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ON SCHRÖDINGER EQUATION WITH PERIODIC POTENTIAL AND CRITICAL SOBOLEV EXPONENT

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1. Introduction

The main purpose of this paper is to establish the existence of a solution of the semilinear Schrödinger equation

$$-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1}$$

involving a critical Sobolev exponent $2^* = 2N/(N-2)$ with $N \geq 4$ and a subcritical nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$.

Throughout this paper it is assumed that

(A) The coefficients V and K are continuous and 1-periodic functions in each variable x_i, $i = 1, \ldots, N$. Moreover, we assume that $K \geq 0$ on $\mathbb{R}^N$.

In this case it is known that the operator $-\Delta + V$ on $L^2(\mathbb{R}^N)$ has a purely continuous spectrum consisting of closed disjoint intervals. In this paper we consider the case:

(B) 0 is in the spectral gap of the operator $-\Delta + V$.

There are many existence results in the case $K \equiv 0$ on $\mathbb{R}^N$ and we refer to the papers [4], [5], [9], [10], [13].

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In particular, Kryszewski–Szulkin [5] and Troestler–Willem [13] proved the existence of a solution using a generalized linking theorem which allowed a decomposition of $H^1(\mathbb{R}^N)$ into two infinite dimensional subspaces. This approach has been simplified by Pankov–Pflüger [9] by using the approximation technique with periodic functions. In this paper we apply this technique to obtain the existence result for the equation (1). The crucial point in the approach presented in our paper lies in the fact that the approximation technique of [9] can be combined with the method developed in our earlier paper [3] to determine the range of level sets of the energy functional for which the Palais–Smale condition holds. This allows us to obtain an approximating sequence of solutions by applying the Linking Theorem of Rabinowitz [10].

We assume that the nonlinearity f satisfies the following conditions:

(f_1) $f(x,u)$ is continuous on $\mathbb{R}^N\times\mathbb{R}$ and 1-periodic in each variable x_i, $i=1,\dots,N$.

(f_2) $|f(x,u)|\le C(1+|u|^{p-1})$ on $\mathbb{R}^N\times\mathbb{R}$ for some constants $C>0$ and $2<p<2N/(N-2)$,

(f_3) $f(x,u)=o(|u|)$ as $u\to 0$ uniformly in $x\in\mathbb{R}^N$,

(f_4) there exists a constant $\theta>2$ such that
$$0<\theta F(x,u)\le uf(x,u)$$
for all $u\ne 0$ and $x\in\mathbb{R}^N$, where $F(x,u)=\int_0^u f(x,s)\,dx$.

Our main existence result will be based on the following critical point theorem [10]:

Theorem A (Linking Theorem). *Let $X=Y\oplus Z$ be a Banach space with $\dim Y<\infty$. Let $R>r>0$ and $z\in Z$ be such that $\|z\|=r$. Define*
$$\begin{aligned} M&=\{u=y+tz:\|u\|\le R,\ t\ge 0,\ y\in Y\},\\ \partial M&=\{u=y+tz:y\in Y,\ \|u\|=R\ \ \textit{and}\ t\ge 0,\ \textit{or}\ \|u\|\le R\ \textit{and}\ t=0\},\\ N&=\{u\in Z:\|u\|=r\}.\end{aligned}$$
Let $I\in C^1(X,\mathbb{R})$ be such that
$$b=\inf_{u\in N}I(u)>a=\max_{u\in\partial M}I(u).\tag{$*$}$$
If I satisfies the $(\mathrm{PS})_c$ condition with
$$c=\inf_{\gamma\in\Gamma}\max_{u\in M}I(\gamma(u)),$$
where $\Gamma=\{\gamma\in C(M,X):\gamma_{|\partial M}=\mathrm{id}\}$, then c is a critical point of I.

In this work we always denote in a given Banach space X a weak convergence by "$\rightharpoonup$" and a strong convergence by "$\to$". The duality pairing between X and its dual X^* is denoted by $\langle\cdot,\cdot\rangle$. We say that a C^1-functional $F:X\to$

$\mathbb{R}$ satisfies the Palais–Smale condition (the $(\mathrm{PS})_c$-condition for short) if each sequence $\{u_m\} \subset X$ such that $F(u_m) \to c$ and $F'(u_m) \to 0$ in X^* is relatively compact in X.

2. Existence result in cubes

The existence of solutions of (1) will be obtained by approximating with solutions in cubes $Q_k \subset \mathbb{R}^N$, with length of edge k, $k \in \mathbb{N}$.

Let $H^1_{\mathrm{per}}(Q_k)$ be the space of $H^1(Q_k)$-functions consisting of k-periodic functions in x_i, $i = 1, \dots, N$. For simplicity we write $E_k = H^1_{\mathrm{per}}(Q_k)$.

To obtain a solution of (1), we first investigate the problem

$$(1_k) \qquad \begin{cases} -\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u) & \text{in } Q_k, \\ u \in E_k. \end{cases}$$

It is known that the operator $-\Delta + V$ on $L^2_{\mathrm{loc}}(Q_k)$ has a discrete spectrum with eigenvalues $\lambda_{k,1} \le \lambda_{k,2} \le \dots$ converging to ∞ as $k \to \infty$. Moreover, for each k the following minima

$$\gamma(k) = \min\{i : \lambda_{k,i} > 0\}$$

are finite and every eigenvalue $\lambda_{k,i}$ is contained in the spectrum of $-\Delta + V$ in the whole space $L^2(\mathbb{R}^N)$. This is a consequence of the Spectral Decomposition Theorem XIII.97 in [11]. Therefore, if $(-\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, stands for the spectral gap around 0, then $\lambda_{k,i} \notin (-\alpha, \beta)$ for every $k, i \in \mathbb{N}$. We denote by $\phi_{k,i}$ the corresponding eigenfunctions. We now observe that every eigenfunction $\phi \in E_k$ is, by periodicity, also in E_{mk} for each $m \in \mathbb{N}$. Consequently, every eigenvalue of $-\Delta + V$ on $L^2_{\mathrm{per}}(Q_k)$ is also an eigenvalue of this operator in $L^2_{\mathrm{per}}(Q_{mk})$, $m \in \mathbb{N}$.

To proceed further we define an orthogonal decomposition of E_k, by

$$E_k = Y_k \oplus Z_k, \quad \text{where } Y_k = \{\phi_{k,1}, \dots, \phi_{k,\gamma(k)-1}\}.$$

The solutions of the problem (1_k) will be found as critical points of the variational functional

$$J_k(u) = \frac{1}{2}\int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,dx - \frac{1}{2^*}\int_{Q_k} K(x)|u|^{2^*}\,dx - \int_{Q_k} F(x,u)\,dx.$$

By $\ell_k : E_k \to \mathbb{R}$ we denote the quadratic part of J_k, that is,

$$\ell_k(u) = \int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,dx.$$

The quadratic part ℓ_k is positive on Z_k and negative on Y_k. We can define a new scalar product $(\,\cdot\,,\,\cdot\,)_k$ on E_k with the corresponding norm $\|\cdot\|_k$ such that

$$\int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,dx = -\|u\|_k^2 \quad \text{for } u \in Y_k,$$
$$\int_{Q_k} (|\nabla u|^2 + V(x)u^2)\,dx = \|u\|_k^2 \quad \text{for } u \in Z_k.$$

Let $P_k : E_k \to Y_k$ and $T_k : E_k \to Z_k$ be orthogonal projections of E_k onto Y_k and Z_k, respectively. With the aid of these projections we can write the variational functional J_k in the form

$$J_k(u) = \frac{1}{2}(\|T_k u\|_k^2 - \|P_k u\|_k^2) - \frac{1}{2^*}\int_{Q_k} K(x)|u|^{2^*}\,dx - \int_{Q_k} F(x,u)\,dx.$$

According to our assumptions on f, the functional J_k is C^1 and

$$\langle J_k'(u), v\rangle = (T_k u, v)_k - (P_k u, v)_k - \int_{Q_k} K(x)|u|^{2^*-2}uv\,dx - \int_{Q_k} f(x,u)v\,dx.$$

We commence by establishing some technical lemmas.

Lemma 1. *The norm $\|\cdot\|_k$ is equivalent to the standard norm $\|\cdot\|_{H^1}$ in $H^1(Q_k)$, that is,*

$$a\|u\|_k \le \|u\|_{H^1} \le b\|u\|_k$$

for all $u \in E_k$, and some constants $a > 0$ and $b > 0$ independent of k.

For the proof we refer to Lemma 2 in [9].

Lemma 2. *There exist constants $\varrho > 0$ and $\alpha > 0$ independent of k such that $\inf_{u\in N_k} J_k(u) \ge \alpha$, where $N_k = \{z \in Z_k : \|z\|_k = \rho\}$.*

Proof. Let $z \in Z_k$, then

$$J_k(z) = \frac{1}{2}\|z\|_k^2 - \frac{1}{2^*}\int_{Q_k} K(x)|z|^{2^*}\,dx - \int_{Q_k} F(x,z)\,dx.$$

It follows from (f_2) and (f_3), that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|F(x,s)| \le \varepsilon s^2 + C_\varepsilon |s|^p$$

for all $s \in \mathbb{R}$. Applying the Sobolev embedding theorem we get that

$$\int_{Q_k} F(x,z)\,dx \le C(\varepsilon\|z\|_k^2 + C_\varepsilon\|z\|_k^p)$$

for some constant $C > 0$ independent of k. Consequently,

$$J_k(z) \ge \frac{1}{2}\|z\|_k^2 - \frac{\|K\|_\infty}{2^*}\|z\|_k^{2^*} - C(\varepsilon\|z\|_k^2 + C_\varepsilon\|z\|_k^p),$$

where $\|K\|_\infty = \sup_{x\in\mathbb{R}^N}|K(x)|$. Choosing $\varepsilon > 0$ and $\rho > 0$ sufficiently small, the result readily follows. □

In the next lemma we find the energy range of the functional J_k for which the Palais–Smale condition holds.

Lemma 3. *Let $\{u_n\}$ be a sequence in E_k such that $J_k'(u_n) \to 0$ and*

$$J_k(u_n) \to c_k \in \left(0, \frac{1}{N} \|K\|_\infty^{-(N-2)/2} S^{N/2}\right)$$

as $n \to \infty$, where S is the best Sobolev constant. Then $\{u_n\}$ is relatively compact in E_k.

Proof. In the first step of the proof we show that the sequence $\{u_n\}$ is bounded in E_k. Indeed, we have

$$\begin{aligned} (2)\quad c_k + o(1) &= J_k(u_n) - \frac{1}{2}\langle J_k'(u_n), u_n\rangle \\ &= \frac{1}{N}\int_{Q_k} K(x)|u_n|^{2^*}\,dx + \frac{1}{2}\int_{Q_k} u_n f(x,u_n)\,dx - \int_{Q_k} F(x,u_n)\,dx \\ &\geq \frac{1}{N}\int_{Q_k} K(x)|u_n|^{2^*}\,dx + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{Q_k} u_n f(x,u_n)\,dx. \end{aligned}$$

We now use the following estimates for f, that easily follow from assumptions (f_2)–(f_4), namely:

$$\begin{aligned} |f(x,u)|^2 &\leq Cuf(x,u) && \text{if } |u| \leq 1 \text{ and } x \in \mathbb{R}^N, \\ |f(x,u)|^{p'} &\leq C|u|^{(p-1)(p'-1)}|f(x,u)| = Cuf(x,u) && \text{if } |u| \geq 1 \text{ and } x \in \mathbb{R}^N, \end{aligned}$$

for some constant $C > 0$. Letting $B_n = \{x \in Q_k : |u_n(x)| \leq 1\}$ we derive from these estimates and (2) that

$$\begin{aligned} c_k \geq{}& \frac{1}{N}\int_{Q_k} K(x)|u_n(x)|^{2^*}\,dx \\ &+ C\left(\int_{B_n} |f(x,u_n)|^2\,dx + \int_{Q_k - B_n} |f(x,u_n)|^{p'}\,dx\right) + o(1) \end{aligned}$$

for some constant $C > 0$. Hence

$$\int_{B_n} |f(x,u_n)|^2\,dx \leq \frac{c_k}{C} \quad \text{and} \quad \int_{Q_k - B_n} |f(x,u_n)|^{p'}\,dx \leq \frac{c_k}{C}$$

for all n. Let $y_n = P_k u_n$ and $z_n = T_k u_n$. Since $\langle J_k'(u_n), y_n\rangle = \varepsilon_n \|y_n\|_k$, with $\varepsilon_n \to 0$, we deduce from the Hölder inequality that

$$\begin{aligned} \|y_n\|_k^2 ={}& -\int_{Q_k} K(x)|u_n|^{2^*-2}u_n y_n\,dx - \int_{Q_k} f(x,u_n)y_n\,dx + \varepsilon_n\|y_n\|_k \\ \leq{}& C_1\|K\|_\infty^{1/2^*}\left(\int_{Q_k} K(x)|u_n|^{2^*}\,dx\right)^{(2^*-1)/2^*}\|y_n\|_k + \varepsilon_n\|y_n\|_k \\ &+ \left(\frac{c_k}{C}\right)^{1/2}\|y_n\|_{L^2} + \left(\frac{c_k}{C}\right)^{1/p'}\|y_n\|_{L^p} \end{aligned}$$

for some constant $C_1 > 0$. This implies that $\{\|y_n\|_k\}$ is bounded. In a similar manner we show that $\{\|z_n\|_k\}$ is bounded. Consequently, $\{\|u_n\|_k\}$ is bounded and we may assume that $u_n \rightharpoonup u$ in E_k. Let $v_n = u_n - u$. According to Brézis–Lieb lemma [2] we have

$$J_k(u_n) = J_k(u) + J_k(v_n) + o(1)$$

and

$$\langle J_k'(u_n), u_n\rangle = \langle J_k'(u), u\rangle + \langle J_k'(v_n), v_n\rangle + o(1) = o(1).$$

Hence

$$c_k + o(1) \geq \frac{1}{2}\int_{Q_k} |\nabla v_n|^2\,dx - \frac{1}{2^*}\int_{Q_k} K(x)|v_n|^{2^*}\,dx. \tag{3}$$

Since $v_n \to 0$ in $L^2(Q_k)$, applying the Sobolev inequality (see [7, p. 69, formula 2.17]) we get

$$\int_{Q_k} |\nabla v_n|^2 dx = \int_{Q_k} K(x)|v_n|^{2^*}\,dx + o(1) \leq \|K\|_\infty \bigg(S^{-1}\int_{Q_k} |\nabla v_n|^2\,dx\bigg)^{2^*/2}. \tag{4}$$

If $\int_{Q_k} |\nabla v_n|^2\,dx \to l > 0$, then we deduce from (4) that

$$l \geq \|K\|_\infty^{-(N-2)/2} S^{N/2},$$

which combined with (3) and (4) gives

$$c_k \geq \frac{l}{N} \geq \frac{\|K\|_\infty^{-(N-2)/2}}{N} S^{N/2}$$

which is impossible. Therefore $l = 0$ and the result follows. $\square$

To verify condition $(*)$ of Theorem A we use a truncated Talenti function (see [12]). Let

$$\psi_\varepsilon(x) = \bigg(\frac{\sqrt{N(N-2)}\varepsilon}{\varepsilon^2 + |x|^2}\bigg)^{(N-2)/2}, \quad \varepsilon > 0.$$

Let $B(x_\circ, 2r) \subset Q_1$, where $x_\circ$ is the centre of the cube Q_1. By $\zeta \in C_\circ^1(\mathbb{R}^N)$ we denote the function that satisfies $\zeta(x) = 1$ in $B(x_\circ, r)$ and $\zeta(x) = 0$ on $\mathbb{R}^N - B(x_\circ, 2r)$ and $0 \leq \zeta(z) \leq 1$ on $\mathbb{R}^N$. We set $\widetilde{\varphi}_\varepsilon(x) = \zeta(x)\psi_\varepsilon(x)$ and extending as a periodic function we have $\widetilde{\varphi}_\varepsilon \in H^1_{\mathrm{per}}(Q_k)$. For $\varphi_\varepsilon(x) = (1/k^N)\widetilde{\varphi}_\varepsilon(x)$ we define

$$Q_k(\varepsilon) = \{y + tT_k\varphi_\varepsilon : y \in Y_k,\ t \geq 0\}.$$

Lemma 4. *There exists $\varepsilon_\circ > 0$ such that $T_k\varphi_\varepsilon \not\equiv 0$ for $0 < \varepsilon \le \varepsilon_\circ$.*

The proof is identical to that of Lemma 5 in [3].

In the sequel we need the following asymptotic estimates of norms of φ_ε:

$$\|\nabla\varphi_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \tag{5}$$

$$\|\varphi_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^{N}), \tag{6}$$

$$\|\varphi_\varepsilon\|_2^2 = \begin{cases} K_1\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \ge 5, \\ K_1\varepsilon^2|\log\varepsilon^2| + O(\varepsilon^2) & \text{if } N = 4, \end{cases} \tag{7}$$

$$\|\varphi_\varepsilon\|_1 \le K_2\varepsilon^{(N-2)/2}, \tag{8}$$

and

$$\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1} \le K_3\varepsilon^{(N-2)/2}, \tag{9}$$

for some constants $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ (see [1]).

To proceed further we introduce two additional assumptions:

(K) $0 < K(x_\circ) = \max_{x\in Q_1} K(x)$ and $K(x) = K(x_\circ) + O(|x - x_\circ|)$ for x near $x_\circ$ and $K(x)$ is bounded from below on Q_1 by a positive constant.

(f_5) there exists a function $\overline{f}$ such that

$$f(x,u) \ge \overline{f}(u) \text{ a.e. for } x \in \omega \text{ and } u \ge 0,$$

where ω is some nonempty open set in Q_1 and the function $\overline{F}(u) = \int_0^u \overline{f}(s)\,dx$ satisfies

$$\lim_{\varepsilon\to 0} \varepsilon^{\min((N+2)/2,p(N-2)/2)} \int_0^{\varepsilon^{-1}} \overline{F}\bigg[\bigg(\frac{\varepsilon^{-1/2}}{1+s^2}\bigg)^{(N-2)/2}\bigg] s^{N-1}\,dx = \infty.$$

If $\overline{F}(s) = |s|^p$, then this condition is satisfied.

The assumption (K) will be only used in the proof of Lemma 5 below. The fact that K attains its maximum at the centre of the cube Q_1 is not essential. We introduce it to have a simple construction of a periodic cut-off function ϕ_ε.

Lemma 5. *We have*

$$\sup_{u\in Q_k(\varepsilon)} J_k(u) < \frac{1}{N}\|K\|_\infty^{-(N-2)/2} S^{N/2}.$$

Proof. We follow some ideas from the paper [3] (see Lemma 6). First we observe that if $u \in E_k$ with $u \not\equiv 0$, then

$$\begin{aligned} J_k(su) &= \frac{s^2}{2}\int_{Q_k} \big(|\nabla u|^2 + V(x)u^2\big)\,dx - \frac{s^{2^*}}{2^*}\int_{Q_k} K(x)|u|^{2^*}\,dx - \int_{Q_k} F(x,su)\,dx \\ &\le \frac{s^2}{2}\int_{Q_k} \big(|\nabla u|^2 + V(x)u^2\big)\,dx - \frac{s^{2^*}}{2^*}\int_{Q_k} K(x)|u|^{2^*}\,dx. \end{aligned}$$

From this estimate we deduce that $\lim_{s\to\infty} J_k(su) = -\infty$. Hence there exists $s_\varepsilon \geq 0$ such that

$$J_k(s_\varepsilon u) = \sup_{t\geq 0} J_k(tu).$$

We may assume that $s_\varepsilon > 0$ and it satisfies

$$s_\varepsilon \int_{Q_k} \big(|\nabla u|^2 + Vu^2\big)\,dx - s_\varepsilon^{2^*-1} \int_{Q_k} K|u|^{2^*}\,dx - \int_{Q_k} uf(x, s_\varepsilon u)\,dx = 0.$$

This equation implies that

$$s_\varepsilon \leq \bigg(\frac{\int_{Q_k}(|\nabla u|^2 + Vu^2)\,dx}{\int_{Q_k} K|u|^{2^*}\,dx}\bigg)^{(N-2)/4} = A.$$

Since the function

$$s \to \frac{s^2}{2}\int_{Q_k}(|\nabla u|^2 + Vu^2)\,dx - \frac{s^{2^*}}{2^*}\int_{Q_k} K|u|^{2^*}\,dx$$

is increasing in the interval $[0, A]$ we see that

$$J_k(su) \leq \frac{1}{N}\bigg[\frac{\int_{Q_k}(|\nabla u|^2 + Vu^2)\,dx}{(\int_{Q_k} K|u|^{2^*}\,dx)^{2/2^*}}\bigg]^{N/2} - \int_{Q_k} F(x, s_\varepsilon u)\,dx. \tag{10}$$

For simplicity we may assume that $K(0) = \max_{x\in Q_1} K(x)$, as J_k is translation invariant. For $u = u^- + tT_k\phi_\varepsilon \in Q_k(\varepsilon)$, with $u^- = P_k u$ and $\|u\|_{2^*,K} = 1$, we write

$$\begin{aligned}\int_{Q_k}(|\nabla u|^2 + Vu^2)\,dx = -\,\|u^-\|_k^2 + \frac{\|\nabla\big(tT_k\varphi_\varepsilon\big)\|_2^2}{\|tT_k\varphi_\varepsilon\|_{2^*,K}^2}\|tT_k\varphi_\varepsilon\|_{2^*,K}^2 \\ + t^2\int_{Q_k} V(T_k\varphi_\varepsilon)^2\,dx.\end{aligned} \tag{11}$$

As in [3] (see formula (20) there) we have the following estimate

$$\bigg|\int_{Q_k} K(|T_k\varphi_\varepsilon|^{2^*} - |\varphi_\varepsilon|^{2^*})\,dx\bigg| \leq C_2\varepsilon^{N-2}$$

for some constant $C_2 > 0$. Using this, (K) and (6) we get

$$\begin{aligned}\|T_k\varphi_\varepsilon\|_{2^*,K}^2 &= (\|T_k\varphi_\varepsilon\|_{2^*,K}^{2^*})^{2/2^*} = (\|\varphi_\varepsilon\|_{2^*,K}^{2^*} + O(\varepsilon^{N-2}))^{(N-2)/N} \\ &= (K(0)S^{N/2} + O(\varepsilon) + O(\varepsilon^{N-2}))^{(N-2)/N} \\ &= K(0)^{(N-2)/N}S^{(N-2)/2} + O(\varepsilon^{(N-2)/N}).\end{aligned} \tag{12}$$

As in [3] (see p. 288) we can derive the following estimate

$$\bigg|\int_{Q_k}|\nabla\varphi_\varepsilon|^2\,dx - \int_{Q_k}|\nabla(T_k\varphi_\varepsilon)|^2\,dx\bigg| = O(\varepsilon^{N-2}). \tag{13}$$

Inserting (12) and (13) into (11) and using (5) we get

$$(14)\qquad \int_{Q_k}(|\nabla u|^2+Vu^2)\,dx = -\|u\|_k^2+(K(0)^{-(N-2)/N}S + O(\varepsilon^{N-2}))\|tT_k\varphi_\varepsilon\|^2_{2^*,K}+t^2\int_{Q_k}V(T_k\varphi_\varepsilon)^2\,dx.$$

As in [3] (see (25), (26) and (27) there) we derive the estimate

$$(15)\qquad \begin{aligned} 1 &= \|u\|^{2^*}_{2^*,K}\\ &\ge \|tT_k\varphi_\varepsilon\|^{2^*}_{2^*,K}+\frac{1}{2}\|u^-\|^{2^*}_{2^*,K}-C_4t^{2^*}\varepsilon^{(N-2)N/(N+2)}\\ &\ge t^{2^*}\|\varphi_\varepsilon\|^{2^*}_{2^*,K}-C_3t^{2^*}\varepsilon^{N-2}-C_4t^{2^*}\varepsilon^{(N-2)N/(N+2)}+\frac{1}{2}\|u^-\|^{2^*}_{2^*,K}\end{aligned}$$

for some constants $C_3>0$ and $C_4>0$. This estimate implies that t is bounded. We now distinguish two cases:

(i) $\|u^-\|^{2^*}_{2^*,K}\le 2C_4t^{2^*}\varepsilon^{(N-2)N/(N+2)}$ or
(ii) $\|u^-\|^{2^*}_{2^*,K}> 2C_4t^{2^*}\varepsilon^{(N-2)N/(N+2)}$.

In the first case we have (see [3] p. 289 formula (26))

$$(16)\qquad \|tT_k\varphi_\varepsilon\|^2_{2^*,K}\le 1+C_5\varepsilon^{N-2}$$

for some constant $C_5>0$. If the case (ii) prevails, then by the first part of the inequality (15) we have

$$(17)\qquad \|tT_k\varphi_\varepsilon\|^{2^*}_{2^*,K}\le 1.$$

Since s_ε satisfies

$$\int_{Q_k}(|\nabla(u^-+tT_k\varphi_\varepsilon)|^2+V(x)(u^-+tT_k\varphi_\varepsilon)^2)\,dx - s_\varepsilon^{2^*-2}\int_{Q_k}K(x)|u^-+tT_k\varphi_\varepsilon|^{2^*}\,dx - \int_{Q_k}\frac{(u^-+tT_k\varphi_\varepsilon)f(x,s_\varepsilon(u^-+tT_k\varphi_\varepsilon))}{s_\varepsilon}\,dx=0$$

we get that

$$\lim_{\varepsilon\to0}\int_{Q_k}|\nabla(u^-+tT_k\varphi_\varepsilon)|^2+V(x)|u^-+tT_k\varphi_\varepsilon|^2\,dx\ge\lim_{\varepsilon\to0}s_\varepsilon^{2^*-2}.$$

In both cases (16) and (17) we deduce from (14) that

$$\lim_{\varepsilon\to0}s_\varepsilon^{2^*-2}\le K(0)^{-(N-2)/N}S$$

and s_ε is bounded for small $\varepsilon > 0$. We now estimate the integral involving F:

$$\begin{aligned}(18)\quad &\left|\int_{Q_k} F(x,u^-+tT_k\varphi_\varepsilon)\,dx-\int_{Q_k} F(x,u^-)\,dx-\int_{Q_k} F(x,tT_k\varphi_\varepsilon)\,dx\right|\\ &\qquad=\left|\int_{Q_k}\left[\int_0^{tT_k\varphi_\varepsilon} f(x,u^-+s)\,ds-\int_0^{tT_k\varphi_\varepsilon} f(x,s)\,ds\right]dx\right|\\ &\qquad\le C_6\bigg[\int_{Q_k}|(tT_k\varphi_\varepsilon)|(1+|u^-+tT_k\varphi_\varepsilon|^{p-1})\,dx\\ &\qquad\quad+\int_{Q_k}|(tT_k\varphi_\varepsilon)|(1+|tT_k\varphi_\varepsilon|^{p-1})\,dx\bigg]\\ &\qquad\le C_6\left[\int_{Q_k}(|u^-|^{p-1}|tT_k\varphi_\varepsilon|+|tT_k\varphi_\varepsilon|+|tT_k\varphi_\varepsilon|^p)\,dx\right].\end{aligned}$$

We deduce from the condition $\|(u^-+tT_k\varphi_\varepsilon)\|_{2^*,K}=1$ that $\|u^-\|_\infty$ is uniformly bounded. As in [3] (see formula (20) there) we have

$$\begin{aligned}\left|\int_{Q_k}(|T_k\varphi_\varepsilon|^p-|\varphi_\varepsilon|^p)\,dx\right| &\le C_7(\|\varphi_\varepsilon\|_{p-1}^{p-1}\|P_k\varphi_\varepsilon\|_\infty+\|P_k\varphi_\varepsilon\|_p^p)\\ &\le(\varepsilon^{N-(N-2)(p-1)/2}\varepsilon^{(N-2)/2}+\varepsilon^{p(N-2)/2})\\ &=O(\varepsilon^{(N-2)/2}).\end{aligned}$$

Therefore it follows from (18) that

$$\left|\int_{Q_k}[F(x,u)-F(x,u^-)-F(x,tT_k\varphi_\varepsilon)]\,dx\right|\le C_8(\varepsilon^{(N-2)/2}+\varepsilon^{N-p(N-2)/2}).$$

Consequently,

$$\begin{aligned}(19)\quad &\int_{Q_k}F(x,s_\varepsilon(u^-+tT_k\varphi_\varepsilon))\,dx\\ &\qquad\ge\int_{Q_k}F(x,s_\varepsilon u^-)\,dx+\int_{Q_k}F(x,s_\varepsilon tT_k\varphi_\varepsilon)\,dx+O(\varepsilon^{(N-2)/2}).\end{aligned}$$

It then follows from (14) and (19) (taking into account both cases (16) and (17)) that

$$\begin{aligned}(20)\quad &J_k(s_\varepsilon(u^-+tT_k\varphi_\varepsilon))\\ &\quad\le\frac{1}{N}K(0)^{-(N-2)/2}S^{N/2}\\ &\qquad+O(\varepsilon^{(N-2)/2})+O(\varepsilon^{N-p(N-2)/2})-\int_{Q_k}F(x,s_\varepsilon u)\,dx\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{N}K(0)^{-(N-2)/2}S^{N/2} + O(\varepsilon^{(N-2)/2}) + O(\varepsilon^{N-p(N-2)/2}) \\
&\quad - \int_{Q_k} F(x, s_\varepsilon u^-)\,dx - \int_{Q_k} F(x, s_\varepsilon t T_k\varphi_\varepsilon)\,dx \\
&\le \frac{1}{N}K(0)^{-(N-2)/2}S^{N/2} + O(\varepsilon^{(N-2)/2}) \\
&\quad + O(\varepsilon^{N-p(N-2)/2}) - \int_{Q_k} F(x, s_\varepsilon t T_k\varphi_\varepsilon)\,dx.
\end{aligned}
$$

We now observe that

$$
\begin{aligned}
(21)\quad &\left|\int_{Q_k} (F(x, s_\varepsilon t T_k\varphi_\varepsilon) - F(x, s_\varepsilon t\varphi_\varepsilon))\,dx\right| \\
&\le \int_{Q_k}\left|\int_{s_\varepsilon t\varphi_\varepsilon}^{s_\varepsilon t T_k\varphi_\varepsilon} f(x,s)\,ds\right| dx \le C(\|T_k\varphi_\varepsilon\|_2^2 + \|T_k\varphi_\varepsilon\|_p^p) = o(\varepsilon^{(N-2)/2}).
\end{aligned}
$$

Therefore by (20), (21) and with the aid of assumption (f_5) we get

$$
\begin{aligned}
J_k(s(u^- + tT_k\varphi_\varepsilon)) &\le \frac{1}{N}K(0)^{-(N-2)/2}S^{N/2} + O(\varepsilon^{(N-2)/2}) \\
&\quad + O(\varepsilon^{N-p(N-2)/2}) - \int_{Q_k} \overline{F}(s_\varepsilon t\varphi_\varepsilon)\,dx \\
&\le K(0)^{-(N-2)/2}S^{N/2} + O(\varepsilon^{(N-2)/2}) + O(\varepsilon^{N-p(N-2)/2}) \\
&\quad - \int_{B(0,R)} \overline{F}\bigg(\frac{A\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}\bigg)\,dx.
\end{aligned}
$$

We now observe that assumption (f_5) implies that

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{(N-2)/2}} \int_{B(0,R)} \overline{F}\bigg(\frac{A\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}\bigg)\,dx = \infty
$$

and

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-p(N-2)/2}} \int_{B(0,R)} \overline{F}\bigg(\frac{A\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}\bigg)\,dx = \infty.
$$

From this we deduce that

$$
J_k(s(u^- + tT_k\varphi_\varepsilon)) < \frac{S^{N/2}}{N}K(0)^{-(N-2)/2}. \qquad \square
$$

3. Main existence result

First we establish the existence result for the problem (1_k).

Lemma 6. *Let $M_k(\varepsilon) = \{y + tT_k\varphi_\varepsilon : \|y + tT_k\varphi_\varepsilon\|_k \le R,\ t \ge 0,\ y \in Y_k\}$, then for $R > 0$ sufficiently large*

$$
c_k = \inf_{h\in\Gamma_k} \sup_{u\in M_k(\varepsilon)} J_k(h(u))
$$

are critical values of J_k.

Proof. Let ρ be a constant from Lemma 2. We claim that for $R > \rho$ sufficiently large $\sup_{u\in\partial M_k(\varepsilon)} J_k(u) = 0$. If $u \in \partial M_k(\varepsilon)$ and $t = 0$, then $J_k(u) \le 0$. So let $R = \|y + tT_k\varphi_\varepsilon\|_k$, with $t > 0$. It follows from assumptions (f_2)–(f_4) that for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$F(x,u) \ge -\eta u^2 + C_\eta |u|^\theta$$

with $2 < \theta < 2^*$. This implies that

$$\int_{Q_k} F(x, y + tT_k\varphi_\varepsilon)\,dx \ge -\eta\|y\|_2^2 - \eta t^2\|T_k\varphi_\varepsilon\|_2^2 + C_\eta\|y + tT_k\varphi_\varepsilon\|_\theta^\theta.$$

By the Sobolev inequality [7] we have

$$\begin{aligned} J_k(y + tT_k\varphi_\varepsilon) \le &-\frac{1}{2}\|y\|_k^2 + \eta C\|y\|_k^2 + \frac{1}{2}t^2\|T_k\varphi_\varepsilon\|_k^2 + C\eta t^2\|T_k\varphi\|_k^2 \\ &- C_\eta\|y + tT_k\varphi_\varepsilon\|_\theta^\theta - \frac{m}{2^*}\|y + tT_k\varphi_\varepsilon\|_{2^*}^{2^*}, \end{aligned}$$

for some constant $C > 0$ and $m = \inf_{x\in\mathbb{R}^N} K(x)$. We now observe that $X_k = Y_k \oplus \mathbb{R}T_k\varphi_\varepsilon$ is continuously embedded in $L^q(Q_k)$ for $2 \le q \le 2^*$ and there exists a continuous projection $\Pi_k : X_k \to \mathbb{R}T_k\varphi_\varepsilon$ such that

$$\|tT_k\varphi_\varepsilon\|_q \le \|\Pi_k\|_q\|y + tT_k\varphi_\varepsilon\|_q \quad \text{and} \quad \|\Pi_k\|_q \ge 1.$$

Choosing η so that $\eta C = 1/4$ we get

$$J_k(y + tT_k\varphi_\varepsilon) \le -\frac{1}{4}\|y\|_k^2 + \frac{3}{4}\|tT_k\varphi_\varepsilon\|_k^2 - C_1(t^\theta\|T_k\varphi_\varepsilon\|_\theta^\theta + t^{2^*}\|T_k\varphi_\varepsilon\|_{2^*}^{2^*}),$$

where $C_1 > 0$ is a constant depending on $\|\Pi_k\|_q$, $\|\Pi_k\|_{2^*}$, m, N and C_η. Consequently, we see that $J_k(y + tT_k\varphi_\varepsilon) \to -\infty$ as $\|y + tT_k\varphi_\varepsilon\|_k \to \infty$ and our claim follows. We now observe that, by Lemma 5, $c_k < (S^{N/2}/N)\|K\|_\infty^{-(N-2)/2}$ and by the virtue of Lemma 3 the Palais–Smale condition holds at the level c_k. Therefore the result follows from Theorem A. □

According to Lemma 6 for each $k \ge 1$ we obtain a solution $u_k \in H^1_{\mathrm{per}}(Q_k)$. Since

$$c_k \le c_{k-1} \le c_1 < \frac{1}{N}\|K\|_\infty^{-(N-2)/2} S^{N/2}$$

we can repeat the argument of the proof of Lemma 3 to establish a uniform bound for the norms $\|u_k\|_k$.

Lemma 7. *Critical points u_k of J_k with $J_k(u_k) = c_k$ satisfy the estimate $\|u_k\|_k \le C$ for some constant independent of k.*

Lemma 8. *There exists $\varepsilon_1 > 0$ independent of k such that $\|u_k\|_k \geq \varepsilon_1$ and $\|u\|_{H^1} \geq \varepsilon_1$ hold for every nontrivial critical points u_k of J_k and u of J. Furthermore, there exists $\varepsilon_2 > 0$ independent k such that $J_k(u_k) \geq \varepsilon_2$ and $J(u) \geq \varepsilon_2$ for every nontrivial critical points u_k of J_k and u of J.*

Proof. As in Lemma 4 in [9] we check that $|\ell_k(u_k)| \geq C\|u_k\|_k^2$ and $\|u_k\|_k^2 \geq C_1$ for some constants $C > 0$ and $C_1 > 0$ independent of k. Since

$$\begin{aligned} J_k(u_k) &\geq \frac{1}{2} l_k(u_k) - \frac{1}{\theta}\int_{Q_k} f(x,u_k)u_k\,dx - \frac{1}{2^*}\int_{Q_k} K|u|^{2^*}\,dx \\ &= \frac{1}{2}\bigg(l_k(u_k) - \int_{Q_k} f(x,u_k)u_k\,dx - \int_{Q_k} K|u_k|^{2^*}\,dx\bigg) \\ &\quad + \bigg(\frac{1}{2} - \frac{1}{\theta}\bigg)\int_{Q_k} f(x,u_k)u_k\,dx + \bigg(\frac{1}{2} - \frac{1}{2^*}\bigg)\int_{Q_k} K|u_k|^{2^*}\,dx \\ &\geq s\ell_k(u_k) \geq sC\|u_k\|_k^2, \end{aligned}$$

where $s = \min(1/2 - 1/\theta, 1/N)$ and the assertion concerning J_k follows. The same argument applies to J. □

We need the following modification of the Concentration–Compactness Lemma [6], whose proof can be found in [9] (see Lemma 5 there).

Lemma 9. *Let Q_n be the cube of the edge length $l_n \to \infty$ as $n \to \infty$ centred at the origin, and $K_r(\xi)$ be the closed cube with the edge length r centred at the point ξ. Let $\{u_n\} \subset H^1_{\mathrm{loc}}(\mathbb{R}^N)$ be sequence of l_n-periodic periodic functions such that $\|u_n\|_{H^1(Q_n)} \leq C$ for some constant $C > 0$ independent of n. Suppose that there exists $r > 0$ such that*

$$\liminf_{n\to\infty} \sup_{\xi} \int_{K_r(\xi)} |u_n|^2\,dx = 0.$$

Then $\|u_n\|_{L^q(Q_n)} \to 0$ as $n \to \infty$ for $q \in (2, 2N/(N-2))$.

Lemma 10. *Let $\{u_k\} \subset E_k$ be a sequence such that*

$$J_k(u_k) = c_k < \frac{1}{N}\|K\|_\infty^{-(N-2)/2} S^{N/2}$$

and $J_k'(u_k) \to 0$ as $k \to \infty$. Then the following alternative holds: either

(i) *$\|u_k\|_k \to 0$ as $k \to \infty$, or*

(ii) *there exist numbers $r, \eta > 0$ and a sequence of points $\{\xi_k\} \subset \mathbb{R}^N$ such that*

$$\lim_{k\to\infty} \int_{K_r(\xi_k)} u_k^2\,dx \geq \eta.$$

Proof. Suppose that (ii) does not hold. Then by virtue of Lemma 9 we have that

$$\|u_k\|_{L^q(Q_k)} \to 0 \quad \text{as } k \to \infty \tag{22}$$

for $2 < q < 2N/(N-2)$. Therefore, by (f_2) and (f_3) we have

$$\int_{Q_k} f(x,u_k)u_k\,dx \to 0 \quad \text{and} \quad \int_{Q_k} F(x,u_k)\,dx \to 0 \tag{23}$$

as $k \to \infty$. As in Lemma 3 we show that $\{u_k\}$ is uniformly bounded in H^1-norm. First we claim that

$$\int_{Q_k} V u_k^2\,dx \to 0 \quad \text{as } k \to \infty. \tag{24}$$

In fact, we have $Q_k = \bigcup_{i=1}^{k^N} Q_k^i$, $Q_k^i \cap Q_k^j = \emptyset$ if $i \neq j$, where Q_k^i, $i = 1,\dots,k^N$ are cubes with the length of edges 1. Since V is 1-periodic, we have $V|_{Q_k^i} = V|_{Q_k^j}$, $i,j = 1,\dots,k^N$. Therefore it follows from the Hölder inequality that

$$\begin{aligned}
\biggl|\int_{Q_k} V(x)u_k^2\,dx\biggr| &\leq \sum_{i=1}^{k^N}\biggl|\int_{Q_k^i} V(x)u_k^2\,dx\biggr| \\
&\leq \sum_{i=1}^{k^N}\biggl(\int_{Q_k^i}|V|^{p/(p-2)}\,dx\biggr)^{(p-2)/p}\biggl(\int_{Q_k^i}|u_k|^p\,dx\biggr)^{2/p} \\
&= \biggl(\int_{Q_k^1}|V|^{p/(p-2)}\,dx\biggr)^{(p-2)/2}\sum_{i=1}^{k^N}\biggl(\int_{Q_k^i}|u_k|^p\,dx\biggr)^{2/p},
\end{aligned}$$

where $2 < p < 2^*$. Since (see [7, p. 66 formula 2.10])

$$\|u_k\|_{L^p(Q_k^i)} \leq A\|u_k\|_{L^2(Q_k^i)}^{1-\sigma}\|u_k\|_{H^1(Q_k^i)}^{\sigma},$$

where $\sigma = N(p-2)/2p$, $0 < \sigma < 1$ and a constant $A > 0$ depends only on p and N, we have

$$\begin{aligned}
\biggl(\int_{Q_k^i}|u_k|^p\,dx\biggr)^{2/p} &\leq A^2\|u_k\|_{L^2(Q_k^i)}^{2(1-\sigma)}\|u_k\|_{H^1(Q_k^i)}^{2\sigma-2}\int_{Q_k^i}(|\nabla u_k|^2 + u_k^2)\,dx \\
&\leq A^2\sup_k\biggl(\int_{Q_1(\psi_k)} u_k^2\biggr)^{1-\sigma}\sup_k\|u_k\|_{H^1(Q_k)}^{2\sigma-2}\int_{Q_k^i}(|\nabla u_k|^2 + u_k^2)\,dx,
\end{aligned}$$

where $Q_1(\psi_k)$ is a cube with centre at ψ_k and the length of edge 1. Consequently,

$$\begin{aligned}
&\sum_{i=1}^{k^N}\biggl(\int_{Q_k^i}|u_k|^p\,dx\biggr)^{2/p} \\
&\qquad\leq A^2\sup_k\biggl(\int_{Q_1(\psi_k)} u_k^2\,dx\biggr)^{1-\sigma}\|u_k\|_{H^1(Q_k)}^2\sup_k\|u_k\|_{H^1(Q_k)}^{2\sigma-2}.
\end{aligned}$$

The right hand side of this inequality goes to 0 by the assumption and the fact that $\|u_k\|_{H^1(Q_k)}$ is bounded uniformly in k. Thus (24) readily follows. Since

$$\begin{aligned}\int_{Q_k} |u_k|^2\,dx &= \sum_{i=1}^{k^N} \int_{Q_k^i} |u_k|^2\,dx \\ &\le \sum_{i=1}^{k^N} |Q_k^i|^{(p-2)/2} \biggl(\int_{Q_k^i} |u_k|^2\,dx\biggr)^{2/p} = \sum_{i=1}^{k^N} \biggl(\int_{Q_k^i} |u_k|^p\,dx\biggr)^{2/p},\end{aligned}$$

we see that $u_k \to 0$ in $L^2(Q_k)$ as $k \to \infty$. Next we prove that

$$\int_{Q_k} K(x)|u_k|^{2^*}\,dx \to 0 \quad \text{as } k \to 0. \tag{25}$$

Argueing by contradiction suppose that $\int_{Q_k} K|u_k|^{2^*}\,dx \to \ell$ as $k \to \infty$. Since u_k satisfies (1_k) and (23), (24) hold we see that

$$\int_{Q_k} |\nabla u_k|^2\,dx = \int_{Q_k} K(x)|u_k|^{2^*}\,dx + o(1) \tag{26}$$

and consequently

$$c_k = \frac{1}{N}\int_{Q_k} K(x)|u_k|^{2^*}\,dx. \tag{27}$$

Since $\|u_k\|_{L^2(Q_k)} \to 0$ as $k \to \infty$, by the Sobolev embedding theorem we have

$$\int_{Q_k} |\nabla u_k|^2\,dx + o(1) \ge S\|u_k\|_{2^*}^2 \ge S\|K\|_\infty^{-2/2^*} (K(x)|u_k|^{2^*}\,dx)^{2/2^*}.$$

Combining this with (26) we derive that

$$\ell \ge S^{N/2}\|K\|_\infty^{-(N-2)/2}.$$

This and (27) imply that $\lim_{k\to\infty} c_k \ge (1/N)\|K\|_\infty^{-(N-2)/2} S^{N/2}$ which is impossible. From the fact that u_k satisfies (1_k) we deduce that

$$\|z_k\|_k^2 = \int_{Q_k} K(x)|u_k|^{2^*-2}u_k z_k\,dx + \int_{Q_k} f(x,u_k)z_k\,dx \tag{28}$$

and

$$\|y_k\|_k^2 = -\int_{Q_k} K(x)|u_k|^{2^*-2}u_k y_k\,dx - \int_{Q_k} f(x,u_k)y_k\,dx, \tag{29}$$

where $z_k = T_k u_k$, $y_k = P_k u_k$. Using (23), (24) and (25) we deduce from (28) and (29) that $|u_k\|_k \to 0$, that is (i) holds. $\square$

We are now in a position to establish the main existence result.

THEOREM B. *Suppose that* (f_1)–(f_2)*,* (A)*,* (B) *and* (K) *hold. Then problem* (1) *has a nontrivial solution.*

PROOF. Let $\{u_k\}$ be the sequence obtained in Lemma 6. By virtue of Lemmas 7 and 8 the sequence of norms $\|u_k\|_k$ is bounded uniformly from above and below by positive constants. Then by Lemma 10, we have

$$\|u_k\|^2_{L^2(K_r(\xi_k))} \geq \frac{\eta}{2}.$$

Hence we can find a sequence $\{b_k\} \subset \mathbb{Z}^N$ and a number $s > 0$ such that the sequence $\{\widetilde{u}_k\}$ defined by $\widetilde{u}_k(x) = u_k(x + b_k)$ satisfies

$$\|\widetilde{u}_k\|_{L^2(K_s(0))} \geq \frac{\eta}{2}. \tag{30}$$

Since V, K and f are translation invariant, we have $J_k(\widetilde{u}_k) = J_k(u_k)$ and $\|{J_k}'(\widetilde{u}_k)\| = \|{J_k}'(u_k)\|$. By virtue of Lemma 7 $\{\widetilde{u}_k\}$ is uniformly bounded in H^1_k- norm. Therefore, we can assume that $\widetilde{u}_k \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. We then have for any test function $\varphi \in C^\infty_\circ(\mathbb{R}^N)$ that

$$\begin{aligned}
\langle J'(u), \varphi\rangle &= \int_{\mathbb{R}^N} [\nabla u \nabla \varphi + +V(x)u\varphi - K(x)|u|^{2^*-2}u\varphi - f(x,u)\varphi]\,dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^N} [\nabla \widetilde{u}_k \varphi + V(x)\widetilde{u}_k\varphi - K(x)|\widetilde{u}_k|^{2^*-2}\widetilde{u}_k\varphi - f(x,\widetilde{u}_k)\varphi]\,dx \\
&= 0,
\end{aligned}$$

which means that u is a weak solution of (1) and by (30) $u \not\equiv 0$. □

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