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TRAJECTORY ATTRACTORS FOR THE 2D NAVIER–STOKES SYSTEM AND SOME GENERALIZATIONS

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To the memory of Juliusz Schauder

Introduction

We are dealing with the non-autonomous 2D Navier–Stokes system

$$\partial_t u + \nu L u + B(u) = g(x,t), \quad (\nabla, u) = 0, \quad u|_{\partial\Omega} = 0, \tag{1}$$

$x \in \Omega \Subset \mathbb{R}^2$, $t \geq 0, u = u(x,t) = (u^1, u^2) \equiv u(t)$, $g = g(x,t) = (g^1, g^2) \equiv g(t)$. Here $Lu = -P\Delta u$ is the Stokes operator, $\nu > 0$, $B(u) = P\sum_{i=1}^{2} u_i \partial_{x_i} u$; P is the orthogonal projector onto the space of divergence-free vector fields (see Section 1).

Consider the autonomous case: $g(x,t) \equiv g(x)$, $g \in H$, to begin with. Suppose for $t = 0$ we are given the initial condition

$$u|_{t=0} = u_0, \quad u_0 \in H. \tag{2}$$

The problem (1), (2) has a unique solution $u(t)$, $t \geq 0$, which can be represented in the form $u(t) = S(t)u_0$. The family of mappings $\{S(t) \mid t \geq 0\}$ forms a *semigroup*: $S(t_1)S(t_2) = S(t_1 + t_2)$ for $t_1, t_2 \geq 0$, $S(0) = \mathrm{Id}$. A set $\mathfrak{A} \subset H$ is said to be an *attractor* of this semigroup (or an attractor of equation (1)) if $\mathfrak{A}$

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is compact in H, $\mathfrak{A}$ is strictly invariant with respect to $\{S(t)\}$: $S(t)\mathfrak{A} = \mathfrak{A}$ for $t \geq 0$, and $\mathfrak{A}$ attracts every bounded set B in H:

$$\operatorname{dist}_H(S(t)B, \mathfrak{A}) \to 0 \quad (t \to \infty)$$

(see, for example, [13], [20], [2], and the references cited there).

The non-autonomous equation (1) has been less studied. Let an external force $g_0(x,t) \equiv g_0(t)$ in (1) depend on t, $t \geq 0$. Assume the function g_0 is translation-compact in $L_2^{\mathrm{loc}}(\mathbb{R}_+; H) \equiv L_2^{\mathrm{loc}}$ (or in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H) \equiv L_{2,w}^{\mathrm{loc}}$). This means that the family of translations $\{g_0(\cdot + h) \mid h \geq 0\}$ forms a precompact set in L_2^{loc} (respectively, in $L_{2,w}^{\mathrm{loc}}$). It is easy to formulate translation-compactness criterions (see Section 1). For example, g_0 is translation-compact in $L_{2,w}^{\mathrm{loc}}$ if and only if the following norm is finite:

$$\|g_0\|_{\mathrm{a}}^2 = \sup_{t \geq 0} \int_t^{t+1} |g_0(s)|^2 \, ds < \infty. \tag{3}$$

Denote by $\mathcal{H}_+(g_0)$ the hull of the function g_0 in the space $L_{2,w}^{\mathrm{loc}}$, i.e.

$$\mathcal{H}_+(g_0) = [\{g(\cdot + h) \mid h \geq 0\}]_{L_{2,w}^{\mathrm{loc}}},$$

where $[\cdot]_X$ means the closure in a topological space X.

Consider the family of equations (1) with external forces $g \in \mathcal{H}_+(g_0) \equiv \Sigma$. Let $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$ be a family of operators (called a *process* in H) such that $U_g(t,\tau)u_\tau = u_g(t)$, $t \geq \tau \geq 0$, where u_g is a solution of equation (1) with the external force g and with the initial condition $u|_{t=\tau} = u_\tau \in H$. Evidently, $U_g(t,\tau) : H \to H$, $U_g(t,\theta)U_g(\theta,\tau) = U_g(t,\tau)$, $U_g(\tau,\tau) = \mathrm{Id}$ for $t \geq \theta \geq \tau \geq 0$. Consider the family $\{U_g(t,\tau) \mid g \in \mathcal{H}_+(g_0)\}$ of processes corresponding to the family of equations (1) with external forces $g \in \mathcal{H}_+(g_0)$. (In the autonomous case, $g_0(t) \equiv g_0$, $\mathcal{H}_+(g_0) = \{g_0\}$, $U_g(t,\tau) = S(t-\tau)$.) It is known that this family has a uniform (with respect to $g \in \Sigma$) attractor $\mathfrak{A}_\Sigma$ in H. More precisely, $\mathfrak{A}_\Sigma$ is compact in H, it attracts every bounded set B in H uniformly with respect to $g \in \Sigma$:

$$\sup_{g \in \Sigma} \operatorname{dist}_H(U_g(t,\tau)B, \mathfrak{A}_\Sigma) \to 0 \quad (t \to \infty) \quad \forall \tau \geq 0,$$

and $\mathfrak{A}_\Sigma$ is a minimal compact, uniformly attracting set (see [9], [6], and [4] dealing with a more restrictive case). In [6], [4] the structure and properties of the uniform attractor for (1) were also studied.

In the present work we introduce and study a *trajectory attractor* $\mathcal{A}_\Sigma$ for equation (1). We point out at once that a trajectory attractor $\mathcal{A}_\Sigma$ is a compact set in the corresponding trajectory space of equations (1) that consists of their solutions $u_g(t), t \geq 0$, considered as functions of t with values in H. In the previous considerations, the attractor $\mathfrak{A}_\Sigma$ was a compact subset of points in H.

Consider as before a fixed external force g_0 which is a translation-compact function in L_2^{loc} (or in $L_{2,w}^{\mathrm{loc}}$) and let $\mathcal{H}_+(g_0) \equiv \Sigma$ be the hull of g_0 in L_2^{loc}. (The case when g_0 is translation-compact in $L_{2,w}^{\mathrm{loc}}$ is studied in Section 1.) Let $H^{\mathbf{r}}(Q_{t_1,t_2}), \mathbf{r} = (2,2,1)$, be the Nikol'skiĭ space in $Q_{t_1,t_2} = \Omega \times]t_1, t_2[$ (see [3]) of functions $\varphi(x,t) = \varphi(t) = (\varphi^1, \varphi^2) \in H$, $t \in]t_1, t_2[$, with a finite norm

$$\|\varphi\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} = \int_{Q_{t_1,t_2}} \Big(\sum_{|\boldsymbol{\alpha}| \le 2} |\partial_x^{\boldsymbol{\alpha}} \varphi(x,t)|^2 + |\partial_t \varphi(x,t)|^2 \Big)\, dx\, dt.$$

To each external force $g \in \mathcal{H}_+(g_0)$ there corresponds a trajectory space $\mathcal{K}_g^+$. The space $\mathcal{K}_g^+$ is the union of all solutions $u(t) = u_g(t)$, $t \ge 0$, of equation (1) in the space $H^{\mathbf{r},\mathrm{loc}}(Q_+) \equiv H^{\mathbf{r},\mathrm{loc}}$, $Q_+ = \Omega \times]0, \infty[$ (i.e. $u \in H^{\mathbf{r}}(Q_{t_1,t_2})$ for all $]t_1,t_2[\subset \mathbb{R}_+$). Let $\mathcal{K}^+ = \bigcup_{g \in \mathcal{H}_+(g_0)} \mathcal{K}_g^+$ be the union of all $\mathcal{K}_g^+$. The translation semigroup $\{T(h) \mid h \ge 0\}$ acts on $H^{\mathbf{r},\mathrm{loc}}$:

$$T(h)\varphi(t) = \varphi(t+h), \quad h \ge 0.$$

Evidently, $T(h)u_g(\cdot) = u_g(\cdot + h) = u_{T(h)g}(\cdot) \in \mathcal{K}^+_{T(h)g}$. Therefore,

$$T(h)\mathcal{K}^+ \subseteq \mathcal{K}^+ \quad \forall h \ge 0 \tag{4}$$

(the inclusion may be strict, see Section 1). It is proved that $\mathcal{K}^+$ is closed in $H^{\mathbf{r},\mathrm{loc}}$. It is clear that the semigroup $\{T(h)\}$ is continuous on $H^{\mathbf{r},\mathrm{loc}}$. Denote by $H^{\mathbf{r},\mathrm{a}}(Q_+) \equiv H^{\mathbf{r},\mathrm{a}}$ the subset of $H^{\mathbf{r},\mathrm{loc}}$ of functions $\varphi(t)$, $t \ge 0$, having a finite norm

$$\|\varphi\|^2_{H^{\mathbf{r},\mathrm{a}}} = \sum_{|\boldsymbol{\alpha}| \le 2} \|\partial_x^{\boldsymbol{\alpha}} \varphi\|^2_{\mathrm{a}} + \|\partial_t \varphi\|^2_{\mathrm{a}} < \infty,$$

where $\|\cdot\|_{\mathrm{a}}$ is defined in (3).

A *trajectory attractor* of the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$ is a set $\mathcal{A}_\Sigma \subseteq \mathcal{K}^+$ which is compact in $H^{\mathbf{r},\mathrm{loc}}$, bounded in $H^{\mathbf{r},\mathrm{a}}$, invariant with respect to $\{T(h)\}$: $T(h)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma$ for $h \ge 0$, and has the following attraction property: for every set $B \subset \mathcal{K}^+$ bounded in $H^{\mathbf{r},\mathrm{a}}$, and for each $[t_1,t_2] \subset \mathbb{R}_+$ the set $T(h)B$ tends to $\mathcal{A}_\Sigma$ in the strong topology of the space $H^{\mathbf{r}}(Q_{t_1,t_2})$, i.e.

$$\mathrm{dist}_{H^{\mathbf{r}}(Q_{t_1,t_2})}(T(h)B, \mathcal{A}_\Sigma) \to 0 \quad (h \to \infty). \tag{5}$$

In Section 2, we construct the trajectory attractor $\mathcal{A}_\Sigma$ of the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$. Section 1 deals with the trajectory attractor $\mathcal{A}_\Sigma$ in the "weak" topology of $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$ under the assumption that g_0 is translation-compact in $L_{2,w}^{\mathrm{loc}}$ only. In this case $T(h)B$ tends to $\mathcal{A}_\Sigma$ in the weak topology of $H^{\mathbf{r}}(Q_{t_1,t_2})$ for all $[t_1,t_2] \subset \mathbb{R}_+$. In Section 3, the structure of the trajectory attractor $\mathcal{A}_\Sigma$ is described.

Trajectory attractors have been constructed for various equations and systems of PDE for which the corresponding Cauchy problem has non-unique solution or for which the uniqueness theorem has not been proved yet (see [7]–[10] and [5]).

In Section 4 we construct a trajectory attractor for the 3D Navier–Stokes system; the structure and some properties of the trajectory attractor are given as well. In particular, the trajectory attractor $\mathcal{A}_\Sigma$ is stable with respect to small perturbations of the external force $g_0(x,t)$; the trajectory attractor $\mathcal{A}_\Sigma^{(N)}$ of the Faedo–Galerkin approximation system of order N tends to $\mathcal{A}_\Sigma$ as $N \to \infty$ in the corresponding topology. Some other unexpected properties are also exhibited.

1. Trajectory attractor for the 2D N–S system with translation-compact external force in $L_{2,w}^{\rm loc}$

We consider the Navier–Stokes system in a bounded domain $\Omega \Subset \mathbb{R}^2$. Excluding the pressure, the system can be written in the form

$$\partial_t u + \nu L u + B(u) = g(x,t), \quad (\nabla, u) = 0, \quad u|_{\partial\Omega} = 0, \quad x \in \Omega,\ t \geq 0, \tag{1.1}$$

where $x = (x_1, x_2)$, $u = u(x,t) = (u^1, u^2)$, $g = g(x,t) = (g^1, g^2)$. L is the Stokes operator: $Lu = -P\Delta u$; $B(u) = B(u,u)$, $B(u,v) = P(u,\nabla)v = P\sum_{i=1}^2 u_i \partial_{x_i} v$, $\nu > 0$ (see [16], [15], [19], [21]). By H, V, and H_2 we denote respectively the closure in $(L_2(\Omega))^2$, $(H^1(\Omega))^2$, and $(H^2(\Omega))^2$ of the set $\mathcal{V}_0 = \{v \mid v \in (C_0^\infty(\Omega))^2, (\nabla, v) = 0\}$. P denotes the orthogonal projector in $(L_2(\Omega))^2$ onto the Hilbert space H. The scalar products in H and in V are $(u,v) = \int_\Omega (u(x), v(x))\,dx$ and $((u,v)) = \langle Lu, v\rangle = \int_\Omega (\nabla u(x), \nabla v(x))\,dx$ and the norms are respectively $|u| = (u,u)^{1/2}$ and $\|u\| = \langle Lu, u\rangle^{1/2}$. The norm in H_2 is $\|\cdot\|_2$.

To describe the external force $g(x,s)$ in (1.1) consider the topological space $L_{2,w}^{\rm loc}(\mathbb{R}_+; H)$. By definition, the space $L_{2,w}^{\rm loc}(\mathbb{R}_+; H) = L_{2,w}^{\rm loc}$ is $L_2^{\rm loc}(\mathbb{R}_+; H) = L_2^{\rm loc}$ endowed with the following local weak convergence topology. The sequence $\{g_n\}$ converges to g as $n \to \infty$ in $L_{2,w}^{\rm loc}$ whenever $\int_{t_1}^{t_2} (g_n(s) - g(s), v(s))\,ds \to 0$ ($n \to \infty$) for all $[t_1, t_2] \subseteq \mathbb{R}_+$ and all $v \in L_2(t_1, t_2; H)$.

Suppose we are given some fixed external force $g_0 \in L_2^{\rm loc}$. Assume it is translation-compact (tr.-c.) in $L_{2,w}^{\rm loc}$, i.e. the set $\{g_0(\cdot + h) \mid h \in \mathbb{R}_+\}$ is precompact in $L_{2,w}^{\rm loc}$. This condition is valid if and only if

$$\|g_0\|^2_{L_2^{\rm a}(\mathbb{R}_+;H)} = \|g_0\|^2_{\rm a} = \sup_{t\geq 0} \int_t^{t+1} |g_0(s)|^2\,ds < \infty \tag{1.2}$$

(see [6]). Denote by $\mathcal{H}_+(g_0)$ the hull of the function g_0 in $L_{2,w}^{\rm loc}$: $\mathcal{H}_+(g_0) = [\{g_0(\cdot + h) \mid h \in \mathbb{R}_+\}]_{L_{2,w}^{\rm loc}}$. Here $[\,\cdot\,]_{L_{2,w}^{\rm loc}}$ means the closure in $L_{2,w}^{\rm loc}$. It can be shown that the set $\mathcal{H}_+(g_0)$, which is a topological subspace of $L_{2,w}^{\rm loc}$, is metrizable and the corresponding metric space is complete. Moreover, every function $g \in \mathcal{H}_+(g_0)$ is tr.-c. in $L_{2,w}^{\mathrm{loc}}$, $\mathcal{H}_+(g) \subseteq \mathcal{H}_+(g_0)$, and $\|g\|_{\mathrm{a}} \leq \|g_0\|_{\mathrm{a}}$.

The translation semigroup $\{T(t) \mid t \geq 0\} = \{T(t)\}$ acts on $\mathcal{H}_+(g_0)$: $T(t)g(s) = g(s+t)$. Evidently, $T(t)$ is continuous in $L_{2,w}^{\mathrm{loc}}$ and $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$ for $t \geq 0$.

We shall study the family of equations (1.1) with various external forces $g \in \mathcal{H}_+(g_0)$.

Denote by Q_{t_1,t_2} the cylinder $\Omega \times [t_1, t_2]$, where $[t_1, t_2] \subset \mathbb{R}_+$.

Consider the space $H^{\mathbf{r}}(Q_{t_1,t_2}), \mathbf{r} = (2,2,1)$ (see [3]), $H^{\mathbf{r}}(Q_{t_1,t_2}) = L_2(t_1, t_2; H_2) \cap \{v \mid \partial_t v \in L_2(t_1, t_2; H)\}$. The norm in $H^{\mathbf{r}}(Q_{t_1,t_2})$ is

$$\|v\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} = \int_{t_1}^{t_2} (\|v(s)\|_2^2 + |\partial_t v(s)|^2)\, ds. \tag{1.3}$$

Let us recall the existence and uniqueness theorem.

THEOREM 1.1. *Let $g \in L_2(t_1, t_2; H)$ and $u_0 \in V$. Then there exists a unique solution u of equation* (1.1) *belonging to the space $H^{\mathbf{r}}(Q_{t_1,t_2})$ such that $u(t_1) = u_0$. Moreover, $u \in C([t_1, t_2]; V)$.*

This theorem is a variant of the classical result (see [14]–[16], [19], [2]). The proof uses the Faedo–Galerkin approximation method.

We shall study equation (1.1) in the semicylinder $Q_+ = \Omega \times \mathbb{R}_+$, where $g \in \mathcal{H}_+(g_0)$.

Consider the space $H^{\mathbf{r},\mathrm{loc}}(Q_+) = L_2^{\mathrm{loc}}(\mathbb{R}_+; H_2) \cap \{v \mid \partial_t v \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)\}$, i.e. $v \in H^{\mathbf{r},\mathrm{loc}}(Q_+)$ if $\|\Pi_{t_1,t_2} v\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} < \infty$ for every $[t_1, t_2] \subset \mathbb{R}_+$, where Π_{t_1,t_2} is the restriction operator to the interval $[t_1, t_2]$. We introduce two different topological spaces $H_s^{\mathbf{r},\mathrm{loc}}(Q_+)$ and $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$ ("strong" and "weak"). The space $H_s^{\mathbf{r},\mathrm{loc}}(Q_+)$ (resp. $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$) is $H^{\mathbf{r},\mathrm{loc}}(Q_+)$ with the following convergence topology. By definition, $v_n \to v$ $(n \to \infty)$ in $H_s^{\mathbf{r},\mathrm{loc}}(Q_+)$ (resp. in $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$) if $\Pi_{t_1,t_2} v_n \to \Pi_{t_1,t_2} v$ $(n \to \infty)$ strongly in $H^{\mathbf{r}}(Q_{t_1,t_2})$ (respectively, $\Pi_{t_1,t_2} v_n \rightharpoonup \Pi_{t_1,t_2} v$ $(n \to \infty)$ weakly in $H^{\mathbf{r}}(Q_{t_1,t_2})$) for all $[t_1, t_2] \subseteq \mathbb{R}_+$. It is easy to prove that the linear topological space $H_s^{\mathbf{r},\mathrm{loc}}(Q_+)$ is metrizable, for example, by means of the Fréchet metric generated by the seminorms $\|\Pi_{n,n+1} v\|_{H^{\mathbf{r}}(Q_{n,n+1})}$, $n = 0, 1, 2, \ldots$ The space $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$ is not metrizable, but it is a Hausdorff and Fréchet–Urysohn space with a countable topology base.

We shall also use the space $H^{\mathbf{r},\mathrm{a}}(Q_+)$, which is a subspace of $H^{\mathbf{r},\mathrm{loc}}(Q_+)$. By definition, $v \in H^{\mathbf{r},\mathrm{a}}(Q_+)$ if the following norm is finite:

$$\|v\|^2_{H^{\mathbf{r},\mathrm{a}}(Q_+)} = \|v\|^2_{\mathbf{r},\mathrm{a}} = \sup_{t \geq 0} \|\Pi_{t,t+1} v\|^2_{H^{\mathbf{r}}(Q_{t,t+1})}. \tag{1.4}$$

Evidently, $H^{\mathbf{r},\mathrm{a}}(Q_+)$ with the norm (1.4) is a Banach space. We shall not use the topology generated by the norm (1.4). We need the Banach space $H^{\mathbf{r},\mathrm{a}}(Q_+)$ to define bounded sets in $H^{\mathbf{r},\mathrm{loc}}(Q_+)$ only.

With any external force $g \in \mathcal{H}_+(g_0)$ we associate the trajectory space $\mathcal{K}_g^+$ that is the union of all solutions $u(s), s \geq 0$, of equation (1.1) in the space $H^{\mathbf{r},\mathrm{loc}}(Q_+)$. Notice that $|B(v)| \leq C|v|^{1/2}\|v\|^2\|v\|_2^{1/2}$; therefore any solution $u \in \mathcal{K}_g^+$ satisfies (1.1) in the strong sense of the space $L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$. By Theorem 1.1, the trajectory space $\mathcal{K}_g^+$ is wide enough for each $g \in \mathcal{H}_+(g_0)$. Define $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)} = \bigcup_{g\in\mathcal{H}_+(g_0)} \mathcal{K}_g^+$.

Lemma 1.1. *If $g_0 \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ satisfies* (1.2) *then $\mathcal{K}^+ \subset H^{\mathbf{r},\mathrm{a}}(Q_+)$.*

This lemma will be proved later on.

Consider the translation semigroup $\{T(t) \mid t \geq 0\}$ acting on $H^{\mathbf{r},\mathrm{loc}}(Q_+)$ by the formula

$$T(t)v(s) = v(s+t), \quad s \geq 0,\ v \in H^{\mathbf{r},\mathrm{loc}}(Q_+).$$

Obviously, the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ of trajectory spaces corresponding to equation (1.1) satisfies the embedding

$$T(t)\mathcal{K}_g^+ \subseteq \mathcal{K}^+_{T(t)g}, \quad \forall t \geq 0. \tag{1.5}$$

In other words, for each $t \geq 0$, the function $u(s+t)$, $s \geq 0$, is a solution of equation (1.1) with a shifted symbol $g(s+t) = T(t)g(s)$ for any solution $u \in \mathcal{K}_g^+$ of equation (1.1) with symbol $g \in \mathcal{H}_+(g_0)$. Hence, the translation semigroup $\{T(t)\}$ takes $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ into itself: $T(t)\mathcal{K}^+ \subseteq \mathcal{K}^+$, $t \geq 0$.

In this section we study the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ attracts every set $T(t)B$ as $t \to \infty$ in the topology of $\Theta_+^{\mathrm{loc}} = H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$, where $B \subset \mathcal{K}^+$ and B is bounded in the Banach space $\mathcal{F}_+^{\mathrm{a}} = H^{\mathbf{r},\mathrm{a}}(Q_+)$.

Definition 1.1. Let Σ be a complete metric space and let Θ be a topological space. Consider a family of sets $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$, $\mathcal{K}_\sigma \subset \Theta$, depending on a parameter $\sigma \in \Sigma$. The family $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ is said to be (Θ, Σ)-*closed* if the graph set $\bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma \times \{\sigma\}$ is closed in the topological space $\Theta \times \Sigma$ with the usual product topology.

Proposition 1.1. *Let Σ be a compact metric space and $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ be (Θ, Σ)-closed. Then the set $\mathcal{K}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma$ is closed in Θ.*

Proof. We use the standard reasoning. Let $u \notin \mathcal{K}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma$. Then $(u, \sigma) \notin \bigcup_{\sigma'\in\Sigma} \mathcal{K}_{\sigma'} \times \{\sigma'\}$ for all $\sigma \in \Sigma$. The set $\bigcup_{\sigma'\in\Sigma} \mathcal{K}_{\sigma'} \times \{\sigma'\}$ is closed in $\Theta \times \Sigma$, so there is a neighbourhood $\mathcal{W}_\sigma \times \mathcal{O}_\sigma$ in $\Theta \times \Sigma$ such that $\mathcal{W}_\sigma \times \mathcal{O}_\sigma \cap (\bigcup_{\sigma'\in\Sigma} \mathcal{K}_{\sigma'} \times \{\sigma'\}) = \emptyset$, $u \in \mathcal{W}_\sigma$, $\sigma \in \mathcal{O}_\sigma$, where $\mathcal{W}_\sigma$ and $\mathcal{O}_\sigma$ are open sets in Θ and Σ respectively. The family $\{\mathcal{O}_\sigma \mid \sigma \in \Sigma\}$ forms an open covering of Σ. Since Σ is compact, there is a finite subcovering $\{\mathcal{O}_{\sigma_i} \mid i = 1, \ldots, N\}$. Put $\mathcal{W}(u) = \bigcap_{i=1}^N \mathcal{W}_{\sigma_i}$. Evidently, $\mathcal{W}(u) \cap \mathcal{K}_\Sigma = \emptyset$. Hence, for every $u \notin \mathcal{K}_\Sigma$ there is a neighbourhood $\mathcal{W}(u)$ with $\mathcal{W}(u) \cap \mathcal{K}_\Sigma = \emptyset$, i.e. $\mathcal{K}_\Sigma$ is closed in Θ. □

LEMMA 1.2. *The family* $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ *of trajectory spaces corresponding to equation* (1.1) *is* $(\Theta_+^{\mathrm{loc}}, \mathcal{H}_+(g_0))$*-closed and* $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ *is closed in* Θ_+^{loc}.

PROOF. Assume that $u_n \in \mathcal{K}_{g_n}$, $g_n \in \mathcal{H}_+(g_0)$, $u_n \to u$ $(n \to \infty)$ in Θ_+^{loc} and $g_n \to g$ $(n \to \infty)$ in $L_{2,w}^{\mathrm{loc}}$. We claim that $u \in \mathcal{K}_g^+$. Indeed, for each fixed $[t_1, t_2] \subset \mathbb{R}_+$ we have $u_n \rightharpoonup u$ $(n \to \infty)$ weakly in $H^{\mathbf{r}}(Q_{t_1,t_2})$. Thus, $\partial_t u_n \rightharpoonup \partial_t u$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; H)$ and $\partial^{\boldsymbol{\alpha}} u_n \rightharpoonup \partial^{\boldsymbol{\alpha}} u$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; H)$ for all $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $|\boldsymbol{\alpha}| \le 2$. In particular, by refining, we may assume that $u_n \to u$ $(n \to \infty)$ almost everywhere in Q_{t_1,t_2} and $B(u_n) \rightharpoonup B(u)$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; H)$ (see the compactness theorems in [16], [19]). Therefore, in the equation

$$\partial_t u_n + \nu L u_n + B(u_n) = g_n(x, t),$$

we may pass to the limit as $n \to \infty$ weakly in $L_2(t_1, t_2; H)$ and get

$$\partial_t u + \nu L u + B(u) = g(x, t),$$

so that $u \in \mathcal{K}_g^+$. Finally, it follows from Proposition 1.1 that $\mathcal{K}^+_{\mathcal{H}_+(g_0)}$ is closed in Θ_+^{loc} since $\Sigma = \mathcal{H}_+(g_0)$ is a compact metric space. □

Consider the translation semigroup $\{T(t)\}$ acting on the metric space $\mathcal{H}_+(g_0)$. Evidently, the semigroup $\{T(t)\}$ is continuous in $\mathcal{H}_+(g_0)$.

DEFINITION 1.2. A set $\mathfrak{A}$ is said to be a *global attractor* of a semigroup $\{S(t)\}$ acting on a complete metric space X if (i) $\mathfrak{A}$ is compact in X and $\mathfrak{A}$ attracts every bounded set B: $\mathrm{dist}_X(S(t)B, \mathfrak{A}) \to 0$ $(t \to \infty)$; (ii) $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \ge 0$.

For the case $X = \Sigma = \mathcal{H}_+(g_0)$ we have

PROPOSITION 1.2. *The translation semigroup* $\{T(t)\}$ *acting on the compact metric space* $\Sigma = \mathcal{H}_+(g_0)$ *has a global attractor* $\mathfrak{A}$ *which coincides with the* ω*-limit set of* Σ*:*

$$\mathfrak{A} = \omega(\Sigma) = \bigcap_{t \ge 0} \Big[\bigcup_{h \ge t} T(h)\Sigma \Big]_\Sigma, \quad \omega(\Sigma) \subseteq \Sigma,$$

where $[\cdot]_\Sigma$ *means the closure in* Σ*. Moreover,* $T(t)\omega(\Sigma) = \omega(\Sigma)$ *for* $t \ge 0$.

This statement follows from well-known theorems from the theory of attractors of semigroups acting in metric spaces (see, for example, [2], [20], [13]).

Consider a more general scheme. Let Σ be a complete metric space. Let also $\mathcal{F}$ be a Banach space. Assume $\mathcal{F} \subseteq \Theta$, where Θ is a Hausdorff topological space. Let a semigroup $\{T(t)\}$ act on Θ: $T(t)\Theta \subseteq \Theta$, $t \ge 0$. Suppose we are given a family of sets $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$, $\mathcal{K}_\sigma \subseteq \mathcal{F}$. Put $\mathcal{K}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma$.

Definition 1.3. A set $P \subseteq \Theta$ is said to be a *uniformly* (with respect to $\sigma \in \Sigma$) *attracting set* for the family $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ in the topology Θ if for every bounded set B in $\mathcal{F}$ and $B \subseteq \mathcal{K}_\Sigma$, the set P attracts $T(t)B$ as $t \to \infty$ in the topology of Θ, i.e. for every neighbourhood $\mathcal{O}(P)$ of P in Θ there exists $t_1 \geq 0$ such that $T(t)B \subseteq \mathcal{O}(P)$ for all $t \geq t_1$.

Definition 1.4. A set $\mathcal{A}_\Sigma \subseteq \Theta$ is said to be a *uniform* (with respect to $\sigma \in \Sigma$) *attractor* of the semigroup $\{T(t)\}$ on $\mathcal{K}_\Sigma$ in the topology Θ if $\mathcal{A}_\Sigma$ is compact in Θ and is a minimal compact uniformly attracting set of $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$, i.e. $\mathcal{A}_\Sigma$ is contained in every compact uniformly attracting set P of $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$.

Let a semigroup $\{T(t)\}$ act on Σ: $T(t)\Sigma \subseteq \Sigma$, $t \geq 0$.

Definition 1.5. The family $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ of trajectory spaces is said to be *translation-coordinated* (tr.-coord.) if for all $\sigma \in \Sigma$ and $u \in \mathcal{K}_\sigma$,

$$T(t)u \in \mathcal{K}_{T(t)\sigma} \quad \forall t \geq 0.$$

It follows from (1.5) that the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ is tr.-coord. with respect to the translation semigroup $\{T(t)\}$.

Proposition 1.3. *Let Σ be a compact metric space and suppose that a continuous semigroup $\{T(t)\}$ acts on Σ and on Θ: $T(t)\Sigma \subseteq \Sigma$, $T(t)\Theta \subseteq \Theta$, $t \geq 0$. Suppose we are given a family of sets $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$, $\mathcal{K}_\sigma \subseteq \mathcal{F}$. Assume that the family $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ is (Θ, Σ)-closed and tr.-coord. Let there exist a uniformly* (*with respect to $\sigma \in \Sigma$*) *attracting set P for $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ in Θ such that P is compact in Θ and P is bounded in $\mathcal{F}$. Then the semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma$ has a uniform* (*with respect to $\sigma \in \Sigma$*) *attractor $\mathcal{A}_\Sigma \subseteq \mathcal{K}_\Sigma \cap P$ in the space Θ, and*

$$T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma \quad \forall t \geq 0. \tag{1.6}$$

Moreover,

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)},$$

where $\mathcal{A}_{\omega(\Sigma)}$ is the uniform (*with respect to $\sigma \in \omega(\Sigma)$*) *attractor of the family $\{\mathcal{K}_\sigma \mid \sigma \in \omega(\Sigma)\}$, $\mathcal{A}_{\omega(\Sigma)} \subseteq \mathcal{K}_{\omega(\Sigma)}$. Here $\omega(\Sigma)$ is the attractor of the semigroup $\{T(t)\}$ on Σ, $T(t)\omega(\Sigma) = \omega(\Sigma)$. The set $\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$ is compact in Θ and bounded in $\mathcal{F}$.*

The proof of Proposition 1.3 is given in [5] (see also [10]).

In application to the Navier–Stokes system (1.1) in this section, $\Sigma = \mathcal{H}_+(g_0)$, $\mathcal{F} = \mathcal{F}_+^{\mathrm{a}} = H^{\mathbf{r},\mathrm{a}}(Q_+)$, $\Theta = \Theta_+^{\mathrm{loc}} = H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$, $\{T(t)\}$ is the translation semigroup, and $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ is the family of trajectory spaces of equation (1.1). In this case a uniform (with respect to $\sigma \in \Sigma$) attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is called a *trajectory attractor* of the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$. In the next section we shall consider the "strong" space $\Theta = \Theta_+^{\rm loc} = H_s^{\mathbf{r},\rm loc}(Q_+)$.

Let us formulate the main result of this section.

Theorem 1.2. *Let g_0 be tr.-c. in $L_{2,w}^{\rm loc}(\mathbb{R}_+;H)$. Then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ has a trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $\Theta_+^{\rm loc} = H_w^{\mathbf{r},\rm loc}(Q_+)$; the set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ attracts every set $B \subseteq \mathcal{K}^+$, bounded in $\mathcal{F}_+^{\rm a} = H^{\mathbf{r},\rm a}(Q_+)$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}^+$, compact in $\Theta_+^{\rm loc}$, and it is invariant with respect to the translation semigroup: $T(t)\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\mathcal{H}_+(g_0)}$ for all $t \geq 0$. Moreover,*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))}, \tag{1.7}$$

where $\mathcal{A}_{\omega(\mathcal{H}_+(g_0))}$ is the trajectory attractor of the family $\{\mathcal{K}_g \mid g \in \omega(\mathcal{H}_+(g_0))\}$, $\mathcal{A}_{\omega(\mathcal{H}_+(g_0))} \subseteq \mathcal{K}_{\omega(\mathcal{H}_+(g_0))}$. Every function $u \in \mathcal{A}_{\mathcal{H}_+(g_0)}$ is tr.-c. in $\Theta_+^{\rm loc}$.

Notice that the topology of the space $H_w^{\mathbf{r}}(Q_{t_1,t_2})$ is stronger than the uniform convergence topology of the space $C([t_1,t_2];H)$, $H_w^{\mathbf{r}}(Q_{t_1,t_2}) \subset C([t_1,t_2];H)$. So, we have

Corollary 1.1. *For every set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}^+$, one has*

$$\operatorname{dist}_{C([0,\Gamma];H)}(\Pi_{0,\Gamma}T(t)B, \Pi_{0,\Gamma}\mathcal{A}_{\mathcal{H}_+(g_0)}) \to 0 \quad (t \to \infty) \quad \forall \Gamma \geq 0.$$

Similarly, from the embedding $H_w^{\mathbf{r}}(Q_{t_1,t_2}) \subset C_w([t_1,t_2];V)$, we obtain

Corollary 1.2. *For every set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}^+$, and for all $v \in V$, one has*

$$\operatorname{dist}_{C([0,\Gamma])}(\Pi_{0,\Gamma}J_vT(t)B, \Pi_{0,\Gamma}J_v\mathcal{A}_{\mathcal{H}_+(g_0)}) \to 0 \quad (t \to \infty) \quad \forall \Gamma \geq 0,$$

where J_v is the mapping from $H_w^{\mathbf{r}}(Q_{t_1,t_2})$ into $C([t_1,t_2])$ given by $J_v(u(\cdot)) = ((u(\cdot),v))$, $((\cdot,\cdot))$ being the scalar product in V.

To prove Theorem 1.2 we use Proposition 1.3. According to (1.5) and Lemma 1.2 we only have to check that the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ of trajectory spaces corresponding to equation (1.1) has a uniformly (with respect to $g \in \mathcal{H}_+(g_0)$) attracting set P compact in $\Theta_+^{\rm loc}$ and bounded in $\mathcal{F}_+^{\rm a}$. This is the most difficult part of the proof. We separate the proof of this fact into a few lemmas.

Lemma 1.3. *For all $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$, the following estimates are valid:*

$$|u(\tau+t)|^2 \leq e^{-\lambda t}|u(\tau)|^2 + C_1\|g\|_{\rm a}^2, \quad t,\tau \geq 0, \tag{1.8}$$

$$\|T(t)u\|^2_{L_\infty(\mathbb{R}_+;H)} \leq e^{-\lambda t}|u(0)|^2 + C_1\|g\|_{\rm a}^2, \tag{1.9}$$

where λ is the first eigenvalue of the operator νL, $C_1 = \lambda^{-1}(1-e^{-\lambda})^{-1}$;

$$\nu \int_t^{t+1} \|u(s)\|^2\,ds \le |u(t)|^2 + C_2 \int_t^{t+1} |g(s)|^2\,ds, \tag{1.10}$$

$$\nu \|T(t)u\|^2_{L_2^{\mathrm{a}}(\mathbb{R}_+;V)} \le e^{-\lambda t}|u(0)|^2 + C_3\|g\|^2_{\mathrm{a}}, \tag{1.11}$$

where $C_2 = \lambda^{-1}$, $C_3 = C_1 + C_2$, $t \ge 0$.

Proof. Taking the scalar product in H of (1.1) with u, we get

$$\frac{d}{dt}|u(t)|^2 + \lambda|u(t)|^2 \le \frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le \lambda^{-1}|g(t)|^2, \tag{1.12}$$

and integrating from τ to $\tau + t$ we obtain

$$|u(\tau+t)|^2 \le e^{-\lambda t}|u(\tau)|^2 + \lambda^{-1}e^{-\lambda(\tau+t)} \int_\tau^{\tau+t} |g(s)|^2 e^{\lambda s}\,ds.$$

Estimating the last expression, we get

$$\begin{aligned}
&\int_\tau^{\tau+t} |g(s)|^2 e^{-\lambda(\tau+t-s)}\,ds \\
&\quad\le \int_{\tau+t-1}^{\tau+t} |g(s)|^2 e^{-\lambda(\tau+t-s)}\,ds + \int_{\tau+t-2}^{\tau+t-1} |g(s)|^2 e^{-\lambda(\tau+t-s)}\,ds + \dots \\
&\quad\le \int_{\tau+t-1}^{\tau+t} |g(s)|^2\,ds + e^{-\lambda}\int_{\tau+t-2}^{\tau+t-1} |g(s)|^2\,ds + e^{-2\lambda}\int_{\tau+t-3}^{\tau+t-2} |g(s)|^2\,ds + \dots \\
&\quad\le \|g\|^2_{\mathrm{a}}(1 + e^{-\lambda} + e^{-2\lambda} + \dots) = \|g\|^2_{\mathrm{a}}(1-e^{-\lambda})^{-1}.
\end{aligned}$$

So, inequality (1.8) is proved. Inequality (1.9) follows directly from (1.8). In the usual way, one derives (1.10) from (1.12). Combining (1.8) and (1.10), we get (1.11). □

Lemma 1.4. *For all $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$,*

$$\sup_{0\le t\le\Gamma} t\|u(\tau+t)\|^2 \le C_1\bigg(\Gamma, |u(\tau)|^2, \int_\tau^{\tau+\Gamma} |g(s)|^2\,ds\bigg), \quad \Gamma,\tau \ge 0, \tag{1.13}$$

where $C_1(\eta_1,\eta_2,\eta_3)$ is a continuous and increasing function with respect to each $\eta_i \ge 0$.

The proof is analogous to one given in [2]. We sketch the main points for convenience of the readers. For brevity, we suppose without loss of generality that $\nu = 1$ and $\tau = 0$. Multiplying equation (1.1) by tLu we get

$$\frac{1}{2}\frac{d}{dt}(t\|u(t)\|^2) - \frac{1}{2}\|u(t)\|^2 + t\|u(t)\|_2^2 + t(B(u), Lu) \le t|g(t)|^2 + \frac{1}{4}t\|u(t)\|_2^2. \tag{1.14}$$

Recall that $(u, Lu) = \|u\|^2$ and $(Lu, Lu) = \|u\|_2^2$. We also have

$$(B(u), Lu) \le |B(u)| \cdot \|u\|_2, \tag{1.15}$$

$$|B(u)| \le c\biggl(\int_\Omega |u|^2 |\nabla u|^2 \, dx\biggr)^{1/2} \le c\|u\|_{0,4} \|u\|_{1,4}, \tag{1.16}$$

$$\|u\|_{0,4} \le c_1 \|u\|^{1/2} |u|^{1/2}, \quad \|u\|_{0,4} \le c_2 \|u\|_2^{1/2} \|u\|^{1/2} \tag{1.17}$$

(see inequalities (1.17) in [15], [21]). It follows from (1.15)–(1.17) that

$$|B(u)| \le c_3 \|u\|_2^{1/2} \|u\| \cdot |u|^{1/2}, \tag{1.18}$$

$$t(B(u), Lu) \le tc_3 \|u\|_2^{3/2} \|u\| \cdot |u|^{1/2} \le \frac{t}{4}\|u\|_2^2 + \frac{tc_4}{2}\|u\|^4 |u|^2. \tag{1.19}$$

Using (1.14) and (1.19) we obtain

$$\frac{d}{dt}(t\|u(t)\|^2) + t\|u(t)\|_2^2 \le \|u(t)\|^2 + 2t|g(t)|^2 + tc_4 \|u(t)\|^4 |u(t)|^2. \tag{1.20}$$

Define $z(t) = t\|u(t)\|^2$. Consequently,

$$z'(t) \le b(t) + \gamma(t) z(t), \quad b(t) = \|u(t)\|^2 + 2t|g(t)|^2, \quad \gamma(t) = c_4 \|u(t)\|^2 |u(t)|^2.$$

Applying the Gronwall inequality, we get

$$z(t) \le \int_0^t b(s) \exp\biggl(\int_s^t \gamma(\theta)\, d\theta\biggr) ds \le \biggl(\int_0^t b(s)\, ds\biggr) \exp\biggl(\int_0^t \gamma(s)\, ds\biggr).$$

Using (1.12), we have

$$|u(t)|^2 + \int_0^t \|u(s)\|^2 \, ds \le |u(0)|^2 + \lambda^{-1} \int_0^t |g(s)|^2 \, ds. \tag{1.21}$$

Therefore

$$\begin{aligned} t\|u(t)\|^2 &\le \biggl(\int_0^t (\|u(s)\|^2 + 2s|g(s)|^2)\, ds\biggr) \exp\biggl(\int_0^t c_4 \|u(s)\|^2 |u(s)|^2 \, ds\biggr) \\ &\le \biggl(|u(0)|^2 + (\lambda^{-1} + 2t) \int_0^t |g(s)|^2 \, ds\biggr) \\ &\quad \times \exp\biggl(c_4\biggl(|u(0)|^2 + \lambda^{-1} \int_0^t |g(s)|^2 \, ds\biggr)^2\biggr). \end{aligned}$$

Finally,

$$\sup_{0 \le t \le \Gamma} t\|u(t)\|^2 \le C_1\biggl(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2 ds\biggr), \tag{1.22}$$

where $C_1(\eta_1, \eta_2, \eta_3) = (\eta_2 + (\lambda^{-1} + 2\eta_1)\eta_3) \exp(c_4(\eta_2 + \lambda^{-1}\eta_3)^2)$. □

Inequality (1.13) implies that

$$\|u(t + \tau + 1)\|^2 \le C_1\biggl(1, |u(t + \tau)|^2, \int_{t+\tau}^{t+\tau+1} |g(s)|^2 \, ds\biggr). \tag{1.23}$$

Taking sup in (1.23) with respect to $\tau \geq 0$, we obtain, according to (1.9),

$$\begin{aligned}\|T(t+1)u\|^2_{L_\infty(\mathbb{R}_+;V)} &\leq C_1(1, \|T(t)u\|^2_{L_\infty(\mathbb{R}_+;H)}, \|T(t)g\|^2_{\mathrm{a}})\\ &\leq C_2(e^{-\lambda t}|u(0)|^2, \|g\|^2_{\mathrm{a}}).\end{aligned}$$

Hence we get

Corollary 1.3. *For all* $u \in \mathcal{K}^+_g$, $g \in \mathcal{H}_+(g_0)$,

$$\|T(t+1)u\|^2_{L_\infty(\mathbb{R}_+;V)} \leq C_2(e^{-\lambda t}|u(0)|^2, \|g\|^2_{\mathrm{a}}), \quad t \geq 0.$$

Lemma 1.5. *For all* $u \in \mathcal{K}^+_g$, $g \in \mathcal{H}_+(g_0)$,

$$\int_\tau^{\tau+\Gamma} (s-\tau)(\|v(s)\|^2_2 + |\partial_t v(s)|^2)\, ds \leq C_3\biggl(\Gamma, |u(\tau)|^2, \int_\tau^{\tau+\Gamma} |g(s)|^2\, ds\biggr), \tag{1.24}$$

$$\begin{aligned}\|T(t+1)u\|^2_{\mathbf{r},\mathrm{a}} &= \sup_{\tau \geq t+1} \int_\tau^{\tau+1} (\|u(s)\|^2_2 + |\partial_t u(s)|^2)\, ds\\ &\leq C_4(e^{-\lambda t}|u(0)|^2, \|g\|^2_{\mathrm{a}}),\end{aligned} \tag{1.25}$$

where τ, t, Γ *are positive and arbitrary.*

Proof. It is sufficient to prove (1.24) for $\tau = 0$ and $\nu = 1$. It follows from (1.20)–(1.22) that

$$\begin{aligned}&\int_0^\Gamma s\|u(s)\|^2_2\, ds\\ &\leq \int_0^\Gamma \|u(s)\|^2\, ds + 2\Gamma \int_0^\Gamma |g(s)|^2\, ds\\ &\quad + c_4(\sup_{0\leq t\leq\Gamma} |u(t)|^2)(\sup_{0\leq t\leq\Gamma} t\|u(t)\|^2) \int_0^\Gamma \|u(s)\|^2\, ds\\ &\leq |u(0)|^2 + \lambda^{-1} \int_0^\Gamma |g(s)|^2\, ds + 2\Gamma \int_0^\Gamma |g(s)|^2\, ds\\ &\quad + c_4\biggl(|u(0)|^2 + \lambda^{-1}\int_0^\Gamma |g(s)|^2\, ds\biggr)^2 C_1\biggl(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2\, ds\biggr)\\ &= C_3'\biggl(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2\, ds\biggr).\end{aligned} \tag{1.26}$$

Now, equation (1.1) implies directly that

$$(1.27)\quad \left(\int_0^\Gamma s|\partial_t v(s)|^2\,ds\right)^{1/2}$$
$$\leq \left(\int_0^\Gamma s\|u(s)\|_2^2\,ds\right)^{1/2} + \left(\int_0^\Gamma s|B(u)|^2\,ds\right)^{1/2} + \left(\int_0^\Gamma s|g(s)|^2\,ds\right)^{1/2}$$
$$\leq C_3'\left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2\,ds\right)$$
$$+\Gamma^{1/2}\left(\int_0^\Gamma |g(s)|^2\,ds\right)^{1/2} + c_3\left(\int_0^\Gamma s\|u(s)\|_2\|u(s)\|^2|u(s)|\,ds\right)^{1/2}.$$

We have used inequality (1.18). At the same time by (1.22) and (1.26), we get

$$(1.28)\quad \int_0^\Gamma s\|u(s)\|_2\|u(s)\|^2|u(s)|\,ds \leq \int_0^\Gamma s\|u(s)\|^4|u(s)|^2\,ds + \int_0^\Gamma s\|u(s)\|_2^2\,ds$$
$$\leq (\sup_{0\leq t\leq\Gamma}|u(t)|^2)(\sup_{0\leq t\leq\Gamma} t\|u(t)\|^2)\int_\tau^\Gamma \|u(s)\|^2\,ds + C_3'\left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2\,ds\right)$$
$$\leq \left(|u(0)|^2 + \lambda^{-1}\int_0^\Gamma |g(s)|^2\,ds\right)^2 C_1\left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2\,ds\right) + C_3'(\cdot).$$

Combining (1.27) and (1.28) we obtain

$$(1.29)\quad \int_0^\Gamma s|\partial_t v(s)|^2\,ds \leq C_3''\left(\Gamma, |u(0)|^2, \int_0^\Gamma |g(s)|^2\,ds\right).$$

Summing (1.26) and (1.29), we derive (1.24). From (1.24) it follows for $\Gamma = 2$ that

$$(1.30)\quad \int_{t+\tau+1}^{t+\tau+2} (\|v(s)\|_2^2 + |\partial_t v(s)|^2)\,ds \leq C_3\left(2, |u(t+\tau)|^2, \int_{t+\tau}^{t+\tau+2} |g(s)|^2\,ds\right).$$

Taking sup in (1.30) with respect to $\tau \geq 0$, we obtain, according to (1.9),

$$\|T(t+1)u\|_{\mathbf{r},\mathrm{a}}^2 \leq C_3(2, \|T(t)u\|_{L_\infty(\mathbb{R}_+;H)}^2, 2\|T(t)g\|_{\mathrm{a}}^2) \leq C_4(e^{-\lambda t}|u(0)|^2, \|g\|_{\mathrm{a}}^2).\ \square$$

Lemma 1.1 follows from a more general

Lemma 1.6. *For all* $u \in \mathcal{K}_g^+$, $g \in \mathcal{H}_+(g_0)$,

$$(1.31)\quad \int_\tau^{\tau+\Gamma} (\|v(s)\|_2^2 + |\partial_t v(s)|^2)\,ds \leq C_5\left(\|u(\tau)\|^2, \int_\tau^{\tau+\Gamma} |g(s)|^2\,ds\right)$$

for $\tau, \Gamma \geq 0$, *and*

$$\|u\|_{\mathbf{r},\mathrm{a}}^2 = \sup_{\tau\geq 0}\int_\tau^{\tau+1} (\|u(s)\|_2^2 + |\partial_t u(s)|^2)\,ds \leq C_6(\|u(0)\|^2, \|g\|_{\mathrm{a}}^2).$$

PROOF. Similarly to (1.20) we get

$$\frac{d}{dt}(\|u(t)\|^2) + \|u(t)\|_2^2 \le 2|g(t)|^2 + c_4\|u(t)\|^4|u(t)|^2,$$
$$z_1'(t) \le b_1(t) + \gamma(t)z_1(t), \quad z_1(t) = \|u(t)\|^2, \quad b_1(t) = 2|g(t)|^2,$$
$$z_1(t) \le \left(z_1(0) + \int_0^t b_1(s)\,ds\right)\exp\left(\int_0^t \gamma(s)\,ds\right).$$

So, using (1.21), we obtain, as above, (1.31). Finally, combining (1.31) with $\tau \in [0,1]$ and (1.25) with $\tau \in]1,\infty[$ we get

$$\|u\|_{\mathbf{r},\mathrm{a}}^2 = \sup_{\tau\ge 0}\int_\tau^{\tau+1}(\|u(s)\|_2^2 + |\partial_t u(s)|^2)\,ds$$
$$\le \max\{C_5(\|u(0)\|^2, \|g\|_{\mathrm{a}}^2), C_4(|u(0)|^2, \|g\|_{\mathrm{a}}^2)\} = C_6(\|u(0)\|^2, \|g\|_{\mathrm{a}}^2). \quad \square$$

Coming back to the proof of Theorem 1.2, we construct a uniformly attracting set P in Θ_+^{loc} for the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$. From (1.25) it follows that

$$\|T(t+1)u\|_{\mathbf{r},\mathrm{a}}^2 \le C_4(e^{-\lambda t}\|u\|_{\mathbf{r},\mathrm{a}}^2, \|g\|_{\mathrm{a}}^2) \le C_4(e^{-\lambda t}\|u\|_{\mathbf{r},\mathrm{a}}^2, \|g_0\|_{\mathrm{a}}^2) \tag{1.32}$$

for $u \in \mathcal{K}^+$, since $\|g\|_{\mathrm{a}} \le \|g_0\|_{\mathrm{a}}$ for all $g \in \mathcal{H}_+(g_0)$. Consider the set

$$P_0 = \{v \in \mathcal{F}_+^{\mathrm{a}} \mid \|v\|_{\mathbf{r},\mathrm{a}}^2 \le C_4(1, \|g_0\|_{\mathrm{a}}^2)\}.$$

Evidently, P_0 is the desired attracting set. Indeed, if $B \subseteq \mathcal{K}^+ \cap \mathcal{F}_+^{\mathrm{a}}$ is a bounded set of trajectories then $e^{-\lambda t}\|u\|_{\mathbf{r},\mathrm{a}}^2 \le 1$ for all $u \in B$ whenever $t \ge t' \gg 1$ and therefore, by (1.32), $T(t+1)B \subseteq P_0$. Hence P_0 is even a uniformly absorbing set. Notice that the set P_0 is bounded in $\mathcal{F}_+^{\mathrm{a}}$ and compact in $\Theta_+^{\mathrm{loc}} = H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$. The latter is true since the topology in $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$ is generated by the weak topology of the Banach spaces $H^{\mathbf{r}}(Q_{t_1,t_2}) = L_2(t_1,t_2;H_2) \cap \{v \mid \partial_t v \in L_2(t_1,t_2;H)\}$. Recall that $u_n \rightharpoonup u$ $(n \to \infty)$ weakly in $H^{\mathbf{r}}(Q_{t_1,t_2})$ whenever $\partial_t u_n \rightharpoonup \partial_t u$ $(n \to \infty)$ weakly in $L_2(t_1,t_2;H)$ and $\partial^{\boldsymbol{\alpha}} u_n \rightharpoonup \partial^{\boldsymbol{\alpha}} u$ $(n \to \infty)$ weakly in $L_2(t_1,t_2;H)$ for all $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$ with $|\boldsymbol{\alpha}| \le 2$. That is, a bounded set in $H^{\mathbf{r}}(Q_{t_1,t_2})$ is weakly compact in $H^{\mathbf{r}}(Q_{t_1,t_2})$.

REMARK 1.1. The set P_0, being a compact subspace of $H_w^{\mathbf{r},\mathrm{loc}}(Q_+)$, is a metrizable space and the corresponding metric space is compact. This follows from the fact that a ball of a separable Banach space endowed with the weak topology of this space is metrizable and compact. The translation semigroup $\{T(t)\}$ is continuous on P_0 and $T(t)$ takes P_0 into itself: $T(t)P_0 \subseteq P_0$ for all $t \ge 0$. So Proposition 1.2 is applicable. In particular, the set $\mathfrak{A} = \omega(P_0)$ is a global attractor of the semigroup $\{T(t)\}$ acting on P_0. Moreover, $\mathfrak{A} = \mathcal{A}_{\mathcal{H}_+(g_0)}$ because P_0 is a uniformly absorbing set of the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ of trajectory spaces. This reasoning proves the first part of Theorem 1.2. To prove property (1.7) we have to use a more subtle reasoning (see [5]).

2. Trajectory attractor for the 2D N–S system with translation-compact external force in $L_2^{\rm loc}$

Now consider the case when the external force $g(x,s)$ in (1.1) is a tr.-c. function in $L_2^{\rm loc}(\mathbb{R}_+;H)$. The space $L_2^{\rm loc}(\mathbb{R}_+;H) = L_2^{\rm loc}$ is endowed with the following local strong convergence topology. A sequence $\{g_n\}$ converges to g as $n\to\infty$ in $L_2^{\rm loc}$ whenever $\int_{t_1}^{t_2}|g_n(s)-g(s)|^2\,ds\to 0$ $(n\to\infty)$ for each $[t_1,t_2]\subseteq\mathbb{R}_+$. The space $L_2^{\rm loc}$ is metrizable and complete. A function $g\in L_2^{\rm loc}$ is tr.-c. in $L_2^{\rm loc}$ whenever the set $\{g(\cdot+h)\mid h\in\mathbb{R}_+\}$ is precompact in $L_2^{\rm loc}$. The criterion of being tr.-c. in $L_2^{\rm loc}$ is given in [6]. We recall that a function $g\in L_2^{\rm loc}$ is tr.-c. in $L_2^{\rm loc}$ if and only if

(i) for every $h\ge 0$ the set $\{\int_t^{t+h} g(\cdot,s)\,ds\mid t\in\mathbb{R}_+\}$ is precompact in H;

(ii) there is a function $\beta(s)>0, s>0$, such that $\beta(s)\to 0+$ $(s\to 0+)$ and

$$\int_t^{t+1}|g(s)-g(s+l)|^2\,ds\le\beta(|l|)\quad\forall t\ge 0.$$

REMARK 2.1. Let us give a simple sufficient condition. A function $g\in L_2^{\rm loc}$ is tr.-c. in $L_2^{\rm loc}$ if

$$\|\Pi_{0,1}g(\cdot+t)\|_{H^\delta(Q_{0,1})}\le M\quad\forall t\ge 0$$

for some $\delta>0$. Here $H^\delta(Q_{0,1})=H^\delta(\Omega\times[t_1,t_2])$ is the Sobolev space of order δ.

Suppose we are given a fixed tr.-c. function g_0 in $L_2^{\rm loc}$. Evidently, g_0 is tr.-c. in $L_{2,w}^{\rm loc}$ as well. Consider the set $\{g_0(\cdot+h)\mid h\in\mathbb{R}_+\}$. Notice that $[\{g_0(\cdot+h)\mid h\in\mathbb{R}_+\}]_{L_{2,w}^{\rm loc}}\equiv[\{g_0(\cdot+h)\mid h\in\mathbb{R}_+\}]_{L_2^{\rm loc}}$ and the corresponding topological subspaces of $L_{2,w}^{\rm loc}$ and $L_2^{\rm loc}$ are homeomorphic. Hence, $\mathcal{H}_+(g_0)=[\{g_0(\cdot+h)\mid h\in\mathbb{R}_+\}]_\Xi$ does not depend on $\Xi=L_{2,w}^{\rm loc}$ or $\Xi=L_2^{\rm loc}$. As usual, the topological space $\mathcal{H}_+(g_0)$ is compact and every function $g\in\mathcal{H}_+(g_0)$ is tr.-c. in $L_2^{\rm loc}$, $\mathcal{H}_+(g)\subseteq\mathcal{H}_+(g_0)$, and $\|g\|_{\rm a}\le\|g_0\|_{\rm a}$.

Now consider the "strong" space $H_s^{\mathbf{r},\rm loc}(Q_+)$ introduced in Section 1. Recall that $v_n\to v$ $(n\to\infty)$ in $H_s^{\mathbf{r},\rm loc}(Q_+)$ if $\Pi_{t_1,t_2}v_n\to\Pi_{t_1,t_2}v$ $(n\to\infty)$ strongly in $H^{\mathbf{r}}(Q_{t_1,t_2})$ with respect to the norm (1.3) for each $[t_1,t_2]\subseteq\mathbb{R}_+$. The linear topological space $H_s^{\mathbf{r},\rm loc}(Q_+)$ is metrizable and complete.

To each $g\in\mathcal{H}_+(g_0)$ there corresponds the trajectory space $\mathcal{K}_g^+$ that is the union of all solutions $u(s), s\ge 0$, of equation (1.1) in the space $H^{\mathbf{r},\rm loc}(Q_+)$. Consider the family $\{\mathcal{K}_g^+\mid g\in\mathcal{H}_+(g_0)\}$ and the union $\mathcal{K}_{\mathcal{H}_+(g_0)}^+=\bigcup_{g\in\mathcal{H}_+(g_0)}\mathcal{K}_g^+$.

In this section we study the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+=\mathcal{K}_{\mathcal{H}_+(g_0)}^+$ in the "strong" topological space $H_s^{\mathbf{r},\rm loc}(Q_+)$. The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ attracts every set $T(t)B$ as $t\to\infty$ in the topology of $\Theta_+^{\rm loc}=H_s^{\mathbf{r},\rm loc}(Q_+)$, where $B\subset\mathcal{K}^+$ and B is bounded in the Banach space $\mathcal{F}_+^{\rm a}=H^{\mathbf{r},\rm a}(Q_+)$.

Theorem 2.1. *Let g_0 be tr.-c. in $L_2^{\rm loc}$. Then the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $H_w^{\mathbf{r},\rm loc}(Q_+)$ of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+$ from Theorem 1.2 serves as the trajectory attractor in $H_s^{\mathbf{r},\rm loc}(Q_+)$ of this semigroup. In particular, for every set $B \subset \mathcal{K}^+$,*

$$\operatorname{dist}_{H^{\mathbf{r}}(Q_{0,\Gamma})}(T(t)B, \mathcal{A}_{\mathcal{H}_+(g_0)}) \to 0 \quad (t \to \infty) \quad \forall \Gamma > 0.$$

The set $\mathcal{A}_{\mathcal{H}_+(g_0)}$ is bounded in $\mathcal{F}^+$, and compact in $H_s^{\mathbf{r},\rm loc}(Q_+)$. Every function $u \in \mathcal{A}_{\mathcal{H}_+(g_0)}$ is tr.-c. in $H_s^{\mathbf{r},\rm loc}(Q_+)$.

Notice that if the trajectory attractor in $H_s^{\mathbf{r},\rm loc}(Q_+)$ exists then it coincides with the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $H_w^{\mathbf{r},\rm loc}(Q_+)$ since the embedding $H_s^{\mathbf{r},\rm loc}(Q_+) \subseteq H_w^{\mathbf{r},\rm loc}(Q_+)$ is continuous and the trajectory attractor is the minimal attracting set. So to apply Proposition 1.3 we have to produce an attracting set P_1 that is compact in $H_s^{\mathbf{r},\rm loc}(Q_+)$ and bounded in $\mathcal{F}_+^{\rm a} = H^{\mathbf{r},a}(Q_+)$.

From the continuous embedding $H_s^{\mathbf{r}}(Q_{t_1,t_2}) \subset C([t_1,t_2];V)$ and from Theorem 2.1, we deduce

Corollary 2.1. *For every set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}^+$, one has*

$$\operatorname{dist}_{C([0,\Gamma];V)}(\Pi_{0,\Gamma}T(t)B, \Pi_{0,\Gamma}\mathcal{A}_{\mathcal{H}_+(g_0)}) \to 0 \quad (t \to \infty) \quad \forall \Gamma \geq 0.$$

Proof of Theorem 2.1. Consider the set $P_0' = P_0 \cap \mathcal{K}^+$, where P_0 is the absorbing set constructed in Section 1. Evidently, P_0' is uniformly absorbing for the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$. Put

$$P_1 = S(1)P_0' = \{\widetilde{u}_g \equiv u_g(s+1),\ s \geq 0 \mid u_g \in \mathcal{K}_g^+ \cap P_0,\ g \in \mathcal{H}_+(g_0)\}.$$

The set P_1 is uniformly absorbing for the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ as well. To complete the proof of Theorem 2.1 we have to establish the following

Lemma 2.1. *The set P_1 is compact in $H_s^{\mathbf{r},\rm loc}(Q_+)$.*

It is easy to prove the following statement using the diagonal process.

Proposition 2.1. *The set B is compact in $H_s^{\mathbf{r},\rm loc}(Q_+)$ if and only if $\Pi_{0,\Gamma}B$ is compact in $H^{\mathbf{r}}(Q_{0,\Gamma})$ for every $\Gamma > 0$.*

Proof of Lemma 2.1. Fix $\Gamma > 0$. Let $\widetilde{u}^n(s) = T(1)u^n(s) = u^n(s+1)$ be any sequence from P_1, $u^n \in \mathcal{K}_{g_n}^+ \cap P_0$, $g_n \in \mathcal{H}_+(g_0)$. Without loss of generality, we may assume that

$$\int_0^{\Gamma+1} |g_n(s) - g(s)|^2\,ds \to 0 \quad (n \to \infty) \tag{2.1}$$

for some $g \in \mathcal{H}_+(g_0)$. Let us show that the sequence $\{\widetilde{u}^n\}$ is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma})$. Since $u^n \in P_0$, we have

$$\|u^n(\cdot)\|_{H^{\mathbf{r}}(Q_{0,\Gamma+1})} \leq M(\Gamma+1) \quad \forall n \in \mathbb{N}, \tag{2.2}$$

where the positive function $M(\theta)$ is non-decreasing. We can represent the function $u^n(s)$ as a sum of two functions:

$$u^n(s) = u_1^n(s) + u_2^n(s), \quad s \geq 0,$$

where $u_1^n(s)$ and $u_2^n(s)$ are solutions of the following problems:

$$\partial_t u_1^n(t) + Lu_1^n(t) = 0, \quad t \geq 0, \tag{2.3}$$

$$u_1^n(0) = u^n(0), \quad u_1^n|_{\partial\Omega} = 0, \quad \|u_1^n(0)\| \leq M_1; \tag{2.4}$$

$$\partial_t u_2^n(t) + Lu_2^n(t) = -B(u^n(t)) + g_n(t), \quad t \geq 0, \tag{2.5}$$

$$u_2^n(0) = 0, \quad u_2^n|_{\partial\Omega} = 0. \tag{2.6}$$

Accordingly, $\widetilde{u}^n(s) = \widetilde{u}_1^n(s) + \widetilde{u}_2^n(s)$.

Since u_1^n is a solution of the Stokes problem (2.3), (2.4), we obtain

$$\int_0^{t+1} (\|u_1^n(s)\|_2^2 + |\partial_t u_1^n(s)|^2)\, ds \leq M_2(t+1, \|u_1^n(0)\|) = M_3(t+1) \tag{2.7}$$

for $0 \leq t \leq \Gamma$. Let $\psi(t)$ be a cut-off function:

$$\psi(t) \equiv 1, \quad t \geq 1; \quad \psi(t) \equiv 0, \quad 0 \leq t \leq 1/2; \quad \psi \in C_0^\infty(\mathbb{R}), \quad \psi(t) \geq 0.$$

It follows from (2.3) that

$$\partial_t(\psi(t)u_1^n(t)) + L(\psi(t)u_1^n(t)) = \psi'(t)u_1^n(t).$$

Differentiating this equation in t and setting $\partial_t(\psi(t)u_1^n(t)) = p^n$, we get

$$\begin{aligned} &\partial_t p^n + Lp^n = \psi''(t)u_1^n(t) + \psi'(t)\partial_t u_1^n(t), \\ &p^n(0) = 0, \quad p^n|_{\partial\Omega} = 0. \end{aligned}$$

So,

$$\begin{aligned} \int_0^{t+1} (|L\partial_t(\psi u_1^n)|^2 + |\partial_t^2(\psi u_1^n)|^2)\, ds \\ &= \int_0^{t+1} (\|p^n(s)\|_2^2 + |\partial_t p^n(s)|^2)\, ds \\ &\leq C\int_0^{t+1} (|u_1^n|^2 + |\partial_t u_1^n|^2)\, ds \leq M_4(t+1). \end{aligned} \tag{2.8}$$

Combining (2.8) and (2.7), we obtain

$$\int_0^{t+1} \psi^2(s)(\|\partial_t u_1^n(s)\|_2^2 + |\partial_t^2 u_1^n(s)|^2)\, ds \leq M_5(t+1). \tag{2.9}$$

Now we apply the operator L to both sides of equation (2.3) and get

$$L^2 u_1^n(t) = -\partial_t L u_1^n(t), \quad Lu_1^n|_{\partial\Omega} = -\partial_t u_1^n|_{\partial\Omega} = 0.$$

Therefore

$$(2.10)\qquad \int_0^{t+1} \psi^2(s)|L^2u_1^n(s)|^2\,ds = \int_0^{t+1} \psi^2(s)|\partial_t(Lu_1^n(s))|^2\,ds \le M_6(t+1).$$

Finally, by virtue of (2.7), (2.9), and (2.10), we conclude that

$$\int_1^{t+1} (\|\partial_t u_1^n(s)\|_2^2 + |\partial_t^2 u_1^n(s)|^2 + \|u_1^n(s)\|_2^2 + \|u_1^n(s)\|_4^2)\,ds \le M_7(t+1).$$

In particular, the sequence $\{u_1^n\}$ is compact in $H^{\mathbf{r}}(Q_{1,\Gamma+1})$ and $\{\widetilde{u}_1^n\}$ is compact in $H^{\mathbf{r}}(Q_{0,\Gamma})$.

Now we shall prove that the sequence $\{u_2^n\}$ is compact in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$ as well. According to (2.5) it is sufficient to prove that the sequence $B(u^n) = B(u^n,u^n)$ is precompact in $L_2(0,\Gamma+1;H)$. (From (2.1) it follows that the sequence $\{g_n\}$ is precompact in $L_2(0,\Gamma+1;H)$.) The sequence $\{u^n\}$ is bounded in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$, hence, by refining, we may assume that $u_n \rightharpoonup u$ $(n\to\infty)$ weakly in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$. Thus, $\partial_t u_n \rightharpoonup \partial_t u$ $(n\to\infty)$ weakly in $L_2(0,\Gamma+1;H)$ and $\partial^{\boldsymbol{\alpha}} u_n \rightharpoonup \partial^{\boldsymbol{\alpha}} u$ $(n\to\infty)$ weakly in $L_2(0,\Gamma+1;H)$ for each $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ with $|\boldsymbol{\alpha}|\le 2$. Let us prove that

$$(2.11)\qquad B(u^n)\to B(u)\quad (n\to\infty)\quad \text{strongly in } L_2(0,\Gamma+1;H).$$

By the Nikol'skiĭ theorem (see [3]),

$$(2.12)\qquad H^{\mathbf{r}}(Q_{0,\Gamma+1}) \subset H_q^{\boldsymbol{\varrho}}(Q_{0,\Gamma+1}),\quad \mathbf{r}=(r_1,r_2,r_3),\quad \boldsymbol{\varrho}=(\varrho_1,\varrho_2,\varrho_3),\quad q\ge 2,$$

whenever

$$(2.13)\qquad \varrho_j/r_j \le 1-(1/2-1/q)(1/r_1+1/r_2+1/r_3),\quad j=1,2,3.$$

Moreover, the embedding (2.12) is compact if the inequalities in (2.13) are strict.

The values $\mathbf{r}=(r_1,r_2,r_3)=(2,2,1)$, $\boldsymbol{\varrho}=(\varrho_1,\varrho_2,\varrho_3)=(1,1,0)$, $q\le 4$ meet the conditions (2.13), since $\varrho_j/r_j\le 1/2\le 1-(1/2-1/q)2$. So we conclude that

$$(2.14)\qquad \left\|\frac{\partial v}{\partial x_i}\right\|_{L_q(Q_{0,\Gamma+1})} \le C_q\|v\|_{H^{\mathbf{r}}(Q_{0,\Gamma+1})},\quad i=1,2,\ 2\le q\le 4.$$

For $q<4$ the embedding $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \Subset H_q^{(1,1,0)}(Q_{0,\Gamma+1})$ is compact. Similarly, taking $\boldsymbol{\varrho}=(\varrho_1,\varrho_2,\varrho_3)=(0,0,0)$, $\varrho_j/r_j=0<1-(1/2-1/q_1)2$ for all $q_1\ge 2$, we obtain

$$\|v\|_{L_{q_1}(Q_{0,\Gamma+1})} \le C'_{q_1}\|v\|_{H^{\mathbf{r}}(Q_{0,\Gamma+1})}$$

and the embedding $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \Subset L_{q_1}(Q_{0,\Gamma+1})$ is compact.

Finally, we get

$$
\begin{aligned}
(2.15)\qquad & \|B(u^n) - B(u)\|_{L_2(0,\Gamma+1;H)} \\
&\equiv \|B(u^n) - B(u)\| \le \|B(u^n - u, u^n)\| + \|B(u, u^n - u)\| \\
&\le C\Big(\int_{Q_{0,\Gamma+1}} |u^n - u|^2 |\nabla u^n|^2 \,dx\,ds\Big)^{1/2} \\
&\quad + C\Big(\int_{Q_{0,\Gamma+1}} |u|^2 |\nabla (u^n - u)|^2 \,dx\,ds\Big)^{1/2} \\
&\le C_1\Big(\int_{Q_{0,\Gamma+1}} |\nabla u^n|^3 \,dx\,ds\Big)^{1/3}\Big(\int_{Q_{0,\Gamma+1}} |u^n - u|^6 \,dx\,ds\Big)^{1/6} \\
&\quad + C_1\Big(\int_{Q_{0,\Gamma+1}} |u|^6 \,dx\,ds\Big)^{1/6}\Big(\int_{Q_{0,\Gamma+1}} |\nabla (u^n - u)|^3 \,dx\,ds\Big)^{1/6}.
\end{aligned}
$$

Since $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \Subset H_3^{(1,1,0)}(Q_{0,\Gamma+1})$ and $H^{\mathbf{r}}(Q_{0,\Gamma+1}) \Subset L_6(Q_{0,\Gamma+1})$, we get

$$
\int_{Q_{0,\Gamma+1}} |\nabla (u^n - u)|^3 \,dx\,ds \to 0, \quad \int_{Q_{0,\Gamma+1}} |u^n - u|^6 \,dx\,ds \to 0 \quad (n \to \infty)
$$

and, by (2.14) and (2.2),

$$
\int_{Q_{0,\Gamma+1}} |\nabla u^n|^3 \,dx\,ds \le M'.
$$

Therefore, the right-hand side of (2.15) tends to zero as $n \to \infty$ and (2.11) is proved.

Thus, the right-hand sides of (2.5) form a precompact set in $L_2(0, \Gamma + 1; H)$ and, hence, the set $\{u_2^n\}$ of solutions is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma+1})$. Consequently, $\{\widetilde{u}_2^n\}$ is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma})$. The sum $\{\widetilde{u}^n\}$ of two precompact sequences $\{\widetilde{u}_1^n\}$ and $\{\widetilde{u}_2^n\}$ is precompact in $H^{\mathbf{r}}(Q_{0,\Gamma})$. Lemma 2.1 is proved. □

3. On the structure of trajectory attractors

In this section we shall describe the structure of the trajectory attractors from Theorems 1.2 and 2.1 in terms of complete trajectories of equation (1.1), i.e. when solutions $u(s), s \in \mathbb{R}$, are determined on the whole time axis $\mathbb{R}$.

Let the function $g_0(x, s)$ satisfy (1.2) and let $\mathcal{H}_+(g_0)$ be the hull of g_0 in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H)$. As usual, $\mathcal{H}_+(g_0)$ is a complete metric space and the translation semigroup $\{T(t)\}$ acts on $\mathcal{H}_+(g_0)$, $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$, $T(t)$ is continuous for all $t \ge 0$. Consider the attractor $\omega(\mathcal{H}_+(g_0))$ of the semigroup $\{T(t)\}$ on $\mathcal{H}_+(g_0)$,

$$
(3.1)\qquad T(t)\omega(\mathcal{H}_+(g_0)) = \omega(\mathcal{H}_+(g_0)) \quad \forall t \ge 0
$$

(see Proposition 1.2).

Similarly to $L_2^{\rm loc}(\mathbb{R}_+;H)$ and $L_2^{\rm a}(\mathbb{R}_+;H)$ we consider the spaces $L_2^{\rm loc}(\mathbb{R};H)$ and $L_2^{\rm a}(\mathbb{R};H)$ of functions on the whole axis. The space $L_2^{\rm a}(\mathbb{R};H)$ has the norm

$$\|\zeta\|^2_{L_2^{\rm a}(\mathbb{R};H)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} |\zeta(s)|^2\,ds < \infty.$$

Consider an external force $g \in \omega(\mathcal{H}_+(g_0))$. The invariance property (3.1) implies that there is a function $g_1 \in \omega(\mathcal{H}_+(g_0))$ such that $T(1)g_1 = g$. Consider the function $\zeta(s), s \geq -1$, $\zeta(s) = g_1(s+1)$. Obviously, $\zeta(s) \equiv g(s)$ for $s \geq 0$, hence, ζ is a prolongation of g on the semiaxis $[-1,\infty[$. Next, there is $g_2 \in \omega(\mathcal{H}_+(g_0))$ such that $T(1)g_2 = g_1, T(2)g_2 = g$. Put $\zeta(s) = g_2(s+2)$ for $s \geq -2$. Evidently, the function ζ is well defined, since $g_2(s+2) = g_1(s+1)$ for $s \geq -1$. Continuing this process, we define $\zeta(s) = g_n(s+n)$ for $s \in [-n,\infty[$, where $g_n \in \omega(\mathcal{H}_+(g_0))$ and $n \in \mathbb{N}$. We have defined a function $\zeta(s), s \in \mathbb{R}$, which is a prolongation of the initial external force $g(s), s \in \mathbb{R}_+$. Moreover, ζ has the following property: $\Pi_+\zeta_t \in \omega(\mathcal{H}_+(g_0))$ for all $t \in \mathbb{R}$, where $\zeta_t(s) = \zeta(t+s)$. Here $\Pi_+ = \Pi_{0,\infty}$ is the restriction operator to the semiaxis $\mathbb{R}_+$. Evidently, $\zeta \in L_2^{\rm a}(\mathbb{R};H)$ and $\|\zeta\|^2_{L_2^{\rm a}(\mathbb{R};H)} \leq \|g_0\|^2_{L_2^{\rm a}(\mathbb{R}_+;H)}$.

Definition 3.1. (i) A function $\zeta \in L_2^{\rm a}(\mathbb{R};H)$ is said to be a *complete external force* in $\omega(\mathcal{H}_+(g_0))$ if $\Pi_+\zeta_t(\cdot) = \Pi_+\zeta(t+\cdot) \in \omega(\mathcal{H}_+(g_0))$, for all $t \in \mathbb{R}$. Let $Z(g_0)$ be the set of all complete external forces in $\mathcal{H}_+(g_0)$.

As shown above, for every symbol $g \in \omega(\mathcal{H}_+(g_0))$ there exists at least one complete external force ζ which is the prolongation of g for negative s. Notice at once that, in general, this prolongation need not be unique.

By analogy to Section 1, for the cylinder $Q = \Omega \times \mathbb{R}$ we introduce the space $H^{\mathbf{r},{\rm loc}}(Q) = L_2^{\rm loc}(\mathbb{R};H_2) \cap \{v \mid \partial_t v \in L_2^{\rm loc}(\mathbb{R};H)\}$, i.e. $v \in H^{\mathbf{r},{\rm loc}}(Q)$ if

$$\|\Pi_{t_1,t_2} v\|^2_{H^{\mathbf{r}}(Q_{t_1,t_2})} < \infty \quad \forall [t_1,t_2] \subseteq \mathbb{R}.$$

We shall use the topological spaces $H_s^{\mathbf{r},{\rm loc}}(Q)$, $H_w^{\mathbf{r},{\rm loc}}(Q)$, and the Banach space $H^{\mathbf{r},{\rm a}}(Q)$ with the norm

$$\|v\|^2_{H^{\mathbf{r},{\rm a}}(Q)} = \|v\|^2_{\mathbf{r},{\rm a}} = \sup_{t\in\mathbb{R}} \|\Pi_{t,t+1} v\|^2_{H^{\mathbf{r}}(Q_{t,t+1})}.$$

Suppose we are given some complete external force $\zeta(s)$, $s \in \mathbb{R}$, in $\omega(\mathcal{H}_+(g_0))$. Consider the equation

$$\partial_t u + \nu L u + B(u) = \zeta(x,t), \quad (\nabla, u) = 0, \quad u|_{\partial\Omega} = 0, \quad x \in \Omega,\ t \in \mathbb{R}. \tag{3.2}$$

Definition 3.2. The *kernel* $\mathcal{K}_\zeta$ of equation (3.2) with the complete external force $\zeta \in Z(g_0)$ is the set of all solutions $u(s), s \in \mathbb{R}$, of equation (3.2) that are in the space $H^{\mathbf{r},{\rm a}}(Q)$.

The following theorem specifies the structure of the trajectory attractor from Theorems 1.2 and 2.1.

THEOREM 3.1. (i) *Let* g_0 *be tr.-c. in* $L^{\mathrm{loc}}_{2,w}(\mathbb{R}_+;H)$. *Then the trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(g_0)}$ *in* $H^{\mathbf{r},\mathrm{loc}}_w(Q_+)$ *of the translation semigroup* $\{T(t)\}$ *acting on* $\mathcal{K}^+ = \mathcal{K}^+_{\mathcal{H}_+(g_0)}$ *can be represented in the form*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))} = \Pi_+\Big(\bigcup_{\zeta\in Z(g_0)} \mathcal{K}_\zeta\Big) = \Pi_+\mathcal{K}_{Z(g_0)}. \tag{3.3}$$

The set $\mathcal{K}_{Z(g_0)}$ *is compact in* $H^{\mathbf{r},\mathrm{loc}}_w(Q)$ *and bounded in* $H^{\mathbf{r},\mathrm{a}}(Q)$. *For all* $\zeta \in Z(g_0)$ *the kernel* $\mathcal{K}_\zeta$ *is non-empty and every function* $u \in \mathcal{K}_\zeta$ *is tr.-c. in* $H^{\mathbf{r},\mathrm{loc}}_w(Q)$.

(ii) *Let* g_0 *be tr.-c. in* $L^{\mathrm{loc}}_2(\mathbb{R}_+;H)$. *Then the set* $\mathcal{K}_{Z(g_0)}$ *is compact in* $H^{\mathbf{r},\mathrm{loc}}_s(Q)$ *and every function* $u \in \mathcal{K}_\zeta$ *for* $\zeta \in Z(g_0)$ *is tr.-c. in* $H^{\mathbf{r},\mathrm{loc}}_s(Q)$.

The proof of Theorem 3.1 is given in [5] and it uses the invariance property (1.6) of the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(g_0)}$: $T(t)\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\mathcal{H}_+(g_0)}$ for $t \geq 0$.

REMARK 3.1. It was mentioned above that, in general, the prolongation ζ of an external force $g \in \omega(\mathcal{H}_+(g_0))$ for $s < 0$ need not be unique. Let us describe an important case when it is unique. Let g_0 be a tr.-c. function in $L^{\mathrm{a}}_2(\mathbb{R}_+;H)$, i.e. the set $\{g_0(\cdot + h) \mid h \in \mathbb{R}_+\}$ is precompact in the Banach space $L^{\mathrm{a}}_2(\mathbb{R}_+;H)$ with the uniform norm (1.2) and, hence, the hull $\mathcal{H}_+(g_0)$ is compact in $L^{\mathrm{a}}_2(\mathbb{R}_+;H)$. It can be proved that there exists a unique function $\widetilde{g}_0(s)$, $s \in \mathbb{R}$, such that $\widetilde{g}_0$ is tr.-c. in $L^{\mathrm{a}}_2(\mathbb{R};H)$ and

$$\int_t^{t+1} |g_0(s) - \widetilde{g}_0(s)|^2\,ds \to 0 \quad (t \to \infty).$$

Therefore, $\omega(\mathcal{H}_+(g_0)) = \mathcal{H}_+(\widetilde{g}_0)$. Tr.-c. functions in $L^{\mathrm{a}}_2(\mathbb{R};H)$ are also called *almost periodic functions in the Stepanov sense.* These functions have all the main properties of usual almost periodic functions (in the Bohr or Bochner–Amerio sense, see [1]). In particular, the translation semigroup $\{T(t)\}$ is invertible on $\mathcal{H}_+(\widetilde{g}_0)$ and $\mathcal{H}_+(\widetilde{g}_0) = \Pi_+\mathcal{H}(\widetilde{g}_0)$, where $\mathcal{H}(\widetilde{g}_0) = [\{\widetilde{g}_0(\cdot + h) \mid h \in \mathbb{R}\}]_{L^{\mathrm{a}}_2(\mathbb{R};H)}$ is the hull of the almost periodic function $\widetilde{g}_0$. Finally, in (3.3), $Z(g_0) = \mathcal{H}(\widetilde{g}_0)$ and every external force $g \in \omega(\mathcal{H}_+(g_0))$ has a unique prolongation for $s < 0$ as an almost periodic function.

To conclude the section we describe the uniform (with respect to $g \in \mathcal{H}_+(g_0)$) attractor $\mathfrak{A}_{\mathcal{H}_+(g_0)}$ for the family $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$, $g \in \mathcal{H}_+(g_0)$, of processes corresponding to equation (1.1). By Theorem 1.1, for every $g \in \mathcal{H}_+(g_0)$, one defines a process $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$ acting on V: $U_g(t,\tau)u_\tau = u_g(t)$, where u_g is a solution of (1.1) with the initial condition $u|_{t=\tau} = u_\tau$, $\tau \geq 0$. Now consider the set $Z(g_0)$. In a similar way, to each $\zeta \in Z(g_0)$ there corresponds a complete process $\{U_\zeta(t,\tau) \mid t \geq \tau,\ \tau \in \mathbb{R}\}$, $U_\zeta(t,\tau)u_\tau(t) = u_\zeta(t)$, where $u_\zeta(t)$ is a solution of (3.2) with the initial condition $u|_{t=\tau} = u_\tau$, $\tau \in \mathbb{R}$. Consider the kernel $\mathcal{K}_\zeta$ corresponding to ζ.

We denote by $\mathcal{K}_\zeta(t)$ the *kernel section at time* $t \in \mathbb{R}$: $\mathcal{K}_\zeta(t) = \{u(t) \mid u(\cdot) \in \mathcal{K}_\zeta\} \subset V$. It is clear that

$$U_\zeta(t,\tau)\mathcal{K}_\zeta(\tau) = \mathcal{K}_\zeta(t) \quad \forall t \geq \tau,\ \tau \in \mathbb{R}.$$

Using Theorem 3.1, Corollary 2.1, and Corollary 1.2 we get

Corollary 3.1. (i) *If* g_0 *is tr.-c. in* $L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ *then the set*

$$\mathfrak{A}_{\mathcal{H}_+(g_0)} = \bigcup_{\zeta \in Z(g_0)} \mathcal{K}_\zeta(0) \tag{3.4}$$

is the uniform (*with respect to* $g \in \mathcal{H}_+(g_0)$) *attractor* $\mathfrak{A}_{\mathcal{H}_+(g_0)}$ *in* V *of the family of processes* $\{U_g(t,\tau) \mid t \geq \tau \geq 0\}$, $g \in \mathcal{H}_+(g_0)$, *and the set* $\mathfrak{A}_{\mathcal{H}_+(g_0)}$ *is compact in* V.

(ii) *If* g_0 *is tr.-c. in* $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H)$ *then the set* $\mathfrak{A}_{\mathcal{H}_+(g_0)}$ *defined in* (3.4) *serves as the uniform* (*with respect to* $g \in \mathcal{H}_+(g_0)$) *attractor in* V_w (*with the weak topology of* V) *and it is bounded in* V. *In particular,* $\mathfrak{A}_{\mathcal{H}_+(g_0)}$ *is the uniform attractor in* $H_{1-\delta}$, $\mathfrak{A}_{\mathcal{H}_+(g_0)} \Subset H_{1-\delta}$, $0 < \delta \leq 1$.

4. Trajectory attractors for the 3D N–S system

In this section we shall construct a trajectory attractor for the non-autonomous Navier–Stokes system in a 3D domain $\Omega \Subset \mathbb{R}^3$. The structure of the trajectory attractor will be described and some properties of the attractor will be given. Only a brief general scheme will be sketched, without proofs and detailed explanations. This part will be expounded in more detail in another publication (see also [7], [10], [18]).

Consider the 3D Navier–Stokes system in the semicylinder $Q_+ = \Omega \times \mathbb{R}_+$:

$$\begin{gathered} \partial_t u + \nu L u + B(u) = g(x,t), \quad (\nabla, u) = 0, \\ u|_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbb{R}^3, \quad t \geq 0, \end{gathered} \tag{4.1}$$

where $x = (x_1, x_2, x_3)$, $u = u(x,t) = (u^1, u^2, u^3)$, $g = g(x,t) = (g^1, g^2, g^3)$. L is the 3D Stokes operator: $Lu = -P\Delta u$; $B(u) = B(u,u)$, $B(u,v) = P(u,\nabla)v = P\sum_{i=1}^3 u_i \partial_{x_i} v$. The spaces H and V are determined similar to the 2D case. Suppose $g \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$.

Let there be given an initial external force $g_0 \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ in (4.1). Assume that g_0 is tr.-c. in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H) \equiv L_{2,w}^{\mathrm{loc}}$, i.e.

$$\|g_0\|^2_{L_2^{\mathrm{a}}(\mathbb{R}_+;H)} = \|g_0\|^2_{\mathrm{a}} = \sup_{t \in \mathbb{R}_+} \int_t^{t+1} |g_0(s)|^2\, ds < \infty. \tag{4.2}$$

Let $\Sigma = \mathcal{H}_+(g_0) \equiv [\{g_0(\cdot + t) \mid t \geq 0\}]_{L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;H)}$ be the hull of the function g_0 in the space $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H)$. It can be proved that $\mathcal{H}_+(g_0)$ is a complete metric space. The translation semigroup $\{T(t)\}$ is continuous on $\mathcal{H}_+(g_0)$ and $T(t)\mathcal{H}_+(g_0) \subseteq \mathcal{H}_+(g_0)$ for all $t \geq 0$; moreover, $\|g\|^2_{\mathrm{a}} \leq \|g_0\|^2_{\mathrm{a}}$ for every $g \in \mathcal{H}_+(g_0)$.

To study the trajectory attractor of equation (4.1) we consider the family of those equations with various external forces $g \in \mathcal{H}_+(g_0)$.

To describe a trajectory space $\mathcal{K}_g^+$ of equation (4.1) with the external force g we shall consider weak solutions of equation (4.1) in the space $L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$. If $u \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$ then equation (4.1) makes sense in the distribution space $D'(\mathbb{R}_+; V')$, where V' is the dual space of V. This is the usual way to define weak solutions of equation (4.1) (see [16]).

Definition 4.1. The trajectory space $\mathcal{K}_g^+$ is the union of all weak solutions $u \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$ of equation (4.1) with the external force g that satisfy the inequality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le (g(t), u(t)), \quad t \in \mathbb{R}_+. \tag{4.3}$$

This inequality should be read as follows: for each $\psi \in C_0^\infty(]0,\infty[)$, $\psi \ge 0$,

$$-\frac{1}{2}\int_0^\infty |u(s)|^2\psi'(s)\,ds + \nu\int_0^\infty \|u(s)\|^2\psi(s)\,ds \le \int_0^\infty (g(s), u(s))\psi(s)\,ds. \tag{4.4}$$

Let us formulate the existence theorem:

Theorem 4.1. *Let $g \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ and $u_0 \in H$. Then there exists a weak solution u of equation* (4.1) *belonging to the space $L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$ such that $u(0) = u_0$ and u satisfies inequality* (4.4).

The existence theorem is a classical result (see [14]–[16], [19]). The proof uses the Faedo–Galerkin approximation method. To get (4.4) one has to pass to the limit in the corresponding a priori equality involving the sequence $\{u_m\}$ of Faedo–Galerkin approximations.

Remark 4.1. For the 3D case, the uniqueness problem is still open. Also, it is not known whether every weak solution of (4.1) satisfies inequality (4.3).

It can be shown that every weak solution $u \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$ of equation (4.1) satisfies

$$\partial_t^{1/4-\varepsilon} u \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H) \quad \forall\varepsilon, 0 < \varepsilon < 1/4,$$

(see [16]), and $\partial_t u \in L_{4/3}^{\mathrm{loc}}(\mathbb{R}_+; V')$ (see [20]). Consider the following space:

$$\begin{aligned}\mathcal{F}_+^{\mathrm{loc}} &= L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)\\ &\qquad \cap \{v \mid \partial_t^{1/4-\varepsilon} v \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V')\} \cap \{v \mid \partial_t v \in L_{4/3}^{\mathrm{loc}}(\mathbb{R}_+; V')\},\end{aligned}$$

where ε is fixed, $0 < \varepsilon < 1/4$. The space $\mathcal{F}_+^{\mathrm{loc}}$ is endowed with the following "weak" convergence topology.

Definition 4.2. A sequence $\{v_n\} \subset \mathcal{F}_+^{\mathrm{loc}}$ converges (in a weak sense) to $v \in \mathcal{F}_+^{\mathrm{loc}}$ as $n \to \infty$ if $v_n \to v$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; V)$, $*$-weakly in $L_\infty(t_1, t_2; H)$, $\partial_t^{1/4-\varepsilon} v_n \to \partial_t^{1/4-\varepsilon} v$ $(n \to \infty)$ weakly in $L_2(t_1, t_2; H)$, and $\partial_t v_n \to \partial_t v$ $(n \to \infty)$ weakly in $L_{4/3}(t_1, t_2; V')$ for all $[t_1, t_2] \subset \mathbb{R}_+$.

The space $\mathcal{F}_+^{\mathrm{loc}}$ with the above weak topology is denoted by Θ_+^{loc}. We shall also use the space

$$\begin{aligned}\mathcal{F}_+^{\mathrm{a}} = {} & L_2^{\mathrm{a}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{a}}(\mathbb{R}_+; H) \\ & \cap \{v \mid \partial_t^{1/4-\varepsilon} v \in L_2^{\mathrm{a}}(\mathbb{R}_+; V')\} \cap \{v \mid \partial_t v \in L_{4/3}^{\mathrm{a}}(\mathbb{R}_+; V')\},\end{aligned}$$

which is a subspace of $\mathcal{F}_+^{\mathrm{loc}}$. If X is a Banach space then $L_p^{\mathrm{a}}(\mathbb{R}_+; X)$ means the subspace of $L_p^{\mathrm{loc}}(\mathbb{R}_+; X)$ having the finite norm

$$\|v\|^p_{L_p^{\mathrm{a}}(\mathbb{R}_+; X)} = \sup_{t \geq 0} \int_t^{t+1} \|v(s)\|_X^p \, ds.$$

Similarly, the space $L_p^{\mathrm{a}}(\mathbb{R}; X)$ has the norm

$$\|v\|^p_{L_p^{\mathrm{a}}(\mathbb{R}; X)} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|v(s)\|_X^p \, ds.$$

Lemma 4.1. (i) $\mathcal{K}_g^+ \subset \mathcal{F}_+^{\mathrm{a}}$ *for all* $g \in \mathcal{H}_+(g_0)$.

(ii) *For every* $u \in \mathcal{K}_g^+$,

$$\|T(t)u(\cdot)\|_{\mathcal{F}_+^{\mathrm{a}}} \leq C\|u(\cdot)\|^2_{L_\infty(0,1;H)} \exp(-\lambda t) + R_0 \quad \forall t \geq 0, \tag{4.5}$$

where λ *is the first eigenvalue of the operator* νL; C *depends on* λ, *and* R_0 *depends on* λ *and* $\|g_0\|^2_{L_2^{\mathrm{a}}(\mathbb{R}_+; H)}$.

Put

$$\mathcal{K}_\Sigma^+ = \bigcup_{g \in \mathcal{H}_+(g_0)} \mathcal{K}_g^+, \quad \Sigma = \mathcal{H}_+(g_0).$$

The translation semigroup $\{T(t) \mid t \geq 0\}$ acts on $\mathcal{K}_\Sigma^+$:

$$T(t)u(s) = u(t+s), \quad s \geq 0.$$

Evidently

$$T(t)u \in \mathcal{K}_{T(t)g}^+ \quad \forall u \in \mathcal{K}_g^+, \quad t \geq 0,$$

so the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ is translation-coordinated. Therefore

$$T(t)\mathcal{K}_\Sigma^+ \subseteq \mathcal{K}_\Sigma^+ \quad \forall t \geq 0.$$

It is clear that every mapping $T(t)$ is continuous in Θ_+^{loc}.

It follows from (4.5) that the ball $B_0 = \{\|v\|_{\mathcal{F}_+^{\mathrm{a}}} \leq 2R_0\}$ serves as a uniformly absorbing set of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+$. The set B_0 is bounded in $\mathcal{F}_+^{\mathrm{a}}$ and it is compact in Θ_+^{loc}.

LEMMA 4.2. *The family* $\{\mathcal{K}_g^+ \mid g \in \Sigma\}$ *is* $(\Theta_+^{\mathrm{loc}}, \mathcal{H}_+(g_0))$*-closed and* $\mathcal{K}_\Sigma^+$ *is closed in* Θ_+^{loc}.

In this way, by Lemmas 4.1 and 4.2, Proposition 1.2 is applicable.

Let $\omega(\mathcal{H}_+(g_0))$ denote the global attractor of the semigroup $\{T(t)\}$ on $\mathcal{H}_+(g_0)$. Here

$$\omega(\mathcal{H}_+(g_0)) = \bigcap_{\tau\ge 0}\Big[\bigcup_{t\ge\tau} T(t)\mathcal{H}_+(g_0)\Big]_{L_{2,w}^{\mathrm{loc}}}$$

is the ω-limit set of $\mathcal{H}_+(g_0)$.

Let $Z(g_0)$ be the set of all complete external forces in $\mathcal{H}_+(g_0)$, i.e. the set of all functions $\zeta \in L_2^{\mathrm{loc}}(\mathbb{R}; H)$ such that $\zeta_t \in \omega(\mathcal{H}_+(g_0))$ for all $t \in \mathbb{R}$, where $\zeta_t(s) = \Pi_+\zeta(s+t), s \ge 0$. Evidently, for every $g \in \omega(\mathcal{H}_+(g_0))$ there is at least one $\zeta \in Z(g_0)$ such that $\zeta(s)$ is a prolongation of $g(s)$ for negative s. To each complete external force $\zeta \in Z(g_0)$ there corresponds the kernel $\mathcal{K}_\zeta$ of equation (4.1). The kernel $\mathcal{K}_\zeta$ consists of all weak solutions $u(s), s \in \mathbb{R}$, of the equation

$$\partial_t u + \nu L u + B(u) = \zeta(x,t), \quad t \in \mathbb{R},$$

that satisfy inequality (4.4) and that are in the space

$$\begin{aligned}\mathcal{F}^{\mathrm{a}} &= L_2^{\mathrm{a}}(\mathbb{R}; V) \cap L_\infty^{\mathrm{a}}(\mathbb{R}; H)\\ &\quad\cap \{v \mid \partial_t^{1/4-\varepsilon} v \in L_2^{\mathrm{a}}(\mathbb{R}; V')\} \cap \{v \mid \partial_t v \in L_{4/3}^{\mathrm{a}}(\mathbb{R}; V')\}.\end{aligned}$$

Let us formulate the main

THEOREM 4.2. *Let* g_0 *be tr.-c. in* $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H)$. *Then the translation semigroup* $\{T(t)\}$ *acting on* $\mathcal{K}_\Sigma^+$ $(\Sigma = \mathcal{H}_+(g_0))$ *has a trajectory attractor* $\mathcal{A}_\Sigma = \mathcal{A}_{\mathcal{H}_+(g_0)}$ *in* Θ_+^{loc}. *The set* $\mathcal{A}_{\mathcal{H}_+(g_0)}$ *is bounded in* $\mathcal{F}_+^{\mathrm{a}}$ *and compact in* Θ_+^{loc}. *Moreover,*

$$\mathcal{A}_{\mathcal{H}_+(g_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(g_0))} = \Pi_+\Big(\bigcup_{\zeta\in Z(g_0)} \mathcal{K}_\zeta\Big) = \Pi_+\mathcal{K}_{Z(g_0)}.$$

The kernel $\mathcal{K}_\zeta$ *is non-empty for all* $\zeta \in Z(g_0)$; *the set* $\mathcal{K}_{Z(g_0)}$ *is bounded in* $\mathcal{F}^{\mathrm{a}}$ *and compact in* Θ^{loc}.

The detailed proof of Lemmas 4.1, 4.2, and Theorem 4.2 is given in [5].

Notice that the following embedding is continuous: $\Theta_+^{\mathrm{loc}} \subset L_2^{\mathrm{loc}}(\mathbb{R}_+; H_{1-\delta})$, $0 < \delta \le 1$, so we get

COROLLARY 4.1. *For every set* $B \subset \mathcal{K}^+$ *bounded in* $\mathcal{F}_+^{\mathrm{a}}$,

$$\mathrm{dist}_{L_2(0,\Gamma;H_{1-\delta})}(\Pi_{0,\Gamma}T(t)B, \Pi_{0,\Gamma}\mathcal{K}_{Z(g_0)}) \to 0 \quad (t\to\infty),$$

where Γ *is fixed and arbitrary.*

In conclusion, we shall formulate some properties of trajectory attractors of the Navier–Stokes system.

(I) Let $g_0(x,s) = g_1(x,s) + a(x,s)$ in (4.1), where g_1 and a are tr.-c. functions in $L_{2,w}^{\rm loc}(\mathbb{R}_+;H)$. Assume that $T(t)a \to 0$ $(t \to \infty)$ in $L_{2,w}^{\rm loc}(\mathbb{R}_+;H)$, i.e.

$$\int_0^1 (a(s+t),\psi(s))ds \to 0 \quad (t \to \infty) \tag{4.6}$$

for all $\psi \in L_2(0,1;H)$. Then the trajectory attractors corresponding to $\Sigma = \mathcal{H}_+(g_1+a)$ and to $\Sigma_1 = \mathcal{H}_+(g_1)$ coincide:

$$\mathcal{A}_{\mathcal{H}_+(g_1+a)} = \mathcal{A}_{\mathcal{H}_+(g_1)}. \tag{4.7}$$

In particular, if $g_1 \equiv 0$ then $\mathcal{A}_{\mathcal{H}_+(a)} = \mathcal{A}_{\mathcal{H}_+(0)} = \{0\}$.

For example, the function $a(x,s) = \varphi(x)\sin(s^2)$ satisfies (4.6) for all $\varphi \in H$. Thus a more and more rapidly oscillating additional term $a(s)$ does not affect the trajectory attractor. The equality (4.7) is valid for 3D just as for 2D N–S systems.

(II) Let $g_0(x,s) = g_{0\varepsilon}(x,s) = g_1(x,s) + \varepsilon g_2(x,s)$ in (4.1), where the g_i are tr.-c. functions in $L_{2,w}^{\rm loc}(\mathbb{R}_+;H)$ and $|\varepsilon| \le 1$. Put $\mathcal{A}(\varepsilon) = \mathcal{A}_{\mathcal{H}_+(g_{0\varepsilon})}$. Then $\mathcal{A}(\varepsilon)$ is lower semicontinuous with respect to ε. More precisely, it can be proved that the ball $B_0 = \{\|v\|_{\mathcal{F}_+^{\rm a}} \le R_1\}$, which is a topological subspace of $\Theta_+^{\rm loc}$, is metrizable, and in this metric

$$\operatorname{dist}_{\Theta_+^{\rm loc}}(\mathcal{A}(\varepsilon),\mathcal{A}(0)) \to 0 \quad (t \to \infty). \tag{4.8}$$

The radius R_1 is large enough to provide the inclusion $\mathcal{A}(\varepsilon) \subseteq B_1$ for all $\varepsilon, |\varepsilon| \le 1$. For the 2D N–S system (1.1) the property (4.8) is also valid with $\operatorname{dist}_{\Theta_+^{\rm loc}}$ being replaced by $\operatorname{dist}_{H^{\mathbf{r},\rm loc}}$ or by $\operatorname{dist}_{H_w^{\mathbf{r},\rm loc}}$ depending on the tr.-c. class the external force belongs to.

(III) Let $\mathcal{A}^{(N)}_{\mathcal{H}_+(P_N g_0)} \equiv \mathcal{A}^{(N)}$ be the trajectory attractor of the Faedo–Galerkin approximation system of order N for equation (4.1), where P_N is the projection onto the finite-dimensional subspace of H spanned by the first N eigenfunctions of the Stokes operator. Then

$$\operatorname{dist}_{\Theta_+^{\rm loc}}(\mathcal{A}^{(N)},\mathcal{A}_{\mathcal{H}_+(g_0)}) \to 0 \quad (t \to \infty).$$

In other words, for each neighbourhood $\mathcal{O}(\mathcal{A}_{\mathcal{H}_+(g_0)})$ of $\mathcal{A}_{\mathcal{H}_+(g_0)}$ in $\Theta_+^{\rm loc}$ there is N_1 such that $\mathcal{A}^{(N)} \subseteq \mathcal{O}(\mathcal{A}_{\mathcal{H}_+(g_0)})$ for all $N \ge N_1$.

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