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THE COINCIDENCE REIDEMEISTER CLASSES OF MAPS ON NILMANIFOLDS

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Introduction

Given a pair of maps $f, g : N_1 \to N_2$ where N_1, N_2 are compact nilmanifolds of the same dimension, in [15], C. K. McCord has very recently shown that $N(f,g) = |L(f,g)|$ where $N(f,g), L(f,g)$ mean the coincidence Nielsen number and Lefschetz coincidence number, respectively. Furthermore, he has also shown that the essential coincidence Nielsen classes have the same coincidence index which is either $+1$ or -1. In the fixed point situation, or even more general in the coincidence case where $N_1 = N_2$, several authors have exploited the relation among $N(f,g), L(f,g)$, $\operatorname{coin}(f_\#, g_\#)$ and $R(f,g)$, where $f_\#$, $g_\#$ are the induced homomorphisms on the fundamental group by f, g, respectively, and $R(f,g)$ is the Reidemeister coincidence number. See for example [2], [7] and [8]. For the general situation $f, g : N_1 \to N_2$, the main part which is missing so far is the relation between $\operatorname{coin}(f_\#, g_\#)$ and $R(f,g)$. The purpose of this work is first to study such relation including the case where the two compact nilmanifolds N_1 and N_2 do not have the same dimension. Finally, to study $\operatorname{coin}(f,g)$ for $g = c$ the constant map, where the two compact nilmanifolds N_1 and N_2 do not have necessarily the same dimension. Then we prove:

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THEOREM 2.5. *Let $f, g : N_1 \to N_2$, where N_1, N_2 are compact nilmanifolds. Then the two conditions below are equivalent*

(a) *The Hirsch lenght of* $\operatorname{coin}(f_\#, g_\#)$ *is* $\dim N_1 - \dim N_2$,
(b) $R(f, g) < \infty$.

Based on this result we can prove:

THEOREM 2.6. *Let $f, g : N_1 \to N_2$, where N_1, N_2 are compact nilmanifolds of the same dimension. Then the three conditions below are equivalent:*

(a) $N(f, g) \neq 0$,
(b) $\operatorname{coin}(f_\#, g_\#) = 1$,
(c) $R(f, g) < \infty$.

If one of the three conditions above holds, then $N(f, g) = R(f, g) = |L(f, g)|$.

Finally, we consider the root case. Let c denote a constant map. We prove:

THEOREM 3.4. *For $f : N_1 \to N_2$ where N_1, N_2 are compact nilmanifolds, the following three conditions are equivalent:*

(a) $N(f, c) \neq 0$,
(b) *the Hirsch lenght of* $\operatorname{Ker} f_\# : \pi_1(N_1) \to \pi_1(N_2)$ *is* $\dim N_1 - \dim N_2$,
(c) *the index of $f_\#(\pi_1(N_1))$ in $\pi_1(N_2)$, i.e. $R(f, c)$ is finite.*

If one of the three conditions above holds, then

$$N(f, c) = R(f, c) = [f_\# \pi(N_1)), \pi_1(N_2)],$$

and $\check{H}^{m-n}(F_i, Z) \neq 0$ for l coincidence Nielsen classes $F_1, \dots, F_l$ and $l = R(f, c)$.

This paper is organized in three sections. In Section 1 we present some general facts about maps on Lie groups and give the background in order to relate our original questions with a question about maps on Lie groups. In Section 2, for given N_1, N_2 compact nilmanifolds, we consider the Lie Groups which are the respectively universal covers. Then we solve the related problem for these Lie Groups and prove Theorems 2.5 and 2.6. In Section 3, we consider the root case, where we show that the number of essential Nielsen classes is precisely the number of Reidemeister classes if this number is finite, and zero otherwise. This is Theorem 3.4.

Certainly the result here will have implication in the coincidence theory of solvmanifolds. This will be analized elsewhere.

Theorem 2.5, has been obtained independently by P. Wong, at the same time, using different method (see [20]).

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1. Preliminaries

Let G_1, G_2 be two simply connected Lie groups, and $\Gamma_1 \hookrightarrow G_1$, $\Gamma_2 \hookrightarrow G_2$ two uniform (discrete and co-compact) subgroups. Suppose $\varphi_1, \varphi_2 : G_1 \to G_2$ are two group homomorphisms such that $\varphi_i(\Gamma_1) \subset \Gamma_2$, $i = 1, 2$, and call $f, g : G_1/\Gamma_1 \to G_2/\Gamma_2$ the maps induced by φ_1, φ_2, respectively, on the homogeneous spaces.

The following elementary fact is true.

Proposition 1.1. *We have that* $\pi_1(G_1/\Gamma_1) = \Gamma_1$, $\pi_1(G_2/\Gamma_2) = \Gamma_2$ *and* $f_\#, g_\# : \pi_1(G_1/\Gamma_1) \to \pi_1(G_2/\Gamma_2)$ *are the homomorphism* $\varphi_1|_{\Gamma_1}$, $\varphi_2|_{\Gamma_1}$, *respectively.*

As before, let G be a simply connected Lie Group.

Definition 1.2. We say that G *has the* P_1 *property*, if the map $\psi : H_1 \times H_2 \to G$ given by $\psi(h_1, h_2) = h_2 h_1^{-1}$ is surjective for any pair of closed Lie subgroups H_1 and H_2 where the two submanifolds H_1, H_2 are in general position (see [10, p. 4] for the definition of general position).

For the next definition we will consider any two connected and simply connected closed Lie subgroups H_1, H_2 of G and uniform subgroups $\Gamma_1 \subset H_1$, $\Gamma_2 \subset H_2$, $\Gamma \subset G$ such that $\Gamma_1, \Gamma_2 \subset \Gamma$.

Definition 1.3. The *Reidemeister classes* of (Γ_1, Γ_2) on Γ, denoted by $\mathcal{R}[\Gamma_1, \Gamma_2; \Gamma]$, are the classes of elements of Γ given by the relation $\alpha \sim h_2 \alpha h_1^{-1}$, for $h_1 \in \Gamma_1$, $h_2 \in \Gamma_2$.

Remark. We can define in a similar way $\mathcal{R}[\Gamma_1, \Gamma_2; \Gamma]$ without the assumption $\Gamma_i \subset \Gamma$, $i = 1, 2$. For this we have the relation $\alpha \sim h_2 \alpha h_1^{-1}$ whenever both elements belong to Γ.

Let the cardinality of $\mathcal{R}[\Gamma_1, \Gamma_2; \Gamma]$ be denoted either by $\#\mathcal{R}[\Gamma_1, \Gamma_2; \Gamma]$ or $R(\Gamma_1, \Gamma_2; \Gamma)$. Recall that $\mathcal{R}[f_\#, g_\#]$ is the same as $\mathcal{R}[f, g]$ which is the set of Reidemeister classes defined by the pair (f, g). So $\#\mathcal{R}[f_\#, g_\#] = \#\mathcal{R}[f, g] = R(f_\#, g_\#) = R(f, g)$.

Definition 1.4. A pair (G, Γ), where G is a simply connected Lie group and $\Gamma \subset G$ is a uniform subgroup, *has the* P_2 *property*, if for any two pairs (H_1, Γ_1), (H_2, Γ_2) as above, $R(\Gamma_1, \Gamma_2; \Gamma) < \infty$ implies that $\psi : H_1 \times H_2 \to G$ is surjective. Finally, we say that G has the P_2 property if (G, Γ) has the P_2 property for all uniform subgroups Γ.

The map ψ plays an important role in our approach and it has the following nice property, which is going to be used later.

PROPOSITION 1.5. *The map ψ has constant rank.*

PROOF. Let us show that the rank of $d\psi_{(e_1,e_2)}$ is the same as the rank of $d\psi_{(h_1,h_2)}$ for any point $(h_1,h_2) \in H_1 \times H_2$. Consider the commutative diagram

$$\begin{array}{ccc} H_1 \times H_2 & \xrightarrow{\psi} & G \\ {\scriptstyle L_{(h_1,h_2)}}\Big\downarrow & & \Big\downarrow{\scriptstyle L_{h_2}.(\ ).R_{h_1^{-1}}.} \\ H_1 \times H_2 & \xrightarrow[\psi]{} & G \end{array}$$

So, at the tangent space level, we have the commutative diagram

$$\begin{array}{ccc} T_{(e_1,e_2)}(H_1 \times H_2) & \xrightarrow{d\psi} & T_eG \\ {\scriptstyle dL_{(h_1,h_2)}}\Big\downarrow & & \Big\downarrow{\scriptstyle d(L_{h_2}.(\ ).R_{h_1^{-1}}).} \\ T_{(h_1,h_2)}(H_1 \times H_2) & \xrightarrow[d\psi]{} & T_{h_2h_1^{-1}}G \end{array}$$

Since the maps $L_{(h_1,h_2)}$ and $L_{h_2}.(\ ).R_{h_1^{-1}}$ are diffeomorphisms, we have that $dL_{(h_1,h_2)}$ and $d(L_{h_2}.(\ ).R_{h_1^{-1}})$ are isomorphisms and the result follows. $\square$

REMARK. By completely analogous argument, if $\varphi_1, \varphi_2 : G_1 \to G_2$ are two group homomorphisms, then the map $\varphi = \varphi_2 \cdot \varphi_1^{-1} : G_1 \to G_2$ has constant rank.

We consider now, the necessary preliminaries on the topology of the Lie groups to show the two main results of this section. Following [12, Théorèm 5, Exposé 22] we have that any simply connected Lie group is topologicaly equivalent to the product of a compact Lie group by an Euclidean space $\mathbb{R}^n$. This also follows from [17, Chapter 1, Theorem 6, Problem 13 and Chapter 6, Theorem 2]. (For the non simply connected case see [12, Théorèm 6, Exposé 22]). We consider the Lie groups G_1 which are homeomorphic to the Euclidean space $\mathbb{R}^n$ for some n. There are many groups which satisfy this condition. By the result above they are the Lie groups which do not contain a non trivial compact subgroup. For example the universal cover of $SL(2,R)$ is homeomorphic to $\mathbb{R}^3$. The families of abelian, nilpotent and solvable Lie groups, also provide us with such examples. For the topology of the abelian, nilpotent and solvable Lie groups, we refer to [11] and [1]. For the topology of the nilmanifolds and solvmanifolds we refer to [13], [16] and [19]. Also in [14, Section 1] one finds some information in a very explicit and suitable form that we may use. A simply connected abelian group is difeomorphic to $\mathbb{R}^n$. For N a simply connected nilpotent Lie group, not only is diffeomorphic to $\mathbb{R}^n$, but the exponencial map at the identity, denoted by exp, provides one diffeomorphism. For S simply connected solvable Lie groups, the situation is more complicated. Nevertheless, we still have that S is diffeomorphic to the Euclidean space. There is a subfamily called the exponential groups. They consist of those groups where the map exp is a diffeomorphism. The compact nilmanifolds and solvmanifolds are Eilenberg–MacLane spaces $K(\pi, 1)$.

Finally, let us consider the family of the properly discontinuous groups Γ, operating in the Euclidean space $\mathbb{R}^n$ for some n, such that the quotient $\mathbb{R}^n/\Gamma$ is compact. Define $l(\Gamma)$ to be the dimension of the Euclidean space. This number $l(\Gamma)$ is well defined. To see this, we have that the orbit space $\mathbb{R}^n/\Gamma$ is a compact manifold. Since its universal covering is contractible, the cohomology of $\mathbb{R}^n/\Gamma$ is the same as the group cohomology of Γ. Hence $l(\Gamma)$ coincides with the maximal dimension where $H^l(\Gamma, Z_2) \neq 0$. So $l(\Gamma)$ is well defined and coincides with the maximal dimension where $H^l(\Gamma, Z_2) \neq 0$. The uniform subgroups of a Lie group homeomorphic to $\mathbb{R}^n$, are examples of groups where our definition of length applies. In particular, if Γ is a finitely generated torsion free nilpotent group, by [13] we have that $l(\Gamma)$ coincides with the Hirsh lenght of Γ. See [18] for the definition of the Hirsh length. Let $\dim(G_1) = m$ and $\dim(G_2) = n$.

Now we are ready to show the two main results of this section. For the purpose of Proposition 1.6, we will assume that G_1 is diffeomorfic to the Euclidean space $\mathbb{R}^n$.

Proposition 1.6. *If $G_1 \times G_2$ satisfies the property* P_1*, then* $l(\mathrm{coin}(f_\#, g_\#)) = m - n$ *implies* $\varphi : G_1 \to G_2$ *is surjective.*

Proof. Let $G = G_1 \times G_2$ and H_1, H_2 the graphs of φ_1, φ_2, respectively. Certainly H_1, H_2 are closed subgroups. Since $\dim H_1 + \dim H_2 = 2\dim G_1 = 2m$, in order to show that H_1, H_2 are transverse at the identity, it suffices to show that $T_e(H_1) \cap T_e(H_2) = T_e(H_1 \cap H_2)$ is a subspace of dimension $m - n$.

Certainly $H_1 \cap H_2 = \{(x, \varphi_1(x)) | x \in \mathrm{coin}(\varphi_1, \varphi_2)\}$.

But $\mathrm{coin}(\varphi_1, \varphi_2)$ is certainly a closed subgroup which has $\mathrm{coin}(f_\#, g_\#)$ as a uniform subgroup (see the proof of Lemma 2.2 in [15]). Therefore $\mathrm{coin}(\varphi_1, \varphi_2)$ has dimension $m - n$, since G_1 has no nontrivial compact subgroup. So H_1, H_2 are in general position and ψ is surjective. So φ is also surjective. □

Proposition 1.7. *If $G_1 \times G_2$ satisfies the property* P_2*, then* $R(f_\#, g_\#) < \infty$ *implies that* $l(\mathrm{coin}(f_\#, g_\#)) = m - n$.

Proof. As before, let H_1, H_2 be the closed Lie subgroups of G which are the graphs of φ_1, φ_2, respectively. It is straightforward to see that the inclusion $G_2 \hookrightarrow G_1 \times G_2$, $i(g) = (e_1, g)$, induces a map $\bar{i} : \mathcal{R}[f_\#, g_\#] \to \mathcal{R}[\Gamma_1', \Gamma_2'; \Gamma_1 \times \Gamma_2]$ which is a bijection where $\Gamma_1' = (\Gamma_1, f_\#(\Gamma_1))$ and $\Gamma_2' = (\Gamma_1, g_\#(\Gamma_1))$. So $R(\Gamma_1', \Gamma_2'; \Gamma_1 \times \Gamma_2) < \infty$. Since $G_1 \times G_2$ satisfies the P_2 condition, it follows by definition that $\psi : H_1 \times H_2 \to G$ is surjective. By Proposition 1.5, ψ has constant rank. Since it is surjective, Sard's Theorem implies that ψ is a submersion. Once we know that $\psi : H_1 \times H_2 \to G = G_1 \times G_2$ is a submersion it remains to notice that $\psi^{-1}(1,1)$ is a closed Lie subgroup,

$$\dim \psi^{-1}(1,1) = \dim (H_1 \times H_2) - \dim (G_1 \times G_2) = 2m - (m+n) = m - n,$$

and

$$\psi^{-1}(1,1) = H_1 \cap H_2 = \{(x, \varphi_1(x)); x \in \mathrm{coin}(\varphi_1, \varphi_2)\}$$

which is homeomorphic to $\mathrm{coin}(\varphi_1, \varphi_2)$. So the result follows. $\square$

2. The nilmanifold case

In this section we specialize for the case where G_1, G_2 are simply connected Nilpotent Lie groups, and prove our main result for compact nilmanifolds.

It is not difficult to see that the next two propositions hold for all commutative groups G. Using induction, in these two propositions we will show the results for the noncommutative Nilpotent Lie groups.

PROPOSITION 2.1. *If G is a simply connected Nilpotent Lie group, then G satisfies the property* P_1.

PROOF. The proof is by induction on the dimension of G. If $\dim G = 2$ then $G = R^2$ with the standart Lie Group structure and the result is clear. So, suppose that the result is true for simply connected Nilpotent Lie Group of dimension less than or equal to n. Let $\dim G = n+1$ and $H_1, H_2 \subset G$ be two closed subgroups which are in general position. Consider the sequence.

$$1 \longrightarrow C(G) \longrightarrow G \xrightarrow{p} G/C(G) \longrightarrow 1,$$

where $C(G)$ is the center of G. It is known that $\dim (C(G)) > 0$ therefore $\dim G/C(G) \leq n$. Since the two closed Lie subgroups $H_1, H_2 \subset G$ are in general position, then the subgroups $H_1' = p(H_1)$, $H_2' = p(H_2) \subset G/C(G)$ are also closed subgroups which are transversal. This follows from the diagram

$$\begin{array}{ccc} T(H_1 \times H_2) & \xrightarrow{d\psi} & TG \\ {\scriptstyle dp}\big\downarrow & & \big\downarrow \\ T(H_1' \times H_2') & \xrightarrow[d\psi']{} & TG/C(G) \end{array}$$

since the right vertical map and the top horizontal map are surjective, where ψ' is the induced map from ψ.

By induction hypothesis we have that $im(\psi') = G/C(G)$. So, it suffices to show that $C(G) \subset \mathrm{Im}\,(\psi)$. Since $d\psi_e : T(H_1 \times H_2) \to T_eG$ is surjective, by the Local Submersion Theorem (see [9, Section 4]) it follows that $\mathrm{Im}\,\psi$ contains an open neighbourhood U of the identity. So $U_C = U \cap C(G)$

is an open neighbourhood of e in $C(G)$. If we show that $\operatorname{Im}\psi \cap C(G)$ is closed under the group operation, it follows by [3, Chapter II, §IV, Theorem 1], that $\operatorname{Im}\psi \cap C(G) = C(G)$. So let us show that $\operatorname{Im}\psi \cap C(G)$ is closed under the group operation. Let $c_1, c_2 \in \operatorname{Im}\psi \cap C(G)$. We have $c_1 = g_2 g_1^{-1}$, $c_2 = h_2 h_1^{-1}$ and $c_1.c_2 = (g_2 g_1^{-1})(h_2 h_1^{-1}) = g_2(g_1^{-1}(h_2 h_1^{-1})) = g_2(h_2 h_1^{-1}) g_1^{-1} = (g_2 h_2).(h_1^{-1} g_1^{-1}) = (g_2 h_2)(g_1 h_1)^{-1}$, where the third equality follows from the fact that $(h_2 h_1^{-1})$ belongs to the center. So the result follows. □

Proposition 2.2. *If G is a simply connected nilpotent Lie group, then G satisfies the property* P_2.

Proof. Let $\Gamma \subset G$ be a uniform subgroup. We will show that (G, Γ) satifies the property P_2. The proof is by induction on the dimension of G. If $\dim G$ is 2 the result is easy. Let us assume that the result is true if $\dim G \leq n$. Let $\dim G = n+1$. In order to apply the induction hypothesis, we will define a Lie subgroup H of $C(G)$ of dimension one.

In order to define H, let Γ_0 be the center of Γ. Take $H_1, H_2 \subset G$ and $\Gamma_1, \Gamma_2 \subset \Gamma \subset G$ where Γ_1, Γ_2, Γ are uniform subgroups of H_1, H_2, G, respectively, and $R(\Gamma_1, \Gamma_2; \Gamma) < \infty$. Since Γ_0 is abelian, denoting $[e] \in \mathcal{R}(\Gamma_1, \Gamma_2; \Gamma)$ the class which contains the identity, we have that $\Gamma_0 \cap [e]$, is a subgroup and the set of classes on Γ_0 given by $\Gamma_0 \cap [g]$ for $[g] \in \mathcal{R}(\Gamma_1, \Gamma_2; \Gamma)$ is finite because $\mathcal{R}(\Gamma_1, \Gamma_2; \Gamma)$ is finite. The fact that $\Gamma_0 \cap [e]$ is a subgroup can be proved as follows: given $c = h_2 h_1^{-1}$ then $c^{-1} = h_1 h_2^{-1} = h_1 h_2^{-1} h_1^{-1} h_2 h_2^{-1} h_1 = h_1^{-1}.h_1 h_2^{-1}.h_2 h_2^{-1} h_1 = h_2^{-1} h_1$. If $c_1 = h_2 h_1^{-1}$, $c_2 = g_2 g_1^{-1}$ then $c_1 c_2 = h_2 h_1^{-1} g_2 g_1^{-1} = h_2 g_2 g_1^{-1} h_1^{-1} = h_2 g_2 (h_1 g_1)^{-1}$. So this is a subgroup. To prove that $c_0(\Gamma_0 \cap [e]) = \Gamma_0 \cap [c_0]$ it remains to notice that $c_0[e] = [c_0]$ for $c_0 \in \Gamma_0 =$ the center of Γ.

So we conclude that $\Gamma_0/{\sim}$ is the coset classes of $\Gamma_0/\Gamma_0 \cap [e]$ where $\sim$ is the relation induced by the one which gives $\mathcal{R}(\Gamma_1, \Gamma_2; \Gamma)$. Since $\Gamma_0/\Gamma_0 \cap [e]$ is finite, it means that we have an element $g \in \Gamma_0 \cap [e]$ where $g \neq e$. So $g = g_2 g_1^{-1}$ for some $g_i \in \Gamma_i$, $i = 1, 2$. If γ_x denote the one-parameter subgroup through x, we define $H = \gamma_g$.

Now let us consider the short exact sequence:

$$1 \longrightarrow H \longrightarrow G \xrightarrow{p} G/H \longrightarrow 1.$$

The subgroups $p(H_1), p(H_2) \subset G/H$ are closed subgroups. For, the projection $p_1 : G \to G/\Gamma$ is the composite of p with projection $p_2 : G/H \to G/\Gamma$. Since H_i/Γ_i is compact, $p_1(H_i)$ is compact and hence $p(H_i) = p_2^{-1}(p_1(H_i)$ is closed. The groups $p(\Gamma_1)$, $p(\Gamma_2)$, $p(\Gamma)$ are discrete subgroups of $p(H_1)$, $p(H_2)$, G/H, respectively. It suffices to show that $p(\Gamma)$ is a discrete subgroup of G/H. Since $p_1 : G \to G/\Gamma$ is a covering map and $p_1(H)$ is compact, there is a neighbourhood V of $p_1(H)$ such that the restriction of p_1 of $p_1^{-1}(V) \to V$ is also a covering. Hence we have that $p(\Gamma)$ is discrete. So we have that $p(\Gamma_1)$, $p(\Gamma_2)$, $p(\Gamma)$ are uniform subgroups of $p(H_1)$, $p(H_2)$, $p(\Gamma)$, respectively. Certainly the projection induces a map $\overline{p} : \mathcal{R}(\Gamma_1, \Gamma_2; \Gamma) \to \mathcal{R}(p(\Gamma_1), p(\Gamma_2); p(\Gamma))$ which is surjective. By induction, hypothesis $\psi : p(H_1) \times p(H_2) \to G/H$ is surjective. So, it suffices to show that $\operatorname{Im}\psi \supset H$.

In order to show that $\operatorname{Im}\psi \supset H$, it suffices to show that $\gamma_g = \gamma_{g_2}.\gamma_{g_1}^{-1}$. First we show that for every rational p/q the two curves coincide. We consider the parameter $t = 1/q$. Let Γ' be the subgroup generated by Γ, w, w_1 and w_2, where $w = \gamma_g(1/q)$, $w_1 = \gamma_{g_1}(1/q)$ and $w_2 = \gamma_{g_2}(1/q)$. The center of Γ', denoted by Γ'_0, clearly contains Γ_0. We would like to show that $w = w_2 w_1^{-1}$. We know that $w^q = w_2^q w_1^{-q}$. By a result of Mal'cev (see [18, Chapter 5, 5.2.19]) the quotient of a Nilpotent group by the center is torsion free. So the quotient of Γ' by the center Γ'_0 is torsion free and we have $[w_1]^q = [w_2]^q$ since $w^q \in \Gamma'_0$. Also, in a torsion free nilpotent group the qth root is unique. Therefore $[w_1] = [w_2]$ and $w_2 w_1^{-1} \in \Gamma'_0$. Since $w^q = w_2^q w_1^{-q} = (w_2 w_1^{-1})^q$, where the last equality follows because $w_2 w_1^{-1} \in \Gamma'_0$, we have $w = w_2 w_1^{-1}$. It follows that $\gamma_g(1/q) = \gamma_{g_2}(1/q)\gamma_{g_1}^{-1}(1/q)$. By a similar and simpler argument we show that in fact the two curves coincide for all $t = p/q$. By continuity, it follows that $\gamma_g = \gamma_{g_2}\gamma_{g_1}^{-1}$ for all t. Therefore, we conclude that $H \subset \operatorname{Im}(\psi)$. So the result follows. $\square$

Let $f, g : N_1 \to N_2$ be two maps between compact nilmanifolds. Now we can prove

PROPOSITION 2.3. *If* $l(\operatorname{coin}(f_\#, g_\#)) = m - n$ *then* $R(f, g) < \infty$.

PROOF. By Lemma 2.7 of [15], maps f, g (up to homotopy) are covered by homomorphisms $\varphi_1, \varphi_2 : G_1 \to G_2$.

By Proposition 1.6, the map $\varphi = \varphi_2 \dot{\varphi}_1^{-1}$ is surjective. So, given any element $y \in \Gamma$, there exists $g \in G_1$ such that $\varphi_1(g)\varphi_2(g^{-1}) = y^{-1}$ or $\varphi_2(g^{-1})y = \varphi_1(g^{-1})$. Therefore $g^{-1} \in \operatorname{coin}(\varphi_2, \varphi_1)$ and consequently $\operatorname{coin}(\varphi_2, \varphi_1)$ is non empty. Therefore, for each Reidemeister class, there is a Nielsen class (non-empty one) which corresponds to this Reidemeister class. Since the number of Nielsen classes is finite, we must have only a finite number of Reidemeister classes and the result follows. $\square$

PROPOSITION 2.4. *If* $R(f, g) < \infty$ *then* $l(\operatorname{coin}(f_\#, g_\#)) = m - n$.

PROOF. As in Proposition 2.3, let $\varphi_1, \varphi_2 : G_1 \to G_2$ be homomorphisms which cover (up to homotopy) f and g, respectively. Since $G_1 \times G_2$ satisfies the property P_2, by Proposition 1.7 the result follows. $\square$

Now we come to the main result.

THEOREM 2.5. *Let* $f, g : N_1 \to N_2$, *where* N_1, N_2 *are compact nilmanifolds. Then, the two conditions below are equivalent*

(a) *The Hirsch length of* $\operatorname{coin}(f_\#, g_\#)$ *is* $\dim N_1 - \dim N_2$,

(b) $R(f,g) < \infty$.

Proof. The equivalence follows imediately from Propositions 2.3 and 2.4.□

Theorem 2.6. *Let $f, g : N_1 \to N_2$, where N_1, N_2 are compact nilmanifolds of the same dimension. Then, the three conditions below are equivalent*

(a) $N(f,g) \neq 0$,
(b) $\mathrm{coin}(f_\#, g_\#) = 1$,
(c) $R(f,g) < \infty$.

If one of the three conditions above holds, then $N(f,g) = R(f,g) = |L(f,g)|$.

Proof. The fact that (a) is equivalent to (b) has been proved in [15]. The equivalence of (b) and (c) follows from Theorem 2.5. Since (a), (b) and (c) are equivalent, let us show that they imply $N(f,g) = R(f,g) = |L(f,g)|$. From either the condition (a) or (c) follows (see [15]) that $N(f,g) = |L(f,g)|$. Finally, if (c) holds, we know that the map ψ is surjective. Therefore every Reidemeister class comes from a non empty Nielsen class. By Lemma 2.6 of [15], we have that this Reidemeister class represents an essential Nielsen class. So we have $N(f,g) = \#\mathcal{R}[f,g]$ and the result follows. □

Comment. In Theorem 2.5 we should expect that the conditions (a) and (b) are also equivalent to say that the pair (f,g) cannot be deformed to coincidence free. The usual type of argument to show this cannot be applied because, at present, there are difficulties to define a suitable Nielsen coincidence number in terms of a local index.

3. The root case

We begin by stating and giving a very nice and simple proof, due to A. Dold, of a classical result due C. Ehresmann (see [5]). Let $f : M \to N$ be a differential map between manifolds.

Theorem 3.1. *If $f : M \to N$ is a submersion which is proper and closed, then f is a fibration.*

Proof. We will prove that f is locally trivial. Denote by F the preimage $f^{-1}(y)$ of a point $y \in N$. Let $r : U \to F$ be a smooth neighbourhood retraction (e.g. the one of the tubular neighbourhood). Then $\tau = (f,r) : U \to N \times F$ is a map over N which is identity on F, hence it is diffeomorphic in an open neighbourhood V of F. Since F is compact the set W of all $w \in N$ such that $w \times F$ is contained in τV is an open neighbourhood of y, and $W \times F$ is in τV. The counterimage of $W \times F$ under τ may be smaller than the counterimage of W under f. We therefore cut down W as follows: the set of all $v \in V$ such that $\tau(v) \in W \times F$ is open in M, its complement C is closed in M, hence fC is closed in N, and its complement CfC is an open neighbourhood of y in N. Now $(W \cap CfC) \times F$ will do; its counterimage under τ coincides with the counterimage of $W \cap CfC$ under f. □

Let $g : N_1 \to N_2$ be a primitive map, i.e. $g_\#(\pi_1(N_1)) = \pi_1(N_2)$, where N_i are nilmanifolds.

Proposition 3.2. *The subset $g^{-1}(y)$ has the property that $\check{H}^{m-n}(g^{-1}(y), Z)$ is different from zero.*

Proof. Call G_1, G_2 the universal covers of N_1, N_2, respectively. We know that g can be covered (up to homotopy) by a homomorphism $\psi : G_1 \to G_2$, i.e. the map induced by ψ, $f : N_1 \to N_2$, is homotopic to g. Since $g_\#(\pi_1(N_1)) = \pi_1(N_2)$, this implies that $f_\#(\pi_1(N_1)) = \pi_1(N_2)$. Hence f is a submersion and, by Theorem 3.1, f is a fibration. The fibre F is certainly a nilmanifold of dimension $m - n$ and $\check{H}^{m-n}(F, Z) \neq 0$. Now we consider the diagram

$$\begin{array}{ccc} & & N_1 \\ & & \big\downarrow f \\ N_1 & \underset{g}{\overset{f}{\rightrightarrows}} & N_2 \end{array}$$

where g is homotopic to f. Since $f : N_1 \to N_2$ is a fibration, by the lifting homotopy property, we have $H : N_1 \times I \to N_1$ where $H(\cdot\,, 0) = \mathrm{id}_{N_1}$. Call $\phi = H(\cdot\,, 1)$. We have that $g^{-1}(y) = \phi^{-1}(F)$. Now, we apply Proposition 10.2 Chapter VIII of [4] for the case where the two manifolds M, M' are equal to N_1. The map f in Proposition 10.2 is ϕ, $K = F$ and L the empty subset. Since ϕ is homotopic to the identity, we have that the transfer is multiplication by one. Since $\check{H}^{m-n}(F, Z) \neq 0$, by definition of the transfer map, follows that $\check{H}^{m-n}(g^{-1}(y), Z) = \check{H}^{m-n}(\phi^{-1}(F), Z) \neq 0$ and the result follows. □

Proposition 3.3. *If $g : N_1 \to N_2$ has the property that*

$$[g_\#\pi_1(N_1)), \pi_1(N_2)] = l < \infty,$$

then $g^{-1}(y)$ is the union of at least l disjoint subsets $F_1, \ldots, F_l$ for $\check{H}^{m-n}(F_i, Z) \neq 0$ for $i = 1, \ldots, l$.

Proof. The map $g : N_1 \to N_2$ admits a lift $\overline{g} : N_1 \to \widetilde{N}_2$ where $\widetilde{N}_2$ is the cover of N_2 which corresponds to the subgroup $g_\#(\pi_1(N_1))$. Then we apply Proposition 3.2 for each point $\overline{y}_i$, $i = 1, \ldots, l$ over the base point y and the result follows. □

Let $N(f, g)$ be the topological Nielsen coincidence number as defined in [14, Section 2].

Theorem 3.4. *Let $f : N_1 \to N_2$ where N_1, N_2 are compact nilmanifolds. Then, the three conditions below are equivalent*

(a) $N(f,c) \neq 0$,
(b) *the Hirsch lenght of* $\operatorname{Ker} f_{\#} : \pi_1(N_1) \to \pi_1(N_2)$ *is* $\dim N_1 - \dim N_2$,
(c) *the index of* $f_{\#}(\pi_1(N_1))$ *in* $\pi_1(N_2)$, *i.e.* $R(f,c)$ *is finite.*

If one of the three conditions above holds, then

$$N(f,c) = R(f,c) = [f_{\#}\pi(N_1)), \pi_1(N_2)],$$

and $\check{H}^{m-n}(F_i, Z) \neq 0$ *for* l *coincidence Nielsen classes* $F_1, \ldots, F_l$ *and* $l = N(f,c)$.

Proof. The equivalence of (b) and (c) follows by Theorem 2.5. The fact that (c) implies (a) follows by Proposition 3.3. So let $\#\mathcal{R}[f,c] = \infty$. Consider the cover $p : \overline{N}_2 \to N_2$ which corresponds to $f_{\#}(\pi_1(N_1))$. $\overline{N}_2$ is a noncompact manifold. Let $\overline{f}$ be a lifting of f and consider a triangulations of N_2 and $\overline{N}_2$ such that the projection is a simplicial map. Since N_1 is compact, then $\overline{f}(N_1) \subset \overline{N}_2$ is also compact. Hence, there is a compact submanifold M of the same dimension as $\overline{N}_2$ (necessarily with boundary) which contains $\overline{f}(N_1)$ and is a subcomplex. Hence there is a retraction of this submanifolds into the $(n-1)$-subcomplex of $\overline{N}_2$. Since we can assume that y is in the interior of a maximal simplex, it follows that $p^{-1}(y)$ lies in the union of the interior of maximal simplexes. Therefore, we can deform $\overline{f}$ into $\overline{N}_2 - p^{-1}(y)$. Hence we can deform f to f' without roots.

Finally, if one of the three conditions holds, then by Proposition 3.3 the result follows. □

Remarks. (1) From the proof of Proposition 3.2, we can see that the map g can be deformed to a map f' such that the set $f'^{-1}(y)$ is a connected manifold of dimension $m-n$. Hence, by routine argument using covering, the map f in Theorem 3.4 can be deformed to f' such that $f'^{-1}(y)$ is the union of l connected submanifolds all of dimension $m-n$.

(2) It would be nice to know if the above result for roots extends to coincidence in general.

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