R}_+ v$, with v an arbitrary "generalized eigenfunction" (i.e., solution of $(P)_{(\alpha,\lambda)}$). Thus, some precise information on "all" these generalized eigenfunctions was necessary to handle the case of resonance from above at C_2. In Section 3, the one-dimensional periodic problem was considered in general situations of resonance (from below or from above) at an arbitrary eigenvalue λ_k, precisely because both the Fučik spectrum and a variational characterization of C_2 were known in this case.

Let us next consider one of the simplest possible domains in $\mathbb{R}^2$, namely the unit disc $\Omega = \{(x,y) \mid x^2+y^2<1\}$. When $\alpha=\lambda=\lambda_k$ (the usual kth eigenvalue of $-\Delta$ on $H^1_0(\Omega)$), it is known (see [Wa] and [A-C]) that $\lambda_k=\nu^2$ where ν is a positive zero of some (unique) Bessel function $J_l(r)$ of first kind ($l=0,1,2,\ldots$). Moreover, λ_k is either a simple or a double eigenvalue depending on whether $l=0$ or $l\geq 1$, respectively. For a simple eigenvalue, a corresponding (radial) eigenfunction is $J_0(\nu r)$, whereas, for a double eigenvalue, two linearly independent eigenfunctions spanning the eigenspace N_k are given by $v_1(r,\theta)=J_l(\nu r)\cos(l\theta)$ and $v_2(r,\theta)=J_l(\nu r)\sin(l\theta)$. Now, in view of the radial symmetry of both the domain Ω and the operator $-\Delta$, given $\widehat{\theta}\in[0,2\pi)$, any of the functions $v_1(\cdot,\widehat{\theta}+\cdot)$, $v_2(\cdot,\widehat{\theta}+\cdot)$ is again a λ_k-eigenfunction (and it is clear from the cosine (or sine) addition formula how these eigenfunctions are expressed in terms of v_1 and v_2). On the other hand, when $\alpha\neq\lambda$, we still have a whole "continuum" of generalized eigenfunctions obtained as above by "rotating" any given generalized eigenfunction $v(r,\theta)$. However, as the set F_k of all these generalized (α,λ)-eigenfunctions is no longer a vector space, it is not hard to imagine the technical difficulties arising from a possible "infinitude" of linearly independent such functions.

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Manuscript received August 30, 1995

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