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POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS WITH DISCONTINUOUS NONLINEARITIES

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In this paper an existence theorem of positive solutions to the Dirichlet problem for elliptic equations having nonlinear terms with an *uncountable* set of discontinuities is established. Some applications to special cases, such as problems with critical Sobolev growth, are also presented. The approach taken is strictly based on set-valued analysis and fixed point arguments.

1. Introduction

A branch in today's literature on elliptic boundary value problems deals with the existence of positive solutions to equations having discontinuous nonlinearities. This field of research exhibits both theoretical interest [3, 5, 6, 17, 21] and applications to specific models arising from mathematical physics [3, 9, 17]. For nonlinearities of a special kind and discontinuous at a finite or countable set of points, very complete and satisfactory results are already available; see for example [6] and the references given there. On the contrary, to the best of our knowledge, no investigation has been devoted to equations having nonlinear terms with an uncountable set of discontinuities, probably because in this case the usual variational techniques are not applicable in a simple way.

The aim of the present paper is to provide a first contribution in the above-mentioned direction. Accordingly, here, we study an elliptic boundary value

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problem of the type

$$\text{(P)} \qquad \begin{cases} -\Delta u = f(u) + h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω denotes a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$, h belongs to $L^p(\Omega)$ for some $p \in \,]n/2, \infty]$, and $f : \mathbb{R} \to \mathbb{R}$ is a function whose set of discontinuity points has *only* Lebesgue measure zero. We look for solutions to (P) that lie in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ if $p < \infty$ or in a suitable subset of $\bigcap_{s>n/2} W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$ if $p = \infty$.

As in [20], the approach we develop is strictly based on set-valued analysis. We first consider an appropriate upper semicontinuous convex-valued regularization $F(x, u)$ of $f(u) + h(x)$ and, through fixed point arguments, we get a positive solution u to the elliptic differential inclusion $-\Delta u \in F(x, u)$, $x \in \Omega$, vanishing on $\partial\Omega$. Next, by using a lemma of [10] and a technical condition, we show that u also satisfies (P). The existence theorem so established (Theorem 3.1) may profitably be employed to study elliptic problems not solvable by means of standard techniques, such as problems with critical Sobolev growth or problems having a nonlinearity of exponential type (see Remark 3.5, Theorem 3.3, and Remark 3.9). Moreover, when applied to more concrete situations, it takes very simple and practical forms. As an example, Theorem 3.2 below immediately yields the following

Theorem. *Let $h \equiv 0$. Suppose f is bounded and has positive infimum on $\mathbb{R}$. Then problem* (P) *admits at least one solution.*

2. Definitions and preliminary results

Let X and Y be two nonempty sets. A *multifunction* Φ from X to Y is a function from X into the family of all subsets of Y. The graph of Φ, denoted by $\mathrm{gr}(\Phi)$, is the set $\{(x, y) \in X \times Y : y \in \Phi(x)\}$. A function $\varphi : X \to Y$ such that $\varphi(x) \in \Phi(x)$ for every $x \in X$ is called a *selection* of Φ. For every set $W \subseteq Y$ we define $\Phi^-(W) = \{x \in X : \Phi(x) \cap W \neq \emptyset\}$. When $(X, \mathfrak{F})$ is a measurable space, Y is a topological space, and for any open subset W of Y one has $\Phi^-(W) \in \mathfrak{F}$, we say that Φ is *measurable*. If X and Y are two topological spaces and the set $\Phi^-(W)$ is closed for every closed set $W \subseteq Y$, the multifunction Φ is said to be *upper semicontinuous*. When Y is a compact Hausdorff space and $\Phi(x)$ is a closed subset of Y for all $x \in X$, Theorems 7.1.15 and 7.1.16 of [15] ensure that Φ is upper semicontinuous if and only if the set $\mathrm{gr}(\Phi)$ is closed in $X \times Y$.

The following result is an immediate consequence of the Ky Fan fixed point theorem; see also [4, Theorem 1 and Remark 1].

THEOREM 2.1. *Let X be a metrizable locally convex topological vector space and let K be a weakly compact convex subset of X. Suppose Φ is a multifunction from K into itself with nonempty convex values and weakly sequentially closed graph. Then there exists $x_0 \in K$ such that $x_0 \in \Phi(x_0)$.*

PROOF. Since $\mathrm{gr}(\Phi)$ is weakly relatively compact and weakly sequentially closed, Theorem 7 of [16], p. 313, shows that it is also weakly compact. Therefore, $\Phi(x)$ is weakly closed for every $x \in K$ and the multifunction Φ is weakly upper semicontinuous. The Ky Fan fixed point theorem [12, Theorem 1] yields the desired conclusion. $\square$

If n is a positive integer, $\mathbb{R}^n$ denotes the real Euclidean n-space, and V is a subset of $\mathbb{R}^n$, we write $\overline{V}$ for the closure of V, ∂V for the boundary of V, $\mathrm{co}(V)$ for the convex hull of V.

Throughout this paper the symbol Ω indicates a nonempty, bounded, open and connected subset of $\mathbb{R}^n$, $n \geq 3$, with a boundary of class $C^{1,1}$, $p \in \,]n/2, \infty[$ or $p = \infty$, and p' is the conjugate exponent of p. The usual norm of $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. Moreover, "measurable" always means Lebesgue measurable and $|E|$ stands for the measure of E. The symbol $\mathcal{L}(\Omega)$ is used for the Lebesgue σ-algebra of Ω, while, for any nonempty open set $A \subseteq \mathbb{R}$, $\mathfrak{B}(A)$ is the Borel σ-algebra of A.

Given a nonnegative integer k and a real number s greater than $n/2$, we denote by $W^{k,s}(\Omega)$ the space of all real-valued functions defined on Ω whose weak partial derivatives up to order k lie in $L^s(\Omega)$, equipped with the usual norm. The symbol $W_0^{1,s}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the space $W^{1,s}(\Omega)$.

Let L be the linear, second order, elliptic differential operator defined by

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i},$$

where: $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$ for every $i, j = 1, \dots, n$, and $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \xi_1^2 + \xi_2^2 + \dots + \xi_n^2$ for all $x \in \Omega$ and $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$; $b_i \in L^\infty(\Omega)$ for every $i = 1, \dots, n$.

It is well known that L is a one-to-one operator from $W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$ onto $L^s(\Omega)$ (see, for instance, [13, Theorem 9.15]). This implies that the set $\{u \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) : Lu \in L^\infty(\Omega)\}$ does not depend on $s \in \,]n/2, \infty[$, as an easy computation shows. We define

$$X_p(\Omega) = \begin{cases} W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) & \text{if } p < \infty, \\ \{u \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) : Lu \in L^\infty(\Omega)\} & \text{if } p = \infty. \end{cases}$$

Denote by ω_n the volume of the unit ball in $\mathbb{R}^n$ and set

$$\beta = \operatorname*{ess\,sup}_{x\in\Omega} \bigg(\sum_{i=1}^{n} \bigg(b_i(x) + \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \bigg)^2 \bigg)^{1/2}.$$

Owing to Theorem 3.1 of [2], for every $u \in X_p(\Omega)$ we obtain

$$\sup_{x\in\Omega} |u(x)| \le B\|Lu\|_p, \tag{1}$$

where

$$B = \frac{1}{n^2\omega_n^{2/n}} \bigg[\int_0^{|\Omega|} \bigg(e^{-\beta(r/\omega_n)^{1/n}} \int_r^{|\Omega|} s^{-2+2/n} e^{\beta(s/\omega_n)^{1/n}}\, ds \bigg)^{p'} dr \bigg]^{1/p'}. \tag{2}$$

When $\beta = 0$ the constant B becomes [23, Theorem 2 and Remark 1]

$$B = |\Omega|^{2/n+1/p'-1} \frac{\Gamma(1+n/2)^{2/n}}{n(n-2)\pi} \bigg[\frac{\Gamma(1+p')\Gamma(n/(n-2)-p')}{\Gamma(n/(n-2))} \bigg]^{1/p'}, \tag{3}$$

Γ being the Gamma function.

Finally, a simple argument based on Lemma 1 of [10] and Lemma 7.7 of [13] yields the following proposition.

Proposition 2.1. *Let $u \in W^{2,s}(\Omega)$ and let E be a measurable subset of $\mathbb{R}$ such that $|E| = 0$. Then $Lu(x) = 0$ for almost every $x \in u^{-1}(E)$.*

3. Existence theorems

We are now in a position to formulate the main result of this paper, which extends Theorem 3.1 of [20] to the setting of positive solutions for elliptic problems with discontinuous nonlinearities.

Theorem 3.1. *Let f be a real-valued function defined on $\Omega \times \mathbb{R}$, having the following properties:*

(a_1) *There exists a set $\Omega_0 \subseteq \Omega$ with $|\Omega_0| = 0$ such that the set*

$$D_f = \bigcup_{x\in\Omega\setminus\Omega_0} \{z \in \mathbb{R} : f(x,\cdot) \text{ is discontinuous at } z\}$$

has measure zero.

(a_2) *The function $x \to f(x,z)$ is measurable for all $z \in \mathbb{R} \setminus D_f$.*

(a_3) *There is $r > 0$ so that the function $m(x) = \inf_{z\in A_r} f(x,z)$, $x \in \Omega$, where $A_r =]0, Br[\cap (\mathbb{R} \setminus D_f)$ and B is given by (2), is almost everywhere nonnegative in Ω and strictly positive in a set $\Omega^* \subseteq \Omega$ with $|\Omega^*| > 0$.*

(a_4) *The function $M(x) = \sup_{z\in A_r} f(x,z)$, $x \in \Omega$, belongs to $L^p(\Omega)$ and its norm in this space is less than r.*

(a_5) *For almost every $x \in \Omega \setminus \Omega^*$ and every $z \in D_f \cap]0, Br[$,*

$$\liminf_{w\in A_r, w\to z} f(x,w) = 0 \quad \textit{implies} \quad f(x,z) = 0.$$

Then there exists a function $u \in X_p(\Omega)$ satisfying $Lu(x) = f(x, u(x))$, $m(x) \leq Lu(x) \leq M(x)$ almost everywhere in Ω, and $u(x) > 0$ for all $x \in \Omega$.

Proof. Assumption (a_1) guarantees that $\mathbb{R} \setminus D_f$ is dense in $\mathbb{R}$. So, there is a countable set $D \subseteq (\mathbb{R} \setminus D_f) \cap]0, Br[$ fulfilling $\overline{D} = [0, Br]$. For every $(x,z) \in \Omega \times]0, Br[$ we define

$$F(x,z) = \bigcap_{k\in\mathbb{N}} \operatorname{co}(\overline{f(x,[z-1/k, z+1/k] \cap D)}).$$

Obviously, $F(x,z)$ is a convex closed subset of $[m(x), M(x)]$. By using hypothesis (a_4) we obtain a set $\Omega_1 \subseteq \Omega$ with $|\Omega_1| = 0$ such that $0 \leq m(x) \leq M(x) < \infty$ for all $x \in \Omega \setminus \Omega_1$. Hence, owing to Cantor's theorem, $F(x,z)$ is also nonempty and compact whenever $x \in \Omega \setminus \Omega_1$. Moreover,

$$F(x,z) = \{f(x,z)\} \quad \text{for every } x \in \Omega \setminus \Omega_0,\ z \in (\mathbb{R} \setminus D_f) \cap]0, Br[, \tag{4}$$

as a simple computation shows.

Now, pick $x \in \Omega$ and $(z,y) \in]0, Br[\times \mathbb{R}$, and choose two sequences $\{z_h\} \subseteq]0, Br[$ and $\{y_h\} \subseteq \mathbb{R}$ satisfying the conditions: $y_h \in F(x, z_h)$ ($h \in \mathbb{N}$), $\lim_{h\to\infty} z_h = z$ and $\lim_{h\to\infty} y_h = y$. Since for any $k \in \mathbb{N}$ there exists a positive integer ν such that

$$\operatorname{co}(\overline{f(x,[z_h - 1/(2k), z_h + 1/(2k)] \cap D)}) \subseteq \operatorname{co}(\overline{f(x,[z-1/k, z+1/k] \cap D)})$$

for every $h \geq \nu$ and $y_h \in \operatorname{co}(\overline{f(x,[z_h - 1/(2k), z_h + 1/(2k)] \cap D)})$ for all $h \in \mathbb{N}$, we get $y \in F(x,z)$. Therefore, the multifunction $z \to F(x,z)$ has a closed graph for every $x \in \Omega$ and is upper semicontinuous when $x \in \Omega \setminus \Omega_1$.

The preceding arguments, combined with Example 1.3 of [11], produce two functions $\varphi, \psi : \Omega \times]0, Br[\to \mathbb{R}_0^+$ having the properties:

$$F(x,z) = [\varphi(x,z), \psi(x,z)] \text{ in } (\Omega \setminus \Omega_1) \times]0, Br[; \tag{5}$$

for each $x \in \Omega \setminus \Omega_1$, $z \to \varphi(x,z)$ is lower semicontinuous and $z \to \psi(x,z)$ is upper semicontinuous.

The next step is to prove that the multifunction F is $\mathcal{L}(\Omega) \otimes \mathfrak{B}(]0, Br[)$-measurable. Let A be an open subset of $\mathbb{R}$. For any $k \in \mathbb{N}$ one has

$$\begin{aligned}
\{(x,z) \in \Omega \times]0, Br[\, : \overline{f(x,[z-1/k, z+1/k] \cap D)} \cap A \neq \emptyset\} \\
= \{(x,z) \in \Omega \times]0, Br[\, : f(x,[z-1/k, z+1/k] \cap D) \cap A \neq \emptyset\} \\
= \bigcup_{w\in D} [\{x \in \Omega : f(x,w) \in A\} \times ([w-1/k, w+1/k] \cap]0, Br[)];
\end{aligned}$$

namely, by the definition of D and assumption (a_2),

$$\{(x,z) \in \Omega \times]0, Br[\, : \overline{f(x, [z-1/k, z+1/k] \cap D)} \cap A \neq \emptyset\} \in \mathcal{L}(\Omega) \otimes \mathfrak{B}(]0, Br[).$$

Consequently, the multifunction $(x,z) \to \overline{f(x, [z-1/k, z+1/k] \cap D)}$, $(x,z) \in \Omega \times]0, Br[$, is $\mathcal{L}(\Omega) \otimes \mathfrak{B}(]0, Br[)$-measurable. Theorem 9.1 and Corollary 4.2 of [14] ensure that the same holds for F. So, in particular, the functions φ and ψ can be supposed to be $\mathcal{L}(\Omega) \otimes \mathfrak{B}(]0, Br[)$-measurable.

Set $q = p$ if $p < \infty$, $q = s \in]n/2, \infty[$ otherwise, and define

$$K = \{v \in L^q(\Omega) : m(x) \leq v(x) \leq M(x) \text{ almost everywhere in } \Omega\}.$$

Evidently, K is a nonempty convex weakly compact subset of $L^q(\Omega)$. Furthermore, owing to (1) and hypothesis (a_4), for any $v \in K$ one has

$$|L^{-1}(v)(x)| \leq B\|v\|_p \leq B\|M\|_p < Br, \quad x \in \Omega.$$

Since $v(x) \geq m(x)$ almost everywhere in Ω and assumption (a_3) holds, Corollary I.2 of [18] implies $L^{-1}(v)(x) > 0$ in Ω. Therefore, it make sense to define

$$\Phi(v) = \{w \in K : w(x) \in F(x, L^{-1}(v)(x)) \text{ for almost every } x \in \Omega\}, \quad v \in K.$$

We claim that $\Phi(v)$ is nonempty. Indeed, the multifunction $x \to F(x, L^{-1}(v)(x))$ is measurable (see, for instance, [24, Theorem 1]) and so, by the Kuratowski and Ryll-Nardzewski selection theorem [14, Theorem 5.1], it has a measurable selection $w : \Omega \to \mathbb{R}$. The inclusion $F(x, L^{-1}(v)(x)) \subseteq [m(x), M(x)]$, $x \in \Omega$, leads to $w \in \Phi(v)$, that is, $\Phi(v) \neq \emptyset$.

Obviously, the set $\Phi(v)$ is convex. Moreover, the multifunction Φ has a weakly sequentially closed graph. To see this, pick $v, w \in K$ and choose two sequences $\{v_h\}$, $\{w_h\}$ in K fulfilling $w_h \in \Phi(v_h)$ for all $h \in \mathbb{N}$ and $\lim_{h\to\infty} v_h = v$, $\lim_{h\to\infty} w_h = w$ weakly in $L^q(\Omega)$. Identity (5) implies

$$\varphi(x, L^{-1}(v_h)(x)) \leq w_h(x) \leq \psi(x, L^{-1}(v_h)(x)) \quad \text{almost everywhere in } \Omega,$$

while the weak convergence of $\{w_h\}$ to w produces

$$\liminf_{h\to\infty} \int_E (\varphi(x, L^{-1}(v_h)(x)) - w(x))\, dx \leq \liminf_{h\to\infty} \int_E (w_h(x) - w(x))\, dx = 0$$

for any measurable set $E \subseteq \Omega$. Bearing in mind the Fatou lemma, we get

$$\liminf_{h\to\infty} (\varphi(x, L^{-1}(v_h)(x)) - w(x)) \leq 0 \quad \text{almost everywhere in } \Omega. \tag{6}$$

Since L^{-1} is a continuous operator from $L^q(\Omega)$ into $W^{2,q}(\Omega)$ (see, for instance, [13, Lemma 9.17]) and $q > n/2$, the Rellich–Kondrashov theorem [1, Theorem 6.2] guarantees that the sequence $\{L^{-1}(v_h)\}$ converges pointwise in Ω to $L^{-1}(v)$. Therefore, due to (6) and the lower semicontinuity of the function $z \to \varphi(x,z)$, $x \in \Omega \setminus \Omega_1$, $\varphi(x, L^{-1}(v)(x)) - w(x) \leq 0$ for almost all $x \in \Omega$.

The same arguments, with ψ in place of φ, yield $w(x) - \psi(x, L^{-1}(v)(x)) \leq 0$ almost everywhere in Ω. Consequently, by (5), $w(x) \in F(x, L^{-1}(v)(x))$ and the assertion follows.

We have thus proved that all the hypotheses of Theorem 2.1 hold. So, there exists a function $v \in K$ such that $v \in \Phi(v)$. The function $u = L^{-1}(v)$ lies in $X_p(\Omega)$ and satisfies the conditions: $u(x) > 0$ in Ω, owing to assumption (a_3) and Corollary I.2 of [18];

$$Lu(x) \in F(x, u(x)) \subseteq [m(x), M(x)] \quad \text{for almost every } x \in \Omega. \tag{7}$$

Furthermore, since $|D_f| = 0$, Proposition 2.1 gives

$$Lu(x) = 0 \quad \text{almost everywhere in } u^{-1}(D_f). \tag{8}$$

Let $\Omega_i \subseteq \Omega, i = 2, 3, 4$, be such that $|\Omega_i| = 0$ for all i, hypothesis (a_5) applies for each $x \in (\Omega \setminus \Omega^*) \setminus \Omega_2$, (7) is true in $\Omega \setminus \Omega_3$, and (8) holds when $x \in u^{-1}(D_f) \setminus \Omega_4$. Define $\widetilde{\Omega} = \bigcup_{i=0}^4 \Omega_i$. Evidently, $|\widetilde{\Omega}| = 0$. Moreover, one has $Lu(x) = f(x, u(x))$ for every $x \in \Omega \setminus \widetilde{\Omega}$. To see this, pick $x \in \Omega \setminus \widetilde{\Omega}$. If $u(x) \notin D_f$, then $Lu(x) = f(x, u(x))$ because $Lu(x) \in F(x, u(x))$ and, by (4), $F(x, u(x)) = \{f(x, u(x))\}$. If $u(x) \in D_f$, hypothesis (a_5), together with (7) and (8), leads to $Lu(x) = 0 = f(x, u(x))$. Hence, in either case, $Lu(x) = f(x, u(x))$, and the proof is complete. □

REMARK 3.1. The set-valued part of the above proof is adapted from that of Theorem 1 of [22]. Moreover, similar arguments can be used to establish Theorem 3.1 with $]0, Br[$ and (a_4) respectively replaced by $]0, Br]$ and

(a_4)′ *The function $M(x) = \sup_{z \in A_r} f(x, z)$, $x \in \Omega$, belongs to $L^p(\Omega)$ and its norm in this space is less than or equal to r.*

In this form, it extends Theorem 2 of [7] to elliptic problems with discontinuous nonlinearities.

The preceding result gains in interest if we make the following remarks, which emphasize some important features of its assumptions.

REMARK 3.2. The conditions assumed in [3, 5, 6, 9, 21] for the function f imply that the set D_f is at most countable. Therefore, hypothesis (a_1) is fulfilled. On the other hand, several functions f have an uncountable set of discontinuity points and, nevertheless, satisfy all the assumptions of Theorem 3.1. For instance, this is the case of $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(z) = \begin{cases} 0 & \text{if } z \in C, \\ 1 & \text{otherwise}, \end{cases}$$

where C denotes the Cantor "middle third" set.

REMARK 3.3. As in [19, Remark 2.1], one can see that hypotheses (a_1) and (a_2) do not guarantee the measurability of the function $x \to f(x, z)$ for all $z \in \mathbb{R}$.

Remark 3.4. Although the conclusion of Theorem 3.1 is no longer true without assuming (a_3), there are many examples of elliptic equations where the nonlinearity does not satisfy (a_3) and, nevertheless, can successfully be treated by other methods; see for instance [5, 6]. However, in these cases the discontinuities are supposed to be of a very special kind.

Remark 3.5. A condition like (a_4) has previously been employed in [7, 19, 20]. Both [19] and [7] deal with the case $D_f = \emptyset$, while [20] is devoted to elliptic equations having discontinuous right-hand side. In those papers the authors showed that this assumption may profitably be used to study elliptic problems not solvable by means of standard techniques, such as problems with critical Sobolev growth or problems having a nonlinearity of exponential type; we refer to Theorem 2.4 and Example 2.1 of [19], Remark 2 and Theorem 3 of [7], Examples 3.1 and 3.2 of [20]. Evidently, a similar comment also holds for the present paper.

Remark 3.6. Conditions can easily be drawn from [20, Remark 3.5] so that the function $x \to \sup_{z\in A_r} f(x,z)$ is measurable for any $r > 0$ and hypothesis (a_4) is fulfilled.

Theorem 3.1 has a variety of interesting special cases. As an example, if $f(x,z) = |z|^{\sigma-1} + h(x,z)$, $(x,z) \in \Omega \times \mathbb{R}$, where σ denotes the critical Sobolev exponent,

$$\sigma = \frac{2n}{n-2},$$

and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies assumptions (a_1)–(a_3) and (a_5) of Theorem 3.1, then a positive solution $u \in X_p(\Omega)$ to the equation

$$Lu = u^{\sigma-1} + h(x,u)$$

can be obtained provided the function $x \to \sup_{z\in A_r} |h(x,z)|$ lies in $L^p(\Omega)$ and one has

$$(Br)^{\sigma-1}|\Omega|^{1-1/p'} + \|\sup_{z\in A_r} |h(\cdot,z)|\|_p < r.$$

For $-L = \Delta$ (the Laplace operator) and h of Carathéodory's type, the preceding equation has been investigated extensively; see for instance [7, 8, 19, 25] and the references given there.

When the function $(x,z) \to f(x,z)$ is independent of x, Theorem 3.1 takes the simpler and practical form given below.

Theorem 3.2. *Let f be a real-valued function defined on $\mathbb{R}$. Assume that:*

(b_1) *The set $D_f = \{z \in \mathbb{R} : f$ is discontinuous at $z\}$ has measure zero.*

(b_2) *There is $r > 0$ so that $\inf_{z\in A_r} f(z) > 0$ and $|\Omega|^{1-1/p'} \sup_{z\in A_r} f(z) < r$, where $A_r =]0, Br[\cap (\mathbb{R} \setminus D_f)$.*

Then there exists a positive function $u \in X_p(\Omega)$ satisfying the equation $Lu = f(u)$ almost everywhere in Ω.

Remark 3.7. Obviously, any function $f : \mathbb{R} \to \mathbb{R}$ with finite variation and positive infimum on $\mathbb{R}$ has properties (b_1) and (b_2).

Remark 3.8. We point out that the above result may be proved without using multifunctions. Indeed, in this case, one can see that the function

$$\varphi(v)(x) = f(L^{-1}(v)(x)), \quad x \in \Omega,$$

is measurable for all $v \in K$, where K is defined as in the proof of Theorem 3.1, and the operator $\varphi : K \to K$ is weakly sequentially continuous. Thus, Theorem 1 of [4] applies.

Finally, consider the problem [6, Section 3]

$$(\mathrm{P}_{a,b}) \qquad \begin{cases} -\Delta = u^{\sigma-1} + bg(u-a) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where a, b are two positive real numbers and the function $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(z) = 1$ if $z \le 0$, $g(z) = 0$ otherwise. From Theorem 3.2 we easily infer the following

Theorem 3.3. *Suppose*

$$b < \frac{4}{n-2}\left(\frac{n-2}{n+2}\right)^{(n+2)/4} \frac{(2n\pi)^{(n+2)/4}}{[|\Omega|\Gamma(1+n/2)]^{(n+2)/(2n)}}. \tag{9}$$

Then, for every $a > 0$, problem $(\mathrm{P}_{a,b})$ admits at least one solution $u \in X_\infty(\Omega)$.

Proof. Pick $a > 0$ and set $f(z) = |z|^{\sigma-1} + bg(z-a)$, $z \in \mathbb{R}$. Since $D_f = \{a\}$, hypothesis (b_1) of Theorem 3.2 is obviously fulfilled. Now, observe that

$$\inf_{z \in]0,\varrho[} f(z) \ge \min\{a^{\sigma-1}, b\}, \qquad \sup_{z \in]0,\varrho[} f(z) \le \varrho^{\sigma-1} + b$$

for any $\varrho > 0$. Therefore, because of (9), to satisfy assumption (b_2) it is sufficient to choose $r = [(n-2)/(n+2)]^{(n-2)/4} B^{-(n+2)/4}$, where B is given by (3) for $p' = 1$. $\square$

Remark 3.9. Problem $(\mathrm{P}_{a,b})$ has previously been investigated in [6] by means of variational techniques. In that paper the authors performed a complete and satisfactory study. In particular, in Theorem 3.1, they established the existence of a constant $b^* > 0$ so that problem $(\mathrm{P}_{a,b})$ is solvable in $X_\infty(\Omega)$ for every $a > 0$ if and only if $b \le b^*$. No estimate of such a constant is provided in [6], whereas Theorem 3.3 above immediately leads to

$$b^* \ge \frac{4}{n-2}\left(\frac{n-2}{n+2}\right)^{(n+2)/4} \frac{(2n\pi)^{(n+2)/4}}{[|\Omega|\Gamma(1+n/2)]^{(n+2)/(2n)}}.$$

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Alvino and G. Trombetti, *Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri e non*, Ricerche Mat. **30** (1981), 15–33.

[3] A. Ambrosetti and R. E. L. Turner, *Some discontinuous variational problems*, Differential Integral Equations **3** (1988), 341–349.

[4] O. Arino, S. Gautier and J. P. Penot, *A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations*, Funkcial. Ekvac. **27** (1984), 273–279.

[5] M. Badiale, *Critical exponent and discontinuous nonlinearities*, Differential Integral Equations **6** (1993), 1173–1185.

[6] M. Badiale and G. Tarantello, *Existence and multiplicity results for elliptic problems with critical growth and discontinuous nonlinearities*, Nonlinear Anal. (to appear).

[7] G. Bonanno, *Existence theorems on the Dirichlet problem for the equation* $\Delta u + f(x,u) = 0$, Proc. Edinburgh Math. Soc. **39** (1996), 31–36.

[8] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[9] G. Cerami, *Soluzioni positive di problemi con parte non lineare discontinua e applicazione a un problema di frontiera libera*, Boll. Un. Mat. Ital. B (6) **2** (1983), 321–338.

[10] E. De Giorgi, G. Buttazzo and G. Dal Maso, *On the lower semicontinuity of certain integral functionals*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl. **74** (1983), 274–282.

[11] K. Deimling, *Multivalued Differential Equations*, de Gruyter Ser. Nonlinear Anal. Appl., vol. 1, de Gruyter, Berlin, 1992.

[12] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121–126.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* 2nd ed., Springer-Verlag, Berlin, 1983.

[14] C. J. Himmelberg, *Measurable relations*, Fund. Math. **87** (1975), 53–72.

[15] E. Klein and A. C. Thompson, *Theory of Correspondences*, Wiley, New York, 1984.

[16] G. Köthe, *Topological Vector Spaces I*, Springer-Verlag, Berlin, 1965.

[17] H. J. Kuiper, *On positive solutions of nonlinear elliptic eigenvalue problems*, Rend. Circ. Mat. Palermo (2) **20** (1971), 113–138.

[18] P. L. Lions, *Problèmes elliptiques du 2-ème ordre non sous forme divergence*, Proc. Roy. Soc. Edinburgh Sect. A **84** (1979), 263–271.

[19] S. A. Marano, *Existence theorems for a semilinear elliptic boundary value problem*, Ann. Polon. Math. **60** (1994), 57–67.

[20] ———, *Elliptic boundary value problems with discontinuous nonlinearities*, Set-Valued Anal. **3** (1995), 167–180.

[21] I. Massabò, *Positive eigenvalues for elliptic equations with discontinuous nonlinearities*, Boll. Un. Mat. Ital. B (5) **15** (1978), 814–827.

[22] O. Naselli Ricceri and B. Ricceri, *An existence theorem for inclusions of the type* $\Psi(u)(t) \in F(t, \Phi(u)(t))$ *and application to a multivalued boundary value problem*, Appl. Anal. **38** (1990), 259–270.

[23] G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), 697–718.

[24] V. Z. Tsalyuk, *Superposition measurability of multivalued functions*, Math. Notes (Mat. Zametki) **43** (1988), 58–60.

[25] M. Zuluaga, *Existence of solutions for some elliptic problems with critical Sobolev exponents*, Rev. Mat. Iberoamericana **5** (1989), 183–193.

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