Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 12, 1998, 309–322

THE INTRINSIC MOUNTAIN PASS PRINCIPLE

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1. Introduction

Recently M. Schechter proposed new ideas in the variational methods usually known under the name "mountain pass". In [9] he proved a quantitative result of mountain pass type giving up the basic geometrical essence of the classical theorem. Let us fix the abstract mountain pass structure (cf. [4]). We are given a functional $f : X \to R$ defined on a complete metric space X, a barrier $B \subset X$, a boundary $A \subset X$ and a family of paths $\Gamma \subset 2^X$ satisfying:

(a) $\gamma \supset A$ for every $\gamma \in \Gamma$,
(b) $\gamma \cap B \neq \emptyset$ for every $\gamma \in \Gamma$,
(c) for any $\gamma \in \Gamma$, $t \geq 0$ and any

$$\eta \in \Xi := \{\theta \in C(X \times [0,1], X) : \eta(x,t) = x \text{ for all } (x,t) \in (X \times \{0\}) \cup (A \times [0,1])\}$$

we have $\eta(\gamma, t) \in \Gamma$, i.e. Γ is stable with respect to Ξ.

Remark 1.1. In case needed one can assume that the family of paths Γ is stable with respect to a proper subset of the class of deformations Ξ.

In the classical mountain pass setting the functional f is C^1 and is "high" on the barrier and "low" on the boundary. In [9] this is violated – the barrier is split into two parts: a "high" part where the functional values are greater

1991 *Mathematics Subject Classification.* 58E05, 58E30.

Key words and phrases. Nonsmooth critical point theory, mountain pass, deformations.

This work is partially supported by Bulgarian Science Foundation - contracts MM-506/95, MM-521/95, MM-612/96 and MM-807/98.

than or equal to its values on the boundary and a "low" part where this is not the case. Nevertheless, the abstract result of [9] implies the existence of a point with an arbitrarily fixed small slope if the "low" part of the barrier is sufficiently far from the boundary. This is important for the application to nonlinear elliptic PDE-s presented in [9].

In this note we extend the results of [9] dropping any smoothness or even continuity assumptions on the considered functional. To do this we had to prove a new version of the deformation lemma. We also present another result in which the boundary is split into a "low" part and a "high" part and derive the same conclusion as before if the "high" part of the boundary is sufficiently far from the barrier. In a recent work [1] J.-N. Corvellec proved similar results building on [9]. In particular he obtained a variant of the result with a split boundary for the case of continuous functionals. Note that splitting the boundary is not the same as splitting the barrier because they do not play symmetric roles in the setting (moreover one can not expect that the deformations involved are homeomorphisms in a general metric space or for a non-Lipschitz functional on a Banach space). The results presented here contain (in contrast to the ones in [1]) estimates for the location of the points with "small slope". Also, we do not impose continuity assumption on the considered functional, thus including the lower semicontinuous case. A different treatment of the lower semicontinuous case is presented in [2] and [3]. Our approach is closely related to the one proposed by A. Ioffe and E. Schwartzman in [5]. Introducing a slightly different definition of steepness, they prove abstract deformation results and apply them to obtain a mountain pass theorem for continuous functionals. The intrinsic mountain pass principle presented here implies directly a mountain pass theorem in the classical setting for discontinuous functionals. We do not impose any additional assumptions (as in [2], condition (4.1) for the lower semicontinuous case), but our Corollary 2.4 and Theorem 4.5 in [2] are not comparable, because the two notions of weak slope are different.

We would like to express our gratitude to Prof. A. Ioffe for the helpful discussions on the topic.

2. Quantitative theorems

We shall always assume that X is a complete metric space with metric d and that f is an arbitrary real-valued functional defined on X. We shall denote by $B(x,\delta)$ the closed ball in X centered at $x \in X$ with radius $\delta > 0$ and

$$\operatorname{dist}(A,B) := \inf\{d(a,b) : a \in A,\ b \in B\},$$

where $A \subseteq X$, $B \subseteq X$.

When $f \in C^1(X,R)$ and X is a Banach space, the slope at a given $x \in X$ is measured by $\|f'(x)\|$. In the present setting a natural substitute for $\|f'(x)\|$ is (cf. [2], [3], [5], [6], [8]) given in

Definition 2.1. The supremum of the numbers $\sigma \geq 0$ such that there exist $\delta > 0$ and a continuous map $H : B(x,\delta) \times [0,\delta] \to X$ satisfying

$$d(H(y,t),y) \leq t \quad \text{and} \quad f(H(y,t)) \leq f(y) - \sigma.t,$$

whenever $(y,t) \in B(x,\delta) \times [0,\delta]$, is called *weak slope* of f at x and is denoted by $|df|(x)$.

As in the C^1 case, $x \in X$ is called critical if $|df|(x) = 0$.

The basic tool in our consequent analysis is the following version of

Lemma 2.2. (Deformation lemma). *Let S be a closed subset of X and Q be an open neighbourhood of S. Let $\sigma > 0$ and $|df|(y) > \sigma$ for every $y \in Q$. Then there exists $\eta \in C(X \times [0,+\infty), X)$ with the properties:*

(i) *$\eta(x,0) = x$ for each $x \in X$,*
(ii) *$\eta(x,t) = x$ for each $x \in X \setminus Q$ and each $t \geq 0$,*
(iii) *$d(\eta(x,t),x) \leq t$ for each $x \in X$ and each $t \geq 0$,*
(iv) *$f(x) - f(\eta(x,t)) \geq \sigma.d(x,\eta(x,t))$ for each $x \in X$ and each $t \geq 0$,*
(v) *for every point $x \in X$ there exists $\lambda_x \in [0+\infty]$ such that*

$$\begin{aligned} d(\eta(x,\lambda_x),\eta(x,\tau)) &\leq \tau - \lambda_x && \textit{for each } \tau \geq \lambda_x, \\ f(\eta(x,\lambda_x)) - f(\eta(x,\tau)) &\geq \sigma.d(\eta(x,\lambda_x),\eta(x,\tau)) && \textit{for each } \tau \geq \lambda_x, \\ f(\eta(x,\tau)) &\leq f(x) - \sigma.\tau && \textit{for each } \tau \in [0,\lambda_x], \end{aligned}$$

and $\eta(x,\lambda_x) \notin S$ whenever $\lambda_x \neq \infty$.

Remark. Conditions (i)–(iv) are standard and are usually met in the classical deformation results dealing with a C^1 functional f. In our setting we need condition (v) because the local deformations H (cf. Definition 2.1) and hence, the global one η *lack* the semigroup property $\eta(\eta(x,t_1),t_2) = \eta(x,t_1+t_2)$.

Proof. Since $|df|(x) > \sigma$ for every $x \in Q$, by Definition 2.1 we obtain that for every $x \in Q$ there exists a positive real δ_x and $H_x \in C(B(x,\delta_x) \times [0,\delta_x], X)$, such that $B(x,\delta_x) \subset Q$ and for every $y \in B(x,\delta_x)$ and every $t \in [0,\delta_x]$ the following two inequalities hold true:

$$d(H_x(y,t),y) \leq t, \tag{1a}$$

$$f(H_x(y,t)) \leq f(y) - \sigma.t. \tag{1b}$$

Let us denote by U_x the open ball with centre x and radius $\delta_x/2$. Then $\{U_x\}_{x\in Q} \cup \{X \setminus S\}$ is an open cover of the metric space X. Let $\{U_\gamma\}_{\gamma\in\Gamma} \cup \{X\setminus S\}$ be a locally finite refinement of this cover and $\{\alpha_\gamma\}_{\gamma\in\Gamma}\cup\alpha$ be a Lipschitz partition of unity subordinated to this refinement. Let $U_\gamma\subset U_{x_\gamma}$, $x_\gamma\in Q$ and for short $\delta_\gamma=\delta_{x_\gamma}$, $H_\gamma=H_{x_\gamma}$. Without loss of generality we can have

$$Q=\bigcup\{U_\gamma:\gamma\in\Gamma\}\supset S.$$

Let $\Gamma=[0,\gamma_0]$ be well ordered. We set $t_x=\min\{\delta_\gamma : x\in\overline{U}_\gamma\}/2$ if $x\in Q$ and $t_x=0$ if $x\in X\setminus Q$. We define inductively the mappings $\{\xi_\gamma(x,t)\}_{\gamma\in[0,\gamma_0]}$:

(a) $\xi_0(x,t)=x$ for every $x\in X$ and $0\le t\le t_x$.,

(b) if γ has a predecessor, then for every $x\in X$ and $t\in[0,t_x]$

$$\xi_\gamma(x,t)=\begin{cases} H_{\gamma-1}(\xi_{\gamma-1}(x,t),\alpha_{\gamma-1}(x).t) & \text{if } x\in U_{\gamma-1},\\ \xi_{\gamma-1}(x,t) & \text{if } x\notin U_{\gamma-1},\end{cases}\tag{2}$$

(c) if γ is a limit ordinal, then

$$\xi_\gamma(x,t)=\lim_{\beta<\gamma}\ \xi_\beta(x,t)\quad \text{for each } x\in X,\ x\in[0,t_x].$$

Next we show that for each $\gamma\in[0,\gamma_0]$ and for $x\in X$, $t\in[0,t_x]$ the mapping $\xi_\gamma(x,t)$ is well defined and continuous and the following properties hold true:

$$d(\xi_\gamma(x,t),x)\le\bigg(\sum_{\beta<\gamma}\alpha_\beta(x)\bigg).t,\tag{3a}$$

$$f(\xi_\gamma(x,t))\le f(x)-\sigma\bigg(\sum_{\beta<\gamma}\alpha_\beta(x)\bigg).t,\tag{3b}$$

We will proceed by induction on γ. For $\gamma=0$ the claim is clear. Let the claim be true for every $\beta<\gamma$.

Case I. γ has a predecessor. If $x\notin U_{\gamma-1}$ then $\xi_\gamma(x,t)$ is clearly well defined. If $x\in U_{\gamma-1}$ then $\alpha_{\gamma-1}(x).t\le t_x\le\delta_{\gamma-1}/2$. Using (3a) for $\gamma-1$ we have $d(\xi_{\gamma-1}(x,t),x)\le t\le t_x\le\delta_{\gamma-1}/2$ and hence $\xi_{\gamma-1}(x,t)\in B(x_{\gamma-1},\delta_{\gamma-1})$ so $\xi_\gamma(x,t)=H_{\gamma-1}(\xi_{\gamma-1}(x,t),\alpha_{\gamma-1}(x).t)$ is well defined. Moreover, whenever $x\in X$ and $t\in[0,t_x]$ we have

$$\begin{aligned} d(\xi_\gamma(x,t),x)&\le d(\xi_\gamma(x,t),\xi_{\gamma-1}(x,t))+d(\xi_{\gamma-1}(x,t),x)\\ &\le\alpha_{\gamma-1}(x).t+\bigg(\sum_{\beta<\gamma-1}\alpha_\beta(x)\bigg).t=\bigg(\sum_{\beta<\gamma}\alpha_\beta(x)\bigg).t\end{aligned}$$

(according to (1a), (2) and inductive assumption) and (3a) is proved. Now

$$\begin{aligned} f(\xi_\gamma(x,t))&=f(\xi_\gamma(x,t))-f(\xi_{\gamma-1}(x,t))+f(\xi_{\gamma-1}(x,t))\\ &\le f(\xi_\gamma(x,t))-f(\xi_{\gamma-1}(x,t))+f(x)-\sigma\bigg(\sum_{\beta<\gamma-1}\alpha_\beta(x)\bigg).t.\end{aligned}$$

If $x \in U_{\gamma-1}$, then by (1b) and (2) we have

$$\begin{aligned} & f(\xi_\gamma(x,t)) - f(\xi_{\gamma-1}(x,t)) \\ &\qquad = f(H_{\gamma-1}(\xi_{\gamma-1}(x,t), \alpha_{\gamma-1}(x).t)) - f(\xi_{\gamma-1}(x,t)) \le -\sigma.\alpha_{\gamma-1}(x).t. \end{aligned}$$

If $x \notin U_{\gamma-1}$, then

$$\alpha_{\gamma-1}(x) = 0 \quad \text{and} \quad f(\xi_\gamma(x,t)) - f(\xi_{\gamma-1}(x,t)) = 0 = -\sigma.\alpha_{\gamma-1}(x).t.$$

Hence

$$\begin{aligned} f(\xi_\gamma(x,t)) &\le -\sigma.\alpha_{\gamma-1}(x).t + f(x) - \sigma\biggl(\sum_{\beta<\gamma-1} \alpha_\beta(x)\biggr).t \\ &= f(x) - \sigma\biggl(\sum_{\beta<\gamma} \alpha_\beta(x)\biggr).t, \end{aligned}$$

thus proving (3b).

Next we establish the continuity of ξ_γ at (x_0, t_0), where $0 \le t_0 \le t_{x_0}$. Let $x_n \to x_0$ and $t_n \to t_0$, where $0 \le t_n \le t_{x_n}$. There are two possibilities: $x_0 \in U_{\gamma-1}$ or $x_0 \notin U_{\gamma-1}$.

If $x_0 \in U_{\gamma-1}$, then $x_n \in U_{\gamma-1}$ for n sufficiently large. As above

$$d(\xi_{\gamma-1}(x_n, t_n), x_n) \le t_n \le t_{x_n} \le \delta_{\gamma-1}/2$$

for every $n \ge n_0$ and for $n = 0$, so

$$\xi_{\gamma-1}(x_n, t_n) \in B(x_{\gamma-1}, \delta_{\gamma-1}), \quad n \ge n_0, \quad \xi_{\gamma-1}(x_0, t_0) \in B(x_{\gamma-1}, \delta_{\gamma-1}).$$

Now the continuity of ξ_γ at (x_0, t_0) follows from (2) and from the continuity of $\xi_{\gamma-1}$, $\alpha_{\gamma-1}$ and $H_{\gamma-1}$ on the set $B(x_{\gamma-1}, \delta_{\gamma-1}) \times [0, \delta_{\gamma-1}]$.

If $x_0 \notin U_{\gamma-1}$, then the sequence $\{x_n\}_{n=1}^\infty$ consists of two subsequences:

$$\{x_{k_n}\}_{n=1}^\infty \subset X \setminus U_{\gamma-1} \quad \text{and} \quad \{x_{l_n}\}_{n=1}^\infty \subset U_{\gamma-1}.$$

For the first subsequence we have $\xi_\gamma(x_{k_n}, t_{k_n}) = \xi_{\gamma-1}(x_{k_n}, t_{k_n})$ and the continuity of $\xi_{\gamma-1}$ implies

$$\lim_{t\to\infty} \xi_\gamma(x_{k_n}, t_{k_n}) = \xi_{\gamma-1}(x_0, t_0).$$

The second subsequence may be finite. If not, $x_0 \in \overline{U_{\gamma-1}}$ and so $t_0 \le \delta_{\gamma-1}/2$, $\xi_{\gamma-1}(x_0, t_0) \in B(x_{\gamma-1}, \delta_{\gamma-1})$. Therefore

$$\begin{aligned} \lim_{t\to\infty} \xi_\gamma(x_{l_n}, t_{l_n}) &= \lim_{t\to\infty} H_{\gamma-1}(\xi_{\gamma-1}(x_{l_n}, t_{l_n}), \alpha_{\gamma-1}(x_{l_n}).t_{l_n}) \\ &= H_{\gamma-1}(\xi_{\gamma-1}(x_0, t_0), \alpha_{\gamma-1}(x_0).t_0) \\ &= H_{\gamma-1}(\xi_{\gamma-1}(x_0, t_0), 0) = \xi_{\gamma-1}(x_0, t_0). \end{aligned}$$

Thus the continuity of ξ_γ is proved because $\xi_\gamma(x_0, t_0) = \xi_{\gamma-1}(x_0, t_0)$ when $x_0 \notin U_{\gamma-1}$.

Case II. γ has not a predecessor. Let $x \in X$ and $B(x, r_x) \cap U_\beta = \emptyset$ for each $\beta \notin \{\gamma_1, \dots, \gamma_s\}$. Denote $\overline{\gamma} = \max\{\gamma_i < \gamma : \ i = 1, \dots, s\} + 1$. Then $\xi_\gamma(y,t) = \xi_{\overline{\gamma}}(y,t)$ for every $y \in B(x, r_x)$ and $t \in [0, t_y]$. Indeed, a simple induction on $\beta \in [\overline{\gamma}, \gamma]$ shows $\xi_\beta(y,t) = \xi_{\overline{\gamma}}(y,t)$ using (2). Case II is done.

Let us denote ξ_{γ_0} by ξ. This map has the properties:

(4a) $d(\xi(x,t), x) \le t$ for each $x \in X$ and $t \in [0, t_x]$,

(4b) $f(\xi(x,t)) \le f(x) - \sigma d(\xi(x,t), x)$ for each $x \in X$ and $t \in [0, t_x]$,

(4c) $f(\xi(x,t)) \le f(x) - \sigma t$ for each $x \in S$ and $t \in [0, t_x]$.

These properties follow from (3a), (3b), because $\sum_{\beta<\gamma} \alpha_\beta(x) = 1$ on S.

In the sequel we shall need the lower semicontinuity of the mapping $x \to t_x \in [0, \infty)$ on X. If $x \notin Q$ we have $t_x = 0$ and the lower semicontinuity follows from $t_y \ge 0$ for every $y \in X$. Now let $x \in Q = \cup_{\gamma \in \Gamma} U_\gamma$. Since $\{U_\gamma\}_{\gamma\in\Gamma}$ is locally finite, there exists a ball $B(x, r_x)$, such that $B(x, r_x) \cap U_\gamma \ne \emptyset$ only for finitely many γ. Without loss of generality

$$B(x, r_x) \cap U_\gamma \ne \emptyset \Leftrightarrow \gamma \in \{\beta \in \Gamma : x \in \overline{U}_\beta\}.$$

If $y \in B(x, r_x)$ then $\{\beta \in \Gamma : y \in \overline{U}_\beta\} \subset \{\beta \in \Gamma : x \in \overline{U}_\beta\}$, i.e. $t_y \ge t_x$ and the lower semicontinuity of $x \to t_x \in [0, \infty)$ is proved. It implies the existence of a continuous function $\tau : X \to [0, \infty)$ such that $\tau(x) \le t_x$ on X and $\tau(x) > 0$ if and only if $t_x > 0$, i.e. $\tau(x) > 0$ if and only if $x \in Q$.

Next we define inductively

$$\eta_k \in C(X \times [0, +\infty), X) \quad \text{and} \quad \tau_k \in C(X) \quad \text{for } k = 0, 1, 2, \dots$$

as follows:

$$\tau_0(x) = 0 \quad \text{on } X,$$
$$\eta_0(x,t) = x \quad \text{for every } x \in X \text{ and } t \ge 0,$$
$$\tau_{k+1}(x) = \tau_k(x) + \tau(\eta_k(x, \tau_k(x))),$$
$$\eta_{k+1}(x,t) = \begin{cases} \eta_k(x,t) & \text{for every } t \in [0, \tau_k(x)], \\ \xi(\eta_k(x, \tau_k(x)), t - \tau_k(x)) & \text{for every } t \in [\tau_k(x), \tau_{k+1}(x)], \\ \xi(\eta_k(x, \tau_k(x)), \tau_{k+1}(x) - \tau_k(x)) & \text{for every } t \ge \tau_{k+1}(x). \end{cases}$$

The following properties of η_k are corollaries of the properties (4a), (4b) and (4c) of ξ:

(5a) $d(\eta_k(x,t), x) \le t$ for each $x \in X$ and $t \ge 0$,

(5b) $f(\eta_k(x,t)) \le f(x) - \sigma d(\eta_k(x,t), x)$ for each $x \in X$ and $t \ge 0$,

(5c) $f(\eta_k(x,t)) \le f(x) - \sigma t$, for each x satisfying $\eta_i(x,\tau_i(x)) \in S$ whenever $i \in \{0,\dots,k-1\}$ and for every $t \in [0,\tau_k(x)]$.

We shall prove only (5c) since (5a) and (5b) are straightforward. Again, we proceed by induction on k. The first step is trivial. Next we estimate from above $f(\eta_{k+1}(x,t))$. If $t \in [0,\tau_k(x)]$ we have

$$f(\eta_{k+1}(x,t)) = f(\eta_k(x,t)) \le f(x) - \sigma.t$$

by the inductive assumption. If $t \in [\tau_k(x),\tau_{k+1}(x)]$, then

$$\begin{aligned} f(\eta_{k+1}(x,t)) &= f(\xi(\eta_k(x,\tau_k(x)), t-\tau_k(x))) \\ &\le f(\eta_k(x,\tau_k(x))) - \sigma.(t-\tau_k(x)) \\ &\le f(x) - \sigma.\tau_k(x) - \sigma.t + \sigma.\tau_k(x) = f(x) - \sigma.t. \end{aligned}$$

The first of the above inequalities is (4c) applied to $\eta_k(x,\tau_k(x)) \in S$ and $t - \tau_k(x) \in [0, t_{\eta_k(x,\tau_k(x))}]$ and the second one is the inductive assumption.

We set

$$P := \{x \in X : \eta_k(x,\tau_k(x)) \in S \quad \text{for every } k = 0,1,2,\dots\}.$$

The (possibly empty) set P is closed in X. For every $x \in Q\setminus P$ there exist a positive integer $s(x)$ and a neighbourhood $V_x \subset Q\setminus P$ of x such that $\eta_k(y,\tau_k(y)) \notin S$ whenever $y \in V_x$. Now $\{V_x\}_{x\in Q\setminus P} \cup \{X \setminus S\}$ is an open cover of $X \setminus P$. Let $\{V_\beta\}_{\beta\in\Theta} \cup \{X\setminus S\}$ be a locally finite refinement of this cover and $\{\Omega_\beta\}_{\beta\in\Theta} \cup \{\Omega\}$ be a Lipschitz partition of unity subordinated to the refinement. We denote

$$\mu(x) = \sum_{\beta\in\Theta} \Omega_\beta(x)\tau_{s_\beta}(x).$$

Since $\{V_\beta\}_{\beta\in\Theta}$ is locally finite, $\mu : X \setminus P \to [0,\infty)$ is continuous. We define $\mu^* : X \to [0,1]$ by

$$\mu^*(x) = \begin{cases} \dfrac{1}{1+\mu(x)} & \text{if } x \notin P, \\ 0 & \text{if } x \in P. \end{cases}$$

Let us consider also $\tau^* : X \to [0,1]$ defined by

$$\tau^*(x) := \frac{1}{1+\sup\{\tau_k(x) : k = 0,1,2,\dots\}} \quad \text{for } x \in X.$$

The function μ^* is lower semicontinuous and τ^* is upper semicontinuous. Moreover, $\tau^* \le \mu^*$ on X. Indeed, if $x \notin P$ we have

$$\begin{aligned} \mu(x) &= \sum_{\beta\in\Theta} \Omega_\beta(x)\tau_{s_\beta}(x) \le \sup\{\tau_k(x) : k=0,1,2,\dots\}.\sum_{\beta\in\Theta}\Omega_\beta(x) \\ &\le \sup\{\tau_k(x) : k=0,1,2,\dots\} \end{aligned}$$

and hence

$$\tau^*(x) := \frac{1}{1+\sup\{\tau_k(x) : k = 0,1,2,\dots\}} \le \frac{1}{1+\mu(x)} = \mu^*(x).$$

If $x \in P$ we shall prove that $\tau_k(x) \xrightarrow[k\to\infty]{} \infty$, thus implying $\tau^*(x) = 0 = \mu^*(x)$. Assuming the contrary, we obtain that

$$\sum_{k=0}^{\infty} (\tau_{k+1}(x) - \tau_k(x))$$

is convergent. On the other hand

$$d(\eta_{k+1}(x, \tau_{k+1}(x)), \eta_k(x, \tau_k(x))) \le \tau_{k+1}(x) - \tau_k(x)$$

implies that the sequence $\{\eta_k(x, \tau_k(x))\}_{k=1}^{\infty}$ is a Cauchy one. As X is complete, there exists $z = \lim_{k\to\infty} \eta_k(x, \tau_k(x))$. Since S is closed, $z \in S$. Therefore, $\tau(z) > 0$ and the continuity of τ yields $\tau(\eta_k(x, \tau_k(x))) > \tau(z)/2$ for k sufficiently large. On the other hand

$$\tau(\eta_k(x, \tau_k(x))) = \tau_{k+1}(x) - \tau_k(x) \xrightarrow[k\to\infty]{} 0,$$

which is a contradiction.

In the sequel we shall need the following

Claim. *There exists a continuous function $g : X \to [0,1]$ such that $\tau^*(x) \le g(x) \le \mu^*(x)$ for every $x \in X$. Moreover, $\tau^*(x) < g(x) < \mu^*(x)$ whenever $\tau^*(x) < \mu^*(x)$.*

Proof. Let $G : X \to [0,1]$ be the multivalued mapping defined by $G(x) = [\tau^*(x), \mu^*(x)]$. Since τ^* is upper semicontinuous and μ^* is lower semicontinuous, G is lower semicontinuous as a multivalued map (i.e. for each $x \in X$ and for each $(a,b) \subset \mathbb{R}$ with $(a,b) \cap G(x) \ne \emptyset$ there is a neighbourhood V of x with $(a,b) \cap G(y) \ne \emptyset$ for each $y \in V$). Applying the Michael selection theorem (cf. [7]) we obtain a continuous $\overline{g} : X \to [0,1]$ such that $\tau^*(x) \le \overline{g}(x) \le \mu^*(x)$ for $x \in X$. As $\mu^* - \overline{g}$ is lower semicontinuous and nonnegative, there exists a continuous $g_\mu : X \to [0,1]$ with $g_\mu(x) \in [0, \mu^*(x) - \overline{g}(x)]$ for $x \in X$ and $\overline{g}(x) > 0$ whenever $\mu^*(x) - \overline{g}(x) > 0$. Similarly, there exists a continuous $g_\tau : X \to [0,1]$ with $g_\tau(x) \in [0, \overline{g}(x) - \tau^*(x)]$ for $x \in X$ and $\overline{g}(x) > 0$ whenever $\overline{g}(x) - \tau^*(x) > 0$. Then the function $g : X \to [0,1]$ defined by $g(x) := [(g_\mu(x) + \overline{g}(x)) + (\overline{g}(x) - g_\tau(x))]/2$ has the desired properties. □

Let $t \ge 0$ and $x \in X$ be such that $t < \sup\{\tau_k(x) : k = 0,1,2,\dots\}$. Then the common value of $\eta_k(x,t)$ where $\tau_k(x) > t$ will be denoted by $\eta^*(x,t)$. Note that if $x \in P$, then $\sup\{\tau_k(x) : k = 0,1,2,\dots\} = \infty$ and so, $\eta^*(x,t)$ is defined for every $t \geq 0$. If $x \in Q \setminus P$, $\tau_{k+1}(x) > \tau_k(x)$ and, hence, $\sup\{\tau_k(x) : k = 0, 1, 2, \dots\} > \mu(x)$. Therefore, $g(x) \in (\tau^*(x), \mu^*(x))$ and

$$\mu(x) < \frac{1}{g(x)} - 1 < \sup\{\tau_k(x) : \; k = 0, 1, 2, \dots\},$$

for $x \in Q \setminus P$. Thus $\eta^*(x, t)$ is well defined for $t \in [0, 1/g(x) - 1]$. The deformation η in the statement of the lemma will be

$$\eta(x, t) = \begin{cases} \eta^*(x, 1/g(x) - 1) & \text{if } (x, t) \in (Q \setminus P) \times [1/g(x) - 1, \infty), \\ \eta^*(x, t) & \text{otherwise.} \end{cases}$$

The continuity of g together with $g \equiv 0$ on P and $g \equiv 1$ on $X \setminus Q$ imply that $\eta \in C(X \times [0, \infty), X)$.

It remains to verify that the so defined mapping η satisfies the properties (i)–(v). The properties (i) and (ii) are straightforward. The properties (iii) and (iv) follow from the corresponding properties of η_k (cf. (5a) and (5b)).

We set λ_x to be ∞ for $x \in P$ and $\lambda_x = \tau_{k(x)}(x)$, where

$$k(x) = \min\{k \in \mathbb{N} \cup \{0\} : \eta_k(x, \tau_k(x)) \notin S\}$$

for $x \in X \setminus P$. Now the first and the second properties in (v) follow easily from (5a) and (5b), applied to $\eta_{k(x)}(x, \tau_{k(x)}(x))$, and the definition of η. The third property comes from (5c). This completes the proof of the deformation lemma. $\square$

The next theorem is a nonsmooth extension of the basic abstract result in [9].

Theorem 2.3. *Let A be a boundary, B a barrier and Γ a family of paths forming an abstract mountain pass structure in X. Assume* $\operatorname{dist}(A, B) > 0$ *and define*

$$c := \inf_{\gamma \in \Gamma} \sup_{x \in \gamma} f(x) \quad \text{and} \quad b := \inf_{x \in B} f(x).$$

Let $B' := \{x \in B : f(x) < c\}$ be the low part of the barrier and $d' := \operatorname{dist}(B', A)$. Assume further $-\infty < b$ and $c < \infty$. Take

(1) $T \in (0, d')$,
(2) $\varepsilon > (c - b)/T$,
(3) $\delta \in (0, [T\varepsilon - (c - b)]/2)$, $\delta < d' - T$ *and* $\delta < \varepsilon.\operatorname{dist}(A, B \setminus B')/2$.

Then there exists $x = x(T, \varepsilon, \delta) \in X$ such that

(i) $x \in cl(f^{-1}([b - \delta, c + \delta]))$,
(ii) $|df|(x) \leq \varepsilon$,
(iii) *either* $\operatorname{dist}(x, B') \leq T$ *or* $\operatorname{dist}(x, B \setminus B') \leq \delta/\varepsilon$.

Remark. In case $B' = \emptyset$ Theorem 2.3 yields immediately (taking $\varepsilon = 1/n$, $\delta = 1/n^2$) a mountain pass result for arbitrary functional f.

Proof. Let us assume the contrary: for every $x \in \operatorname{cl}(f^{-1}([b-\delta, c+\delta]))$ we have either $|df|(x) > \varepsilon$ or $\operatorname{dist}(x, B') > T$ and $\operatorname{dist}(x, B \setminus B') > \delta/\varepsilon$. We set

$$\begin{aligned} S_1 &:= \{x \in X : \operatorname{dist}(x, B') \le T\}, \\ S_2 &:= \{x \in X : \operatorname{dist}(x, B \setminus B') \le \delta/\varepsilon\}, \\ S &:= \operatorname{cl}(f^{-1}([b-\delta, c+\delta])) \cap (S_1 \cup S_2). \end{aligned}$$

Let $x \in S$. Then $|df|(x) > \varepsilon$. Let U_x be an open neighbourhood of x such that $U_x \subset [(S_1)_\delta \cup (S_2)_{\delta/\varepsilon}]$ and $|df|(y) > \varepsilon$ for every $y \in U_x$. We set $Q := \bigcup_{x \in S} U_x$. Note that $Q \cap A = \emptyset$. There exists $\gamma_\delta \in \Gamma$ such that

$$c \le \sup_{x \in \gamma_\delta} f(x) < c + \delta. \tag{6}$$

Let η be the deformation given by the deformation lemma. Denote $\gamma = \eta(\gamma_\delta, T)$ in Γ. Clearly,

$$\sup_{x \in \gamma} f(x) \ge \sup_{x \in \gamma \cap B} f(x) \ge \inf_{x \in B} f(x) = b. \tag{7}$$

Let $y \in \gamma \cap B$. Then there exists $x \in \gamma_\delta$ such that $y = \eta(x, T)$. The following cases are possible:

Case 1. $\eta(x, \tau) \in S$ for every $\tau \in [0, T]$. Then (v) of the deformation lemma and (6) imply

$$f(\eta(x, T)) \le f(x) - \varepsilon.T \le c + \delta - \varepsilon.T \le b - \delta$$

(the last inequality is based on (3)). Since $y \in B$, we have $f(y) \ge b$, a contradiction to (7).

Case 2. There exists $t \in [0, T]$ such that $\eta(x, t) \notin S$. The property (v) of η (cf. the deformation lemma) implies the existence of $\lambda_x \ge 0$ such that $z := \eta(x, \lambda_x) \notin S$ and

$$f(\eta(x, \tau)) \le f(x) - \varepsilon.\tau \qquad \text{for } \tau \in [0, \lambda_x], \tag{8a}$$

$$f(z) - f(\eta(x, \tau)) \ge \varepsilon.d(z, \eta(x, \tau)) \quad \text{for } \tau \ge \lambda_x, \tag{8b}$$

$$d(z, \eta(x, \tau)) \le \tau - \lambda_x \qquad \text{for } \tau \ge \lambda_x. \tag{8c}$$

Case 2.1. $\lambda_x \ge T$. According to (8a) we obtain a contradiction as in Case 1.

Case 2.2.1. $\lambda_x < T$ and $f(z) < b - \delta$. According to (8b) we have

$$f(y) = f(\eta(x, T)) \le f(z) < b - \delta$$

which contradicts $y \in B$.

Case 2.2.2.1. $\lambda_x < T$ and $f(z) \ge b - \delta$ and $y \in B'$ (i.e. $f(y) < c$). Here (8c) yields

$$T \ge T - \lambda_x \ge d(z, \eta(x, T)) = d(z, y) \ge \operatorname{dist}(z, B'). \tag{9}$$

From (iv) of the deformation lemma and $x \in \gamma_\delta$ we obtain

$$b - \delta \leq f(z) = f(\eta(x, \lambda_x)) \leq f(x) < c + \delta.$$

Hence, $z \in \mathrm{cl}(f^{-1}([b-\delta, c+\delta]))$. But $z \notin S$, so $z \notin S_1 \cup S_2$ which means $\mathrm{dist}(B', z) > T$, a contradiction to (9).

Case 2.2.2.2. $\lambda_x < T$ and $f(z) \geq b - \delta$ and $y \notin B'$ (i.e. $f(y) \geq c$). Since $f(y) \geq c$ and $f(z) \leq f(x) < c + \delta$, we have $f(z) - f(y) < c + \delta - c = \delta$. On the other hand from (8b) we obtain

$$f(z) - f(y) = f(z) - f(\eta(x, T)) \geq \varepsilon.d(z, \eta(x, T)) = \varepsilon.d(y, z).$$

So,

$$\frac{\delta}{\varepsilon} > \frac{f(z) - f(y)}{\varepsilon} \geq d(z, y). \tag{10}$$

Because of $z \notin S$ and $z \in \mathrm{cl}(f^{-1}([b-\delta, c+\delta]))$, we conclude that $z \notin S_1 \cup S_2$ which means

$$d(z, y) \geq \mathrm{dist}(z, B \setminus B') > \delta/\varepsilon,$$

a contradiction to (10). Theorem 2.3 is thus proved. □

Corollary 2.4. *If $B' = \emptyset$ then for every positive integer n there exists $x_n \in X$ with the properties:*

(i) $x_n \in \mathrm{cl}(f^{-1}([c - 1/n^2, c + 1/n^2]))$,
(ii) $|df|(x_n) \leq 1/n$,
(iii) $\mathrm{dist}(x_n, B) < 1/n$.

Remark. If f is continuous, (i) yields $f(x_n) \xrightarrow[n\to\infty]{} c$. Imposing a suitable Palais–Smale condition on f, one obtains a critical point of f.

Corollary 2.5. *Let A_n, B_n and Γ_n form an abstract mountain pass structure in X for every positive integer n. Assume* $\mathrm{dist}(A_n, B_n) > 0$ *and define*

$$c_n := \inf_{\gamma \in \Gamma_n} \sup_{x \in \gamma} f(x) \quad \text{and} \quad b_n := \inf_{x \in B_n} f(x).$$

Define further $c := \liminf c_n$ and $b := \limsup b_n$ and assume $-\infty < b \leq c < \infty$. Let $B'_n := \{x \in B_n : f(x) < c_n\}$ and $d'_n := \mathrm{dist}(B'_n, A_n)$. If $d'_n \xrightarrow[n\to\infty]{} \infty$, then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $|df|(x_n) \xrightarrow[n\to\infty]{} 0$ and $x_n \in$ $\mathrm{cl}(f^{-1}([b - 1/n, c + 1/n]))$ for every n.

Proof. For every positive integer n we set set $T_n := d'_n/2$,

$$\varepsilon_n = (c_n - b_n)/2 + 3\delta_n/\mathrm{dist}(A_n, B_n),$$

where δ_n is chosen to be sufficiently small in order that $3\delta_n/\mathrm{dist}(A_n, B_n) \xrightarrow[n\to\infty]{} 0$. Then Theorem 2.3 provides the desired conclusion. □

Finally we present a result with the boundary split into a "low" and a "high" part. It is worth noting how this result compares to Theorem 2.3.

Theorem 2.6. *Let A, B and Γ form an abstract mountain pass structure in X and* $\operatorname{dist}(A,B)>0$. *Let again*

$$c := \inf_{\gamma\in\Gamma}\sup_{x\in\gamma} f(x), \quad b := \inf_{x\in B} f(x), \quad -\infty < b \quad \textit{and} \quad c < \infty.$$

Let $A' := \{x \in A : f(x) > b\}$ be the high part of the boundary and $d' :=$ $\operatorname{dist}(A',B)$. *Take*

(1) $T \in (0,d')$,
(2) $\varepsilon > (c-b)/T$,
(3) $\delta \in (0,[T\varepsilon-(c-b)]/2)$, $\delta < d'-T$ *and* $\delta < \varepsilon.\operatorname{dist}(B, A\setminus A')$.

Then there exists $x = x(T,\varepsilon,\delta) \in X$ such that

(i) $x \in \operatorname{cl}(f^{-1}([b-\delta,c+\delta]))$,
(ii) $|df|(x) \le \varepsilon$,
(iii) $\operatorname{dist}(x,A') \ge \delta$,
(iv) *either* $\operatorname{dist}(x,A\setminus A') \ge \operatorname{dist}(B,A\setminus A') - \delta/\varepsilon$ *or* $x \in \operatorname{cl}(f^{-1}([b+\delta,\infty)))$.

Proof. Let us assume the contrary, i.e. $|df|(x) > \varepsilon$ for every

$$x \in S := \operatorname{cl}(f^{-1}([b-\delta,c+\delta])) \cap \{y\in X : \operatorname{dist}(y,A') \ge \delta\} \cap (S_1 \cup S_2)$$

where

$$\begin{aligned} S_1 &:= \{y \in X : \operatorname{dist}(y, A\setminus A') \ge \operatorname{dist}(B,A\setminus A') - \delta/\varepsilon\},\\ S_2 &:= \operatorname{cl}(f^{-1}([b+\delta,\infty))). \end{aligned}$$

Note that $S_\mu \cap A = \emptyset$ where $\mu = \min\{\delta/3,\delta/\varepsilon\}$. Using this and the lower semicontinuity of $|df|$ we can find an open set Q containing S with $Q\cap A=\emptyset$ as well. Now we apply the deformation lemma to the so defined S, Q and ε. Let us choose a suboptimal path γ_δ, that is

$$\sup_{x\in\gamma_\delta} f(x) < c+\delta. \tag{11}$$

Denote $\gamma = \eta(\gamma_\delta, T) \in \Gamma$. Clearly

$$\sup_{x\in\gamma} f(x) \ge \sup_{x\in\gamma\cap B} f(x) \ge \inf_{x\in B} f(x) = b.$$

Let $y \in \gamma\cap B$. Then there exists $x\in\gamma_\delta$ such that $y = \eta(x,T)$. The following cases are possible:

Case 1. $\eta(x,\tau) \in S$ for every $\tau\in[0,T]$. Then (v) of the deformation lemma and (11) imply

$$f(\eta(x,T)) \le f(x) - \varepsilon.T \le c+\delta-\varepsilon.T \le b-\delta. \tag{12}$$

(the last inequality is based on (3)). Since $y \in B$, we have $f(y) \geq b$, a contradiction to (12).

Case 2. There exists $t \in [0, T]$ such that $\eta(x, t) \notin S$. The property (v) of η (cf. the deformation lemma) implies the existence of $\lambda_x \geq 0$ such that $z := \eta(x, \lambda_x) \notin S$ and

$$f(\eta(x,\tau)) \leq f(x) - \varepsilon.\tau \quad \text{for } \tau \in [0, \lambda_x], \tag{13a}$$

$$f(z) - f(\eta(x,\tau)) \geq \varepsilon.d(z, \eta(x,\tau)) \quad \text{for } \tau \geq \lambda_x, \tag{13b}$$

$$d(z, \eta(x,\tau)) \leq \tau - \lambda_x \quad \text{for } \tau \geq \lambda_x. \tag{13c}$$

Case 2.1. $\lambda_x \geq T$. According to (13a) we obtain a contradiction as in Case 1.

Case 2.2.1. $\lambda_x < T$ and $f(z) < b - \delta$. According to (13b) we have

$$f(y) = f(\eta(x, T)) \leq f(z) < b - \delta$$

which contradicts $y \in B$.

Case 2.2.2. $\lambda_x < T$ and $f(z) \geq b - \delta$. From (iv) of the deformation lemma and $x \in \gamma_\delta$ we obtain

$$b - \delta \leq f(z) = f(\eta(x, \lambda_x)) \leq f(x) < c + \delta.$$

Hence, $z \in \operatorname{cl}(f^{-1}([b - \delta, c + \delta]))$. Let us assume that $\operatorname{dist}(z, A') < \delta$. Then (13c) and the choice of δ yield

$$\begin{aligned} T &\geq T - \lambda_x \geq d(z, \eta(x,T)) = d(z,y) \geq \operatorname{dist}(z, B) \\ &\geq \operatorname{dist}(A', B) - \operatorname{dist}(z, A') > \operatorname{dist}(A', B) - \delta = d' - \delta > T, \end{aligned}$$

a contradiction. Thus

$$z \in \operatorname{cl}(f^{-1}([b - \delta, c + \delta])) \cap \{y \in X : \operatorname{dist}(y, A') \geq \delta\}.$$

But $z \notin S$, so $z \notin S_1 \cup S_2$. From (13b) we obtain

$$f(z) - f(y) = f(z) - f(\eta(x, T)) \geq \varepsilon.d(z, \eta(x, T)) = \varepsilon.d(z, y).$$

So, by the choice of δ and $z \notin S_1$, we have:

$$\frac{f(z) - f(y)}{\varepsilon} \geq d(z, y) \geq \operatorname{dist}(z, B) \geq \operatorname{dist}(A \setminus A', B) - \operatorname{dist}(z, A \setminus A') > \delta/\varepsilon.$$

Therefore $f(z) > f(y) + \delta \geq b + \delta$, i.e. $z \in S_2$, a contradiction to $z \notin S_1 \cup S_2$. Theorem 2.6 is thus proved. $\square$

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Manuscript received October 20, 1998

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