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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SUPERQUADRATIC NONCOOPERATIVE VARIATIONAL ELLIPTIC SYSTEMS

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1. Introduction and statement of the main results

Let us consider the noncooperative elliptic system

$$\text{(ES)} \qquad \begin{cases} -\Delta u = \alpha u - \delta v + F_u(u,v) & \text{in } \Omega, \\ \Delta v = -\delta u - \gamma v + F_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary, $\alpha \geq 0$, $\delta \geq 0$, $\gamma \geq 0$ are three real parameters and $F \in C^1(\mathbb{R}^2, \mathbb{R})$.

The solutions of (ES) represent the steady state solutions of reaction-diffusion systems which are derivedfrom several applications, such as mathematical biology or chemical reactions (see for instance [18] and [22]). The following examples are, for instance, particular cases of (ES).

$\lambda - \omega$ **systems.** This kind of system has been widely used as a prototype of reaction-diffusion system

$$(\lambda - \omega) \qquad \begin{cases} -\Delta u = \lambda(r)u - \omega(r)v & \text{in } \Omega, \\ \Delta v = -\omega(r)u - \lambda(r)v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $r = u^2 + v^2$ and $\lambda(r)$, $\omega(r)$ are given functions.

FitzHugh–Nagumo system. The following system provides a diffusive extension of the FitzHugh–Nagumo reduction of the Hodgkin–Huxley model for electrical signaling by nerve cells,

$$\text{(FN)} \qquad \begin{cases} -\Delta u = -\delta v + f(u) & \text{in } \Omega, \\ \Delta v = -\delta u - \gamma v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(u) = u(a-u)(u-1)$ and $0 < a < 1$ is a fixed constant. We will refer to (FN) as to a FitzHugh–Nagumo type system when considering a nonlinearity $f(u)$, different from the above mentioned example.

Note that since we impose that the first order interaction factor (represented by δ) is the same for the two variables, then (ES) presents a variational structure; nevertheless, such a restriction seems to be a reasonable geometric assumption in applications and, furthermore, we allow for different interaction ratios in the higher order terms given by the potential of the nonlinearity $F(u,v)$.

Under such hypotheses it is well known that solutions of (ES) are critical points of the functional $I : H_0^1 \times H_0^1 \to \mathbb{R}$ defined by

$$\begin{aligned} I(u,v) = {} & \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{\alpha}{2}\int_\Omega u^2\,dx \\ & + \frac{\gamma}{2}\int_\Omega v^2\,dx + \delta\int_\Omega uv\,dx - \int_\Omega F(u,v)\,dx. \end{aligned}$$

The existence and multiplicity of solutions for elliptic systems of the (ES) type have been found by several authors under various hypotheses on the parameters and the nonlinearity F, (see for instance [1], [2], [5]–[8], [11], [12] and [7], [9], [13], [20], [21] in the particular case in which $F_u(u,v) = f(u)$ and $F_v(u,v) = g(v)$).

That is, the existence of a nontrivial solution has been shown, under various hypotheses at infinity on the nonlinearity $F(u,v)$, which roughly can be thought of as belonging to one of the following classes: a weakening of the Rabinowitz growth condition at infinity (see (F1) below) as for instance in [5], [7], [21], an imposition of a different growth in the two variables (cf. [8]) or a consideration of critical Sobolev exponent growth (cf. [12]). In all these approaches the kind of hypotheses brought difficulties both for the compactness condition and for the geometry of the problem. A second kind of investigation is related to the multiplicity of nontrivial solutions. In this case, to the best of our knowledge, the known results, except for [16], concern the simplified problem in which $F_u(u,v) = f(u)$ and $F_v(u,v) \equiv 0$, i.e. the FitzHugh–Nagumo type problem, and in such a situation several strong results were proved for instance in [9] and [21]. In this paper, we will extend the result found in [16] for the case of $\gamma = 0 = \delta$, by proving the multiplicity of solutions of the complete system (ES) when the nonlinearity satisfies the usual Rabinowitz superquadratic growth condition by exploiting the classical linking structure. Other comparisons with the literature will be given in the remarks following main Theorems.

We suppose that $F \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies

(F1) $\exists \mu : 2 < \mu < 2N/(N-2)$ if $N > 2$, $\mu > 2$ otherwise, such that
$$0 < \mu F(u,v) \leq u F_u(u,v) + v F_v(u,v) \text{ for every } (u,v) \neq (0,0).$$

Furthermore, suppose that

(F2) $\exists a_1 > 0$, s.t. $|F_u(u,v)| + |F_v(u,v)| \leq a_1(|u|^r + |v|^r)$ where $r = \mu - 1$.

Let's that from (Fi), $i = 1,2$ it follows

(F3)
$$\frac{1}{\|u\|^2 + \|v\|^2} \int_\Omega F(u,v) \to 0 \quad \text{for } \|u\| + \|v\| \to 0,$$

(see Remark 3.7), thus (F3) is redundant, but is used throughout the proofs and hence listed explicitly.

Examples. (1) Let $N = 5$. $F(u,v) = (u^2 + 5v^2)^{3/2}$ satisfies hypotheses (Fi) for $i = 1,2$ with $\mu = 3$. Note that in this case we have a different interaction between the two species (for example) when the number of individuals in the two species is "large". If $F(u,v) = (u^2 + v^2)^{3/2}$ is used, (ES) becomes a $\lambda - \omega$ system.

(2) Let $N = 3$. $F(u,v) = u^4 + v^4$. Such a function satisfies (Fi) for $i = 1,2$ with $\mu = 4$.

Our main results are stated below.

Theorem A. *We suppose that* (Fi), $i = 1,2$ *hold. If* α, γ, δ *are such that*

(a1) $0 \leq \alpha < \lambda_1$,

(a2) *there exists* $j \geq 1$ *such that* $\lambda_j < \gamma < \lambda_{j+1}$,

(a3) $0 \leq \delta < \delta_0 = \min\{\lambda_1 - \alpha, (\gamma - \lambda_j)\lambda_1/\lambda_j\}$,

hold, then there exists at least one nontrivial solution of (ES). *Furthermore, there exists* $\alpha^* \in \mathbb{R}$, $\alpha^* < \lambda_1$ *such that for any* $\alpha \in (\alpha^*, \lambda_1)$, $\delta \in (0, \delta_0/2)$, (ES) *admits at least two nontrivial solutions.*

Remark 1.1. (a) We see that in the general FitzHugh–Nagumo type problem, with f regular, our parameter α coincides with $f'(0)$.

(b) By the monotonicity with respect to the domain of the eigenvalues of the Laplace operator, the hypothesis $\alpha < \lambda_1$ can be read in this contest as a compatibility condition between the existence of two nontrivial solutions for (ES) and the size of the domain.

(c) In [21] the case $F_u(u,v) = f(u)$, $F_v(u,v) \equiv 0$ and $f'(0) = 0$ (i.e. $\alpha = 0$) has been considered. Under such hypotheses and assuming some kind of superquadratic growth on f (more general than the usual Rabinowitz condition and which doesn't impose a sign condition on the primitive) and assuming a condition on the primitive of f in the direction of $t\phi_1$, the author proved the existence of one solution under the same restriction on the range of possible δ's.

(d) In [16] the existence of two nontrivial solutions for (ES) was proved for the case $\gamma = 0 = \delta$.

(e) For the limited aim of proving the existence of one nontrivial solution, it would be sufficient to require

(F1)* $\quad \exists M > 0,\ \exists\, \mu : 2 < \mu < 2N/(N-2)$ if $N > 2$, $\mu > 2$ otherwise,

such that $0 < \mu F(u,v) \le uF_u(u,v) + vF_v(u,v)$, for every $u^2 + v^2 \ge M$,

condition (F3) and the sign condition $F(0,v) \ge 0$ for every $v \in \mathbb{R}$, hence showing the existence of one nontrivial solution for the FitzHugh–Nagumo problem. A similar result has been proved in [7]. For example, if $N = 3$ the nonlinearity $F(u,v) = u^4$ satisfies conditions (F1)* and (F2).

THEOREM B. *Let's suppose that* (Fi), $i = 1,2$ *hold. If* α, γ, δ *are such that*

(b1) *there exists* $k \ge 1$ *such that* $\lambda_k < \alpha < \lambda_{k+1}$,

(b2) *there exists* $j \ge 1$ *such that* $\lambda_j < \gamma < \lambda_{j+1}$,

(b3) $0 \le \delta < \delta_0 = \min\{\lambda_{j+1} - \gamma, \lambda_{k+1} - \alpha, \lambda_1(\alpha - \lambda_k)/\lambda_k, \lambda_1(\gamma - \lambda_j)/\lambda_j\}$,

(b4) $k \le j$,

hold, then there exists at least one nontrivial solution of (ES). *Furthermore, there exist an* $\alpha^* \in \mathbb{R}$ *with* $\alpha^* < \lambda_{k+1}$ *and a* $\gamma^* \in \mathbb{R}$ *with* $\lambda_j < \gamma^*$ *such that* (ES) *admits at least two nontrivial solutions provided* $\alpha \in (\alpha^*, \lambda_{k+1})$ *or* $\gamma \in (\lambda_j, \gamma^*)$, *when* $\delta \in (0, \delta_0/2)$.

REMARK 1.2. (a) Such range of parameters has not been considered in [21], even when considering the special case $F_u(u,v) = f(u)$, $F_v(u,v) \equiv 0$, since there, when $\gamma > \lambda_1$, it is assumed that $\alpha = 0$.

(b) Here and in all the following results for proving the existence of one solution, it would be sufficient to assume (F1)* and the sign condition $F(u,v) \ge 0$ everywhere, which is classical in the linking geometry situation. For example, if $N = 3$, $F(u,v) = u^2v^2$ satisfies this sign condition and (F1)*.

(c) When $k = j$ and $\alpha = \gamma$, this result is applicable to $\lambda - \omega$ systems.

(d) We note that for $\alpha \ne \gamma$ and, for instance, $F(u,v) = (u^2 + v^2)^{3/2}$ the system (ES) describes a "generalized $\lambda - \omega$ system" given by

$$\begin{cases} -\Delta u = \lambda_1(r)u - \omega(r)v & \text{in } \Omega, \\ \Delta v = -\omega(r)u - \lambda_2(r)v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_1, λ_2, ω are given functions.

Theorem C. *Let's suppose that* (Fi), $i = 1, 2$ *hold. If* α, γ, δ *are such that*

(c1) *there exists* $k \geq 1$ *such that* $\lambda_k < \alpha < \lambda_{k+1}$,

(c2) $0 \leq \gamma < \lambda_1$,

(c3) $0 \leq \delta < \delta_0 = \min\{\lambda_1 - \gamma, (\alpha - \lambda_k)\lambda_1/\lambda_k\}$,

hold, then there exists at least one nontrivial solution of (ES). *If furthermore,*

(c4) $\delta < \min\{\delta_0, \lambda_{k+1} - \alpha\}$,

then there exists $\alpha^* \in \mathbb{R}$, $\alpha^* < \lambda_{k+1}$ *such that, for any* $\alpha \in (\alpha^*, \lambda_{k+1})$, (ES) *admits at least two nontrivial solutions.*

Remark 1.3. If one supposes, as in [21], [9], [13], [7], [20], that $F_u = f(u)$, $F_v \equiv 0$, the system becomes the FitzHugh–Nagumo type system. As pursued in [20], [13], [9], in this case the second equation can be solved with respect to v, reducing in such a way the study of an integral differential equation. Several results have been obtained in this context. For example, in [9] the existence of a positive and a negative solution is shown for a suitable range of parameters, while in [21] the existence of three nontrivial solutions, under suitable growth conditions on f and F and the hypotheses $\gamma + \delta < \lambda_1$, $\gamma + 2\delta < \widehat{\lambda}_1 < \lambda_1 + \delta^2/(\lambda_1 - \gamma)$, where $f'(0) = \widehat{\lambda}_1$ (i.e. $\widehat{\lambda}_1 = \alpha$), has been proved.

The main results follow from a critical point theorem (see Theorem 2.1) whose proof is based on the use of the limit relative category defined in [10] and provides a simplified form, suitable for our applications, of Theorem 2.1 of [16].

Remark 1.4. The main idea underlying the critical point result is due to Marino, Micheletti and Pistoia (see [17, Theorem 8.4]) in the case in which one of the two spaces is finite dimensional.

2. Basic definitions and the critical point result

We now recall some basic definitions, the critical point result, and set up some terminology.

Let H be an Hilbert space, together with a sequence of closed subspaces H_n. If A is any subset of H we denote by A_n the set $A \cap H_n$, and given $I \in C^1(H, \mathbb{R})$ we set $I_n = I_{|H_n}$.

Definition 2.1. Given $c \in \mathbb{R}$ we say that I satisfies the Palais–Smale condition at level c, with respect to the sequence H_n, if every sequence $\{x_n\}$ satisfying

$$x_n \in H_n, \quad dI_n(x_n) \to 0, \quad I_n(x_n) \to c,$$

has a convergent subsequence which converges in H to a critical point of I. We will then say that $\{x_n\}$ satisfies the $(\mathrm{PS})^*_c$ condition with respect to H_n.

Let's suppose that $H = W \oplus Z$ and let $H_n = W_n \oplus Z_n$ be a sequence of closed subspaces of H such that

$$1 \le \dim H_n < \infty \quad \text{for each } n \in \mathbb{N}. \tag{2.1}$$

Moreover, we suppose that there exist $e_1 \in \bigcap_{n=1}^{\infty} Z_n$ and $e_2 \in \bigcap_{n=1}^{\infty}(Z \ominus \mathbb{R}e_1)_n$ with $\|e_1\| = \|e_2\| = 1$.

For any Y subspace of H, we consider $B_\rho(Y) := \{u \in Y \mid \|u\| \le \rho\}$ and denote by $\partial B_\rho(Y)$ the boundary of $B_\rho(Y)$ relative to Y. Furthermore, we define, for any $e \in H$,

$$Q_R(Y, e) := \{u + ae \in Y \oplus \mathbb{R}e \mid u \in Y \ a \ge 0,\ \|u + ae\| \le R\},$$

and denote by $\partial Q_R(Y)$ its boundary relative to $Y \oplus \mathbb{R}e$.

Let us restate, in simpler terms, the critical point result proved in [16].

Theorem 2.1. *We suppose that I satisfies the* (PS)* *condition with respect to H_n. In addition, we assume that there exist ρ_i, R_i, $i = 1, 2$, such that $0 < \rho_i < R_i$ and*

$$\sup_{\partial Q_{R_1}(W,e_1)} I < \inf_{\partial B_{\rho_1}(Z)} I, \tag{2.2}$$

$$\sup_{Q_{R_1}(W,e_1)} I < \infty, \qquad \inf_{B_{\rho_1}(Z)} I > -\infty, \tag{2.3}$$

$$\sup_{\partial Q_{R_2}(W\oplus\mathbb{R}e_1,e_2)} I < \inf_{\partial B_{\rho_2}(Z\ominus\mathbb{R}e_1)} I, \tag{2.4}$$

$$\sup_{Q_{R_2}(W\oplus\mathbb{R}e_1,e_2)} I < \infty, \qquad \inf_{B_{\rho_2}(Z\ominus\mathbb{R}e_1)} I > -\infty. \tag{2.5}$$

If $R_1 < R_2$, then there exist at least 3 *critical levels of I. Moreover, the critical levels satisfy the following inequalities*

$$\begin{aligned}\inf_{B_{\rho_1}(Z)} I \le c_1 \le \sup_{\partial Q_{R_1}(W,e_1)} I < \inf_{\partial B_{\rho_1}(Z)} I \le c_2 \le \sup_{Q_{R_1}(W,e_1)} I \\ \le \sup_{\partial Q_{R_2}(W\oplus\mathbb{R}e_1,e_2)} I < \inf_{\partial B_{\rho_2}(Z\ominus\mathbb{R}e_1)} I \le c_3 \le \sup_{Q_{R_2}(W\oplus\mathbb{R}e_1,e_2)} I.\end{aligned}$$

Proof. It is presented in [16] and it is based on the remark that in any linking geometrical situation one can define two different classes of admissible minimax sets

$$\Gamma_1 = \{A \mid A \text{ closed, } Y_1 = \partial Q_R \subset A,\ \operatorname{cat}^{\infty}_{H,Y_1}(A) \ge 1\},$$

and

$$\Gamma_2 = \{A \mid A \text{ closed, } Y_2 = \partial B_\rho \subset A,\ \operatorname{cat}^{\infty}_{H,Y_2}(A) \ge 1\},$$

and thus one gets two distinct critical levels, (see [14, Theorem 1]).

In our situation, hence the first set of linking hypotheses (2.2), (2.3), gives rise to the existence of two critical levels $\widehat{c}_1$, $\widehat{c}_2$ such that

$$\inf_{B_{\rho_1}} I \leq \widehat{c}_1 \leq \sup_{\partial Q_{R_1}} I < \inf_{\partial B_{\rho_1}} I \leq \widehat{c}_2 \leq \sup_{Q_{R_1}} I, \tag{2.6}$$

and analogously the second set of linking hypotheses (2.4), (2.5), gives rise to the existence of $\check{c}_1$, $\check{c}_2$ satisfying

$$\inf_{B_{\rho_2}} I \leq \check{c}_1 \leq \sup_{\partial Q_{R_2}} I < \inf_{\partial B_{\rho_2}} I \leq \check{c}_2 \leq \sup_{Q_{R_2}} I. \tag{2.7}$$

But since $R_1 < R_2$ one gets

$$\sup_{Q_{R_1}} I \leq \sup_{\partial Q_{R_2}} I,$$

and hence, combining (2.6) and (2.7) we get the result. □

3. Proof of the main results

We recall that a (PS) type condition is needed in order to get a deformation lemma, which will allow us to use the minimax principle. In our situation we use, as minimax classes, sets with limit relative category bigger than one, and hence a $(\mathrm{PS})^*$ condition is needed to get the right kind of deformation lemma.

Let $H = H^1_0(\Omega) \times H^1_0(\Omega)$. It is very easy to see that if we denote by $-\boldsymbol{\Delta} = (-\Delta, \Delta)$, the eigenvalues of $-\boldsymbol{\Delta}$ on H are given by $\lambda^{\pm}_i = \pm\lambda_i(-\Delta)$ and the associated eigenfunctions by $e^+_i = (e_i, 0)$, $e^-_i = (0, e_i)$, where $\lambda_i(-\Delta)$ denotes the i-th eigenvalue of the Laplace operator on H^1_0, with the associated eigenfunction e_i.

Proposition 3.1. *If* (F1) *and* (F2) *hold, then for any $\alpha \geq 0$, $\gamma \geq 0$, $\delta \geq 0$ the functional I satisfies the* $(\mathrm{PS})^*$ *condition with respect to the family of finite dimentional subspaces*

$$E_n = \mathrm{Span}\{e^-_n, \dots, e^-_1, e^+_1, \dots, e^+_n\}, \quad \text{for } n \geq 1.$$

Proof. The proof of the a priori bounds and the strong convergence of the $(\mathrm{PS})^*$ sequence follows the standard superquadratic growth arguments, see for instance [19] or [14]. That the strong limit is a critical point of the complete functional follows the argument in Proposition 7 of [16]. □

We remark that, as shown in [1], in order to get a deformation lemma suitable to get all the limit relative category results, one could have assumed a weaker kind of (PS) condition parallel in this context to the one introduced in [3], which would have allowed a weaker kind of hypotheses on the nonlinearity introduced in [4] in the case of a single equation and hence applied in [5], [7], [1].

Now we only have to prove that the geometrical conditions of Theorem 2.1 are satisfied for suitable choices of the splitting of the space H. To this end let us consider the quadratic form associated to I,

$$Q(u,v) = \frac{1}{2}\|u\|^2 - \frac{\alpha}{2}\int_\Omega u^2\,dx - \frac{1}{2}\|v\|^2 + \frac{\gamma}{2}\int_\Omega v^2\,dx + \delta\int_\Omega uv\,dx. \tag{Q}$$

Geometrical conditions of Theorem A. We define $W = \{0\} \times V^+$ and $Z = H_0^1 \times V^-$, where $V^- = \operatorname{Span}\{e_1, \dots, e_j\}$ and $V^+ = H_0^1 \ominus V^-$.

Proposition 3.2. *If the parameters α, γ, δ satisfy*

(a1) $0 \le \alpha < \lambda_1$,

(a2) *there exists* $j \ge 1$ *such that* $\lambda_j < \gamma < \lambda_{j+1}$,

(a3) $0 \le \delta < \delta_0 = \min\{\lambda_1 - \alpha, (\gamma - \lambda_j)\lambda_1/\lambda_j\}$,

then Q is positive definite on $Z = H_0^1 \times V^-$ and negative definite on $W = \{0\} \times V^+$.

Proof. In fact, on W and taking into account condition (a2) and the Poincarè inequality we get

$$Q(0,v) \le \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 \le 0.$$

On the other hand, on Z we have by conditions (ai) for $i = 1,2,3$, by means of the Poincarè and Young inequalities, that

$$Q(u,v) \ge \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_1} - \frac{\delta}{\lambda_1}\bigg)\|u\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_j} - \frac{\delta}{\lambda_1}\bigg)\|v\|^2,$$

and hence the result follows by condition (a3). □

Proposition 3.3. *If conditions* (a2) *and* (F1) *hold, then*

$$\sup_W I \le 0.$$

Proof. Indeed, by Proposition 3.2 and (F1) one has

$$I(0,v) = Q(0,v) - \int_\Omega F(0,v)\,dx \le 0.$$ □

Proposition 3.4. *If the conditions* (ai)*, with $i = 1,2$ and* (F1) *hold, then for any subspace $X \subset V^-$ there exists $R^* > 0$ such that for every $R > R^*$*

$$\sup_{\partial B_R(W \oplus Z^*)} I < 0,$$

where $Z^* = X \times \{0\}$.

Proof. First of all, by (F1) we can deduce that there exists a positive constant b_1 such that for every $(u,v) \in \mathbb{R}^2$ it holds that

$$F(u,v) \ge b_1(|u|^\mu + |v|^\mu). \tag{3.1}$$

Since all the norms are equivalent on X and $\int_\Omega uv = 0$, for every $u \in X$, for every $v \in V^+$ by applying (3.1) one has for each $(u,v) \in W \oplus Z^*$,

$$\begin{aligned} I(u,v) &= Q(u,v) - \int_\Omega F(u,v)\,dx \leq \\ &\leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 - b_1\int_\Omega |v|^\mu - b_1\int_\Omega |u|^\mu \\ &\leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 - c_1\|u\|^\mu. \end{aligned}$$

Hence, we have that

$$I(u,v) \to -\infty \quad \text{for } \|u\|^2 + \|v\|^2 \to \infty. \qquad \square$$

PROPOSITION 3.5. *If* (F1), (F2) *and conditions* (ai) *with* $i = 1,2,3$ *hold, then there exists* $\rho > 0$ *such that*

$$\inf_{\partial B_\rho(Z)} I > 0.$$

PROOF. Indeed, by (F3), one knows that for any $z = (u,v) \in Z$ and for any $\varepsilon > 0$ there exists $\rho = \rho(\varepsilon) > 0$ such that

$$\|u\| + \|v\| \leq \rho \Rightarrow \int_\Omega F(u,v)\,dx \leq \varepsilon(\|u\|^2 + \|v\|^2). \tag{3.2}$$

By Proposition 3.2 one gets

$$I(u,v) \geq \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_1} - \frac{\delta}{\lambda_1}\bigg)\|u\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_j} - \frac{\delta}{\lambda_1}\bigg)\|v\|^2 - \int_\Omega F(u,v)\,dx.$$

Since, by condition (a3),

$$A = \frac{1}{2}\min\bigg\{1 - \frac{\alpha}{\lambda_1} - \frac{\delta}{\lambda_1}, -1 + \frac{\gamma}{\lambda_j} - \frac{\delta}{\lambda_1}\bigg\} > 0,$$

we get, by applying (3.2) with any $0 < \varepsilon^{1/2} < A/2$, that there exists a $\rho > 0$ such that for any $(u,v) \in Z$ with $\|u\| + \|v\| = \rho$

$$I(u,v) \geq (A - \varepsilon^{1/2})(\|u\|^2 + \|v\|^2) \geq (A/2)\rho^2 > 0. \tag{3.3}$$

$\square$

THEOREM 3.6. *If* (F1), (F2) *and* (ai), $i = 1,2,3$ *hold, then there exists at least one non trivial solution of* (ES).

PROOF. In fact, Proposition 3.4 and 3.5 imply the geometrical conditions (2.2), (2.3) for suitable ρ_1 and R_1. Thus, by the linking theorem, there exists at least two critical levels, thus a nontrivial solution. $\square$

REMARK 3.7. It is obvious that the result of Proposition 3.5 will hold on any subset $\widehat{Z}$ of Z. Furthermore, it is clear that for γ fixed, there exists $\alpha_1 \in (0, \lambda_1)$

such that, for any $\alpha \in (\alpha_1, \lambda_1)$ one has $\delta_0 = \lambda_1 - \alpha$. Hence, for $\delta \in (0, \delta_0/2)$, we can say that (3.3) holds with

$$\frac{\lambda_1 - \alpha}{4\lambda_1} < A \leq \frac{\lambda_1 - \alpha}{2\lambda_1}. \tag{3.4}$$

Furthermore, by (F1) and (F2), we can precisely estimate that

$$\frac{1}{\|u\|^2 + \|v\|^2} \int_\Omega F(u,v) \leq C_1(\|u\| + \|v\|)^{\mu-2},$$

with $C_1 = \mu^{-1} a_1 2^{3-\mu}$ and hence, fixing $\varepsilon^{1/2}$, the corresponding ρ is given by

$$\rho = \varepsilon^{1/2(\mu-2)}/C_1.$$

Therefore, for $0 < \varepsilon = A^2/4$, we can estimate precisely (3.3) by means of (3.4) as

$$\inf_{\partial B_\rho(Z \ominus \mathbb{R}e_1^+)} I(u,v) \geq \frac{1}{8\lambda_1 C_1^2} (\lambda_1 - \alpha)^{(\mu-1)/(\mu-2)} > 0.$$

Proposition 3.8. *If* (Fi), (ai), $i = 1, 2$ *hold, then there exists* $\alpha^* \in \mathbb{R}$, $0 < \alpha^* < \lambda_1$ *such that for any* $\alpha \in \mathbb{R}$, $\alpha \in (\alpha^*, \lambda_1)$ *and* $\delta \in (0, \delta_0/2)$, *there exists a* $\rho > 0$ *such that*

$$\inf_{\partial B_\rho(Z \ominus \mathbb{R}e_1^+)} I > \sup_{W \oplus \mathbb{R}e_1^+} I.$$

Proof. In fact, by (3.1), for any $a \in \mathbb{R}$, for every $v \in V^+$, since $\int_\Omega e_1 v\, dx = 0$ one has

$$\begin{aligned} I(ae_1, v) &\leq \frac{a^2}{2}\|e_1\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 - \frac{\alpha}{2}\int_\Omega (ae_1)^2\, dx \\ &\quad - b_1 \int_\Omega |ae_1|^\mu\, dx - b_1 \int_\Omega |v|^\mu\, dx \\ &\leq \frac{a^2}{2}\bigg(1 - \frac{\alpha}{\lambda_1}\bigg)\|e_1\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 - b_1|a|^\mu \int_\Omega |e_1|^\mu \\ &\leq + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 + a^2 K_1 - |a|^\mu K_2, \end{aligned}$$

where

$$K_1 = \frac{(\lambda_1 - \alpha)}{2\lambda_1}\|e_1\|^2 \quad \text{and} \quad K_2 = b_1 \int_\Omega |e_1|^\mu.$$

Let us consider, for any $\alpha \in (0, \lambda_1)$, the smooth real valued function g_α defined by

$$g_\alpha(a) = K_1 a^2 - |a|^\mu K_2.$$

Since, for any $\alpha \in (0, \lambda_1)$, $g_\alpha(a) \to -\infty$ for $a \to \infty$, it will admit a maximum point

$$a^\alpha_{\max} = \bigg(\frac{2K_1}{\mu K_2}\bigg)^{1/(\mu-2)},$$

and we get

$$\sup_{W\oplus\mathbb{R}e_1^+} I \le g(a_{\max}^\alpha) = \left(\frac{2}{\mu K_2}\right)^{2/(\mu-2)}\left(1-\frac{2}{\mu}\right)K_1^{\mu/(\mu-2)} = C_2(\lambda_1-\alpha)^{\mu/(\mu-2)},$$

with

$$C_2 = \left(\frac{2}{\mu K_2}\right)^{2/(\mu-2)}\left(1-\frac{2}{\mu}\right)\frac{\|e_1\|^{2\mu/(\mu-2)}}{(2\lambda_1)^{\mu/(\mu-2)}}.$$

Thus

$$g_\alpha(a_{\max}^\alpha) = C_2(\lambda_1-\alpha)^{\mu/(\mu-2)} \searrow 0 \quad \text{for } \alpha \nearrow \lambda_1^-, \tag{3.5}$$

with the same speed of $(\lambda_1-\alpha)^{\mu/(\mu-2)}$. On the other hand, by the Remark 3.7 it is clear that, for any $\alpha \in (\alpha_1, \lambda_1)$, there exists $\rho > 0$ such that, denoting by $C_3 = 1/(8\lambda_1 C_1^2)$

$$\inf_{B_\rho(Z\ominus\mathbb{R}e_1^+)} I \ge C_3(\lambda_1-\alpha)^{(\mu-1)/(\mu-2)} > 0,$$

and hence, by (3.5), the lemma follows for α^* sufficiently near to λ_1, since

$$\mu/(\mu-2) > (\mu-1)/(\mu-2). \qquad \square$$

Proof of Theorem A. By Theorem 3.6 we can find $R_1 > 0$ and $\rho_1 > 0$ such that (2.2) and (2.3) hold. On the other hand, by Propositions 3.4, 3.5, Remark 3.7 and Proposition 3.8, we can find $R_2 > 0$ and $\rho_2 > 0$ such that (2.4) and (2.5) hold and, by Proposition 3.4, we can fix $R_1 < R_2$, and thus the result follows. $\square$

Geometrical conditions of Theorem B. We define $W = U^- \times V^+$ and $Z = U^+ \times V^-$, where $U^- = \mathrm{Span}\{e_1, \dots, e_k\}$, $U^+ = \mathrm{Span}\{e_{k+1}, \dots\}$, and $V^- = \mathrm{Span}\{e_1, \dots, e_j\}$, $V^+ = \mathrm{Span}\{e_{j+1}, \dots\}$.

Lemma 3.9. *If the parameters α, γ, δ satisfy the following hypotheses*

(b1) *there exists $k \ge 1$ such that $\lambda_k < \alpha < \lambda_{k+1}$,*

(b2) *there exists $j \ge 1$ such that $\lambda_j < \gamma < \lambda_{j+1}$,*

(b3) $0 \le \delta < \delta_0 = \min\{\lambda_{j+1}-\gamma, \lambda_{k+1}-\alpha, \lambda_1(\alpha-\lambda_k)/\lambda_k, \lambda_1(\gamma-\lambda_j)/\lambda_j\}$,

then Q is negative definite on $W = U^- \times V^+$ and positive definite on $Z = U^+ \times V^-$.

Proof. By the Poincarè inequality and (b3) for all $(u,v) \in W$ we get

$$Q(u,v) \le \frac{1}{2}\left(1-\frac{\alpha}{\lambda_k}+\frac{\delta}{\lambda_1}\right)\|u\|^2 + \frac{1}{2}\left(-1+\frac{\gamma}{\lambda_{j+1}}+\frac{\delta}{\lambda_{j+1}}\right)\|v\|^2 < 0,$$

while, for every $(u,v) \in Z$, it hold that

$$\begin{aligned} Q(u,v) &\geq \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_{k+1}} - \frac{\delta}{\lambda_{k+1}}\bigg)\|u\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_j} - \frac{\delta}{\lambda_1}\bigg)\|v\|^2 \\ &> A(\|u\|^2 + \|v\|^2) > 0, \end{aligned}$$

where

$$A = \frac{1}{2}\min\bigg\{1 - \frac{\alpha}{\lambda_{k+1}} - \frac{\delta}{\lambda_{k+1}}, -1 + \frac{\gamma}{\lambda_j} - \frac{\delta}{\lambda_1}\bigg\},$$

thus the result is found. □

We now have to analyze the behaviour of the complete functional.

PROPOSITION 3.10. *If* (F1), (F2), (bi), $i = 1,2,3$ *and*

(b4) $k \leq j$

hold, then $\sup_W I \leq 0$ *and there exists* $\rho > 0$ *such that* $\inf_{\partial B_\rho(Z)} I > 0$.

PROOF. Indeed for all $(u,v) \in W$, by Lemma 3.9 and (F1),

$$I(u,v) = Q(u,v) - \int_\Omega F(u,v) \leq 0.$$

On the other hand, by (3.2) and Lemma 3.9, we get, for any $0 < \varepsilon^{1/2} < A/2$ there exists $\rho > 0$ such that for any $(u,v) \in Z$, $\|u\| + \|v\| = \rho$,

$$I(u,v) = Q(u,v) - \int_\Omega F(u,v) \geq (A - \varepsilon^{1/2})(\|u\|^2 + \|v\|^2) \geq (A/2)\rho^2 > 0. \quad \square$$

REMARK 3.11. Following ideas similar to those in Remark 3.7 one can get that there exists $\alpha_2 < \lambda_{k+1}$ such that for $\alpha \in (\alpha_2, \lambda_{k+1})$ and $\delta \in (0, \delta_0/2)$, the estimate

$$\frac{\lambda_{k+1} - \alpha}{4\lambda_{k+1}} < A \leq \frac{\lambda_{k+1} - \alpha}{2\lambda_{k+1}},$$

holds and hence for $0 < \varepsilon = A^2/4$ there exists $\rho > 0$ such that

$$\inf_{\partial B_\rho(Z \ominus \mathbb{R}e_{k+1}^+)} I(u,v) \geq \frac{\lambda_{k+1} - \alpha}{8\lambda_{k+1}}\rho^2 \geq \frac{1}{8\lambda_{k+1}C_1^2}(\lambda_{k+1} - \alpha)^{(\mu-1)/\mu-2)} > 0,$$

where C_1 is defined in Remark 3.7.

Similarly, there exists $\gamma_1 > \lambda_j$ such that, for any $\gamma \in (\lambda_j, \gamma_1)$ one has

$$\frac{\gamma - \lambda_j}{4\lambda_j} < A \leq \frac{\gamma - \lambda_j}{2\lambda_j},$$

and hence for $0 < \varepsilon = A^2/4$ there exists $\rho > 0$ such that

$$\inf_{\partial B_\rho(Z \ominus \mathbb{R}e_j^-)} I(u,v) \geq \frac{\gamma - \lambda_j}{8\lambda_j C_1^2}(\gamma - \lambda_j)^{(\mu-1)/(\mu-2)} > 0.$$

To end the proof of the linking theorem we have to find a suitable direction $z_1 \in Z$ such that it is possible to build the set $Q(W, z_1)$ on which (2.2) is satisfied. We prove

PROPOSITION 3.12. *Let* (bi), $i = 1, 2, 3, 4$ *and* (F1), (F2) *hold. If we consider* $e^+_{k+1} \in Z$ *and* $e^-_j \in Z$*; then there exists* $R > 0$ *such that*

$$\sup_{\partial Q_R(W, e^+_{k+1})} I \le 0, \tag{3.6$_a$}$$

and

$$\sup_{\partial Q_R(W, e^-_j)} I \le 0. \tag{3.6$_b$}$$

Furthermore,

$$\sup_{W \oplus \mathbb{R}e^+_{k+1}} I \searrow 0 \quad \text{for } \alpha \nearrow \lambda^-_{k+1}, \tag{3.7$_a$}$$

and

$$\sup_{W \oplus \mathbb{R}e^-_j} I \searrow 0 \quad \text{for } \gamma \searrow \lambda^+_j. \tag{3.7$_b$}$$

PROOF. The idea behind the proof of (3.6$_a$) and (3.6$_b$) is the same. For simplicity we will just show the first case. First, we suppose that $k < j$. One knows that $\int_\Omega (u^- + e_{k+1})v = 0$, for every $v \in V^+$, $a \in \mathbb{R}$, $u^- \in U^-$ by (b4). Therefore, for every $(u, v) \in W \oplus \mathbb{R}e^+_{k+1}$ we get

$$\begin{aligned} I(u,v) = I(u^- + ae_{k+1}, v)\frac{1}{2}\|u^-\|^2 + \frac{a}{2}\|e_{k+1}\|^2 - \frac{\alpha}{2}\int_\Omega (u^- + ae_{k+1})^2 \\ + \frac{1}{2}\bigg(-\|v\|^2 + \gamma\int_\Omega |v|^2\bigg) - \int_\Omega F(u,v). \end{aligned}$$

Hence, by (F1) and the Hölder inequality, we get

$$\begin{aligned} &I(u^- + ae_{k+1}, v) \\ &\quad\le \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_k}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 \\ &\qquad + \frac{a^2}{2}\bigg(1 - \frac{\alpha}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 - b_1\int_\Omega |u^- + ae_{k+1}|^\mu - b_1\int_\Omega |v|^\mu \\ &\quad\le \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_k}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma}{\lambda_j}\bigg)\|v\|^2 \\ &\qquad + \frac{a^2}{2}\bigg(1 - \frac{\alpha}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 - b_1^*\bigg(\int_\Omega |u^-|^2 + a^2|e_{k+1}|^2\bigg)^{\mu/2}, \end{aligned} \tag{3.8}$$

where b_1^* is a suitable positive constant. Therefore

$$\begin{aligned} I(u^- + ae_{k+1}, v) &\leq \frac{a^2}{2}\biggl(1 - \frac{\alpha}{\lambda_{k+1}}\biggr)\|e_{k+1}\|^2 - b_1^*|a|^\mu\biggl(\int_\Omega |e_{k+1}|^2\biggr)^{\mu/2} \\ &\leq a^2 K_1 - |a|^\mu K_2, \end{aligned}$$

where

$$K_1 = \frac{1}{2}\biggl(\frac{\lambda_{k+1} - \alpha}{\lambda_{k+1}}\biggr)\|e_{k+1}\|^2 \quad \text{and} \quad K_2 = b_1^*\biggl(\int_\Omega |e_{k+1}|^2\biggr)^{\mu/2}, \tag{3.9}$$

are positive constants. Hence from (3.8)

$$I(u^- + ae_{k+1}) \to -\infty \quad \text{for } \|u + ae_{k+1}\| + \|v\| \to \infty,$$

and hence this proves (3.6_a) in the case where $k < j$.

On the other hand, the case $k = j$ can be handled in a similar way, taking into account that in this case it is not true that $\int_\Omega (u^- + ae_{k+1})v = 0$ for every $v \in V^+$. In conclusion, in this case one gets that for every $(u^- + ae_{k+1}, v) \in W \oplus \mathbb{R}e_{k+1}^+$ it holds

$$\begin{aligned} I(u^- + ae_{k+1}, v) \leq{}& \frac{1}{2}\biggl(1 - \frac{\alpha}{\lambda_k}\biggr)\|u^-\|^2 + \frac{a^2}{2}\biggl(1 - \frac{\alpha}{\lambda_k}\biggr)\|e_{k+1}\|^2 \\ &+ \frac{1}{2}\biggl(-1 + \frac{\gamma}{\lambda_{k+1}}\biggr)\|v\|^2 + \frac{\gamma}{2}\int_\Omega v^2 \\ &+ \delta\int_\Omega (u^- + ae_{k+1})v - \int_\Omega F(u,v) \\ \leq{}& \frac{1}{2}\biggl(1 - \frac{\alpha}{\lambda_k}\biggr)\|u^-\|^2 + \frac{a^2}{2}\biggl(1 - \frac{\alpha}{\lambda_{k+1}} + \frac{\delta}{\lambda_{k+1}}\biggr)\|e_{k+1}\|^2 \\ &+ \frac{1}{2}\biggl(-1 + \frac{\gamma}{\lambda_{k+1}} + \frac{\delta}{\lambda_{k+1}}\biggr)\|v\|^2 - b_1^*|a|^\mu\biggl(\int_\Omega |e_{k+1}|^2\biggr)^{\mu/2} \\ \leq{}& a^2 K_1 - |a|^\mu K_2, \end{aligned}$$

where in this case, by (b3)

$$0 < K_1 = \frac{1}{2}\biggl(1 - \frac{\alpha}{\lambda_{k+1}} + \frac{\delta}{\lambda_{k+1}}\biggr)\|e_{k+1}\|^2 < \frac{\lambda_{k+1} - \alpha}{\lambda_{k+1}}, \tag{3.10}$$

and K_2 is given as before and, as before, this proves (3.6_a).

The idea behind the proof of (3.7_a) and (3.7_b) is the same. For simplicity we will just show the case a. As in the proof of Proposition 3.11 denoting by

$$g_\alpha(a) = K_1 a^2 - K_2|a|^\mu,$$

where K_2 is given in (3.9) and K_1 in (3.10), we take into account the last inequality in (3.10)

$$\begin{aligned}\sup_{W\oplus\mathbb{R}e_{k+1}^+} I &\le g(a_{\max}^\alpha) = K_1^{2/(\mu-2)}\left(\frac{2}{\mu K_2}\right)^{2/(\mu-2)}\left(1-\frac{2}{\mu}\right) \\ &< C_3(\lambda_{k+1}-\alpha)^{\mu/(\mu-2)},\end{aligned}\tag{3.11}$$

where

$$C_3 = \left(\frac{2}{\mu K_2}\right)^{2/(\mu-2)}\left(1-\frac{2}{\mu}\right)\frac{1}{\lambda_{k+1}^{\mu/(\mu-2)}}.$$

By (3.9) or (3.10) it is then clear that

$$g_\alpha(a_{\max}^\alpha) = C_3(\lambda_{k+1}-\alpha)^{\mu/(\mu-2)} \searrow 0 \quad \text{for } \alpha \nearrow \lambda_{k+1}^-,\tag{3.12}$$

and hence (3.7_a) follows. □

Theorem 3.13. *If* (F1), (F2) *and* (bi), *for* $i = 1,2,3,4$ *hold, then there exists at least one solution of* (ES).

Proof. Proposition 3.10 and (3.6) provide the necessary estimates to apply the linking theorem, which will give rise to the existence of a nontrivial solution. □

Proposition 3.14. *If we suppose* (F1), (F2) *and* (bi) *with* $i = 1,2,3,4$ *hold, then there exists* $\alpha^* < \lambda_{k+1}$ *such that for any* $\alpha^* < \alpha < \lambda_{k+1}$ *there exists a* $\rho \in \mathbb{R}$ *for which*

$$\inf_{\partial B_\rho(Z\ominus\mathbb{R}e_{k+1}^+)} I > \sup_{W\oplus\mathbb{R}e_{k+1}^+} I,\tag{3.13}$$

and there exists a $\gamma^* > \lambda_j$ *such that any* $\lambda_j < \gamma < \gamma^*$ *there exists a* $\rho \in \mathbb{R}$ *for which*

$$\inf_{\partial B_\rho(Z\ominus\mathbb{R}e_j^-)} I > \sup_{W\oplus\mathbb{R}e_j^-} I.\tag{3.14}$$

Proof. Inequality (3.13) follows immediately from (3.7_a) and the second inequality in Remark 3.11, while (3.14) follows from (3.7_b) and the last inequality in Remark 3.11. □

Proposition 3.15. *If* (F1) *and* (bi) *with* $i = 1,2,3,4$ *hold, then there exists* R^* *such that for every* $R > R^*$

$$\sup_{\partial B_R(W\oplus\mathbb{R}e_{k+1}^+\oplus\mathbb{R}e_j^-)} I \le 0.$$

Proof. If $k < j$, by ($b4$) for every $u^- \in U^-$, $v \in V^+$, $a \in \mathbb{R}$

$$\int_\Omega (u^- + ae_{k+1})v = 0 = \int_\Omega u^- e_j.$$

By the Poincaré and Hölder inequalities and (F1) we have

$$\begin{aligned}
I(u^- &+ae_{k+1}, be_j + v) \\
&\le \frac{1}{2}\bigg(1-\frac{\alpha}{\lambda_k}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1+\frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 \\
&\quad + \frac{a^2}{2}\bigg(1-\frac{\alpha}{\lambda_{k+1}}+\frac{\delta}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 + \frac{b^2}{2}\bigg(-1+\frac{\gamma}{\lambda_j}+\frac{\delta}{\lambda_j}\bigg)\|e_j\|^2 \\
&\quad - b_1\bigg(\int_\Omega |u^-|^2 + a^2e_{k+1}^2\bigg)^{\mu/2} - b_1\bigg(\int_\Omega b^2e_j^2 + |v|^2\bigg)^{\mu/2} \\
&\le \frac{1}{2}\bigg(1-\frac{\alpha}{\lambda_k}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1+\frac{\gamma}{\lambda_{j+1}}\bigg)\|v\|^2 + K_1a^2\|e_{k+1}\|^2 \\
&\quad + K_2b^2\|e_j\|^2 - K_3|a|^\mu\|e_{k+1}\|^\mu - K_4|b|^\mu\|e_j\|^\mu,
\end{aligned}$$

where K_1, K_2, K_3, K_4 are suitable positive constants and hence, taking into account (b1) and (b2),

$$I(u^- + ae_{k+1}, be_j + v) \to -\infty \quad \text{for } \|u^- + ae_{k+1}\| + \|be_j + v\| \to \infty,$$

and thus the result is found. On the other hand, if $k = j$, for every $(u,v) \in W \oplus \mathbb{R}e_{k+1}^+ \oplus \mathbb{R}e_k^- = (U^- \oplus \mathbb{R}e_{k+1} \times (\mathbb{R}e_k \oplus U^+)$ one has

$$\int_\Omega uv = \int_\Omega \Big(u^- + ae_{k+1}\Big)\Big(be_k + v^+\Big) = b\int_\Omega u^-e_k + a\int_\Omega e_{k+1}v^+,$$

thus, with the usual kind of computations one gets

$$\begin{aligned}
I(u,v) \le{}& \frac{1}{2}\bigg(1-\frac{\alpha}{\lambda_k}+\frac{\delta}{\lambda_k}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1+\frac{\gamma}{\lambda_{k+1}}+\frac{\delta}{\lambda_{k+1}}\bigg)\|v^+\|^2 \\
&+ \frac{a^2}{2}\bigg(1-\frac{\alpha}{\lambda_{k+1}}+\frac{\delta}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 + \frac{a^2}{2}\bigg(-1+\frac{\gamma}{\lambda_k}+\frac{\delta}{\lambda_k}\bigg)\|e_k\|^2 \\
&- b_1\int_\Omega |u^- + ae_{k+1}|^\mu - b_1\int_\Omega |be_k + v^+|^\mu,
\end{aligned}$$

and as before $I(u,v) \to -\infty$ for $\|u^- + ae_{k+1}\| + \|be_k + v^+\| \to \infty$ and hence the result is inferred. □

Proof of Theorem B. By Theorem 3.13 we can find $R_1 > 0$ and $\rho_1 > 0$ such that (2.2) and (2.3) hold. On the other hand, by Propositions 3.10, 3.14 and 3.15, we can find $R_2 > 0$ and $\rho_2 > 0$ such that (2.4) and (2.5) hold, and by Proposition 3.15 we can fix $R_1 < R_2$, and thus we find the result. Obviously one has to consider (3.13) for obtaining the result relative to $\alpha \nearrow \lambda_{k+1}^-$, while (3.14) for the result relative to $\gamma \searrow \lambda_j^+$. □

Geometrical conditions of Theorem C. We define $W = U^- \times H_0^1$, $Z = U^+ \times \{0\}$, where $U^- = \mathrm{Span}\,\{e_1, \ldots, e_k\}$ while $U^+ = H_0^1 \ominus U^-$.

Proposition 3.16. *If* (F1), (F3) *and*

(c1) $\lambda_k < \alpha < \lambda_{k+1}$,

(c2) $0 \le \gamma < \lambda_1$,

(c3) $0 \le \delta < \delta_0 = \min\{\lambda_1 - \gamma, (\alpha - \lambda_k)\lambda_1/\lambda_k\}$,

hold, then $\sup_W I \le 0$ *and there exists* $\rho > 0$ *such that* $\inf_{\partial B_\rho(Z)} > 0$.

Proof. In fact (F1), (c1)–(c3) imply that for every $(u,v) \in W$ it holds that

$$I(u,v) \le \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_k} + \frac{\delta}{\lambda_1}\bigg)\|u\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma+\delta}{\lambda_1}\bigg)\|v\|^2 - \int_\Omega F(u,v) \le 0,$$

while (F3), (c1) and (c3) imply that for every $0 < \varepsilon^{1/2} = (\lambda_{k+1} - \alpha)/(4\lambda_{k+1})$ there exists a $\rho > 0$ such that for every $(u,0) \in Z$ with $\|u\| = \rho$ one gets

$$I(u,0) \ge \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_{k+1}}\bigg)\|u\|^2 - \varepsilon^{1/2}\|u\|^2 = \frac{1}{4\lambda_{k+1}}(\lambda_{k+1} - \alpha)\|u\|^2 > 0,$$

and hence the proof is completed. □

Proposition 3.17. *Let* (ci), $i = 1,2,3$ *and* (F1) *hold. If we consider* $e^+_{k+1} \in Z$*, then there exists* $R > 0$ *such that*

$$\sup_{\partial Q_R(W, e^+_{k+1})} I(u,v) \le 0, \tag{3.15}$$

and if

(c4) $0 \le \delta < \min\{\delta_0, \lambda_{k+1} - \alpha\}$,

holds, then

$$\sup_{W \oplus \mathbb{R}e^+_{k+1}} I(u,v) \searrow 0 \quad \text{for } \alpha \nearrow \lambda^-_{k+1}. \tag{3.16}$$

Proof. Indeed for any $u^- \in U^-$, $a \in \mathbb{R}$, $v \in H^1_0$ it holds, using the Poincaré and Hölder inequalities,

$$\begin{aligned}
I(u^-&+ae_{k+1}, v) \\
&\le \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_k} + \frac{\delta}{\lambda_1}\bigg)\|u^-\|^2 + \frac{a^2}{2}\bigg(\|e_{k+1}\|^2 + (\delta - \alpha)\int_\Omega e^2_{k+1}\bigg) \\
&\quad + \frac{1}{2}\bigg(-1 + \frac{\gamma+\delta}{\lambda_1}\bigg)\|v\|^2 - b_1\int_\Omega |u + ae_{k+1}|^\mu \le \\
&\le \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_k} + \frac{\delta}{\lambda_1}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma+\delta}{\lambda_1}\bigg)\|v\|^2 \\
&\quad + \frac{a^2}{2}\bigg(1 - \frac{\alpha}{\lambda_{k+1}} + \frac{\delta}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 - b^*_1\bigg(\int_\Omega |u^- + ae_{k+1}|^2\bigg)^{\mu/2} \\
&\le \frac{1}{2}\bigg(1 - \frac{\alpha}{\lambda_k} + \frac{\delta}{\lambda_1}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1 + \frac{\gamma+\delta}{\lambda_1}\bigg)\|v\|^2 + a^2K_1 - |a|^\mu K_2,
\end{aligned}$$

where b_1^* is suitable positive constant and

$$K_1 = \frac{1}{2}\bigg(\frac{\lambda_{k+1}-\alpha+\delta}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 \quad \text{and} \quad K_2 = b_1^*\bigg(\int_\Omega |e_{k+1}|^2\bigg)^{\mu/2}. \tag{3.17}$$

Hence (3.15) follows for $\|u^- + ae_{k+1}\| + \|v\| \to \infty$.

To prove (3.16), we define $g(a) = a^2K_1 - |a|^\mu K_2$ where K_1 and K_2 are given in (3.17) and by (c4) one has $0 < K_1 < (\lambda_{k+1} - \alpha)/\lambda_{k+1}$. Thus the usual argument gives the needed result. □

THEOREM 3.18. *If* (F1) *and* (ci), *for* $i = 1, 2, 3$ *hold, then there exists at least one solution of* (ES).

PROOF. Proposition 3.16 and (3.15) provide the necessary estimates to apply the linking theorem. □

PROPOSITION 3.19. *If* (F1) *and* (ci) *with* $i = 1, 2, 3, 4$ *hold, then there exists* R^* *such that for every* $R > R^*$

$$\sup_{\partial B_R(W\oplus\mathbb{R}e^+_{k+1}\oplus\mathbb{R}e^+_{k+2})} I \le 0.$$

PROOF. Completing the usual estimates for every $(u, v) \in W\oplus\mathbb{R}e^+_{k+1}\oplus\mathbb{R}e^+_{k+2}$ we get that

$$\begin{aligned}
I(u^-&+ae_{k+1}+be_{k+2}, v)\\
&\le \frac{1}{2}\bigg(1-\frac{\alpha}{\lambda_{k+1}}+\frac{\delta}{\lambda_1}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1+\frac{\gamma}{\lambda_1}+\frac{\delta}{\lambda_1}\bigg)\|v\|^2\\
&\quad + \frac{a^2}{2}\bigg(1-\frac{\alpha}{\lambda_{k+1}}+\frac{\delta}{\lambda_{k+1}}\bigg)\|e_{k+1}\|^2 + \frac{b^2}{2}\bigg(1-\frac{\alpha}{\lambda_{k+2}}+\frac{\delta}{\lambda_{k+2}}\bigg)\|e_{k+2}\|^2\\
&\quad - b_1^*\bigg(\int_\Omega |u^- + ae_{k+1} + be_{k+2}|^2\bigg)^{\mu/2}\\
&\le \frac{1}{2}\bigg(1-\frac{\alpha}{\lambda_{k+1}}+\frac{\delta}{\lambda_1}\bigg)\|u^-\|^2 + \frac{1}{2}\bigg(-1+\frac{\gamma}{\lambda_1}+\frac{\delta}{\lambda_1}\bigg)\|v\|^2\\
&\quad + a^2K_1 - |a|^\mu K_2 + b^2K_3 - |b|^\mu K_4,
\end{aligned}$$

where K_i, $i = 1, 2, 3, 4$ are suitable positive constants. From the above the result follows directly. □

PROPOSITION 3.20. *We suppose* (F1), (F2) *and* (ci) *with* $i = 1, 2, 3$ *hold. Then there exists* $\alpha^* < \lambda_{k+1}$ *such that for any* $\alpha \in (\alpha^*, \lambda_{k+1})$ *there exists* $\rho \in \mathbb{R}$ *for which*

$$\inf_{\partial B_\rho(Z\ominus\mathbb{R}e^+_{k+1})} I > \sup_{W\oplus\mathbb{R}e^+_{k+1}} I. \tag{3.16}$$

Proof. It follows immediately from (3.16), and Proposition 3.16 and the usual comparison between the different speed with which the two quantities go to zero. □

Proof of Theorem C. By Theorem 3.18 we can find $R_1 > 0$ and $\rho_1 > 0$ such that (2.2) and (2.3) hold. On the other hand, by Proposition 3.20 we can find $R_2 > 0$ and $\rho_2 > 0$ such that (2.4) and (2.5) hold and, by Proposition 3.19, we can fix $R_1 < R_2$, and thus the result is found. □

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