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ON CLASSICAL SOLVABILITY OF THE FIRST INITIAL-BOUNDARY VALUE PROBLEM FOR EQUATIONS GENERATED BY CURVATURES

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Dedicated to Jürgen Moser

1. Main theorem and estimations in $\mathbb{C}^2$

The aim of this paper is to prove the existence theorem announced in [5]. The proof is based on á priori estimates which were done in [6]–[8] for solutions to equations including the equations from [5]. We have to add to these estimates the estimates of Hölder constants for u_t and $u_{x_ix_j}$. Section 2 is devoted to this purpose.

We study the problems

$$M_m[u] = 3D - \frac{u_t}{\sqrt{1+u_x^2}} + f_m(k[u]) = 3Dg \quad \text{in } Q_T = 3D\Omega \times (0,T), \tag{1.1}$$

$$u = 3D\varphi \quad \text{on } \partial' Q_T,\ m \in [2,n], \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\partial' Q_T = 3D\partial'' Q_T \cup \Omega(0)$, $\partial'' Q_T = 3D\partial\Omega \times [0,T]$, $\Omega(0) = 3D\{z = 3D(x,t) \mid x \in \Omega, t = 3D0\}$. Functions $g : \overline{Q}_T \to \mathbb{R}^1$ and $\varphi : \overline{\Omega} \to \mathbb{R}^1$ are known, and function

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$u : \overline{Q}_T \to \mathbb{R}^1$ is to be found. Besides, $k = 3D(k_1, \dots , k_n) \in \mathbb{R}^n$,

$$f_m(k) = 3DS_m^{1/m}(k) \quad \text{and} \quad S_m(k) = 3D \sum_{i_1<\dots<i_m} k_{i_1} \dots k_{i_m}.$$

We consider f_m on the cone

$$\Gamma_m^{(n)} = 3D\{k \in \mathbb{R}^n \mid S_l(k) > 0,\ l = 3D1, \dots , m\}. \tag{1.3}$$

The numbers $k_i[u](x, t)$, $i = 3D1, \dots , n$, forming

$$k[u](x, t) = 3D(k_1[u], \dots , k_n[u])(x, t),$$

are the principal curvatures in the point (x, t) of the hypersurface $\mathcal{T}_t \subset \mathbb{R}^{n+1}$, given by equation

$$x_{n+1} = 3Du(x, t), \quad x \in \overline{\Omega}. \tag{1.4}$$

Number t plays here the role of a parameter.

Let us define by K_m the set of all functions v continuous with their derivatives v_{x_i}, $v_{x_ix_j}$ in $\overline{Q}_T$ and such that $k[v](z)$ belong to $\Gamma_m^{(n)}$ for all $z \in \overline{Q}_T$. We will say that $v : \overline{Q}_T \to \mathbb{R}^1$ is *admissible* for M_m if v belongs to K_m, and v is an admissible solution of (1.1) if v belongs to K_m, has v_t belonging to $C(\overline{Q}_T)$ and satisfies (1.1).

In this paper we deal only with admissible solutions and sometimes will omit the word "admissible". In many places we write Q instead of Q_T.

We will use more abridged notation then in [14], they are close to the notation in [13]. Namely, $C(Q_T)$ and $C(\overline{Q}_T)$ are sets of functions continuous on Q_T or $\overline{Q}_T$ correspondingly. The norm in $C(\overline{Q}_T)$ will be denoted by $\|\cdot\|_{\infty,Q_T}$. $C^\alpha(Q_T)$ and $C^\alpha(\overline{Q}_T)$, $\alpha \in (0, 1)$, are sets of functions from $C(Q_T)$ or $C(\overline{Q}_T)$ correspondingly which are α-Hölder continuous in Q_T or $\overline{Q}_T$ with respect to parabolic distance

$$\rho(z, z') = 3D|x - x'| + |t - t'|^{1/2}, \quad z = 3D(x, t),\ z' = 3D(x', t').$$

The norm in $C^\alpha(\overline{Q}_T)$ will be denoted by

$$|u|_{C^\alpha,Q_T} = 3D|u|_{\infty,Q_T} + \langle u\rangle^{(\alpha)}_{Q_T},$$

where

$$\langle u\rangle^{(\alpha)}_{Q_T} = 3D \sup_{z,z'\in Q_T} \frac{|u(z) - u(z')|}{\rho(z, z')}.$$

The number $\langle u\rangle^{(\alpha)}_{Q_T} < u^{(\alpha)}_{Q_T}$ is named Hölder constant for u and α — its Hölder power. $C^2(\overline{Q}_T)$ and $C^{2+\alpha}(\overline{Q}_T)$ are Banach spaces of all elements u of $C(\overline{Q}_T)$ for which u_x, u_{xx} and u_t belong to $C(\overline{Q}_T)$ or $C^\alpha(\overline{Q}_T)$ respectively.

The norm of u in $C^2(\overline{Q}_T)$ is determined by equality

$$|u|_{C^2,Q_T} = 3D\|u\|_{\infty,Q_T} + \|u_x\|_{\infty,Q_T} + \|u_{xx}\|_{\infty,Q_T} + \|u_t\|_{\infty,Q},$$

and the norm of u in $C^{2+\alpha}(\overline{Q}_T)$ is determined by similar equality in which $\|\cdot\|_{\infty,Q_T}$ is replaced by $|\cdot|_{C^\alpha,Q_t}$.

We suppose that boundary $\partial\Omega$ has $C^{4+\alpha}$- smoothness and $\partial\Omega \in \Gamma_m^{(n-1)}$. This means that in a small vicinity of any point x^0 of $\partial\Omega$ the surface $\partial\Omega$ in $\mathbb{R}^n$ can be presented as

$$(1.6_1)\qquad x_n = 3D\omega(\widetilde{x}), \quad \widetilde{x} = 3D(x_1,\dots,x_{n-1}) \in \overline{B}_d(x^0) = 3D\{\widetilde{x}\|\,|\widetilde{x}| \le d\},$$

in Cartesian coordinates $(\widetilde{x}, x_n)$ corresponding to x^0. The latter means that x^0 is the origin of these coordinates (i. e. $x^0 = 3D(0,\dots,0)$) and the axis x_n is directed along the inner normal to $\partial\Omega$ at the point x^0. (The function ω depends on x^0 but we do not indicate this explicitly). Moreover, we will take axis $x_1,\dots,x_{n-1}$ such that

$$(1.6_2)\qquad \omega(\widetilde{x}) = 3D\frac{1}{2}\sum_{\alpha=3D1}^{n-1}\lambda_\alpha(x^0)x_\alpha^2 + O(|\widetilde{x}|^3),$$

for $\widetilde{x} \in \overline{B}_d(x^0)$. Obviously, the numbers $\lambda_1(x^0),\dots,\lambda_{n-1}(x^0)$ are the principal curvatures of $\partial\Omega$ in x^0. The hypotheses $\partial\Omega \subset C^k$ or $C^{k+\alpha}$ imply that ω belongs to $C^k(\overline{B}_d(x^0)$ or $C^{k+\alpha}(\overline{B}_\alpha(x^0))$ for all $x^0 \in \partial\Omega$ and the hypothesis $\partial\Omega \in \Gamma_m^{(n-1)}$ implies that vectors $\lambda(x^0) = 3D(\lambda_1(x^0),\dots,\lambda_{n-1}(x^0))$ belong to $\Gamma_m^{(n-1)}$ for all $x^0 \in \partial\Omega$. Here

$$\Gamma_m^{(n-1)} = 3D\{k \in \mathbb{R}^{n-1} \mid S_l(k) > 0,\ l = 3D1,\dots,m\},$$

is a convex cone in $\mathbb{R}^{n-1}$.

Number $|\partial\Omega|_{C^k}$ is $\sup\limits_{x^0\in\partial\Omega}|\omega|_{C^k,B_d(x^0)}$, $|\partial\Omega|_{C^{k+\alpha}}$ is $\sup\limits_{x^0\in\partial\Omega}|\omega|_{C^{k+\alpha},B_d(x^0)}$ with a number $\alpha \in (0,1)$, number $d > 0$ being common for all $x^0 \in \partial\Omega$.

To find an admissible solution u of problem (1.1), (1.2) we use the continuation by parameter $\tau \in [0,1]$ in the following form. We consider the family of problems

$$(1.1_\tau)\qquad M_m[u^\tau] = 3Dg^\tau \quad \text{in } Q_T,$$

$$(1.2_\tau)\qquad u^\tau = 3D\varphi^\tau \quad \text{on } \partial' Q_T,$$

where $\varphi^0(x) = 3D\varphi(x,0)$, $\varphi^\tau(x,t) = 3D\tau\varphi(x,t) + (1-\tau)\varphi^0(x)$, $g^\tau(x,t) = 3D\tau g(x,t) + (1-\tau)g^0(x)$, $g^0(x) = 3Df_m(k[\varphi^0])$.

For $\tau = 3D1$ this problem coincides with problem (1.1), (1.2) and for $\tau = 3D0$ the problem (1.1_τ), (1.2_τ) has a unique solution $u^0(x,t) = 3D\varphi^0(x)$. Besides, we have to suppose that $\varphi^0 \in K_m$. The other necessary conditions for the existence of an admissible solution to (1.1), (1.2) are the compatibility conditions of zero and first orders. The first condition we include in the conjection that φ belongs to $C^{2+\alpha}(\overline{Q}_T)$ and $u = 3D\varphi$ on $\partial' Q_T$. The second condition is expressed in the standard form

$$(1.7)\quad -\frac{\varphi_t}{\sqrt{1+\varphi_x^2}} + f_m(k[\varphi]) = 3Dg \quad \text{on } \partial^2_{tx} Q = 3D\{(x,t) \mid x \in \partial\Omega,\ t = 3D0\}.$$

It is easy to see that for problem (1.1_τ), (1.2_τ) with any $\tau \in [0,1]$ the compatibility conditions of zero and first orders are fulfilled.

Now we formulate our main theorem.

Theorem 1.1. *Problem* (1.1), (1.2) *has a unique admissible solution u belonging to $C^{2+\beta}(\overline{Q}_T)$ with some $\beta \in (0,\alpha]$ and having the derivatives u_x, u_{xx}, u_t belonging to $C^{2+\alpha}(Q_T)$ if the following conditions are met.*

(a) $\partial\Omega \in \Gamma^{m-1} \cap C^{4+\alpha}$, $\varphi \in C^{4+\alpha}(\overline{Q}_T)$, $\varphi^0 \in K_m$, $g \in C^{2+\alpha}(\overline{Q}_T)$ *and the compatibility condition* (1.7) *is satisfied,*

(b) $\inf_{Q_T} g \geq 0$, $\inf_{\partial' Q_T} u_t + \inf_{Q_T} g \equiv \nu_1$ *and* $g_t \leq 0$ *on* $\overline{Q}_T$ *with a* $\nu_1 > 0$,

(c) *for all admissible solutions u^τ of problems* (1.1_τ), (1.2_τ), $\tau \in [0,1]$ *there is a common minorant ν_2 in inequality*

$$(1.8)\qquad \inf_{\tau\in[0,1]} \inf_{\partial'' Q_T} \frac{\partial u^\tau}{\partial n} \geq \nu_2.$$

Each problem (1.1_τ), (1.2_τ) *also has a unique admissible solution u^τ with the same smoothness as the solution u of problem* (1.1), (1.2).

In (1.8) $\frac{\partial u^\tau}{\partial n}(x^0,t)$, $x^0 \in \partial\Omega$, $t \in [0,T]$ is the derivative of u^τ at point (x^0,t) along the inner normal $\vec{n}$ to $\partial\Omega$ in x^0.

It is easy to check that for each problem (1.1_τ), (1.2_τ) all requirements of (a) and (b) are satisfied, but minorant $\widetilde{\nu}_1$ for

$$\min_{\partial' Q_T} u_t^\tau + \inf_{Q_T} g^\tau,$$

is equal to $\min\{\nu_0;\nu_1\}$, where

$$\nu_0 = 3D \inf g^0 = 3D \inf_{\Omega} f_m(k[\varphi^0]).$$

Thus, for (1.1_τ), (1.2_τ) the inequalities

(b_τ) $\inf_{Q_T} g^\tau \geq 0$, $\quad \inf_{\partial' Q_T} u_t^\tau + \inf_{Q_T} g^\tau \geq \widetilde{\nu}_1 = 3D\min\{\nu_0;\nu_1\} > 0, g_t^\tau \leq 0$,

follow from (b).

Theorem 1.1 is a slightly improved version of Theorem 1 from [5]. Its proof will be given here.

The statement about uniqueness holds due to the following known comparison theorem (see, for example, [8]).

Theorem 1.2. *If u and v belong to $C^2(Q_T) \cap C(\overline{Q}_T) \cap K_m$ and $M_m[u] \geq M_m[v]$ in Q_T then*

$$\sup_{Q_T}(u - v) = 3D \sup_{\partial' Q_T}(u - v).$$

This theorem was used when we derived estimates for derivatives of admissible solutions u. In Section 3 of paper [8], devoted to the estimation of $|u_x|$, we have formulated a sufficient condition on data when we could find a minorant ν_2 for $\partial u/\partial n|_{\partial'' Q_T}$. It has the form

$$\overline{g} \equiv \max_{\partial'' Q_T} g < \mu, \tag{1.9}$$

where μ is a constant determined by data. For $m \in [2, n-1]$

$$\mu = 3D \inf_{x^0 \in \partial\Omega} S_m^{1/m}(\lambda)(x^0). \tag{1.9'}$$

For $m = 3Dn$ majorant μ depends not only on $\lambda(x^0)$, $x^0 \in \partial\Omega$, but on majorants for $|g_x|_{\infty,\widetilde{Q}}$, $|\varphi|_{C^2,\widetilde{Q}}$ and $|\partial\Omega|_{C^3}$, where $\widetilde{Q} = 3D\widetilde{\Omega}_d \times [0, T]$ and $\widetilde{\Omega}_d = 3D\{x \in \Omega \mid \mathrm{dist}\{x; \partial\Omega\} \leq d\}$. It depends also on majorant $\widetilde{\mu}$ for $\sup_{\widetilde{Q}} |\varphi - u|$. The minorant ν_2 for $\partial u/\partial n|_{\partial'' Q_T}$ depends on $\widetilde{\mu}$ in all cases $m \in [2, n]$. But, as we show below, a minorant $\widetilde{\mu}$ is easily calculated (see (1.11) and (1.12)).

For problems (1.1_τ), (1.2_τ) the results of [8] lead to the following statement.

Proposition 1.3. *Let the conditions* (a) *and* (b) *of Theorem* 1.1 *be fulfilled. If*

$$\max_{\tau \in (0,1]} \max_{\partial'' Q_T} g^\tau \equiv \max_{\partial'' Q_T} \max\{g(x, t); \max f_m(k[\varphi^0])(x)\} < \mu,$$

where μ is a constant determined by data then there is a common minorant ν_2 in (1.8) *that can be calculated. For $n \in [2, m-1]$ number μ is given in* (1.9′).

We remark that here we have applied results of [8] to problems (1.1_τ), (1.2_τ), $\tau \in [0, 1]$, and used (1.9′).

As it is well known, the most difficult part of proving the existence theorems is obtaining proper á priori estimates for all possible solutions of the problem under investigation and of some auxiliary problems connected with it. In our case the role of these auxiliary problems play problems (1.1_τ), (1.2_τ), $\tau \in [0, 1]$. For all admissible solutions u^τ of these problems it is desirable to find a majorant c in the inequality

$$\sup_{\tau \in [0,T]} |u^\tau|_{C^{2+\beta}, Q_T} \leq c, \tag{1.10}$$

with a $\beta > 0$. Such estimate is sufficient for the validity of the last affirmation of Theorem 1.1 on solvability of problems (1.1_τ), (1.2_τ).

We explain this in detail in Section 3.

In papers [6]–[8] we obtained estimates of $|u|_{C^2,Q_T}$ for solutions u of problem (1.1), (1.2) using only the properties of data φ, g and $\partial\Omega$ indicated in Theorem 1.1. Since all these properties are valid for data φ^τ, g^τ and $\partial\Omega$ of problems (1.1_τ), (1.2_τ), $\tau \in [0,1]$, we can apply results of [6]–[8] to these problems. Let us remind those which are useful for our purposes now.

As it was pointed out in Section 2 of [6], the hypothesis $g_t \le 0$ implies the estimate $u_t(z) \ge \min_{\partial' Q_T} u_t$ for all $z \in \overline{Q}_T$. For g^τ we have $g^\tau_t = 3D\tau g_t \le 0$ and therefore

$$u^\tau_t(z) \ge \min_{\partial' Q_T} u^\tau_t \ge \inf_{\tau\in[0,1]} \min_{\partial' Q_T} u^\tau_t \equiv \nu_3. \tag{1.11}$$

This gives the following minorants for u^τ

$$u^\tau(x,t) \ge \varphi^0(x) + \int_0^t u^\tau_\xi(x,\xi)\,d\xi \ge \inf_\Omega \varphi^0 + \inf_{[0,T]} (\nu_3 t) \equiv \nu_4. \tag{1.12}$$

To estimate u^τ and $\partial u^\tau/\partial n|_{\partial'' Q}$ from above we keep in mind the inequalities

$$S_1(k[u^\tau])(z) > 0, \quad z \in \overline{Q}_T, \tag{1.13}$$

which are valid for any element of K_m and for admissible solutions in particular. These inequalities and the comparison principle for operator S_1 guarantee the inequalities

$$u^\tau(x,t) \le \widetilde{u}(x,t), \quad x \in \widetilde{\Omega}, \tag{1.14_1}$$

and

$$\frac{\partial u^\tau}{\partial n} \le \frac{\partial \widetilde{u}}{\partial n} \quad \text{on } \partial'' Q_T, \tag{1.14_2}$$

where $\widetilde{u}$ are solutions of problems

$$S_1(k[\widetilde{u}])(x,t) = 3D0, \quad x \in \Omega, \qquad \widetilde{u}(x,t) = 3D\varphi(x,t), \quad x \in \partial\Omega. \tag{1.14_3}$$

In [20] (see also [3]) the classical solvability of this problem was obtained for any smooth function φ if only $\partial\Omega \in \Gamma_1^{(n-1)}$. In particular, there was done an estimate for $|\widetilde{u}(\,\cdot\,,t)|_{C^1,\Omega}$. We can use these estimates as condition $\partial\Omega \subset \Gamma_1^{(n-1)}$ follows from our assumption $\partial\Omega \subset \Gamma_m^{(n-1)}$. Thus, the estimates

$$\sup_Q u^\tau \le \sup_Q \widetilde{u} \equiv \nu_5, \tag{1.15_1}$$

and

$$\sup_{\partial'' Q} \frac{\partial u^\tau}{\partial n} \le \sup_{\partial'' Q} \frac{\partial \widetilde{u}}{\partial n} \equiv \nu_6, \tag{1.15_2}$$

follow from (1.14_k), $k = 3D1, 2, \dots$. From (1.15_k), (1.8) and (1.12) we draw the conclusions

$$\sup_{\tau\in[0,1]} \|u^\tau\|_{\infty,Q} \le \nu_7, \quad \sup_{t\in[0,1]} \|u^\tau_x\|_{\infty,\partial' Q} \le \nu_7. \tag{1.16}$$

Now we can use Theorem 1.2 from [8]. It guarantees the estimate

$$\|u^\tau_x\|_{\infty,Q} \le \Phi(c_0^{-1}, \|u^\tau\|_{\infty,Q}, \|u^\tau_x\|_{\infty,\partial' Q}, \|g^\tau\|_{\infty,Q}, \|g^\tau_x\|_{\infty,Q}) \le \nu_8, \tag{1.17}$$

where Φ is a continuous nondecreasing function of indicated arguments. Theorem 2.1 of [8] gives an analogous estimate for $\|u^\tau_x\|_{\infty,Q}$ with a majorant Φ, which does not depend of T but depends on c_0^{-1}, where c_0 is taken from the inequality

$$\frac{\partial f_m(k)}{\partial k_1} \ge c_0 > 0 \quad \text{for all } k \in \Gamma^{(n)}_m \text{ with } k_1 \le 0. \tag{1.18}$$

Inequality (1.18) can be extracted from several papers. In [1] it was proved for k satisfying additional inequalities $o < \nu \le f_m(k) \le \mu$. But in fact, it is true for any $k \in \Gamma^{(n)}_m$ with $k_1 \le 0$ due to the following known inequalities:

$$\begin{aligned} \widetilde{S}^{1/m}_m(k) &< \widetilde{S}^{\frac{1}{m-1}}_{m-1}(k) && \text{for } \widetilde{S}_m(k) = 3D \binom{n}{m}^{-1} S_m(k), \\ \widetilde{S}^{1/(m-1)}_{m-1}(k) &\le \widetilde{S}^{1/(m-1)}_{m-1}(0, k_2, \dots, k_n) && (\text{as } k_1 \le 0), \end{aligned}$$

and hence

$$\widetilde{S}^{1/m}_m(k) \le \widetilde{S}^{1/(m-1)}_{m-1}(0, k_2, \dots, k_n).$$

On the other hand

$$\frac{\partial}{\partial k_1}\widetilde{S}^{1/m}_m(k) = 3D\frac{1}{m}\widetilde{S}^{(1-m)/m}_m(k)\widetilde{S}_{m-1}(0, k_2, \dots, k_n),$$

and therefore

$$\frac{\partial}{\partial k_1}\widetilde{S}^{1/m}_m(k) \ge \frac{1}{m}. \tag{1.18$'$}$$

From $(1.18')$ follows (1.18) with $c_0 = 3D\frac{1}{m}\binom{n}{m}^{-1/m}$.

Theorem 2 from [6] and (1.17) give a majorant ν_9 for u^τ_t in Q, which with (1.11) gives the inequalities

$$\nu_3 \le u^\tau_t(z) \le \nu_9, \quad z \in \overline{Q}, \ \tau \in [0, 1]. \tag{1.19}$$

The estimate

$$\|u^\tau_{xx}\|_{\infty,\partial'' Q_T} \le \nu_{10}, \tag{1.20}$$

is actually proved in [7]. It follows from Theorem 1.1 of [7] since all conditions of the theorem are valid for problems (1.1_τ), (1.2_τ). Among these conditions there is the requirement on the existence of a positive minorant for

$$j^\tau \equiv u_t^\tau + g^\tau \sqrt{1+(u_x^\tau)^2} \quad \text{in } Q_T.$$

Due to (b_τ) and (1.11) the number $\widetilde{\nu}_1$ is the minorant for j^τ. All other conditions of Theorem 1.1 from [7] are also fulfilled.

The estimate

$$\sup_{\tau\in[0,1]} \|u_{xx}^\tau\|_{\infty,Q} \le \nu_{11}, \tag{1.21}$$

follows from the results of [6], Section 3. The condition (3.53) of [6] is fulfilled and we know some frontiers for $f_m(k[u^\tau])$ (the latter ones are denoted in [6] by ν_4 and ν_5). Namely, due to (1.1_τ)

$$f_m(k[u^\tau]) = 3Dg^\tau + \frac{u_t^\tau}{\sqrt{1+(u_x^\tau)^2}},$$

and due to (1.11), (1.17) and (b_τ)

$$f_m(k[u^\tau]) \ge \frac{1}{\sqrt{1+(u_x^\tau)^2}}\Big[\min_{\partial' Q} u_t^\tau + \inf_Q g^\tau\Big] \ge \frac{\widetilde{\nu}_1}{\sqrt{1+\nu_8^2}} \equiv \nu_{12}. \tag{1.21}$$

As a majornat for $f_m(k[u^\tau])$, we can take

$$\nu_{13} = 3D\nu_9 + \sup_{Q_T} \max\{g(x,t); g^0(x)\}.$$

The fulfilment of condition (3.53) for $f_m = 3DS_m^{1/m}$ has been checked in the end of [6].

So, we can assume that constants ν_{11}, ν_{12} and ν_{13} are known and they are positive.

Because of this, equations (1.1_τ) are uniformly parabolic on u^τ and we know the positive constants ν_{14} and ν_{15} in inequalities

$$\nu_{14}\xi^2 \le \frac{\partial}{\partial u_{x_i x_j}^\tau}[f_m(k[u^\tau])\sqrt{1+(u_x^\tau)^2}]\xi_i\xi_j \le \nu_{15}\xi^2 \tag{1.23}$$

for all $\xi \in \mathbb{R}^n$. The information about u^τ which is now available is sufficient to find majorants for Hölder constants $\langle u_t^\tau\rangle_Q^{(\beta)}$, and $\langle u_{x_i x_j}^\tau\rangle_Q^{(\beta)}$ with $\beta > 0$. Next section is devoted to these problems.

2. Estimation of Hölder constant for u_t^τ and $u_{x_i x_j}^\tau$

The estimates which we will obtain here are proved identically for solutions u of (1.1), (1.2) and solutions u^τ of (1.1_τ), (1.2_τ), and as to g and φ we will use only information about majorants of some of their norms. Therefore, we can restrict ourselves to the study of solutions u to (1.1), (1.2). It is known (see, for example, any of [6]–[8] or preceding papers [1], [4] devoted to stationary equations (1.1)) that equation (1.1) can be represented in the form

$$-u_t + F_m(u_{(xx)}) = 3Dg\sqrt{1+u_x^2}, \tag{2.1}$$

where $F_m(A) = 3DS_m^{\frac{1}{m}}(A)$, $S_m(A)$ is the trace of order m of symmetric matrix A, $u_{(xx)} = 3D\mathcal{T} u_{xx} \mathcal{T}$, where $\mathcal{T} = 3D\mathfrak{g}^{-1/2}(u_x)$ and $\mathfrak{g}(u_x)$ is the metric tensor of the surface $\mathcal{T}_t$ determined in (1.4). The elements g_{ij} of $\mathfrak{g}(u_x)$ are equal $\delta_i^j + u_{x_i} u_{x_j}$. Numbers $k_i[u](z)$, $i = 3D1, \dots, n$, are eigenvalues of matrix $(1+u_x^2)^{-1/2} u_{(xx)}(z)$, so that

$$F_m(u_{(xx)}) = 3Df_m(k[u])\sqrt{1+u_x^2}. \tag{2.2}$$

The cone $\Gamma_m^{(n)}$ determined in (1.3) corresponds to the cone K_m in the space $M_s^{n\times n}$ of all symmetric matrices $n \times n$. This cone is determined as follows

$$K_m = 3D\{A \mid A \in M_s^{t\times n}, S_l(A) > 0,\ l = 3D1, \dots, m\}. \tag{2.3}$$

This cone, as the cone $\Gamma_m^{(n)}$, is a convex set. On it

$$\frac{\partial F_m(A)}{\partial A_{ij}} \xi_i \xi_j > 0 \quad \text{for all } A \in K_m,\ \xi \in \mathbb{R}^n \backslash \{0\}, \tag{2.4}$$

where A_{ij} are elements of A, and F_m is concave, so that

$$\frac{\partial^2 F_m(A)}{\partial A_{ij} \partial A_{kl}} \zeta_{ij} \zeta_{kl} \le 0 \quad \text{for all } \zeta = 3D(\zeta_{ij}) \in M_s^{n\times n}. \tag{2.5}$$

By virtue of (2.5) and convexity of K_m the inequality

$$\frac{\partial F_m(A)}{\partial A_{ij}} (B_{ij} - A_{ij}) \ge F_m(B) - F_m(A), \tag{2.6}$$

holds for all A and B from K_m. Since F_m is 1-homogeneous the inequality (2.6) is equivalent to

$$\frac{\partial F_m(A)}{\partial A_{ij}} B_{ij} \ge F_m(B) = 3D\frac{\partial F_m(B)}{\partial B_{ij}} B_{ij}. \tag{2.6$'$}$$

We will use also the equalities

$$F(B) - F(A) = 3D\int_0^1 \frac{d}{d\xi} F(A^\xi)\, d\xi = 3D\int_0^1 \frac{\partial F(A^\xi)}{\partial A_{ij}^\xi}\, d\xi (B_{ij} - A_{ij}), \tag{2.7}$$

where $A^\xi = 3D\xi B + (1-\xi)A$ and A, B belong to K_m.

Let us remark that the stationary parts of equations (1.1) do not satisfy the conditions which were imposed in the papers of Erance [2], N. V. Krylov [12] and in the book [3] by D. Gilbarg and N. Trudinger on the equations

$$F(u_{xx}, u_x, u, x) = 3Dg(x),$$

when they derived Hölder estimates for $u_{x_i x_j}$. In all these papers and in the book [13] by N. V. Krylov, devoted to parabolic equations

$$-u_t + F(u_{xx}, u_x, u, x, t) = 3Dg(x, t), \tag{2.8}$$

the authors supposed that functions F are determined on the whole space $\mathcal{R} \equiv M_s^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^k$ ($k = 3Dn$ or $k = 3Dn + 1$) and satisfy the condition of ellipticity

$$\frac{\partial F(A, p, u, z)}{\partial A_{ij}} \xi_i \xi_j > 0, \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \tag{2.4$'$}$$

and the condition of convexity

$$\frac{\partial^2 F(A, p, u, z)}{\partial A_{ij} \partial A_{kl}} \zeta_{ij} \zeta_{kl} \le 0, \quad \text{for all } \zeta \in M_s^{n\times n}, \tag{2.5$'$}$$

also on the whole $\mathcal{R}$.

But equations (1.1) as well as equations in our previous papers [9]–[11] do not satisfy these hypothesis and therefore we had to find majorants for $\langle u_t \rangle_{Q_T}^{(\beta)}$ and $\langle u_{x_i x_j} \rangle_{Q_T}^{(\beta)}$. It was done in [9] for solutions of some class of nontotally parabolic equations. In [4] the author attempted to adapt some considerations of N. V. Krylov to the stationary equations (1.1) and on the base of these adaptations in [10] Theorem 2 was announced. In this theorem we asserted the possibility to estimate $\langle u_{x_i x_j} \rangle_{Q_T}^{(\beta)}$ for solutions u of equations (2.8) if the inequalities (2.4$'$) and (2.5$'$) take place only on the investigated solution $u(z)$ (i. e. for $A = 3Du_{xx}(z)$, $p = 3Du_x(z)$, $u = 3Du(z)$, $z \in \overline{Q}_T$)) and if a majorant c for $|u|_{C^2, Q_T}$ is known. But soon after the publication of [10] we found a mistake in [4] (see pages 884–885), and therefore we had to state that our Theorem 2 from [10] has no proof. Maybe it is not even true.

So we had in [9] and [11] to find majorants of $\langle u_{x_i x_j} \rangle_{Q}^{(\beta)}$ for solutions u of equations $-u_t + F(u_{xx}) = 3Dg$. Here we do this for equations (2.1), using some proposals from [15].

Remark. In Theorem 6.1 of [9] taken from the paper [15] (see also [16] or [17]) there are two misprints (in [15]–[17] all is correct)):

(1) after inequality (6.9) it is written "for all $k > 0$" but has to be "for a $k > 0$";
(2) in (6.10) instead of "inf" should be "$\inf_{Q_\rho}$".

Let us introduce the abbreviations $u_i = 3Du_{x_i}$, $u_{ij} = 3Du_{x_i x_j}$, so that u_{ij} are elements of matrix u_{xx}. The elements of matrix $u_{(xx)} = 3D\mathcal{T} u_{xx} \mathcal{T}$ we will denote by $u_{(ij)}$. For them we have the representations $u_{(ij)} = 3D\tau_{ik}\tau_{jl}u_{kl}$ where $\tau_{ik} = 3D\tau_{ki}$ are elements of matrix $\tau = 3D\mathfrak{g}^{-1/2}(u_x) \equiv \mathcal{T}(u_x)$.

We start with the evaluation of $\langle u_t \rangle_Q^{(\beta)}$ with some $\beta > 0$. For this purpose we differentiate (2.1) with respect to t and get

$$-u_{tt} + \frac{\partial F_m(u_{(xx)})}{\partial u_{(ij)}} u_{(ij)t} = 3D(g\sqrt{1+u_x^2})_t. \tag{2.9}$$

Let us introduce the notations

$$\left[\frac{\partial F_m(u_{(xx)})}{\partial u_{(ij)}} \tau_{ik}(u_x)\tau_{jl}(u_x)\right](z) \equiv a_{kl}(z), \tag{2.10$_1$}$$

and remark that for all $z \in \overline{Q}_T$ and all $\xi \in \mathbb{R}^n$ we have inequalities

$$\nu\xi^2 \le a_{kl}(z)\xi_k\xi_l \le \mu\xi^2, \tag{2.10$_2$}$$

with some known positive constants ν and μ.

The estimates (1.20) and (1.23) guarantee (2.10$_2$). The relation (2.9) we consider as a linear equation for u_t:

$$-\partial_t u_t + a_{kl}u_{tx_kx_l} + b_k u_{tx_k} = 3Dg_t\sqrt{1+u_x^2}. \tag{2.11}$$

The form of b_k is not significant for us. For us only majorants for $\|b_k\|_{\infty,Q}$ and $\|g_t\sqrt{1+u_x^2}\|_{\infty,Q}$ are important. Besides this, we can calculate explicitly a majorant for $\|u_t\|_{\partial' Q}^{(\alpha)}$. This information is enough to find a majorant for $\langle u_t \rangle_Q^{(\beta)}$ with a $\beta \in (0, \alpha]$ (see [15–17]). Thus, the estimate

$$\langle u_t \rangle_Q^{(\beta)} \le c, \tag{2.12}$$

is in our hands.

In the next stage we will find a majorant for $\langle u_{ij} \rangle_{\partial'' Q}^{(\beta)}$ on the boundary $\partial'' Q$. We do this as in Section 5 in [9], using the following Lemma for a linear operator

$$L_u = 3D - \partial_t + a_{kl}\partial^2_{x_kx_l}, \tag{2.13}$$

with a_{kl} from (2.10$_1$).

Lemma 2.1. *Let $v \in C(\overline{Q}) \cap C^2(Q)$, $v_x \in C(\overline{Q})$, $v|_{\partial'' Q} = 3D0$, $v|_{t=3D0} \in C^\alpha(\overline{\Omega})$ and let v satisfy the inequality $|L_u v(z)| \le c_1$ in Q. Then*

$$|v_{x_i}|_{\partial'' Q}^{(\beta)} \le c_2, \quad i = 3D1, \dots, n$$

with some $\beta \in (0,1)$. The numbers β and c_2 are determined by numbers ν^{-1}, μ, c_1, n and $|v(\,\cdot\,0)|_\Omega^{(\alpha)}$.

This lemma follows from the results of [18] and from [19] (see, also [13]). Let all above relations be written in cartesian coordinates $x = 3D(\widetilde{x}, x_n)$ corresponding to $x^0 \in \partial\Omega$. We apply Lemma 2.1 to any of $v(x,t) = 3Dv^k(x,t)\zeta(x)$, $k = 3D1, \dots, n-1$, in the cylinder $Q(x^0,d) = 3D[\Omega \cap B_d(x^0)] \times (0,T)$ with $d \ll 1$. The function ζ is a cut-function for $B_d(x^0)$. It is equal to 1 for $x \in B_{d/2}(x^0)$, to zero near $\partial B_d(x^0)$ and $\zeta(x) \in [0,1]$. The functions v^k are determined by equalities

$$v^k(x,t) = 3D(u(x,t) - \varphi(x,t))_{x_k} + (u(x,t) - \varphi(x,t))_{x_n}\omega_{x_k}(x), \quad k < n,$$

where $\omega(x) = 3D\omega(\widetilde{x})$ for $x = 3D(\widetilde{x}, x_n)$. It is easy to see that $v^k\zeta$ are zero on $\partial''Q(x^0,d)$,

$$|L_u(v^k\zeta)| \le c \quad \text{in } \overline{Q}(x^0,d), \tag{2.14}$$

and some majorants for $|v^k\zeta|_{t=3D0}|^{(\alpha)}_{\Omega}$ are known. To prove (2.14) we differentiate (2.1) with respect to x_k, represent the result in the form $L_u u_k = 3D\Phi_k$ and remark that we know a majorant for all $\|\Phi_k\|_{\infty,Q(x^0,d)}$, $k = 3D1, \dots, n$.

Due to the above said and Lemma 2.1 we know on the part $\Sigma_d \times [0,T]$ of boundary $\partial''Q(x^0,d)$ the estimates $|v^k_{x_i}|^{(\beta)}_{\Sigma_d\times[0,T]} \le c_1$, $i = 3D1, \dots, n$, from which we conclude that

$$|u_{ki} + u_{ni}\omega_{x_k}|^{(\beta)}_{\Sigma_d\times[0,T]} \le c_2, \tag{2.15$_1$}$$

with β which is minimal of 1/2 and previous β. Because of (2.15$_1$), for any $z^0 = 3D(x^0,t) \in Q(x^0,d)$ and any $z' = 3D(x',t') \in \Sigma_d \times [0,T]$, we have the inequalities

$$|u_{ki}(z') - u_{ki}(z^0,t)| \le c_3\rho^\beta(z',z^0) + |u_{ni}(z')\omega_{x_k}(x')| \le c_4\rho^\beta(z',z^0), \tag{2.15$_2$}$$

for $k \in [1, n-1]$ and $i \in [1,n]$. Here we have taken into account that

$$\omega_{x_k}(x^0) = 3D0 \quad \text{and} \quad |\omega_{x_k}(x')| \le c|x' - x^0|.$$

For the estimation of $|u_{nn}(z') - u_{nn}(z^0)|$ we consider the difference of equations (2.1) at points z' and z

$$\begin{aligned} j &= 3DF_m(u_{(xx)}(z')) - F_m(u_{(xx)}(z^0)) \\ &= 3Du_t(z') - u_t(z^0) + (g\sqrt{1+u_x^2})(z') - (g\sqrt{1+u_x^2})(z^0). \end{aligned} \tag{2.16}$$

Now we can conclude that the absolute value of the right-hand side of (2.16) does not exceed $c\rho^\beta(z',z^0)$ with some c and $\beta > 0$. The left-hand side of (2.16) we represent, using (2.7), in the form

$$
\begin{aligned}
(2.17)\qquad j &= 3D\widehat{a}_{ij}[u_{(ij)}(z') - u_{(ij)}(z^0)]\\
&= 3D\widehat{a}_{ij}[(u_{kl}\tau_{ik}\tau_{jl})(z') - (u_{kl}\tau_{ik}\tau_{jl})(z^0)]\\
&= 3D\widehat{a}_{ij}\tau_{ik}(z^0)\tau_{jk}(z^0)[u_{kl}(z') - u_{kl}(z^0)]\\
&\quad + \widehat{a}_{ij}[(\tau_{ik}\tau_{jl})(z') - (\tau_{ik}\tau_{jl})(z^0)]u_{kl}(z')\\
&= 3D = 3D\widehat{\widehat{a}}_{ij}[u_{kl}(z') - u_{kl}(z^0)] + O(\rho^\beta(z', z^0)),
\end{aligned}
$$

where

$$\widehat{a}_{ij} = 3D\int_0^1 \frac{\partial F_m(u^\xi_{(xx)})}{\partial u^\xi_{(ij)}}\,d\xi \quad \text{and} \quad u^\xi_{(xx)} = 3D\xi u_{(xx)}(z') + (1-\xi)u_{(xx)}(z^0).$$

In this connection we used convexity of the cone K_m and known estimates for u. Additionally, for quadratic forms $\widehat{a}_{ij}\xi_i\xi_j$ and $\widehat{\widehat{a}}_{ij}\xi_i\xi_j$ we have inequalities (2.10_2) with positive constants ν and μ under control. From this follows the estimates $\widehat{\widehat{a}}_{nn} \geq \nu$ and $|\widehat{\widehat{a}}_{ij}| \leq \mu$, $i, j = 3D1, \dots, n$. These inequalities permit draw from (2.15_k)–(2.17) the conclusion

$$(2.18)\qquad \nu|u_{nn}(z') - u_{nn}(z^0)| \leq c\rho^\beta(z', z^0).$$

Since z^0 is an arbitrary point of $\partial'' Q_T$, we obtain from (2.15_k), $k = 3D1, 2$, and (2.18) the desirable estimates

$$(2.19)\qquad \langle u_{x_i x_j}\rangle^{(\beta)}_{\partial'' Q_T} \leq c, \quad i, j = 3D1, \dots, n,$$

with some c and $\beta > 0$. For estimation of $\langle u_{ij}\rangle^{(\beta)}_Q$ we use Theorem 6.2 from [9].

Lemma 2.2. *Let functions $v^k : \overline{Q}_T \to \mathbb{R}^1$, $k = 3D1, \dots, N$, belong to $W^{2,1}_{n+1}(Q_T) \cap C(\overline{Q}_T)$ and satisfy inequalities $|v^k|_{C^\alpha, \partial' Q_T} \leq c_1$ and*

$$(2.20)\qquad \|(-L_u v^k)_+\|_{n+1, Q_T} \leq c_2,$$

with some c_1 and c_2 where L_u has the form (2.13) with a_{kl} satisfying (2.10_2). Let also the inequalities

$$(2.21)\qquad \delta\sum_{i=3D1}^N [v^k(z_2) - v^k(z_1)]_+ \leq \sum_{i=3D1}^N [v^k(z_2) - v^k(z_1)]_- + c_3\rho^\alpha(z_1, z_2),$$

be fulfilled for all $z_1, z_2 \in Q_T$ with some positive δ and c_3. Then for all $k = 3D1, \dots, N$, v^k belong to $C^\beta(\overline{Q}_T)$ and $|v^k|_{C^\beta, Q_T} \leq c_4$, where c_4 is determined by δ^{-1} and c_i, $i = 3D1, 2, 3$. Number β belongs to $(0, \alpha]$ and is also determined by δ^{-1} and c_i, $i = 3D1, 2, 3$.

In [9] we did a proof of this statement and pointed out that it is a generalization of some statements from [13].

The collection $\{v^k\}_{k=3D1}^N$ of functions to which we will apply Lemma 2.2 has no relation with functions v^k, $k=3D1,\dots,n-1$, used before for proving (2.19). Here we construct v^k, $k=3D1,\dots,N$, with the help of second derivatives of u having the form $u^k\equiv u_{\gamma_k\gamma_k}=3Du_{ij}\cos(\gamma_k,x_i)\cos(\gamma_k,s_j)$, $k=3D1,\dots,N$, where $\gamma_1,\dots,\gamma_N$ is a collection of unite vectors including vectors $e_1,\dots,e_n$, their combinations $e_{ij}^{\pm}=3D(e_i\pm e_j)/\sqrt{2}$, $i\neq j$, and some other vectors. We will describe it below. Due to relations

$$\partial^2_{x_ix_j}=3D\frac{1}{2}(\partial_{x_i}+\partial_{x_j})^2-\frac{1}{2}\partial^2_{x_ix_i}-\frac{1}{2}\partial^2_{x_jx_j},$$

each second derivative u_{ij} can be represented as a sum of u^k, $k=3D1,\dots,N$. Each of u^k satisfies a certain differential inequality. Namely, let us differentiate (2.1) along the direction γ. It gives

$$-u_{\gamma t}+\frac{\partial F_m(u_{(xx)})}{\partial u_{(ij)}}u_{(ij)\gamma}=3D(g\sqrt{1+u_x^2})_\gamma, \tag{2.22}$$

where $u_\gamma=3Du_i\cos(\gamma,x_i)$. Now, we differentiate (2.20) along the same γ and reject in the result the nonpositive number

$$\frac{\partial^2F_m(u_{(xx)})}{\partial u_{(ij)}\partial u_{(kl)}}u_{(ij)\gamma}u_{(kl)\gamma},$$

(see (2.5)). It gives the inequality

$$-u_{\gamma\gamma t}+\frac{\partial F_m(u_{(xx)})}{\partial u_{(ij)}}u_{(ij)\gamma\gamma}\geq(g\sqrt{1+u_x^2})_{\gamma\gamma}. \tag{2.23}$$

This relation, taking for all $\gamma=3D\gamma_k$, we rewrite in the form

$$L_uu^k+\sum_{l=3D1}^N\sum_{i=3D1}^N c_{li}^ku_{x_i}^l\geq\Phi^i,\quad k=3D1,\dots,N. \tag{2.24}$$

where c_{li}^k and Φ^k are continuous functions of $z\in\overline{Q}_T$ for which we know majorants of their modulus. Operator L_u is taken from (2.13) and the inequalities (2.10_2) hold four its coefficients.

As it was understood (see for example [14]) while evaluating of Hölder constants for derivatives u_{x_i} of solutions u to the quasilinear elliptic and parabolic equations, we have to pass from the collection $\{u^k\}_{k=3D1}^N$ to another collection $\{v^k\}_{k=3D1}^N$ which has the following two properties:

(1) any v^k satisfies an inequality

$$L_uv^k\geq c_5,\quad k=3D1,\dots,N, \tag{2.25}$$

(2) quantities $\langle u^k\rangle_Q^{(\beta)}$, $k=3D1,\dots,N$, can be majorized by quantities $\langle v^k\rangle_Q^{(\beta)}$ and $\|u^k\|_{\infty,Q}$.

Such collection is constructed in the following way: at first we "normalized" u^k taking $\widetilde{u}^k(z) = 3D(u^k(z) - m_k^-)/(m_k^+ - m_k^-) \in [0,1]$ with $m_k^+ = 3D\sup_Q u^k$ and $m_k^- = 3D\inf_Q u^k$ instead of u^k. We have for them the inequalities (2.24) only with others coefficients c_{li}^k and other Φ^k. Since these changes are not important for our main purpose we will suppose that u^k themselves have the properties $u^k(z) \in [0,1]$. Using such u^k we construct the collection

$$v^k(z) = 3Du^k(z) + \varepsilon \sum_{l=3D1}^{N} [u^l(z)]^2, \quad k = 3D1, \dots, N, \tag{2.26}$$

with a small positive ε ($\varepsilon \in (0,1]$). Due to (2.24), functions v^k satisfy the inequalities (2.25) with some $c_5 < \infty$ if only $\varepsilon > 0$. Besides this, the differences $\widetilde{u}^m = 3Du^m(z_2) - u^m(z_1)$ and $\widetilde{v}^m = 3Dv^m(z_2) - v^m(z_1)$ are connected by equalities

$$\widetilde{v}^k = 3D\widetilde{u}^k + \varepsilon \sum_{l=3D1}^{N} a_l \widetilde{u}^l,$$

where $a_l = 3Du^l(z_2) + u^l(z_1) \in [0,2]$. They imply

$$\sum_{k=3D1}^{N} a_k \widetilde{v}^k = 3D \sum_{k=3D1}^{N} a_k \widetilde{u}^k \Big(1 + \varepsilon \sum_{l=3D1}^{N} a_l \widetilde{v}^l\Big), \quad k = 3D1, \dots, N,$$

and therefore

$$\widetilde{u}^k = 3D\widetilde{v}^k - \frac{\varepsilon}{1 + \varepsilon \sum_{l=3D1}^{N} a_l} \sum_{l=3D1}^{N} a_l \widetilde{v}^l, \quad k = 3D1, \dots, N. \tag{2.26$'$}$$

Due to this the estimates $|\widetilde{v}^l| \le c\rho^\alpha(z_1, z_2)$ give the analogues estimates for $|\widetilde{u}^k|$.

Now we show that the inequalities (2.21) hold for $\{v^k\}_{k=3D1}^N$ if the collection of directions $\gamma_k = 3D\sum_{i=3D1}^n \gamma_{ki} e_i$, $k = 3D1, \dots, N$, is selected so that all $a_{ij}(z)$ can be represented in the form

$$a_{ij}(z) = 3D \sum_{m=3D1}^{N} \beta_m(z) \gamma_{mi} \gamma_{mj}, \quad i,j = 3D1, \dots, n, \tag{2.27}$$

where $\beta_k(z)$ belong to $[\nu^*, \mu^*]$ with some positive numbers ν^* and μ^* determined by n, μ and ν from (2.10$_2$). A possibility of such a representation is guaranteed by Wasov-Motzkin Lemma (see, for example, [9] or [13]).

Let us take relation (2.16) for arbitrary points z_1 and z_2 from Q:

$$j(z_1, z_2) = 3Du_t(z_2) - u_t(z_1) + (g\sqrt{1+u_x^2})(z_2) - (g\sqrt{1+u_x^2})(z_1), \tag{2.28$_1$}$$

where

$$j_1(z_1, z_2) = 3DF_m(u_{(xx)}(z_2)) - F_m(u_{(xx)}(z_1)), \tag{2.28$_2$}$$

and, using (2.6), evaluate j from below

$$\begin{aligned} j(z_1,z_2) \ge{}& \frac{\partial F_m(u_{(xx)})}{\partial u_{(ij)}}(z_2)[u_{(ij)}(z_2)-u_{(ij)}(z_1)]+j_1(z_1,z_2)=3D \\ &=3Da_{kl}(z_2)[u_{kl}(z_2)-u_{kl}(z_1)]+j_1(z_1,z_2), \end{aligned}$$

where

$$j_1(z_1,z_2) = 3D\frac{\partial F_m(u_{(xx)})}{\partial u_{(ij)}}(z_2)u_{kl}(z_2)[(\tau_{ik}\tau_{jl})(z_2)-(\tau_{ik}\tau_{jl})(z_1)].$$

By virtue of (2.27)

$$j(z_1,z_2)\ge\sum_{m=3D1}^{N}\beta_m(z_2)[u^m(z_2)-u^m(z_1)]+j_1(z_1,z_2). \tag{2.29}$$

It is easy to see that the sum of absolute values of j_1 and the right-hand side of (2.28_1) does not exceed $c\rho^\alpha(z_1,z_2)$ with some c and $\alpha>0$, and therefore we can conclude from (2.28_1) and (2.29) that

$$\sum_{m=3D1}^{N}\beta_m(z_2)[u^m(z_2)-u^m(z_1)]\le c_6\rho^\alpha(z_1,z_2)\equiv c_6\rho^\alpha. \tag{2.30}$$

Because of $\beta_m(z_2)\in[\nu^*,\mu^*]$ we have from (2.30) the inequalities

$$\nu^*\sum_{m=3D1}^{N}[\widetilde{u}^m]_+-\mu^*\sum_{m=3D1}^{N}[\widetilde{u}^m]_-\le c_6\rho^\alpha,$$

where $\widetilde{u}^m=3Du^m(z_2)-u^m(z_1)$, $[a]_+=3D\max\{0;a\}$, $[a]_-=3D\max\{0;-a\}$. From these ineqaulities we deduce

$$\delta_1\sum_{m=3D1}^{N}[\widetilde{u}^m]_+\le\sum_{m=3D1}^{N}[\widetilde{u}^m]_-+c_7\rho^\alpha,\quad \delta_1=3D\frac{\nu^*}{\mu^*}. \tag{2.31}$$

For $\widetilde{v}^m$ determined after (2.26), we have, using (2.31), the following

$$\begin{aligned} &[\widetilde{v}^m]_+\le[\widetilde{u}^m]_++2\varepsilon\sum_{l=3D1}^{N}[\widetilde{u}^l]_+ \\ &\qquad\Rightarrow\sum_{m=3D1}^{N}[\widetilde{v}^m]_+\le c_8\sum_{m=3D1}^{N}[\widetilde{u}^m]_+,\quad c_8=3D1+2\varepsilon N, \\ &[\widetilde{u}^m]_-=3D[\widetilde{v}^m-\varepsilon\sum_{l=3D1}^{N}a_l\widetilde{u}^l]_-\Rightarrow\sum_{m=3D1}^{N}[\widetilde{u}^m]_-\le\sum_{m=3D1}^{N}[\widetilde{v}^m]_-+2N\varepsilon\sum_{l=3D1}^{N}[u^l]_+ \\ &\qquad\le\sum_{m=3D1}^{N}[\widetilde{v}^m]_-+2N\varepsilon\delta_1^{-1}\sum_{m=3D1}^{N}[\widetilde{u}^m]_-+2N\varepsilon\delta_1^{-1}c_7\rho^\alpha. \end{aligned}$$

For $\varepsilon < \delta_1(4N)^{-1}$ the last inequality gives the relations

$$\sum_{m=3D1}^{N}[\widetilde{u}^m]_- \le 2\sum_{m=3D1}^{N}[\widetilde{v}^m]_- + c_9\rho^\alpha, \quad c_9 = 3D4N\varepsilon c_7\delta_1^{-1}. \tag{2.32}$$

From these relations and from (2.31) we obtain the inequalities

$$\begin{aligned}\sum_{m=3D1}^{N}[\widetilde{v}^m]_+ &\le c_8\sum_{m=3D1}^{N}[\widetilde{u}^m]_+ \le c_8\delta_1^{-1}\sum_{m=3D1}^{N}[\widetilde{u}_m]_- + c_7c_8\delta_1^{-1}\rho^\alpha\\ &\le 2c_8\delta_1^{-1}\sum_{m=3D1}^{N}[\widetilde{v}^m]_- + c_8(c_7+c_9)\delta_1^{-1}\rho^\alpha,\end{aligned}$$

which are just the desirable inequalities (2.21) with

$$\delta = 3D\delta_1(2c_8)^{-1} = 3D\nu^*[2\mu^*(1+2\varepsilon N)]^{-1} \quad \text{and} \quad c_3 = 3D(c_7+c_9)/2.$$

The condition (2.20) of Lemma 2.2 is also fulfilled for our v^k because

$$[-L_u v^k]_+ \le \max\{0; -c_5\}, \tag{2.25$'$}$$

due to (2.25). Thus, Lemma 2.2 guarantees the estimates

$$|v^k(z_2) - v^k(z_1)| \le c\rho^\beta(z_1,z_2), \quad k = 3D1,\dots,N, \tag{2.33$_1$}$$

with some c and $\beta \in (0,\alpha)$ for any $z_1, z_2 \in \overline{Q}_T$.

As it was explained above (see (2.26′)), from (2.33$_1$) we can get the estimates

$$|u^k(z_2) - u^k(z_1)| \le c\rho^\beta(z_1,z_2), \quad k = 3D1,\dots,N, \tag{2.33$_2$}$$

and from them the estimates

$$|u_{x_ix_j}(z_2) - u_{x_ix_j}(z_1)| \le c\rho^\beta(z_1,z_2), \quad i,j = 3D1,\dots,n, \tag{2.33$_3$}$$

with some other constant c. All this is true for solutions u^τ of problems (1.1$_\tau$), (1.2$_\tau$). So, we have proved

Theorem 2.1. *For all admissible solutions u^τ of problems* (1.1$_\tau$), (1.2$_\tau$), $\tau \in [0,1]$, *for which $\|u^\tau\|_{C^2,Q_T} \le c$, there exist a constant $\beta \in (0,\alpha]$ and a majorant c_1 such that*

$$\sup_{\tau\in[0,T]}\|u^\tau\|_{C^{2+\beta},Q_T} \le c_1. \tag{2.34}$$

They are determined by c, $|\varphi|_{C^3,Q_T}$, $|g|_{C^2,Q_T}$, and $|\partial\Omega|_{C^3}$.

Thus, the estimate (1.10) is proved. For this we have used that $u^\tau \in C^2(\overline{Q}_T)$ and u^τ_x, u^τ_{xx}, and u^τ_t belong to $C^2(Q_T)$. It is known that to prove the existence theorem for problems (1.1$_\tau$), (1.2$_\tau$) in the functional space indicated in Theorem 1.1 we have to check two facts:

(1) If problem (1.1_τ), (1.2_τ) is solvable for τ_1, $\tau_2, \dots$, and $\tau_k \to \overline{\tau}$ then it is solvable for $\tau = 3D\overline{\tau}$ also. It is proved easily, since due to (1.10), $u^{\tau_{k_i}}$ for some subsequence $\{\tau_{k_i}\}$, $k_i \to \infty$, converges in $C^2(\overline{Q}_T)$ to a function $\overline{u} \in C^{2+\beta}(\overline{Q}_T)$ and this function $\overline{u}$ will be a solution of problem for $\tau = 3D\overline{\tau}$ (we know preliminary that problem (1.1_τ), (1.2_τ) for any $\tau \in [0,1]$ can have not more then one solution). The belonging of the derivatives $\overline{u}_x$, $\overline{u}_{xx}$, $\overline{u}_t$ of $\overline{u}$ to $C^{2+\alpha}(Q_T)$ is proved on the base of the linear theory of parabolic equations.

(2) The second property of the family of problems (1.1_τ), (1.2_τ), $\tau \in [0,1]$, which we have to verify, is the possibility to find solutions in a vicinity of any $\overline{\tau} \in [0,1)$ for which we know the solution $\overline{u} = 3Du^{\overline{\tau}}$. Such possibility is proved often with the help of proper approximations and the existence theorem for contractive mappings. In the monograph [13, Chapter I, Section 3], this way was used for the second order nonlinear parabolic equations of general form, but under condition that the studied problem is "strong" compatible with the first initial-boundary values problem for the heat equation with the same φ on $\partial' Q_T$. It means that function $\varphi : \overline{Q}_T \to \mathbb{R}^1$ has to satisfy not only necessary compatibility conditions but also the condition

$$-\varphi_t + \Delta\varphi = 3D0. \tag{2.35}$$

This requirement is caused not by the essence of problem but by technical reasons — choosing of auxiliary problems. We use other auxiliary problems — the problems (1.1_τ), (1.2_τ), $\tau \in [0,1]$, and do not put on φ the condition (2.35). Each problem (1.1_τ), (1.2_τ) satisfies the compatibility conditions of the zero and first orders. Let us explain how to prove the solvability of problems (1.1_τ), (1.2_τ) for all $\tau+\varepsilon$ with $\varepsilon \in (0,\varepsilon_0)$, $\varepsilon_0 \ll 1$, if we know its solvability (in the same space) for τ.

Let us introduce linear operators L^τ

$$L^\tau v = 3D - v_t + a_{ij}^\tau v_{ij} + a_i^\tau v_i, \tag{2.36}$$

corresponding to the equations in variations for equations (1.1_τ) written in the form

$$-u_t^\tau + F_m(u_{(xx)}^\tau) - g^\tau \sqrt{1+(u_x^\tau)^2} = 3D0. \tag{2.37}$$

In (2.36)

$$a_{ij}^\tau = 3D\frac{\partial \mathcal{F}_m(u_{xx}^\tau, u_x^\tau)}{\partial u_{ij}^\tau}, \quad \text{where } \mathcal{F}_m(u_{xx}, u_x) \equiv F_m(u_{(xx)}),$$

and

$$a_i^\tau = 3D\frac{\partial \mathcal{F}_m(u_{xx}^\tau, u_x^\tau)}{\partial u_i^\tau} - g^\tau \frac{u_i^\tau}{\sqrt{1+(u_x^\tau)^2}}.$$

Let us consider the linear problems

$$(2.38_1) \qquad L^\tau v = 3DB^\tau[u^\tau, w] + (g^{\tau+\varepsilon} - g^\tau)\sqrt{1+w_x^2} \quad \text{in } Q_T,$$

$$(2.38_2) \qquad v = 3D\varphi^{\tau+\varepsilon} \quad \text{on } \partial' Q_T,$$

where $B^\tau[u^\tau, w] = 3Da_{ij}^\tau w_{ij} + a_i^\tau w_i - F_m(w_{(xx)}) + g^\tau\sqrt{1+w_x^2}$.

A unique solution v of the problem (2.38_k), $k = 3D1, 2$, corresponds to each w from $C^{2+\beta}(\overline{Q}_T) \cap K_m$. If the solution v coincides with w then v will be a solution of $(1.1_{\tau+\varepsilon})$, $(1.2_{\tau+\varepsilon})$.

In order that v enters in $C^{2+\beta}(\overline{Q}_T)$ it is necessary that the compatibility conditions of the zero and first orders for problem (2.38_k), $k = 3D1, 2$, are fulfilled. To obtain this we will consider problem (2.38_k), $k = 3D1, 2$, only for w belonging to the set

$$A(\delta,\tau,\varepsilon) = 3D\{w \in C^{2+\beta}(\overline{Q}_T) \mid |w - u^\tau|_{C^{2+\beta}, Q_T} \le \delta,\ w = 3D\varphi^{\tau+\varepsilon} \quad \text{on } \partial' Q_T\}.$$

If $w \in A(\delta,\tau,\varepsilon)$ with a $\delta > 0$ the solution v of (2.38_k), $k = 3D1, 2$, for such w belongs to $A(\widetilde{\delta},\tau,\varepsilon)$, may be with $\widetilde{\delta} > \delta$. This gives us the mapping

$$\Phi^{\tau,\varepsilon} \colon w \in A(\delta,\tau,\varepsilon) \to v \in A(\widetilde{\delta},\tau,\varepsilon).$$

If δ and ε are sufficiently small then $\Phi^{\tau,\varepsilon}$ maps $A(\delta,\tau,\varepsilon)$ in itself. To control this we consider the difference of the equation (2.38_1) and (2.37) and write the result as an equation for $v - u^\tau$

$$(2.39_1) \qquad L^\tau(v - u^\tau) = 3DB^\tau[u^\tau, w - u^\tau] + \varepsilon(g - g^0)\sqrt{1+w_x^2} \equiv j_1 \quad \text{in } Q_T.$$

We also have

$$(2.39_2) \qquad v - u^\tau = 3D\varphi^{\tau+\varepsilon} - \varphi^\tau = 3D\varepsilon(\varphi - \varphi^0) \quad \text{on } \partial' Q_T.$$

It is not difficult to calculate that for $w \in A(\delta,\tau,\varepsilon)$

$$(2.40) \qquad |j_1|_{C^\beta, Q_T} \le c[\varkappa(\delta)\delta + \varepsilon],$$

where $\varkappa : [0, \delta_1] \to \mathbb{R}^1$ is a continuous function equal zero in the end $\delta = 3D0$ of $[0, \delta_1]$.

Because of this and Schauder's estimates for solution $v - u^\tau$ of problem (2.39_k), $k = 3D1, 2$,

$$|v - u^\tau|_{C^{2+\beta}, Q_T} \le c_1[\varkappa(\delta)\delta + \varepsilon],$$

and therefore v will belong to $A(\delta,\tau,\varepsilon)$ if only δ and ε are so small that

$$(2.41) \qquad c_1[\varkappa(\delta)\delta + \varepsilon] \le \delta.$$

The contraction property of $\Phi^{\tau,\varepsilon}$ for small τ and ε is proved analogously. (For this purpose it is necessary to consider the difference of equations (2.38_1) for two different w from $A(\delta,\tau,\varepsilon)$.) This guarantees the existence of a fixed point v for

$\Phi^{\tau,\varepsilon}$ and, by the same token, the solvability of problems $(1.1_{\tau+\varepsilon})$, $(1.2_{\tau+\varepsilon})$ for small ε. It is easy to see that in all our steps we did not leave the cone K_m and for solutions $u^{\tau+\varepsilon}$ the estimates (2.34) hold. Thus, we have tested the fulfillment of both conditions (1) and (2) which permit to use the continuation with respect to parameter $\tau \in [0,1]$ and prove Theorem 2.1.

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