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VARIATIONAL THEOREMS OF MIXED TYPE AND ASYMPTOTICALY LINEAR VARIATIONAL INEQUALITIES

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1. Introduction

To introduce the problem we are going to deal with, let us consider a bounded open subset Ω of $\mathbb{R}^N$, which will be assumed to be connected and with smooth boundary, a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory's conditions and a measurable function $\psi : \Omega \to]-\infty, \infty]$.

We also consider the convex set K_ψ defined by

$$K_\psi = \{u \in W_0^{1,2}(\Omega) \mid u \geq \psi \quad a.e.\}.$$

We are interested in finding solutions of the variational inequality

$$(P) \quad \begin{cases} \displaystyle\int_\Omega (DuD(v-u) - g(x,u)(v-u) + h(v-u))\,dx \geq 0 & \text{for all } v \text{ in } K_\psi, \\ u \in K_\psi, \end{cases}$$

where h is a given function in $L^2(\Omega)$. It is well known that, if $\psi \in W^{2,2}(\Omega)$ and g fulfills some suitable growth conditions with respect to s, then the variational

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inequality (P) is equivalent to the problem

$$\begin{cases} \Delta u + g(x,u) \le h & \text{a.e. in } \{x \in \Omega \mid u(x) = \psi(x)\}, \\ \Delta u + g(x,u) = h & \text{a.e. in } \{x \in \Omega \mid u(x) > \psi(x)\}, \\ u \ge \psi & \text{a.e. in } \Omega, \\ u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega). \end{cases}$$

In all what follows we shall impose on ψ the obvious condition

$$(K) \qquad K_\psi \ne \emptyset.$$

Now we state the assumption that characterizes our problem:

$$(g,\alpha) \qquad \lim_{s\to\infty} \frac{g(x,s)}{s} = \alpha \quad \text{for a.e. } x \text{ in } \Omega.$$

More precisely, we want to estimate the number of solutions of (P) in dependence on the value of the parameter α. At this point it seems natural to suppose that:

$$(g) \qquad \begin{cases} |g(x,s)| \le a|s| + b(x) & \text{for a.e. } x \text{ in } \Omega, \text{ for all } s \text{ in } \mathbb{R}, \\ & \text{where } a \in \mathbb{R},\ b \in L^2(\Omega). \end{cases}$$

For technical reasons we shall assume $g(x,s)$ to be Lipschitz continuous in s, uniformly with respect to x or, more generally, that:

$$\sup_{\substack{x\in\Omega \\ s_1 \ne s_2}} \frac{g(x,s_2) - g(x,s_1)}{s_2 - s_1} < \infty.$$

Set $G(x,s) = \int_0^s g(x,\sigma)\,d\sigma$. We shall use the above written condition on g in the equivalent form

$$(G) \qquad \begin{cases} G(x,s) \le G(x,r) + g(x,r)(s-r) + q|s-r|^2 \\ \text{for every } s, r \text{ in } \mathbb{R}, \text{ for a.e. } x \text{ in } \Omega, \text{ where } q \in \mathbb{R}. \end{cases}$$

We shall investigate the properties of (P) exploring the right hand side space h along the straight lines $h = h_0 + te_1$, where $h_0 \in L^2(\Omega)$, $t \in \mathbb{R}$ and e_1 is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ chosen in such a way that $e_1 > 0$. We denote by (P_t) the problem (P) with $h = h_0 + te_1$:

$$(P_t) \qquad \begin{cases} \displaystyle\int_\Omega (DuD(v-u) - g(x,u)(v-u) + (h_0 + te_1)(v-u))\,dx \ge 0, \\ u \in K_\psi. \end{cases}$$

According to the asymptotic nature of the assumptions we shall study problem (P_t) for $t \gg 0$. As well known the problem has a variational nature: consider the functional $f_t : K_\psi \to \mathbb{R}$ defined by

$$f_t(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx + \int_\Omega u(h_0 + te_1)\,dx,$$

then it is easy to see that the solutions of (P_t) are the "lower critical points" for f_t on the constraint K_ψ (in Section 2 we have synthetically recalled the basic definitions and the main results in subdifferential analysis needed to deal with f_t on K_ψ).

Now let us make some considerations for a better understanding of the nature of the problem. For this let us denote by $(\lambda_i)_i$ $(i = 1, 2, \dots)$ the eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$: $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$. We start with observing that, if $\alpha < \lambda_1$ and for instance $g(x,s) = \alpha s$, then, for all real numbers t, (P_t) has a unique solution.

In the case where $\alpha > \lambda_1$, it is easy to check that (P_t) has no solutions for $t \ll 0$. If $\lambda_1 < \alpha < \lambda_2$, then in [18] it is proved that if $g(x,u) = \alpha u$, $\psi = 0$ and $h > 0$, the problem (P) has at least two solutions, while in [12] it is proved that for all h_0 in $L^2(\Omega)$ there exists $\overline{t}$ such that, if $t > \overline{t}$, then (P_t) has at least two solutions, if $t = \overline{t}$, (P_t) has at least one solution and if $t < \overline{t}$, (P_t) has no solutions. If $g(x,s)$ is convex in s then one can replace all the"at least" by "exactly".

Moreover, if $\alpha > \lambda_2$ in [12] it is proved that, in the case where $g(x,s)$ is linear in s, there exist at least four solutions for $t \gg 0$.

The first result of this paper consists in showing that in the previous result the linearity assumption on g is not actually needed.

Theorem 1.1. *Let (g), (g,α), (G) and (K) hold. Then for all $\alpha > \lambda_2$, there exists $\overline{t}$ in $\mathbb{R}$ such that for $t \ge \overline{t}$ problem (P_t) has at least four solutions.*

In [12] the main technique was to pass to an auxiliary constrained problem. Such an approach seems to be not applicable in a direct way when g is not linear. For this reason, we apply a completely different technique, based on another kind of constraint, which has been recently introduced by Marino ans Saccon in [13]. Our main result is the following

Theorem 1.2. *Let $\lambda_k > \lambda_2$. There exists $\sigma > 0$ such that for all α in $]\lambda_k, \lambda_k + \sigma]$, if (g), (g,α), (G) and (K) hold, then there exists $\overline{t}$ in $\mathbb{R}$ such that for $t \ge \overline{t}$, problem (P_t) has at least six solutions.*

For proving both the above results we have used some variational theorems, which we call "of mixed type" (the "∇-theorems" 3.8 and 3.9), where there are both assumptions on the values of a functional (on some suitable sets), and on the values of its gradient.

Actually, in the proofs of Theorems 1.1 and 1.2, we first examine the case of g linear (see Sections 4–6) and show that under the assumptions of Theorems 1.1 and 1.2 the conditions for applying the "∇-theorems" are fulfilled. Next we prove (see Section 7) that such conditions do persist passing from the linear case to the general one, for $t \gg 0$.

We conclude this introduction making some comparison between problem (P_t) and the classical problem of "jumping nonlinearities", where one studies the equation

$$\begin{cases} \Delta u + g(x,u) = h_0 + te_1, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$\lim_{s\to-\infty} \frac{g(x,s)}{s} = \beta, \qquad \lim_{s\to\infty} \frac{g(x,s)}{s} = \alpha,$$

and, for instance, condition (g) holds. Strong analogies with problem (P_t) are present: in some sense, in our case, g is such that $g(x,s) = -\infty$ for $s < \psi(x)$. Actually the results presented in this paper are analogous to those of the jumping problem in the case where $\beta < \lambda_1$ (see for instance [11], [13] and the numerous references therein).

Among other open problems concerning (P_t) let us point out one which we feel interesting, also in the case of jumping nonlinearities, that is a more precise estimate of the number of solutions of (P_t) as t varies in $\mathbb{R}$, under suitable assumptions (to be individuated) on the function g.

2. Some recalls of subdifferential analysis

In this section we recall some notions and results of subdifferential analysis. We point out that we will study a convex functional under a convex constraint. For more details the reader is referred to [2], [4]–[6], [8], [14].

Throughout this section H will denote a Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and norm $\|\cdot\|$. Let W be a subset of H and $f : W \to \mathbb{R} \cup \{\infty\}$ a function and set $D(f) = \{u \in W \mid f(u) < \infty\}$.

Definition 2.1. Let $u \in D(f)$. We call *subdifferential* of f at u the set $\partial^- f(u)$ of all $\varphi \in H$ such that

$$\liminf_{v\to u} \frac{f(v) - f(u) - \langle \varphi, v-u\rangle}{\|v-u\|} \geq 0.$$

Since $\partial^- f(u)$ is convex and closed, we denote by $\operatorname{grad}^- f(u)$ the element of $\partial^- f(u)$ having minimal norm.

If $0 \in \partial^- f(u)$ we say that u is a *lower critical point* of f. A value $c \in \mathbb{R}$ is said to be *critical* for f if there exists $u \in W$ such that $0 \in \partial^- f(u)$ and $f(u) = c$. If value c is not critical, we say that it is *regular*.

If $W = H$ and f is convex, the notion of $\partial^- f$ coincides with the usual notion of subdifferential in convex analysis.

If $E \subset H$, we define the indicator function $I_E : H \to \mathbb{R} \cup \{\infty\}$ by

$$I_E(u) = \begin{cases} 0 & \text{if } u \in E, \\ \infty & \text{if } u \in H \setminus E. \end{cases}$$

For every u in E the set $\partial^- I_E(u)$ is a closed convex cone and is called the *outward normal cone* to E at u. If $u \in E$ and E is a convex set, then

$$\partial^- I_E(u) = \{\varphi \in H \mid \langle \varphi, v - u\rangle \le 0, \text{ for all } v \text{ in } E\}.$$

The indicator function allows us to consider the lower critical points of f on a constraint E.

Definition 2.2. We say that $u \in E$ is a *lower critical point* of f on E if $0 \in \partial^-(f + I_E)(u)$.

Now we introduce a class of functions which has important properties and, on the other hand, seems to be well fit to our problem.

Definition 2.3. Let p and q be two real continuous functions defined on $D(f)$. We say that f is *of class* $C(p,q)$ if

$$f(v) \ge f(u) + \langle \varphi, v - u\rangle - [p(u)\|\varphi\| + q(u)]\|v - u\|^2,$$

for all v in W, whenever u in $D(f)$ and $\varphi \in \partial^- f(u)$.

We point out that no condition is required at the points u such that $\partial^- f(u) = \emptyset$. For a more general class see [8] and [14]. A significative class of $C(p,q)$ functions will be introduced in Theorem 2.6.

The next theorem, concerning the functions f on a manifold M, besides giving us an interesting example of function of class $C(p,q)$, is useful to clarify the meaning of the constrained critical points. First we give a definition.

Definition 2.4. If A and B are two subsets of H, we say that they are *tangent* at the point u of $A \cap B$ if and only if

$$\partial^- I_A(u) \cap (-\partial^- I_B(u)) \ne \{0\}.$$

In the sequel we assume M to be the closure of an open subset of H with boundary of class $C^{1,1}$.

Remark 2.5. Let K be a convex subset of H and u in $K \cap \partial M$. It is easy to verify that K and M are not tangent at u if and only if there exists $\overline{u}$ in K such that $\langle \nu(u), \overline{u} - u\rangle < 0$, where $\nu(u)$ is the outward normal at u with $\|\nu(u)\| = 1$.

Theorem 2.6. *Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function of class $C(0, q_0)$ where q_0 is a real valued continuous function defined on $D(f)$.*

If $D(f)$ and M are not tangent at any point, then

(a) *for every u in $M \cap D(f)$*

$$\partial^-(f + I_M)(u) = \partial^- f(u) + \partial^- I_M(u).$$

(Hence, if $\partial^-(f + I_M)(u) \neq \emptyset$, then $\partial^- f(u) \neq \emptyset$.)

(b) *There exist two continuous functions $C_1, C_2 : \partial M \cap D(f) \to \mathbb{R}$ such that for every u in $\partial M \cap D(f)$, if $\varphi \in \partial^-(f + I_M)(u)$ and $\lambda \geq 0$ is such that $\varphi - \lambda\nu(u) \in \partial^- f(u)$, then $\lambda \leq C_1(u)\|\varphi\| + C_2(u)$.*

(c) *There exist two continuous functions $p, q : M \cap D(f) \to \mathbb{R}$ such that $f + I_M$ is a function of class $C(p, q)$.*

The proof of this result can be obtained from [4, Theorem 1.13], making simple adaptations (see [9]).

In the following sections we study problems (P_t) and $(\overline{P})$ using suitable variational theorems concerning functionals of class $C(p, q)$. To this end a suitable version of the classical Deformation Lemma applied to this class of functional will be used. We start by a definition extending the Palais–Smale property to the functional of class $C(p, q)$.

Definition 2.7. Let c be a real number and $f : H \to \mathbb{R} \cup \{\infty\}$ be a function. We say that f verifies the Palais–Smale condition at level c (briefly $(\text{P.S.})_c$), if for every sequence (u_n) in $D(f)$ and (φ_n) in H with

$$\lim_n f(u_n) = c, \quad \text{for all } n \in \mathbb{N}, \quad \varphi_n \in \partial^- f(u_n), \quad \lim_n \varphi_n = 0,$$

there exists a subsequence (u_{h_k}) converging to an element u (with the properties: $f(u) = c$ and $0 \in \partial^- f(u)$).

Remark 2.8. If f is a lower semicontinuous function of class $C(p, q)$, then the two last properties in Definition 2.7 immediately follow.

In the sequel we set, for every c in $\mathbb{R}$,

$$f^c = \{u \in D(f) \mid f(u) \leq c\}, \qquad K_c = \{u \in H \mid 0 \in \partial f(u),\ f(u) = c\}.$$

Theorem 2.9 (Deformation Theorem). *Let $c \in \mathbb{R}$. Assume that $f : H \to \infty$ is a lower semicontinuous function of class $\mathcal{C}(p, q)$ which satisfies* $(\text{P.S.})_c$ *Then, given a neighbourhood U of K_c (if $K_c = \emptyset$, we allow $U = \emptyset$), there exist $\varepsilon > 0$ and a continuous map $\eta : f^{c+\varepsilon} \times [0, 1] \to f^{c+\varepsilon}$ such that for every $u \in H$ and $t \in [0, 1]$ we have:*

(a) *$f(\eta(u, t)) \leq f(u)$,*

(b) *$\eta(f^{c+\varepsilon} \setminus U, 1) \subseteq f^{c-\varepsilon}$.*

Theorem 2.10 (Noncritical Interval Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function of class $\mathcal{C}(p, q)$. Assume that f satisfies* (P.S.)$_c$ *for every c in $[a, b]$ and there are no lower critical points of f in $f^{-1}([a, b])$. Then there exists a continuous map $\eta : f^b \times [0, 1] \to f^b$ such that*

$$\begin{aligned} \eta(u,0) &= u && \text{for all } u \text{ in } f^b, \\ f(\eta(u,t)) &\le f(u) && \text{for all } u \text{ in } f^b \text{ and for all } t \text{ in } [0,1], \\ \eta(u,1) &\in f^a && \text{for all } u \text{ in } f^b. \end{aligned}$$

The proofs easily follow (in "the classical way") from the next theorem.

Theorem 2.11 (Evolution Theorem). *Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function of class $\mathcal{C}(p, q)$. Then for every u_0 in $D(f)$ there exist $T > 0$ and an unique absolutely continuous function $U : [0, T] \to H$ such that $U(0) = u_0$, $U(t) \in D(f)$ for every t in $[0, T]$, $f \circ U$ is non increasing and*

$$\begin{aligned} cU'(t) &= -\operatorname{grad}^- f(U(t)) && \text{for a.e. } t \text{ in } [0,T], \\ f \circ U(t_2) - f \circ U(t_1) &= -\int_{t_1}^{t_2} \|U'(s)\|^2 \, ds && \text{for all } t_1, t_2 \text{ in } [0,T]. \end{aligned}$$

Moreover, if (u_n) and (t_n) are two sequences such that $u_n \in D(f)$, $t_n \in [0, T]$, $u_n \to u$, $\sup_n f(u_n) < \infty$ and $t_n \to t$, then for n large the solutions U_n of (2.1) *starting from u_n are defined in $[0, t_n]$, and*

$$U_n(t_n) \to U(t) \quad \text{and} \quad f(U_n(t_n)) \to f(U(t)).$$

For the proof see for example [8], [14]. For more general cases see [7].

3. Some variational theorems of mixed type

In this section we wish to expose the variational theorems we are going to use for the multiplicity results of Sections 6 and 7. As we said in the introduction, some of these theorems, which we call "∇-theorems", contain hypotheses of "mixed type" concerning both the values of functional and the values of its gradient.

The following Theorem 3.1 plays a key role in the proof of these results. This statement is the generalization to the case of nonsmooth functional f of a very well-known and nice result of K. C. Chang (see Theorem 3.5 of Chapter 3 of [3]) which relates the homology and coohomology groups of the sublevels of f with the number of its critical points. The idea of ∇-theorems consists mainly in the introduction of an additional constraint: this constraint on one side increases the topological complexity of the sublevels (so Theorem 3.1 can be used), on the other side they are "fictitious" due to the hypothesis on the gradient of the functional. Some arguments of this kind have been also used by J. Q. Liu [10] (even if not explicitly) and by C. Bertocchi and M. Degiovanni [1], for a functional defined on a sphere.

Let A be a topological space and B be a subspace of A. We will denote by $H_p(A,B)$ and $H^p(A,B)$ respectively the pth-group of relative homology and the pth-group of relative coomology of the pair (A,B) with coefficients in an assigned field. The symbol $\cap$ will be denote the "cap product" defined in [17].

Theorem 3.1. *Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function of class $C(\widetilde{p}, \widetilde{q})$. Let a, b be regular values of f with $a < b$ and let $p, q \in \mathbb{N}$ such that $p \geq 1$ and $q \geq 1$. Assume that there exist τ_1 in $H_p(f^b, f^a)$, τ_2 in $H_{p+q}(f^b, f^a)$ and ω in $H^q(f^b)$ such that $\tau_1 \neq 0$ and $\tau_1 = \tau_2 \cap \omega$. Moreover, assume that f satisfies* $(\mathrm{P.S.})_c$ *for all c in $[a,b]$. Then f has at least two lower critical points in $f^{-1}([a,b])$.*

We will use this theorem in the proof of the next lemma.

Lemma 3.2. *Let m, n in $\mathbb{N}$ be such that $m \geq 1$ and $n \geq 1$. Let $g : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function of class $C(p,q)$. Let S, Σ, X, Y be four subsets of H such that $S \subset \Sigma$, $Y \subset X$, $S \subset H \setminus X$, $\Sigma \subset H \setminus Y$. Assume that*

(1) $a' = \sup g(S) < \inf g(X) = a''$, $b' = \sup g(\Sigma) < \inf g(Y) = b''$,
(2) *there exist τ_1 in $H_n(\Sigma, S)$, τ_2 in $H_{n+m}(\Sigma, S)$ with $\tau_1 \neq 0$, ω in $H^m(\Sigma)$ such that $\tau_1 = \tau_2 \cap \omega$,*
(3) *the inclusion $i : (\Sigma, S) \to (H \setminus Y, H \setminus X)$ is such that $i_* : H_*(\Sigma, S) \to H_*(H \setminus Y, H \setminus X)$ is injective in dimension n and $n+m$, $i^* : H^*(H \setminus Y) \to H^*(\Sigma)$ is surjective in dimension m.*

Let $a \in \,]a', a''[$, $b \in \,]b', b''[$ and suppose that g verifies $(\mathrm{P.S.})_c$ *for all c in $[a,b]$. Then g has at least two lower critical points in $g^{-1}([a,b])$.*

Proof. We have

$$(\Sigma, S) \xrightarrow{i_1} (g^b, g^a) \xrightarrow{i_2} (H \setminus Y, H \setminus X),$$

where i_1 and i_2 are the inclusions. Since $i = i_2 \circ i_1$, then $i_* = i_{2*} \circ i_{1*}$ and $i^* = i_1^* \circ i_2^*$; therefore i_{1*} is injective (in dimension n and $n+m$) and i_1^* is surjective (in dimension m). Now we take $\tau_1' = i_{1*}(\tau_1)$, $\tau_2' = i_{1*}(\tau_2)$ and ω' such that $\omega = i_1^*(\omega')$. Then

$$\tau_2' \cap \omega' = i_{1*}(\tau_2) \cap i_1^*(\omega) = i_{1*}(\tau_2 \cap \omega) = i_{1*}(\tau_1) = \tau_1'.$$

The proof is achieved applying Theorem 3.1. □

Remark 3.3.

(a) Hypothesis (2) of the previous theorem is, in particular, satisfied if the pair (Σ, S) is homeomorphic to $(B^n \times S^m, \partial B^n \times S^m)$ where B^n is the n-dimensional ball and S^m is the m-dimensional sphere. (In fact it is sufficient to take $\Sigma = \Phi(B^n \times S^m)$, $S = \Phi(\partial B^n \times S^m)$ where $\Phi : B^n \times S^m \to H$ is a continuous function such that Φ_* is injective in dimension n and $n+m$ and Φ^* is surjective in dimension m.)

(b) Hypothesis (3) is verified, in particular, if Σ and S are respectively deformation retracts of $H \setminus Y$ and $H \setminus X$.

Now we are going to apply the previous lemma in a concrete situation. First of all let us introduce some notations.

Notations 3.4. Let X_1, X_2, X_3 be three closed subspaces of H such that

$$H = X_1 \oplus X_2 \oplus X_3.$$

Furthermore for $i, j = 1, 2, 3$ and $\rho > 0$ we set

$$S_i(\rho) = \{u \in X_i \mid \|u\| = \rho\}, \qquad B_i(\rho) = \{u \in X_i \mid \|u\| < \rho\},$$
$$S_{ij}(\rho) = \{u \in X_i \oplus X_j \mid \|u\| = \rho\}, \quad B_{ij}(\rho) = \{u \in X_i \oplus X_j \mid \|u\| < \rho\}.$$

Theorem 3.5. *Assume that* $1 \le \dim X_1 < \infty$ *and* $1 \le \dim X_2 < \infty$. *Let* $g : H \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuous function of class* $C(p,q)$. *Let* ρ, ρ_1, R *be such that* $\rho_1 > 0$ *and* $0 < \rho < R \le \infty$. *Let* $\Sigma = B_1(\rho_1) \times S_2(\rho)$, $S = S_1(\rho_1) \times S_2(\rho)$, $X = (X_1 \oplus X_3) \cup B_{23}(R)$ *and* $Y = (X_1 \oplus X_3) \cup S_{23}(R)$. (*If* $R = \infty$ *then we set* $S_{23}(R) = \emptyset$ *and* $B_{23}(R) = X_2 \oplus X_3$). *Assume that*

$$a' = \sup g(S) < \inf g(X) = a'',$$
$$b' = \sup g(\Sigma) < \inf g(Y) = b''.$$

Let $a \in \,]a', a''[$, $b \in \,]b', b''[$ *and suppose that* g *verifies* $(\mathrm{P.S.})_c$ *for all* c *in* $[a, b]$. *Then* g *has at least two lower critical points in* $g^{-1}([a, b])$.

Proof. If $\dim X_2 = 1$, then g^b has two connected components and in each one the "splitting sphere" principle (see [11, Theorem 8.1]) can be applied; therefore we can limit ourselves to the case $\dim X_2 \ge 2$.

We set $n = \dim X_1$, $m = \dim X_2 - 1$ and we suppose that R is finite. We wish to apply Lemma 3.2. By (a) of Remark 3.3 we can deduce that the pair (Σ, S) verifies hypothesis (2) of Lemma 3.2. Let $Z = B_{23}(R) \times X_1$. It is easy to prove that the inclusion

$$j : ((H \setminus Y) \cap Z, (H \setminus X) \cap Z) \to (H \setminus Y, H \setminus X)$$

generates an isomorphism j_* in the relative homology group using the excision property (we excise $(H \setminus X) \setminus Z$). Moreover, since

$$(H \setminus Y) \cap Z = Z \setminus Y = (B_{23}(R) \setminus B_3(R)) \times X_1$$

and

$$(H \setminus X) \cap Z = Z \setminus X = (B_{23}(R) \setminus B_3(R)) \times (X_1 \times \{0\}),$$

it can be easily seen that Σ and S are respectively deformation retracts of $(H \setminus Y) \cap Z$ and $(H \setminus X) \cap Z$. Therefore also the inclusion $i : (\Sigma, S) \to (H \setminus Y, H \setminus X)$ induces an isomorphism $i_* : H_*(\Sigma, S) \to H_*(H \setminus Y, H \setminus X)$. On the other hand, we notice that $\Sigma \subset H \setminus Y \subset H \setminus (X_1 \oplus X_3)$ and Σ is a deformation retract of $H \setminus (X_1 \oplus X_3)$, then $i^* : H^*(\Sigma) \to H^*(H \setminus Y)$ is an epimorphism. At this point, using Lemma 3.2 the assertion follows.

If $R = \infty$, we can repeat the proof without using the excision property. $\square$

Another concrete situation in which Lemma 3.2 is used is the next theorem.

Theorem 3.6. *Assume that* $1 \leq \dim X_1 < \infty$ *and* $1 \leq \dim X_2 < \infty$. *Let* $g : H \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuous function of class* $C(p, q)$, e *in* $X_1 \setminus \{0\}$, $R_1 > 0$ *and* $\rho > 0$. *Set*

$$\Sigma = Q \times S_2(\rho) \quad \textit{and} \quad S = (\partial_{X_1} Q) \times S_2(\rho),$$

where $Q = \{te + u \mid 0 \leq t \leq 1,\ u \in X_1,\ \langle u, e\rangle = 0,\ \|u\| \leq R_1\}$. *Let* $S' = \{u \in \operatorname{span}(e) \oplus X_2 \oplus X_3 \mid \|u\| = R\}$ *with* $\rho < R < \|e\|$. *Suppose that*

$$\begin{aligned} a' &= \sup g(S) < \inf g(S' \cup (X_1 \oplus X_3)) = a'', \\ b' &= \sup g(\Sigma) < \inf g(X_1 \oplus X_3) = b''. \end{aligned}$$

Let $a \in\,]a', a''[$, $b \in\,]b', b''[$ *and suppose that* g *verifies* $(\mathrm{P.S.})_c$ *for all* c *in* $[a, b]$. *Then* g *has at least two lower critical points in* $g^{-1}([a, b])$.

Proof. If $\dim X_2 = 1$, then g^b has two connected components and in each one there is a critical point, obtained by linking argument (see [16]); therefore we can limit ourselves to the case $\dim X_2 \geq 2$.

We set $n = \dim X_1$, $m = \dim X_2 - 1$. We also set $\widehat{X}_1 = \{u \in X_1 \mid \langle u, e\rangle = 0\}$, $\widehat{B}_1(r) = \{u \in \widehat{X}_1 \mid \|u\| < r\}$, $X = S' \cup (X_1 \oplus X_3)$, $Y = X_1 \oplus X_3$ and we introduce the projections $\widehat{P} : H \to \widehat{X}_1$, $P_{e23} : H \to \operatorname{span}(e) \oplus X_2 \oplus X_3$. We wish to apply Lemma 3.2. It is easy to prove that the pair (Σ, S) verifies hypothesis (2) of Lemma 3.2; let us prove that hypothesis (3) is satisfied.

We define, for every u in $H \setminus Y$ and t in $[0, 1]$, the function $\eta : (H \setminus Y, H \setminus X) \times [0, 1] \to (H \setminus Y, H \setminus X)$ by

$$\eta(t, u) = \frac{(1 - t)P_{e23}(u) + tP_2(u)}{\|(1 - t)P_{e23}(u) + tP_2(u)\|}\|P_{e23}(u)\| + \widehat{P}(u).$$

Then we have

$$
\begin{aligned}
&\eta(u,0) = u && \text{for all } u \text{ in } H \setminus Y,\\
&\eta(u,t) = u && \text{for all } u \text{ in } (\widehat{X}_1 \oplus X_2) \setminus \widehat{X}_1,\\
&\eta(u,1) \in \widehat{X}_1 \oplus X_2 \setminus \widehat{X}_1 && \text{for all } u \text{ in } H \setminus Y,\\
&\eta(u,1) \in \widehat{X}_1 \oplus X_2 \setminus (\widehat{X}_1 \cup S_2(R)) && \text{for all } u \text{ in } H \setminus X.
\end{aligned}
$$

Moreover, $\eta(\,\cdot\,,1)$ is an homeomorphism from Σ to Σ', where

$$\Sigma' = (\overline{B_2(\rho_1)} \setminus B_2(\rho)) \times \widehat{B}_1(R_1)$$

with $\rho_1 = \sqrt{\rho^2 + \|e\|^2}$, ($\rho_1 > R$) and $\eta(S,1) = S' = \partial_{\widehat{X}_1 \oplus X_2}\Sigma'$. Now it is simple to verify that Σ' and S' are respectively deformation retracts of $(\widehat{X}_1 \oplus X_2) \setminus \widehat{X}_1$ and $(\widehat{X}_1 \oplus X_2) \setminus (\widehat{X}_1 \cup S_2(R))$. This concludes the proof. $\square$

Now we want to deduce from Theorems 3.6 and 3.6 two propositions ("∇-theorems") that we will use in Section 6 and 7.

In these theorems we will make some assumptions on the gradient of the functional f which allow us to weaken some inequalities on the values of f.

Definition 3.7. Let γ be a real number such that $\gamma > 0$ and let X be a closed subspace of H, we set

$$C_\gamma(X) = \{u \in H \mid \operatorname{dist}(u, X) \geq \gamma\}.$$

Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$ and $f : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function. We say that the condition $(\nabla)(f, C_\gamma(X), a, b)$ holds if

$$\inf\left\{\|\varphi\| \mid \varphi \in \partial^-(f + I_{C_\gamma(X)})(u),\quad \begin{array}{l} f(u) \in [a,b] \\ u \in D(f) \cap \partial C_\gamma(X) \end{array}\right\} > 0. \tag{3.7.1}$$

In some sense we are requiring that $f_{|C_\gamma(X)}$ has no critical points u with $u \in \partial C_\gamma(X)$ and $a \leq f(u) \leq b$ with "some uniformity".

Theorem 3.8 (∇-Theorem A). *Assume that* $1 \leq \dim X_1 < \infty$ *and* $1 \leq \dim X_2 < \infty$. *Let* $f : H \to \mathbb{R} \cup \{\infty\}$ *be a lower semicontinuous function of class* $C(0,q)$. *Assume that there exist* γ, ρ, R, a, b *in* $\mathbb{R}$, *such that* $0 < \gamma < \rho < R \leq \infty$ *and, if* $C_\gamma = C_\gamma(X_1 \oplus X_3)$,

(1) $a' = \sup f(S_{12}(\rho) \cap \partial C_\gamma) < \inf f(B_{23}(R) \cap C_\gamma) = a''$,
(2) $b' = \sup f(S_{12}(\rho) \cap C_\gamma) < \inf f(S_{23}(R) \cap C_\gamma) = b''$,
(3) $a' < a < a''$, $b' < b < b''$, *the condition* $(\nabla)(f, C_\gamma, a, b)$ *holds and* $D(f)$ *and* C_γ *are not tangent at any point,*
(4) f *satisfies* $(\mathrm{P.S.})_c$ *for all* $c \in [a,b]$.

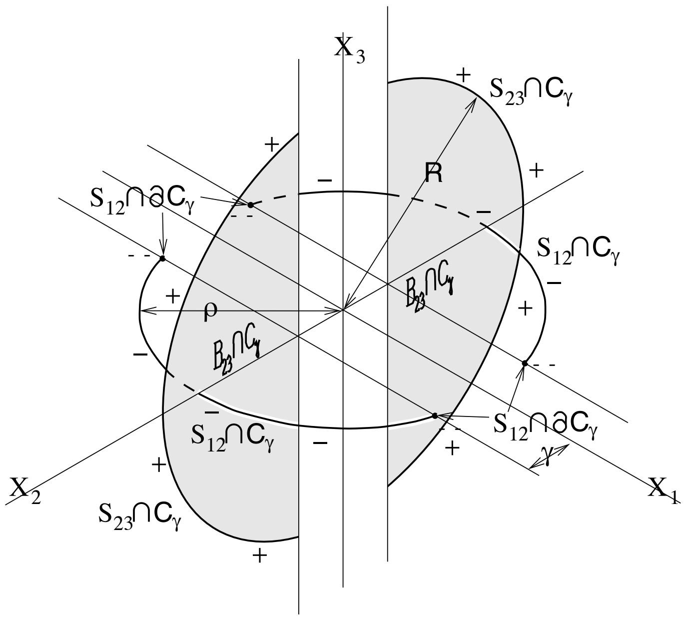

Figure 1. The topological situation of ∇-Theorem A

Then f has at least two lower critical points in $f^{-1}([a,b])$.

Proof. Set $g = f + I_{C_\gamma}$. By Theorem 2.6 g is a function of class $C(p,q)$. Since f satisfies $(\mathrm{P.S.})_c$ for all $c \in [a,b]$ and the condition $(\nabla)(f, C_\gamma, a, b)$ holds, we deduce that g satisfies $(\mathrm{P.S.})_c$ for all $c \in [a,b]$.

In order to apply Theorem 3.5, we set $\Sigma = S_{12}(\rho) \cap C_\gamma$, $S = S_{12}(\rho) \cap \partial C_\gamma$, $Y = (X_1 \oplus X_3) \cup S_{23}(R)$ and $X = (X_1 \oplus X_3) \cup B_{23}(R)$.

In view of Theorem 3.5 (with simple adaptations) we deduce that there exist two lower critical points u_1, u_2 for g in $g^{-1}([a,b])$. Since the condition $(\nabla)(f, C_\gamma, a, b)$ holds, then $u_1, u_2 \in$ int C_γ and so they are critical points of f.□

Theorem 3.9 (∇-Theorem B). *Assume that* $1 \leq \dim X_1 < \infty$, $1 \leq \dim X_2 < \infty$. *Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function of class $C(0,q)$. Let e in $X_1 \setminus \{0\}$, $R_1 > 0$, $\rho > 0$ we set*

$$\Sigma = Q \times S_2(\rho) \quad \textit{and} \quad S = (\partial_{X_1} Q) \times S_2(\rho),$$

where $Q = \{te + u \mid 0 \leq t \leq 1,\ u \in X_1, \langle u, e\rangle = 0,\ \|u\| \leq R_1\}$. *We set* $S' = \{u \in \mathrm{span}(e) \oplus X_2 \oplus X_3 \mid \|u\| = R\}$ *with* $\rho < R < \|e\|$. *Let a, b in $\mathbb{R}$ and suppose that, if* $C_\gamma = C_\gamma(X_1 \oplus X_3)$,

(1) $a' = \sup f(S) < \inf f(S') = a''$,
(2) $b' = \sup f(\Sigma) < \infty$,

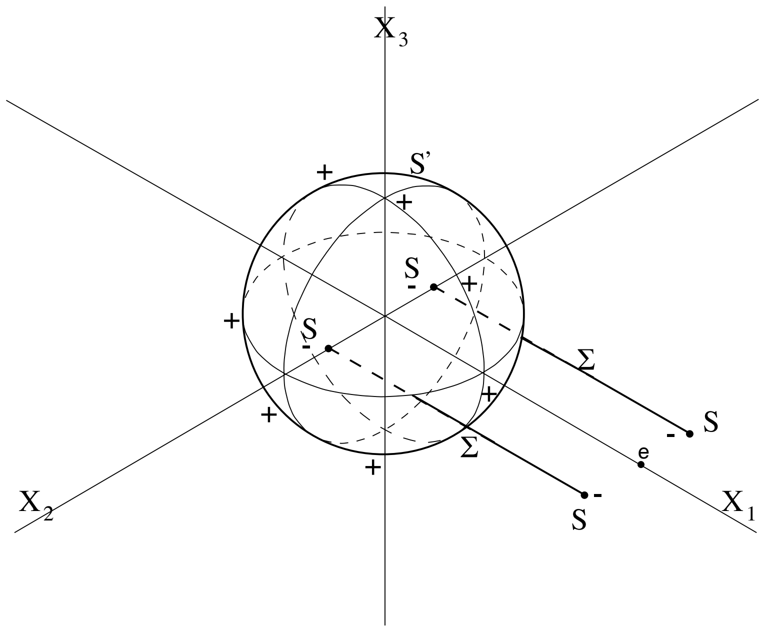

Figure 2. The topological situation of ∇-Theorem B

(3) *$a' < a < a''$, $b' < b < \infty$, and there exists γ in $]0,\rho[$ such that the condition $(\nabla)(f, C_\gamma, a, b)$ holds, $D(f)$ and C_γ are not tangent at any point,*

(4) *f satisfies* $(\mathrm{P.S.})_c$ *for all c in $[a,b]$.*

Then f has at least two lower critical points in $f^{-1}([a,b])$.

Proof. Set $g = f + I_{C_\gamma}$. By Theorem 2.6 g is a function of class $C(p,q)$. Since f satisfies $(\mathrm{P.S.})_c$ for all $c \in [a,b]$ and the condition $(\nabla)(f, C_\gamma, a, b)$ holds, we deduce that g satisfies $(\mathrm{P.S.})_c$ for all $c \in [a,b]$. By Theorem 3.6, g has at least two lower critical points u_1, u_2 in $g^{-1}([a,b])$; since the condition $(\nabla)(f, C_\gamma, a, b)$ holds, then $u_1, u_2 \in \operatorname{int} C_\gamma$ and so they are critical points of f. $\square$

4. The asymptotic problem and some notations

As mentioned in the introduction, in order to study the problem (P_t) with $t \gg 0$, we introduce the following asymptotic problem

$$(\overline{P}) \qquad \begin{cases} \displaystyle\int_\Omega (DuD(v-u) - \alpha(v-u) + e_1(v-u))\,dx \geq 0 \quad \text{for all } v \in K_0, \\ u \in K_0, \end{cases}$$

where $K_0 = \{u \in W_0^{1,2}(\Omega) \mid u \geq 0 \ a.e.\}$ and Ω is an open, connected, bounded subset of $\mathbb{R}^N$ with smooth boundary. More precisely we consider the functional

$f_\alpha : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by

$$f_\alpha(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Du|^2\,dx - \dfrac{\alpha}{2}\int_\Omega u^2\,dx + \int_\Omega ue_1\,dx & \text{if } u \in K_0, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus K_0. \end{cases}$$

As usual we consider $L^2(\Omega)$ endowed with the inner product $\langle u, v\rangle = \int_\Omega uv\,dx$ and the norm $\|u\|^2 = \int_\Omega u^2\,dx$. Then the solutions of the problem $(\overline{P})$ are the lower critical points (see Definition 2.1) of f_α; indeed if $u \in K_0$ and $\varphi \in L^2(\Omega)$, then $\varphi \in \partial^- f_\alpha(u)$ if and only if for all v in K_0

$$\int_\Omega (DuD(v-u) - \alpha u(v-u) + e_1(v-u))\,dx \geq \int_\Omega \varphi(v-u)\,dx. \tag{4.0.1}$$

Moreover, it is clear that the functional f_α is of class $C(0, \alpha/2)$ (see Definition 2.3).

We will use also the norm $\|u\|_W^2 = \int_\Omega |Du|^2\,dx$. We notice that if e_1 is the first eigenfunction of the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

with $e_1 > 0$ and λ_1 is the first eigenvalue, then for $\alpha > \lambda_1$, the function $\overline{e}_1 = e_1/(\alpha - \lambda_1)$ is a solution of $(\overline{P})$. The solutions we are going to find "branch" from $\overline{e}_1$. This motivates the interest for the increment $f_\alpha(\overline{e}_1 + z) - f_\alpha(\overline{e}_1)$; with easy computation one finds:

$$f_\alpha(\overline{e}_1 + z) - f_\alpha(\overline{e}_1) = Q_\alpha(z) \quad \text{for all } z \text{ in } \widetilde{K}, \tag{4.0.2}$$

where $Q_\alpha(z) = \frac{1}{2}\int_\Omega (|Dz|^2 - \alpha z^2)\,dx$ and $\widetilde{K} = \{w \in W_0^{1,2}(\Omega) \mid w \geq -\overline{e}_1\}$.

Finally, we denote by $(\lambda_i)_{i\geq 1}$ and $(e_i)_{i\geq 1}$ respectively the sequence of eigenvalues and eigenfunctions of problem (Δ) ($0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ and $\|e_i\| = 1$). Let $i, j \in \mathbb{N}$ with $i, j \geq 1$ and $i < j$, we set

$$H_i = \operatorname{span}(e_1, \ldots, e_i), \quad H_i^\perp = \{u \in L^2(\Omega) \mid \langle u, v\rangle = 0, \text{ for all } v \text{ in } H_i\},$$

and we consider the orthogonal projections

$$P_i : H \to H_i,\ P_{ij} : H \to H_i \oplus H_j^\perp,\ P_{ij}^* : H \to \operatorname{span}(e_{i+1}, \ldots, e_j).$$

Moreover, for every $k \geq 1$, $\rho > 0$, we set

$$\begin{cases} \mathbf{S}_k(\overline{e}_1, \rho) = \{u \in H_k \mid \|u - \overline{e}_1\| = \rho\}, \\ \mathbf{B}(0, \rho) = \{u \in L^2(\Omega) \mid \|u\| < \rho\}, \\ \mathbf{S}(0, \rho) = \{u \in L^2(\Omega) \mid \|u\| = \rho\}, \\ \mathbf{B}_k^*(0, \rho) = \{u \in \operatorname{span}(e_2, \ldots, e_k) \mid \|u\| < \rho\}, \\ \mathbf{S}_k^*(0, \rho) = \{u \in \operatorname{span}(e_2, \ldots, e_k) \mid \|u\| = \rho\}. \end{cases} \tag{4.0.3}$$

5. The conditions (P.S.) and (∇)

Lemma 5.1. *Assume that $\alpha > \lambda_1$. Let (u_n), (α_n), (φ_n) be three sequences such that $u_n \in K_0$, $\alpha_n \in \mathbb{R}$ and converges to α, $\varphi_n \in \partial^- f_{\alpha_n}(u_n)$ for every n. Suppose that*

$$\sup_n \left| \int_\Omega \varphi_n e_1 \, dx \right| < \infty \quad \textit{and} \quad \sup_n \left| \frac{\int_\Omega \varphi_n u_n \, dx}{\|u_n\|_W} \right| < \infty.$$

Then (u_n) is bounded in $W_0^{1,2}(\Omega)$.

Proof. We argue by contradiction and suppose that $\lim_{n\to\infty} \|u_n\|_W = \infty$. In view of (4.0.1) it turns out

$$\int_\Omega (Du_n D(v-u_n) - \alpha_n u_n (v-u_n))\, dx + \int_\Omega e_1 (v-u_n)\, dx \geq \int_\Omega \varphi_n (v-u_n)\, dx \quad \text{for all } v \text{ in } K_0. \tag{5.1.1}$$

We set $z_n = u_n/\|u_n\|_W \in K_0$. Taking $v = u_n + e_1$ in (5.1.1), and dividing by $\|u_n\|_W$ we obtain

$$\int_\Omega (Dz_n De_1 - \alpha_n z_n e_1)\, dx + \int_\Omega \frac{e_1^2}{\|u_n\|_W}\, dx \geq \int_\Omega \frac{\varphi_n e_1}{\|u_n\|_W}\, dx.$$

Up to a subsequence, $z_n \to z$ in $L^2(\Omega)$ and $z_n \rightharpoonup z$ in $W_0^{1,2}(\Omega)$ for some z in K_0, then $\int_\Omega (DzDe_1 - \alpha z e_1)\, dx \geq 0$ which implies $z = 0$. On the other hand, taking $v = 0$ in (5.1.1) and dividing by $\|u\|_W^2$ we obtain

$$-1 + \int_\Omega \alpha_n z_n^2 \, dx - \int_\Omega \frac{e_1 z_n}{\|u_n\|_W}\, dx \geq -\int_\Omega \frac{\varphi_n u_n}{\|u_n\|_W^2}\, dx.$$

If $n \to \infty$, we have a contradiction. $\square$

The following proposition is an immediate consequence of the previous lemma.

Proposition 5.2. *Assume that $\alpha > \lambda_1$. Then the functional f_α satisfies the* $(\text{P.S.})_c$ *for all $c \in \mathbb{R}$.*

The following statements concern the verification of the (∇) condition (see Definition 3.7) for the functional f_α.

Lemma 5.3. *Let i, j in $\mathbb{N}$ be such that $1 \leq j \leq i$, $\lambda_j < \alpha \leq \lambda_{i+1}$, let $u \in K_0 \cap (H_j \oplus H_i^\perp)$ be a lower critical point of f_α on $H_j \oplus H_i^\perp$. Then*

(a) *either $f_\alpha(u) \leq \sup f_\alpha(\partial_{H_j}(K_0 \cap H_j))$ or*
if $\alpha < \lambda_{i+1}$ then $u = \overline{e}_1$,
if $\alpha = \lambda_{i+1}$ then $u = \overline{e}_1 + e$ with e such that $\Delta e + \lambda_{i+1} e = 0$.
In any case $f_\alpha(u) = f_\alpha(\overline{e}_1)$,

(b) *if* $j = 1$ *then either* $u = 0$ *or*
if $\alpha < \lambda_{i+1}$, *then* $u = \overline{e}_1$,
if $\alpha = \lambda_{i+1}$, *then* $u = \overline{e}_1 + e$ *with* e *such that* $\Delta e + \lambda_{i+1} e = 0$.

In particular, either $f_\alpha(u) \leq \sup f_\alpha(\partial_{H_j}(K_0 \cap H_j))$ *or* $f_\alpha(u) = f_\alpha(\overline{e}_1)$.

Proof. Using (4.0.2), if $u = \overline{e}_1 + z$ is a lower critical point of f_α on $H_j \oplus H_i^\perp$ then

$$Q'_\alpha(z)(w - z) \geq 0 \quad \text{for all } w \text{ in } \widetilde{K}_0 \cap (H_j \oplus H_i^\perp), \tag{5.3.1}$$

where $\widetilde{K}_0 = \{w \in W_0^{1,2}(\Omega) \mid w \geq -\overline{e}_1\}$. Suppose that $z = z_1 + z_2$ with z_1 in H_j and z_2 in $H_i^\perp$.

If $z_1 = 0$, then $Q_\alpha(z) = Q_\alpha(z_2) \geq 0$. On the other hand taking $w = 0$ in (5.3.1) we deduce that $Q_\alpha(z) \leq 0$. Therefore $Q_\alpha(z_2) = 0$.

If $z_1 \neq 0$, then there exists $t > 0$ such that $tz_1 \in \widetilde{K}_0$. Taking $w = tz_1$ in (5.3.1) we obtain

$$tQ_\alpha(z_1) \geq Q_\alpha(z) \geq Q_\alpha(z_1)$$

which implies $t \leq 1$. Assume $\overline{t} = \sup\{t \in \mathbb{R} \mid tz_1 \in \widetilde{K}_0\}$, then $\overline{t} \leq 1$. Taking $w = \overline{t} z_1$ we have

$$Q_\alpha(z) \leq \frac{1}{\overline{t}} Q_\alpha(w) \leq Q_\alpha(w).$$

Since $w \in \partial_{H_j}(K_0 \cap H_j)$ the assertion follows. Furthermore, if $j = 1$, since $z_1 = e_1(\int_\Omega z e_1\, dx) \geq -\overline{e}_1$ then $z_1 \in \widetilde{K}_0$. This implies $\overline{t} \geq 1$ that is $z = -\overline{e}_1$. Since $z \in \widetilde{K}_0$ then $z_2 = 0$. $\square$

Lemma 5.4. *Let* i, j *in* $\mathbb{N}$ *be such that* $1 \leq j < k$ *and* $\alpha \in \mathbb{R}$. *If* $\gamma > 0$ *we set* $C_\gamma = C_\gamma(H_j \oplus H_k^\perp)$ (*see* 3.7). *Then*

(a) *if* $\gamma > 0$, K_0 *and* C_γ *are not tangent at any point* u *in* $K_0 \cap C_\gamma$,

(b) *if* $\gamma > 0$, *for all* u *in* $K_0 \cap \partial C_\gamma$

$$\partial^-(f_\alpha + I_{C_\gamma})(u) = \partial^- f_\alpha(u) + \{\lambda P^*_{jk}(u) \mid \lambda \leq 0\},$$

(c) *there exist two real numbers* c_1, c_2 *such that if* $\gamma > 0$, $u \in \partial C_\gamma \cap K_0$, $\varphi_0 \in \partial^- f_\alpha(u)$ *and* $\lambda \leq 0$, *then*

$$\|\lambda P^*_{j,k}(u)\| \leq c_1 \|\varphi_0 + \lambda P^*_{jk}(u)\| + c_2(1 + \gamma + \|u\|_W).$$

Moreover, c_1, c_2 *do not depend on* α *if* α *varies in a fixed bounded interval.*

Proof. Let us prove (a). Let $u \in \partial C_\gamma \cap K_0$, we take $\overline{u} = 2u \in K_0$ and we have

$$\int_\Omega (\overline{u} - u)(P^*_{jk}(u))\, dx = \int_\Omega (P^*_{jk}(u))^2\, dx > 0.$$

By Remark 2.5 the conclusion follows.

(b) is a consequence of Theorem 2.6. Let us prove (c). If $\varphi_0 \in \partial^- f_\alpha(u)$ then

$$f_\alpha(v) \geq f_\alpha(u) + \langle \varphi_0, v-u \rangle - \frac{\alpha}{2}\|v-u\|^2 \quad \text{for all } v \text{ in } K_0.$$

If $v = u + (P^*_{jk}(u))^+$, we have

$$|\lambda| \, \|(P^*_{jk}(u))^+\|^2 \leq \|\varphi_0 + \lambda P^*_{jk}(u)\| \|(P^*_{jk}(u))^+\| + \frac{\alpha}{2}\|(P^*_{jk}(u))^+\|^2 + f_\alpha(v) - f_\alpha(u).$$

On the other hand,

$$\inf\left\{\frac{\|w^+\|}{\|w\|}, w \in \operatorname{span}(e_{j+1}, \dots, e_k)\right\} = \varepsilon > 0,$$

hence

$$\|\lambda P^*_{jk}(u)\| \leq \frac{1}{\varepsilon^2}\left\{\|\varphi_0 + \lambda P^*_{jk}(u)\| + \frac{\alpha}{2}\|P^*_{jk}(u)\| + \frac{|f_\alpha(v) - f_\alpha(u)|}{\|P^*_{jk}(u)\|}\right\}$$

Finally,

$$\begin{aligned}
\frac{f_\alpha(v) - f_\alpha(u)}{\|P^*_{jk}(u)\|} &= \frac{Q'_\alpha(u)((P^*_{jk}(u))^+) + Q_\alpha((P^*_{jk}(u))^+) + \int_\Omega e_1 (P^*_{jk}(u))^+ \, dx}{\|P^*_{jk}(u)\|} \\
&\leq \frac{c(\|u\|_W \|(P^*_{jk}(u))^+\|_W + \|(P^*_{jk}(u))^+\|_W^2 + \lambda_1 \|P^*_{jk}(u)^+\|_W)}{\|P^*_{jk}(u)\|} \\
&\leq \frac{\lambda_k c(\|u\|_W \|P^*_{jk}(u)\| + \lambda_k \|P^*_{jk}(u)\|^2 + \lambda_1 \|P^*_{jk}(u)\|)}{\|P^*_{jk}(u)\|} \\
&= \lambda_k c \|u\|_W + \lambda_k^2 c\gamma + c\lambda_1\lambda_k.
\end{aligned}$$

The assertion follows. □

Lemma 5.5. *Assume that $\alpha > \lambda_1$ and $1 \leq j < k$. Let (α_n) be a sequence in $\mathbb{R}$ which converges to α, (γ_n) be a bounded sequence of positive real numbers. Let (u_n) and (φ_n) be two sequences such that $u_n \in K_0 \cap \partial C_{\gamma_n}(H_j \oplus H_k^\perp)$, $\varphi_n \in \partial^-(f_{\alpha_n} + I_{C_{\gamma_n}(H_j \oplus H_k^\perp)})(u_n)$ for all n and $\sup_n \|\varphi_n\| < \infty$. Then (u_n) is bounded in $W_0^{1,2}(\Omega)$.*

Proof. By Lemma 5.4 there exists a sequence $(\lambda_n)_n$ such that $\lambda_n \leq 0$, $\varphi_{0,n} = \varphi_n - \lambda_n P^*_{jk}(u_n) \in \partial^- f_{\alpha_n}(u_n)$ and for suitable c_1, c_2 in $\mathbb{R}^+$

$$\|\lambda_n P^*_{jk}(u_n)\| \leq c_1\|\varphi_n\| + c_2(1 + \gamma_n + \|u_n\|_W).$$

We wish to apply Lemma 5.1. We have

$$\sup_n \left| \int_\Omega \varphi_{0,n} e_1 \, dx \right| = \sup_n \left| \int_\Omega \varphi_n e_1 \, dx \right| < \infty.$$

Moreover,

$$\int_\Omega \frac{\varphi_{0,n} u_n}{\|u_n\|_W} \, dx = \int_\Omega \frac{\varphi_n u_n}{\|u_n\|_W} \, dx - \lambda_n \int_\Omega \frac{u_n P^*_{jk}(u_n)}{\|u_n\|_W} \, dx.$$

We notice that

$$\begin{aligned}\lambda_n \int_\Omega \frac{u_n P^*_{jk}(u_n)}{\|u_n\|_W}\,dx &= \lambda_n \int_\Omega \frac{\|P^*_{jk}(u_n)\|^2}{\|u_n\|_W}\,dx \\ &\le \frac{c_1\|\varphi_n\| + c_2(1+\gamma_n+\|u_n\|_W)}{\|u_n\|_W}\|P^*_{jk}(u_n)\|\end{aligned}$$

and the last term is bounded. Applying Lemma 5.1 we deduce the assertion. □

Lemma 5.6. *Let i, j in $\mathbb{N}$ be such that $1 \le j \le i$. Let $\varepsilon > 0$, $\delta > 0$ there exist $\sigma > 0$, $\gamma_0 > 0$ such that for every $\alpha \in [\lambda_j + \delta, \lambda_{i+1} + \sigma]$ and for every γ in $]0, \gamma_0]$ the condition $(\nabla)(f_\alpha, C_\gamma(H_j \oplus H_i^\perp), \sup f_\alpha(\partial_{H_j}(K_0 \cap H_j)) + \varepsilon, f_\alpha(\overline{e}_1) - \varepsilon)$ holds.*

Proof. We argue by contradiction and suppose that there exist $\varepsilon > 0$, $\delta > 0$, and four sequences (α_n), (γ_n), (u_n) (φ_n) such that $\alpha_n \to \alpha$ in $[\lambda_j + \delta, \lambda_{i+1}]$, $\gamma_n \to 0$, $(u_n)_n \in K_0 \cap \partial C_{\gamma_n}(H_j \oplus H_i^\perp)$ and $(\varphi_n) \in \partial^-(f_{\alpha_n} + I_{C_\gamma(H_j \oplus H_i^\perp)})(u_n)$ for all n, $\varphi_n \to 0$ and $\sup f_\alpha(\partial_{H_j}(K_0 \cap H_j)) + \varepsilon \le f_{\alpha_n}(u_n) \le f_{\alpha_n}(\overline{e}_1) - \varepsilon$.

By Lemma 5.5, (u_n) is bounded in $W_0^{1,2}(\Omega)$; hence, up to a subsequence, we can suppose that $u_n \to u$ in $L^2(\Omega)$ and $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ for some u in $K_0 \cap (H_j \oplus H_i^\perp)$. In view of Lemma 5.4, there exists $\lambda_n \le 0$ such that $\varphi_n - \lambda_n P^*_{ji}(u_n) \in \partial^- f_{\alpha_n}(u_n)$ and $\lambda_n P^*_{ji}(u_n)$ is bounded. We can suppose, up to a subsequence, that $\lambda_n P^*_{ji}(u_n)$ converges to a vector $\nu \in \operatorname{span}(e_{j+1}, \ldots, e_i)$. Then

$$f_{\alpha_n}(v) \ge f_{\alpha_n}(u) + \langle \varphi_n - \lambda_n P^*_{ji}(u_n), v - u_n\rangle - \frac{\alpha_n}{2}\|v - u_n\|^2 \tag{5.6.1}$$

for all v in K_0. If $v \in H_j \oplus H_i^\perp$, then $\langle \nu, v - u\rangle = 0$ therefore u is a lower critical point of f_α on $H_j \oplus H_i^\perp$. On the other hand taking $v = u$ in (5.6.1) we have

$$\lim_{n\to\infty} f_{\alpha_n}(u_n) = f_\alpha(u)$$

from which we deduce that $f_\alpha(u) \in [\sup f_\alpha(\partial_{H_j}(K_0 \cap H_j)) + \varepsilon, f_\alpha(\overline{e}_1) - \varepsilon]$. This is impossible in view of Lemma 5.3. □

6. Multiple solutions of the asymptotic problem

In this section we consider the asymptotic problem and we state two theorems which play the same role of Theorems 1.1 and 1.2.

Proposition 6.1. *For every $\alpha \in \mathbb{R}^+$, the origin is a local minimum point for the functional f_α.*

Proof. Let us consider the cone $K^* = \{v \in K_0 \mid Q_\alpha(v) \le \|v\|_W^2/4\}$. In $K_0 \setminus K^*$ the thesis is clear. On the other hand we claim that

$$\inf\left\{ \int_\Omega v e_1\,dx \,\middle|\, v \in K^*, \|v\|_W = 1\right\} = c > 0.$$

By contradiction, suppose that there exists a sequence $(v_k)_k$ in K^* with $\|v_k\|_W = 1$ and $\int_\Omega v_k e_1\,dx \to 0$, then we can suppose that $v_k \to v$ in $L^2(\Omega)$ for some v in K^*. Since $\frac{1}{2\alpha} < \int_\Omega v^2\,dx$ then $v \neq 0$ which is impossible. Moreover, if $v \in K^*$, we have

$$f_\alpha(v) = \int_\Omega e_1 v\,dx + Q_\alpha(v) \geq c\|v\|_W + Q_\alpha(v) \geq c\sqrt{\lambda_1}\|v\| - \frac{\alpha}{2}\|v\|_L^2.$$

If $\|v\|$ is small, $f(v) \geq 0$. □

We notice that since Ω has smooth boundary, then $e_1 \in \mathrm{int}_{H_i}(K_0 \cap H_i)$ for $i \geq 1$ (this is due to the Hopf maximum principle).

Lemma 6.2. *Let $k \geq 2$ and $\alpha > \lambda_k$. Then, for every $\varepsilon > 0$, there exist $\tau_0 > 0$, $T_0 > 0$ such that for all τ in $]0,\tau_0]$, for all T in $]0,T_0]$ there exists $\rho > 0$ such that*

$$\sup f_\alpha(\{\tau e_1, Te_1\} \times \mathbf{S}_k^*(0,\rho)) < \varepsilon \tag{6.1}$$

$$\sup f_\alpha(\partial_{H_k}([\tau e_1, Te_1] \times \mathbf{B}_k^*(0,\rho))) < f_\alpha(\overline{e}_1) \tag{6.2}$$

proof. Since $f_\alpha(0) = 0$ and $\lim_{t\to+\infty} f_\alpha(te_1) = -\infty$ then, for all $\tau > 0$ small and all $T > 0$ large, $f_\alpha(te_1) < \varepsilon$ and $f_\alpha(Te_1) < \varepsilon$. Moreover, if τ and T are fixed there exists $\rho > 0$ such that $[\tau e_1, Te_1] \times \mathbf{B}_k^*(0,\rho) \subset K_0$ and (6.1) holds. By (4.2), taking account that $\alpha > \lambda_k$ we deduce (6.2). □

Now we are able to prove the asymptotic version of Theorem 1.1.

Theorem 6.3. *If $\alpha > \lambda_2$, then f_α has at least four lower critical points.*

Proof. The origin and the function $\overline{e}$ are lower critical points of f_α. Let k be an integer such that $k \geq 2$ and $\lambda_k < \alpha \leq \lambda_{k+1}$. By Lemma 6.1 there exists $R > 0$ such that

$$f_\alpha(0) < \inf\{f_\alpha(u) \mid u \in \mathbf{S}(0,R)\} = a''.$$

Let $0 < a < a''$. By Lemma 6.2, there exist $\rho > 0$, τ, T such that $0 < \tau < R < T$, $a' = \sup f_\alpha(\{\tau e_1, Te_1\} \times \mathbf{S}_k^*(0,\rho)) < a$ and (6.2) holds.

Set $b' = \sup f_\alpha([\tau e_1, Te_1] \times \mathbf{S}_k^*(0,\rho)) < f_\alpha(\overline{e}_1)$, take $b' < b < f_\alpha(\overline{e}_1)$, then, by Lemma 5.6, there exists $0 < \gamma < \rho$ such that the condition $(\nabla)(f_\alpha, C_\gamma(H_1 \oplus H_k^\perp), a, b)$ holds. By Lemma 5.4, C_γ and K_0 are not tangent at any points. Moreover, by Proposition 5.2, f_α satisfies $(\mathrm{P.S.})_c$ for all c in $\mathbb{R}$. Using Theorem 3.9 we deduce that there exist at least two lower critical points in $f_\alpha^{-1}([a,b])$. □

Remark 6.4. Use the same notations of the previous theorem. Then for $\alpha > \lambda_{k+1}$, α close to λ_{k+1}, we still have

$$\sup f_\alpha(\{\tau e_1, Te_1\} \times \mathbf{S}_k^*(0,\rho)) < \inf f_\alpha(\mathbf{S}(0,R)),$$
$$(\nabla)(f_\alpha, C_\gamma(H_1 \oplus H_k^\perp), a, b)\text{holds.}$$

Moreover, if $b' = \sup f_\alpha([\tau e_1, T e_1] \times \mathbf{S}^*_k(0,\rho)) < b < f_\alpha(\overline{e}_1)$, then, by Theorem 3.9, f_α still has two lower critical points in $f_\alpha^{-1}([a,b])$ $(0 < a < a'' = \inf f_\alpha(\mathbf{S}(0,R)))$.

Lemma 6.5. *Let $i \geq 1$ and $\alpha \in \mathbb{R}$. Then there exists* $\min f_\alpha(\overline{e}_1 \oplus H_i^\perp)$ *and it continuously depends on α.*

Proof. In view of (4.2) we can evaluate the minimum of Q_α on the convex set $\{u \geq -\overline{e}_1 \mid u \in H_i^\perp\}$. This minimum coincides with the minimum in

$$K_{i,\alpha} = \{u \in H_0^1(\Omega) \mid Q_\alpha(\leq)0,\ u \geq -\overline{e}_1, u \in H_i^\perp\}.$$

We notice that $K_{i,\alpha}$ is bounded in $L^2(\Omega)$, therefore it is bounded in $W_0^{1,2}(\Omega)$ since $\int_\Omega |Du|^2\,dx \leq \alpha \int_\Omega u^2\,dx$. In fact, by contradiction, suppose that there exists a sequence (u_h) in $K_{i,\alpha}$ such that $\|u_h\| \to +\infty$. Then $z_h = u_h/\|u_h\|$ is such that $\|z_h\|_W \leq \alpha$, therefore we can suppose, up to a subsequence, that (z_h) converges in $L^2(\Omega)$ to some z with $\|z\| = 1$, $z \geq 0$ and $z \in H_i^\perp$. This is impossible. At this point it is easy to deduce the thesis. $\square$

Lemma 6.6. *Let k, j be two integers such that $1 \leq j < k$ and $\lambda_j < \lambda_{j+1} = \lambda_k < \lambda_{k+1}$. Let $\rho > 0$ be such that $\mathbf{S}_k(\overline{e}_1, \rho) \subset K_0$. Then there exists $\sigma > 0$ such that for every $\lambda_k < \alpha \leq \lambda_k + \sigma$ we have*

$$\sup f_\alpha(\mathbf{S}_j(\overline{e}_1,\rho)) < \inf f_\alpha(\overline{e}_1 \oplus H_j^\perp), \tag{6.3}$$

$$\sup f_\alpha(\mathbf{S}_k(\overline{e}_1,\rho)) < \inf f_\alpha(\overline{e}_1 \oplus H_k^\perp). \tag{6.4}$$

Proof. If $\lambda_k < \alpha \leq \lambda_{k+1}$ using (4.2) we have

$$\sup f_\alpha(\mathbf{S}_k(\overline{e}_1,\rho)) < f_\alpha(\overline{e}_1) = \inf f_\alpha(\overline{e}_1 \oplus H_k^\perp).$$

Analogously, if $\alpha = \lambda_{j+1} = \lambda_k$, we have

$$\sup f_\alpha(\mathbf{S}_j(\overline{e}_1,\rho)) < \inf f_\alpha(\overline{e}_1 \oplus H_j^\perp).$$

By Lemma 6.5, there exists $\sigma > 0$ such that, if $\lambda_{j+1} = \lambda_k \leq \alpha \leq \lambda_k + \sigma$, the previous inequality holds. The assertion follows. $\square$

Now we are able to prove the asymptotic version of Theorem 1.2.

Theorem 6.7. *If $\lambda_k > \lambda_2$, there exists $\sigma > 0$ such that for every α such that $\lambda_k < \alpha \leq \lambda_k + \sigma$, f_α has at least six lower critical points.*

Proof. We can suppose that $\lambda_k < \lambda_{k+1}$ and choose j such that $\lambda_j < \lambda_{j+1} = \lambda_k$. By Remark 6.4, there exists $b < f_\alpha(\overline{e}_1)$ such that, if σ is enough small, for every α in $]\lambda_{j+1}, \lambda_{j+1} + \sigma]$ there exist two solutions of $(\overline{P})$ in $f_\alpha^{-1}([a,b])$ with $a > f_\alpha(0) = 0$.

On the other hand, by Lemma 6.6, up to shrinking σ, (6.3) and (6.4) hold. By Lemma 5.6, the condition $(\nabla)(f_\alpha, C_\gamma(H_j\oplus H_k^\perp), \widetilde{a}, \widetilde{b})$ holds, where

$$\sup f_\alpha(\mathbf{S}_j(\overline{e}_1,\rho)) < \widetilde{a} < \inf f_\alpha(\overline{e}_1\oplus H_j^\perp),$$
$$\sup f_\alpha(\mathbf{S}_k(\overline{e}_1,\rho)) < \widetilde{b} < \inf f_\alpha(\overline{e}_1\oplus H_k^\perp) = f_\alpha(\overline{e}_1).$$

(Naturally $C_\gamma(H_j\oplus H_k^\perp)$ and K_0 are not tangent at any points and (P.S.)$_c$ holds for every $c\in[\widetilde{a},\widetilde{b}]$.) Using Theorem 3.8, we deduce that there exist two lower critical points in $f_\alpha^{-1}([\widetilde{a},\widetilde{b}])$. If σ is small enough, $\inf f_\alpha(\overline{e}_1\oplus H_j^\perp)$ is close to $f_\alpha(\overline{e}_1)$ which is greater than b: hence we can suppose that $\widetilde{a} > b$. This concludes the proof. □

7. Multiple solutions of problem (P_t)

Now we want to show how the results found in Section 6 for the asymptotic problem $(\overline{P})$ still hold for the problem (P_t) for $t>0$ sufficiently large. More precisely, we will prove that, in the hypothese considered in the introduction, the inequalities stated for the functional f_α in Section 6, also hold for the functional f defined in the introduction. Moreover, we will show that the conditions (P.S.) and (∇) hold for f. The (∇)-theorems of Section 3 will give us the conclusion.

We will use the same notations of the introduction. In particular, if $t>0$

$$K_{\psi/t} = \{u\in W_0^{1,2}(\Omega) \mid u\geq \psi/t \ a.e.\}.$$

In the sequel we will assume that the hypothesis (K) is fulfilled and we will denote by $\overline{u}$ a fixed function of K_ψ.

Proposition 7.1. *$K_{\psi/t}\to K_0$ in the sense of Mosco (see [15]), namely*

1. *if (t_n), (u_n) are two sequences such that $(t_n)_n\to\infty$, $u_n\in K_{\psi/t_n}$ and $(u_n)_n$ weakly converges in $W_0^{1,2}(\Omega)$ to some u, then $u\in K_0$,*
2. *for every u in K_0, there exists a sequence u_n in K_{ψ/t_n} which strongly converges in $W_0^{1,2}(\Omega)$ to u.*

Proof. (1) is very easy. Let us prove (2). If $u\in K_0$, we take $u_n = u\vee\overline{u}/t_n$. Then $u_n\in K_{\psi/t_n}$ and $u_n\to u$ in $W_0^{1,2}(\Omega)$. □

For $t>0$, it will be convenient to consider the "rescaled" functional $h_t : L^2(\Omega)\to\mathbb{R}\cup\{\infty\}$ defined by

$$h_t(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Du|^2\,dx - \frac{1}{t^2}\int_\Omega G(x,tu)\,dx + \int_\Omega ue_1\,dx & \text{if } u\in K_{\psi/t},\\ \infty & \text{otherwise.}\end{cases}$$

Under the hypothese (g) and (G), it is easy to show that given u in $K_{\psi/t}$, then $\varphi \in \partial^- h_t(u)$ if and only if

$$\begin{aligned} (7.1.1) \qquad & \int_\Omega DuD(v-u)\,dx - \frac{1}{t}\int_\Omega g(x,tu)(v-u)\,dx \\ & \qquad + \int_\Omega e_1(v-u)\,dx \geq \int_\Omega \varphi(v-u)\,dx \quad \text{for all } v \text{ in } K_{\psi/t}. \end{aligned}$$

Therefore, u is a lower critical point of h_t if and only if tu verifies (P_t) (that is tu is a lower critical point of f). Under the hypothesis (G) and (g), the functional h_t is of class $C(0,q)$ (see Definition 2.3) for all t in $\mathbb{R}$: that is

$$(7.1.2) \qquad \begin{cases} h_t(v) \geq h_t(u) + \langle \varphi, v-u\rangle - q\|v-u\|^2 \\ \text{for all } u,v \text{ in } K_{\psi/t} \text{ and for all } \varphi \text{ in } \partial^- h_t(u). \end{cases}$$

The following statement is immediate.

Lemma 7.2. *Assume that (g,α) and (g) hold. Then for all c in $\mathbb{R}$ we have*

$$\lim_{t\to\infty} \sup_{\substack{u\in K_{\psi/t}\cap K_0 \\ \|u\|_W \leq c}} |h_t(u) - f_\alpha(u)| = 0.$$

Now we check the conditions (P.S.) and (∇) for the functional h_t.

Lemma 7.3. *Assume that $\alpha > \lambda_1$. Let (t_n), (u_n), (φ_n) be three sequences such that $(t_n) \in \mathbb{R}$ and $\inf_n t_n > 0$, $(u_n)_n \in K_{\psi/t_n}$, $\varphi_n \in \partial^- h_{t_n}(u_n)$ for every n. Suppose that*

$$\sup_n \left| \int_\Omega \varphi_n e_1\,dx \right| < \infty \quad \textit{and} \quad \sup_n \left| \frac{\int_\Omega \varphi_n u_n\,dx}{\|u_n\|_W} \right| < \infty.$$

Then (u_n) is bounded in $W_0^{1,2}(\Omega)$.

Proof. One can proceed as in the proof of Lemma 5.1. □

Using this lemma we can deduce the next proposition.

Lemma 7.4. *Suppose that (g,α) and (g) are fulfilled and assume that $\alpha > \lambda_1$. Let $t > 0$, then the functional h_t satisfies $(\mathrm{P.S.})_c$ for all c in $\mathbb{R}$.*

Lemma 7.5. *Assume that (g) and (G) are fulfilled and let i, j in $\mathbb{N}$ be such that $1 \leq j < k$, $t > 0$ and $\gamma > 0$. Set $\widetilde{h}_t = h_t + I_{C_\gamma(H_j \oplus H_k^\perp)}$. Then*

(a) *for every u in $K_{\psi/t} \cap \partial C_\gamma(H_j \oplus H_k^\perp)$, $K_{\psi/t}$ and $C_\gamma(H_j \oplus H_k^\perp)$ are not tangent at u and*

$$\partial^- \widetilde{h}_t(u) = \partial^- h_t(u) + \{\lambda P_{jk}^*(u) \mid \lambda \leq 0\},$$

(b) *there exist two real numbers* c_1, c_2 *such that if* $u \in \partial C_\gamma(H_j \oplus H_k^\perp) \cap K_{\psi/t}$, $\varphi_0 \in \partial^- h_t(u)$ *and* $\lambda \leq 0$, *then*

$$\|\lambda P_{j,k}^*(u)\| \leq c_1 \|\varphi_0 + \lambda P_{jk}^*(u)\| + c_2\left(1 + \gamma + \frac{1}{t} + \|u\|_W\right),$$

(c) *there exist two continuous functions* $\widetilde{p}, \widetilde{q} : W_0^{1,2}(\Omega) \to \mathbb{R}$ *strongly continuous in* $L^2(\Omega)$ *such that for all* $t \geq 1$,

$$\widetilde{h}_t(v) \geq \widetilde{h}_t(u) + \langle \varphi, v - u \rangle - [\widetilde{p}(u)\|\varphi\| + \widetilde{q}(u)]\|v - u\|^2$$

for all u, v *in* $D(\widetilde{h}_t)$ *and all* φ *in* $\partial^- \widetilde{h}_t(u)$.

Proof. Let us prove (a). Let $u \in K_{\psi/t} \cap \partial C_\gamma(H_j \oplus H_k^\perp)$, we take $\overline{u} = u + (P_{jk}^*(u))^+$ $(\in K_{\psi/t})$ and we have

$$-\int_\Omega (-P_{jk}^*(u))(\overline{u} - u)\, dx = \int_\Omega (P_{jk}^*(u))(P_{jk}^*(u))^+\, dx > 0.$$

By Remark 2.5 the conclusion follows. For the equality of subdifferentials one can proceed as in Theorem 2.6.

(b) can be proved as in Lemma 5.4. (c) can be proved as in Theorem 2.6, by noting that for $t \geq 1$ the constants are independents of t (in some sense $K_{\psi/t}$ and C_γ are "uniformly non tangent"). □

Lemma 7.6. *Assume that* (g, α), (g) *and* (G) *are fulfilled, let* $\alpha > \lambda_1$, *and* $1 \leq j < k$. *Let* (t_n) *be a sequence in* $\mathbb{R}$ *such that* $\inf_n t_n > 0$ *and* $\gamma > 0$. *Let* (u_n) *and* (φ_n) *be two sequences such that* $u_n \in K_{\psi/t_n} \cap \partial C_{\gamma_n}(H_j \oplus H_k^\perp)$, $\varphi_n \in \partial^-(h_{t_n} + I_{C_\gamma(H_j \oplus H_k^\perp)})(u_n)$ *for every* n *and* $\sup_n \|\varphi_n\| < \infty$. *Then* $(u_n)_n$ *is bounded in* $W_0^{1,2}(\Omega)$. (*We can replace* γ *with a bounded sequence* γ_n *and the assertion still hold.*)

Proof. One can proceed as in the proof of Lemma 5.5. □

Lemma 7.7. *Assume that* (g, α), (g) *and* (G) *are fulfilled, let* $\alpha > \lambda_1$, $1 \leq j < k$ *and* $\gamma > 0$. *Let* (t_n), (u_n), (φ_n) *be three sequences such that* $t_n \to \infty$, $u_n \in K_{\psi/t_n} \cap C_\gamma(H_j \oplus H_k^\perp)$, $u_n \to u$ *in* $L^2(\Omega)$, $\varphi_n \in \partial^-(\widetilde{h}_{t_n})(u_n)$ *and* $\varphi_n \rightharpoonup \varphi$ *in* $L^2(\Omega)$. *Then* $u_n \to u$ *in* $W_0^{1,2}(\Omega)$ *and* $\varphi \in \partial^-(\widetilde{f}_\alpha(u))$. (*We are using the notations* $\widetilde{h}_{t_n} = h_{t_n} + I_{C_\gamma(H_j \oplus H_k^\perp)}$, $\widetilde{f}_\alpha = f_\alpha + I_{C_\gamma(H_j \oplus H_k^\perp)}$.)

Proof. By Lemma 7.6, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ and, by Lemma 7.5, there exist two sequences $\varphi_{0,n}$ in $\partial^-(h_{t_n})(u_n)$ and $(\lambda_n)_n$ in $\mathbb{R}$ such that $\varphi_n = \varphi_{0,n} + \lambda_n P_{jk}^*(u_n)$.

In view of (b) of Lemma 7.5, up to a subsequence, $\lambda_n P_{jk}^*(u_n) \to \nu = \lambda P_{jk}^*(u)$ and $\varphi_{0,n}$ is bounded in $L^2(\Omega)$; therefore, up to a subsequence, $\varphi_{0,n} \rightharpoonup \varphi_0$, hence $\varphi = \varphi_0 + \lambda P_{jk}^*(u)$. For every v in $K_{\psi/t_n} \cap C_\gamma(H_j \oplus H_k^\perp)$

$$h_{t_n}(v) \geq h_{t_n}(u_n) + \langle \varphi_{0,n}, v - u_n \rangle - q\|v - u_n\|^2. \tag{7.7.1}$$

Fix a point v in K_0. By Proposition 7.1 there exists a sequence $(v_n)_n$ in K_{ψ/t_n} such that $v_n \to v$ in $W_0^{1,2}(\Omega)$. Then $h_{t_n}(v_n) \to f_\alpha(v)$ as $n \to \infty$. Taking $v = v_n$ in (7.7.1) and using Proposition 7.1 we deduce that

$$f_\alpha(v) \geq f_\alpha(u) + \langle \varphi_0, v - u\rangle - q\|v - u\|^2.$$

Then $\varphi_0 \in \partial^- f_\alpha(u)$ that is $\varphi \in \partial^- \widetilde{f}_\alpha(u)$. On the other hand, by Proposition 7.1, there exists $\overline{u}_n$ in K_{ψ/t_n} such that $\overline{u}_n \to u$ in $W_0^{1,2}(\Omega)$. Taking $v = \overline{u}_n$ in (7.7.1) we deduce that

$$h_{t_n}(\overline{u}_n) \geq h_{t_n}(u_n) + \langle \varphi_{0,n}, \overline{u}_n - u_n\rangle - q\|\overline{u}_n - u_n\|^2$$

and therefore $f_\alpha(u) \geq \limsup_n h_{t_n}(u_n)$. Since $\liminf_n h_{t_n}(u_n) \geq f_\alpha(u)$ we have

$$\lim_{n\to\infty} h_{t_n}(u_n) = f_\alpha(u)$$

from which we deduce that $u_n \to u$ in $W_0^{1,2}(\Omega)$. $\square$

Lemma 7.8. *Let $\alpha > \lambda_1$, $a, b, \gamma \in \mathbb{R}$ be such that $\gamma > 0$, $a < b$. Let j, k in $\mathbb{N}$ be such that $1 \leq j < k$. If $(\nabla)(f_\alpha, C_\gamma(H_j \oplus H_k^\perp), a, b)$ holds, then for t large enough $(\nabla)(h_t, C_\gamma(H_j \oplus H_k^\perp), a, b)$ holds.*

Proof. The assertion follows from Lemmas 7.1, 7.6 and 7.7. $\square$

Now we are able to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let k, R, ρ, τ, T, a, b, γ as in the proof of Theorem 6.3. We divide the proof into several steps.

(I) We have

$$\liminf_{t\to\infty}(\inf h_t(\mathbf{S}(0,R))) \geq \inf f_\alpha(\mathbf{S}(0,R)) > 0.$$

In fact let $(t_n)_n$ be a sequence such that $t_n \to \infty$ and $(u_n)_n$ in $\mathbf{S}(0,R)$ be such that

$$\liminf_{t\to\infty}(\inf h_t(\mathbf{S}(0,R))) = \lim_{n\to\infty}(\inf h_{t_n}(\mathbf{S}(0,R))) = \lim_{n\to\infty} h_{t_n}(u_n).$$

If this limit is equal to ∞ we have finished. Otherwise, since $(u_n)_n$ is bounded in $L^2(\Omega)$ and h_{t_n} is bounded, it is also bounded in $W_0^{1,2}(\Omega)$ and then, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ with $u \in \mathbf{S}(0,R)$. We easily deduce that

$$\lim_{n\to\infty} h_{t_n}(u_n) \geq f_\alpha(u) \geq \inf f_\alpha(\mathbf{S}(0,R)) > 0.$$

(II) For $t > 0$ we define $\pi_t : W_0^{1,2}(\Omega) \to K_{\psi/t}$ putting $\pi_t(u) = u \vee \overline{u}/t$. It is easy to prove that if $t_n \to \infty$ and $u_n \to u$ in $L^2(\Omega)$ (respectively in $W_0^{1,2}(\Omega)$), then $\pi_{t_n}(u_n) \to u^+$ in $L^2(\Omega)$ (respectively, in $W_0^{1,2}(\Omega)$). Moreover, since $\pi_t(0) \to 0$ in $W_0^{1,2}(\Omega)$, we have

$$\lim_{t\to\infty} h_t(\pi_t(0)) = 0.$$

(III) Set

$$\begin{aligned}
\widetilde{S} &= \partial_{H_k}([\tau e_1, Te_1] \times \mathbf{B}_k{}^*(0,\rho)),\\
\Sigma &= [\tau e_1, Te_1] \times \mathbf{S}_k^*(0,\rho),\\
S &= \{\tau e_1, Te_1\} \times \mathbf{S}_k^*(0,\rho)(\subset \widetilde{S}).
\end{aligned}$$

For $t > 0$ set $\Sigma_t = \pi_t(\Sigma)$, $S_t = \pi_t(S)$, $\widetilde{S}_t = \pi_t(\widetilde{S})$.

Now we prove that, up to shrinking ρ, there exists $\overline{t}$ such that for every $t > \overline{t}$ the pair (Σ_t, S_t) is homeomorphic to the pair (Σ, S) (in $L^2(\Omega)$) and moreover $\widetilde{S}_t$ is homeomorphic to $\widetilde{S}$. To this end we prove that for t large, $\pi_{t|\widetilde{S}}$ is injective.

Let $\widetilde{\Omega}$ be an open subset of Ω; we can suppose that ρ is such that

$$\inf_{\substack{x\in\widetilde{\Omega}\\ u\in\widetilde{S}}} u(x) = \eta > 0.$$

Moreover, there exist $M > 0$ and $E \subset \widetilde{\Omega}$ with $\mathcal{L}^n(E) > 0$ such that $\overline{u}(x) \leq M$ for all x in E. Therefore, if $\overline{t} = M/\eta$ for $t > \overline{t}$, u in $\widetilde{S}$, x in E,

$$\pi_t(u)(x) = u(x).$$

Hence, if $u_1, u_2 \in \widetilde{S}$ with $\pi_t(u_1) = \pi_t(u_2)$, then $u_1 = u_2$ in E that implies $u_1 = u_2$.

(IV) For all $t > \overline{t}$ (up to increasing $\overline{t}$) we have

$$S_t \cap \mathbf{S}(0,R) = \emptyset, \qquad \Sigma_t \subset C_\gamma(H_j \oplus H_k^\perp).$$

Let us prove the first equality. Assume, by contradiction that there exist $t_n \to \infty$ and $(u_n)_n$ in $S_{t_n} \cap \mathbf{S}(0,R)$. Therefore there exists $(v_n)_n$ in S such that $u_n = \pi_{t_n}(v_n)$. Since S is compact, up to a subsequence, then $v_n \to v$ in $L^2(\Omega)$ with $v \in S \subset K_0$. Therefore (see (II)) $u_n \to v^+ = v$ and $v \in \mathbf{S}(0,R) \cap S$. This is impossible because $\mathbf{S}(0,R) \cap S = \emptyset$. The rest is similar.

(V) We have

$$\begin{aligned}
\limsup_{t\to\infty}\, (\sup h_t)(\Sigma_t) &\leq \sup f_\alpha(\Sigma),\\
\limsup_{t\to\infty}\, (\sup h_t)(S_t) &\leq \sup f_\alpha(S),\\
\limsup_{t\to\infty}\, (\sup h_t)(\widetilde{S}_t) &\leq \sup f_\alpha(\widetilde{S}).
\end{aligned}$$

We prove, for example the first inequality. Let (t_n), (u_n) be two sequences such that $t_n \to \infty$, $u_n \in \Sigma_{t_n}$ and $\lim_{n\to\infty} h_{t_n}(u_n) = \limsup_{t\to\infty}(\sup h_t(\Sigma_t))$. For every n there exists v_n in Σ such that $u_n = \pi_{t_n}(v_n)$. Since Σ is a compact in $W_0^{1,2}(\Omega)$, up to a subsequence $v_n \to v \in \Sigma$, hence $u_n \to v \vee 0 = v$ in $W_0^{1,2}(\Omega)$. Moreover, since $v_n \in K_0$, then $u_n \in K_0$. By Lemma 7.2 we have

$$\lim_{n\to\infty} |h_{t_n}(u_n) - f_\alpha(u_n)| = 0.$$

Therefore

$$\lim_{n\to\infty} h_{t_n}(u_n) = \lim_{n\to\infty} f_\alpha(u_n) + (h_{t_n}(u_n) - f_\alpha(u_n)) = f_\alpha(v) \le \sup_\Sigma f_\alpha$$

from which the assertion follows.

(VI) We have

$$\lim_{t\to\infty} (\inf h_t(\overline{e}_1 \oplus H_k^\perp)) = f_\alpha(\overline{e}_1).$$

If $t > 0$, we take $u_t = \overline{e}_1 + P_k^\perp(\overline{u})/t$ (where $P_k^\perp$ is the projection on $H_k^\perp$). Then $u_t \in (\overline{e}_1 \oplus H_k^\perp)$, $u_t \to \overline{e}_1$ in $W_0^{1,2}(\Omega)$ and $u_t \in K_{\psi/t}$ for $t > 0$ large. (In fact $u_t = \overline{e}_1 - P_k(\overline{u})/t + \overline{u}/t$ and $\overline{e}_1 > P_k(\overline{u})/t$ for t large). Since $h_t(u_t) \to f_\alpha(\overline{e}_1)$ we deduce that

$$\limsup_{t\to\infty} (\inf h_t(\overline{e}_1 \oplus H_k^\perp)) \le f_\alpha(\overline{e}_1).$$

Let (t_n), (u_n) be two sequences such that $t_n \to \infty$, $u_n \in (\overline{e}_1 \oplus H_k^\perp) \cap K_{\psi/t_n}$ and

$$\lim_{n\to\infty} h_{t_n}(u_n) = \liminf_{n\to\infty} (\inf h_{t_n}(\overline{e}_1 \oplus H_k^\perp)).$$

We prove that $(u_n)_n$ is bounded in $W_0^{1,2}(\Omega)$. By contradiction, we suppose that $\lim_n \|u_n\|_W = \infty$ and we consider $z_n = u_n/\|u_n\|_W$. Up to a subsequence, $z_n \to z$ in $L^2(\Omega)$ and $z_n \rightharpoonup z$ in $W_0^{1,2}(\Omega)$ for some z; then $z \in K_0 \cap H_k^\perp$ that implies $z = 0$. On the other hand

$$\frac{1}{\|u_n\|_W{}^2} h_{t_n}(u_n) = h_{t_n\|u_n\|_W}(z_n) + \int_\Omega z_n \frac{e_1}{\|u_n\|_W}\,dx - \int_\Omega z_n e_1\,dx$$

from which $h_{t_n\|u_n\|_W}(z_n) \to 0$. Since $h_{t_n\|u_n\|_W}(z_n) = 1/2 - o(1)$ we get a contradiction. Hence up to a subsequence, $u_n \to u$ $L^2(\Omega)$ and $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ for some $u \in K_0 \cap (\overline{e}_1 \oplus H_k^\perp)$. Then

$$\lim_n h_{t_n}(u_n) \ge f_\alpha(u) \ge f_\alpha(\overline{e}_1).$$

By (7.8.1) the assertion follows.

(VII) By Lemma 7.8 there exists $\overline{t} > 0$ such that for all $t \ge \overline{t}$ the condition $(\nabla)(h_t, C_\gamma(H_1 \oplus H_k^\perp), a, b)$ holds and if t is large enough, h_t verifies $(\mathrm{P.S.})_c$ for every $c \in [a,b]$.

(VIII) Now we are able to prove the existence of four solutions. By (I) and (II) we deduce that h_t has a local minimum point u_1 in $\mathbf{B}(0,R)$ for a suitable t and $h_t(u_1) < a$. Moreover,

$$\sup \widetilde{h}_t(S_t) = \sup h_t(S_t) < a < \inf h_t(\mathbf{S}(0,R)) \le \inf \widetilde{h}_t(\mathbf{S}(0,R))$$

(where $\widetilde{h}_t = h_t + I_{C_\gamma(H_1 \oplus H_k^\perp)}$) and

$$\sup \widetilde{h}_t(\Sigma_t) = \sup h_t(\Sigma_t) < \infty.$$

By Theorem 3.9 we deduce that there exist two lower critical points u_2, u_3 of $\widetilde{h}_t$ with $a \le \widetilde{h}_t(u_j) \le b$ $(j = 2,3)$. By (VII) u_2, u_3 are not in $\partial C_\gamma(H_1 \oplus H_k^\perp)$, therefore they are lower critical points of h_t. Since $h_t(u_1) < a$ this points are distinct. If t is large, we have

$$\sup h_t(\widetilde{S}_t) < b < \inf h_t(\overline{e}_1 \oplus H_k^{\perp}) = b_1 < \infty.$$

By the classical Saddle Theorem (see [16]) there exists a lower critical point u_4 of h_t with $h_t(u_4) > b_1$. Since $b_1 > b$, $u_4 \neq u_j$ $(j = 1, 2, 3)$. $\square$

Now we want to conclude with a sketch of the proof of Theorem 1.2.

Sketch of the proof of Theorem 1.2. Let k, j, R, ρ, γ, a, b, $\widetilde{a}$, $\widetilde{b}$ as in the proof of Theorem 6.7. We set

$$\begin{aligned} S &= \{\tau e_1, T e_1\} \times \mathbf{S}_j^*(0, \rho), \\ \Sigma &= [\tau e_1, T e_1] \times \mathbf{S}_j^*(0, \rho), \\ \widetilde{S}_j &= \partial_{H_j}([\tau e_1, T e_1] \times \mathbf{B}_j^*(0, \rho)). \end{aligned}$$

Proceeding, as in the previous proof, one can prove (see notations 4.0.3) that for t large (π_t is the same function introduced in the previous proof at step (II))

$$\min h_t(\overline{\mathbf{B}(0, R)}) < a$$

and the local minimum point is in the interior of $\mathbf{B}(0, R)$. Moreover, one can prove that

$$\begin{aligned} \sup h_t(\pi_t(S)) &< a < \inf h_t(\mathbf{S}(0, R)), \\ \sup h_t(\pi_t(\Sigma)) &\leq \sup h_t(\pi_t(\widetilde{S}_j)) < b, \\ \sup h_t(\pi_t(S_k(\rho) \cap \partial C_\gamma)) &< \widetilde{a} < \inf h_t(\overline{e}_1 \oplus H_j^{\perp}), \\ \sup h_t(\pi_t(S_k(\rho))) &< \widetilde{b} < \inf h_t(\overline{e}_1 \oplus H_k^{\perp}). \end{aligned}$$

Furthermore, the conditions $(\nabla)(h_t, C_\gamma(H_1 \oplus H_j^{\perp}), a, b)$ and $(\nabla)(h_t, C_\gamma(H_j \oplus H_k^{\perp}), \widetilde{a}, \widetilde{b})$ hold and the functional h_t satisfies $(\mathrm{P.S.})_c$ for all $c \in \mathbb{R}$. Arguing as in Theorem 6.7 the assertion follows. $\square$

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