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COVERING MANIFOLDS FOR ANALYTIC FAMILIES OF LEAVES OF FOLIATIONS BY ANALYTIC CURVES

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To Jürgen Moser for his seventieth birthday

Introduction

This paper deals with the foliations of Stein manifolds by analytic curves. A single fiber of such a foliation is a Riemann surface which may be parabolic or hyperbolic. The universal covering over this fiber is either a complex line, or a disk. Now let us take an analytic family of fibers, that is, the saturation of an analytic cross-section by fibers. The main problem in the context, still unsolved, is to find a uniformization of the fibers analytic with respect to the parameter. Precise definitions look like follows.

Definition 1. A *skew cylinder* is a tuple (M, B, π), where M is a complex manifold, B is a complex hypersurface in M, $\pi : M \to B$ is an analytic retraction with the constant rank equal to dim $M - 1$ and with simply connected fibers. The manifold M is called a *total space* of the skew cylinder.

Definition 2. A skew cylinder is *standard*, if $M \subset B \times \widehat{\mathbb{C}}$, and π is the retraction to $B \times \{0\}$ along the second factor.

Remark. Different standard skew cylinders may be conformally nonequivalent. The simplest example is provided by a ball and a bidisk foliated by parallel lines.

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Definition 3. Two skew cylinders (M, B, π), $(\widetilde{M}, \widetilde{B}, \widetilde{\pi})$ are *equivalent*, if there exist two biholomorphic maps $\rho : B \to \widetilde{B}$ and $H : M \to \widetilde{M}$, such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{H} & \widetilde{M} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\widetilde{\pi}} \\ B & \xrightarrow[\rho]{} & \widetilde{B} \end{array}$$

The simultaneous uniformization problem for skew cylinders looks like follows:

Problem. *Is it true that a skew cylinder with the Stein total space is equivalent to a standard one?*

Remark. It is not difficult to find skew cylinders with nonStein total space, which are nonuniformizable, that is, nonequivalent to any standard skew cylinder.

Fortunately, the skew cylinders that occur as covering manifolds mentioned in the title are always Stein. This is the main result of the present paper; the accurate statement is Theorem 1.2 below.

The other main result of the paper is the Fibers Connection Lemma. We will prove it in Section 4 below. Here we state the following

Corollary. *Let K in $\mathbb{C}^N$ be a polynomially convex domain with a smooth boundary and a foliation by parallel complex lines, p be an arbitrary point of tangency of the foliation with ∂K. We take a ball B centered at p; denote φ_a the leaf of the foliation through a, and*

$$\psi_a = B \cap K \cap \varphi_a.$$

Let $Q \subset K$ be a ball that intersects ψ_p. For any $b \in Q$, we denote by ξ_b the connected component of ψ_b that contains b. Then the union

$$K^{\mathrm{tr}} = \bigcup_{b \in Q} \xi_b,$$

is polynomially convex (tr *of truncated*).

The sketch of the proof of the main theorem was given in [I]. Here we give a complete proof.

1. Definition and existence of covering manifolds

Consider a Stein manifold X and a foliation with singularities of X by analytic curves. By definition, this means that there exists an analytic subset $\Sigma \subset X$ of codimension greater than one and a foliation of $X - \Sigma$ by analytic curves that cannot be extended to a neighbourhood of any point of Σ. We denote by φ_p a fiber of this foliation through $p \in X - \Sigma$.

Definition 1.1. Let $\mathcal{F}$ be a foliation by analytic curves with singularities, and B a transversal cross-section. A covering manifold $\widetilde{M}$ over the leaves of the union

$$M = \bigcup_{p \in B} \varphi_p,$$

is the total space $\widetilde{M}$ of the skew cylinder $(\widetilde{M}, \pi, B)$ with the following properties:

- there exists a locally biholomorphic projection $\widetilde{\pi} : \widetilde{M} \to M$,
- for any fiber $\widetilde{\varphi}_p = \pi^{-1}p, p \in B$, the restriction of $\widetilde{\pi}$ to $\widetilde{\varphi}_p$ is the universal covering map over φ_p, with the base point p,
- $\widetilde{\pi}$ restricted to B is identity.

Theorem 1.1 ([I]). *A covering manifold exists for any analytic foliation with singularities of a Stein manifold and any cross-section.*

The sketch of the proof is given at the end of this section. The main result of the paper is

Theorem 1.2. *A covering manifold corresponding to a codimension one Stein cross-section for an analytic foliation by curves with singularities of* $\mathbb{C}^n$*, is a Stein manifold itself.*

This theorem is proved in the next three sections. It may be considered as a partial solution to the following

Serr Problem. *Give sufficient conditions for the total space of a fibration with the Stein base and Stein leaves to be a Stein manifold itself.*

The sufficient condition for the case of one-dimensional connected and simply connected leaves provided by Theorem 1.2 is: there exists n such that the regarded space is a covering manifold for a foliation of $\mathbb{C}^n$ by analytic curves corresponding to a Stein cross-section.

Sketch of the proof of Theorem 1.1. The manifold $\widetilde{M}$ may be constructed as follows. For any point $p \in B$ and $q \in \varphi_p$ the class of curves on φ_p starting at p and ending at q homotopic on φ_p represents a point $\widetilde{q} \in \widetilde{\varphi}_p$. Thus the set of points of $\widetilde{M}$ is well defined.

The projection $\widetilde{\pi} : \widetilde{M} \to M$ is defined by: $\widetilde{\pi}(\widetilde{q}) = q$. The topology on $\widetilde{M}$ is defined in the following way. For any point $\widetilde{q} \in \widetilde{\varphi}_p$ represented by a class of homotopic curves $[\gamma_{pq} \subset \varphi_p]$ we consider a small neighbourhood U of q in $\mathbb{C}^n$ such that the following holds. Any point $q' \in U$ may be connected with some point $p' \in B$ by a curve $\gamma_{p'q'} \subset \varphi_{p'}$ in such a way that the curves $\gamma_{p'q'}$ depend continuously on $q' \in U$. The set of points represented by the homotopy classes of the curves $\gamma_{p'q'}$ on the corresponding fibers $\varphi_{p'}$ forms, by definition, the neighbourhood $\widetilde{U}$ of $\widetilde{q}$.

The manifold $\widetilde{M}$ thus constructed is a Hausdorff one. Indeed, if two curves on the same fiber of $\mathcal{F}$ are nonhomotopic than the nearby curves on nearby fibers are nonhomotopic on this fibers as well. This is the consequence of the analyticity of the foliation. Namely, if the sequence of loops γ_n homotopically trivial on the fibers φ_n tends to the loop γ on the fiber φ, than γ is homotopically trivial of φ as well. Really, the films that span γ_n on φ_n are one-dimensional complex analytic sets. They have minimal area amidst all the films spanning φ_n in $\mathbb{C}^n$ with the induced metrics. The areas of these films form a bounded sequence, hence a limit of this sequence of films exists. The limit film spans the loop γ on φ.

Thus $\widetilde{M}$ is a Hausdorff manifold. The complex structure on $\widetilde{M}$ is inherited from $\mathbb{C}^n$ as a pullback under the projection $\widetilde{\pi} : \widetilde{M} \to M \subset \mathbb{C}^n$.

2. Structure of the Riemann domain on the covering manifold

We consider a covering manifold from Theorem 1.2. That is, let $\mathcal{F}$ be a foliation with singularities of $\mathbb{C}^n$ by analytic curves. Let $B \subset \mathbb{C}^n$ be a Stein hypersurface transversal to the leaves of the foliation. Let $\widetilde{M}$ be the corresponding covering manifold, M and $\widetilde{\pi} : \widetilde{M} \to M$ be the same as in Definition 1.1.

Lemma 2.1. *The manifold $\widetilde{M}$ is a Riemann domain over M.*

Proof. The manifold M is a domain in $\mathbb{C}^n$ because B is a hypersurface in $\mathbb{C}^n$. By Definition 1.1, $\widetilde{\pi} : \widetilde{M} \to M$ is a local holomorphism. The only property in the definition of Riemann domains to be checked is that holomorphic functions separate points on $\widetilde{M}$. To do that, we use the main idea of this paper: holomorphic functions on $\widetilde{M}$ produce holomorphic 1-forms; these forms integrated along the leaves produce new holomorphic functions. Let us pass to the detailed proof.

Let's suppose the converse: there exist points on $\widetilde{M}$ that cannot be separated by the functions from $\mathcal{O}(\widetilde{M})$. These points have the same image under the projection $\widetilde{\pi}$; otherwise the coordinate functions lifted to $\widetilde{M}$ would separate them.

Now, let b_1, b_2 be nonseparable points on $\widetilde{M}$, $a = \widetilde{\pi} b_1 = \widetilde{\pi} b_2$. We consider a fiber $\varphi_a \subset M$ and the fundamental group $\pi_1(\varphi_a, a)$. The points b_j belong to the fibers $\pi^{-1}(\pi b_j) = \widetilde{\varphi}_j$ of the skew cylinder $\widetilde{M}$. Let $p_j = \pi b_j$. Then the points b_j are represented by the curves $\gamma_j \subset \varphi_a$ beginning at p_j and landing at a. These curves may be chosen in such a way that their lifts to $\widetilde{\varphi}_j$ will be simple and smooth; but the curves themselves may have selfintersections. Let

$$\mu = \gamma_1^{-1} \gamma_2, \tag{2.1}$$

be the curve on φ_a beginning at p_1 and ending at p_2.

The construction leading to the contradiction will be the same in different cases; we begin with the simplest ones.

Let ω be a polynomial 1-form on M. The form $\omega^* = \widetilde{\pi}^*\omega$, a pullback of the form ω, is a holomorphic 1-form on $\widetilde{M}$. For any point $q \in \widetilde{M}$ let $p = \pi(q)$ and γ_q be a curve on the fiber $\widetilde{\varphi}_p = \pi^{-1}p$ that joins p with q. The function

$$I : q \mapsto \int_{\gamma_q} \omega^*, \tag{2.2}$$

is well defined on $\widetilde{M}$ because the fibers of π are simply connected; it is holomorphic on $\widetilde{M}$.

This construction, after slight modifications, will prove Lemma 2.1. We consider several particular cases.

Case 1. The curve (2.1) is simple nonclosed. Then, by the theorem of Bishop, any continuous function in μ may be uniformly approximated by a polynomial. This theorem follows from the Stolzenberg theorem quoted below in Section 4.

By a homotopic deformation on a leaf, μ may be chosen to be smooth. Let s be the arc length parameter on μ. Choosing appropriate coordinates, we may assume that $d\zeta/ds \neq 0$, where ζ is the restriction of z_1 onto μ. We take $g = 1/(d\zeta/ds)$ on μ. Let P be a polynomial that approximates g uniformly on μ and take

$$\omega = P\, dz_1.$$

Obviously, the integral of ω over μ is close to the length of μ denoted by $|\mu|$.

Let the pullback ω^* be the same as above: $\omega^* = \widetilde{\pi}^*\omega$. Then the function (2.2) is holomorphic on $\widetilde{M}$ and separates the points b_1 and b_2, a contradiction. Indeed,

$$I(b_2) - I(b_1) = \int_\mu \omega \neq 0. \tag{2.3}$$

Case 2. The curve μ is simple and closed; hence, $p_1 = p_2$. Moreover, $[\mu] \neq 0$ in $H_1(\varphi_a)$. By theorem of Bishop, the following dichotomy holds:

- either any continuous function on μ may be uniformly approximated by a polynomial,
- or μ is a boundary of a compact one-dimensional analytic set.

We will prove that the second option never takes place in our case. Then the first possibility holds, and the proof follows the same lines as before: we find a function $I \in \mathcal{O}(M)$ for which (2.3) holds.

We shall prove that for μ defined above, the first case holds. Let's suppose, on the contrary, that the second case takes place. Then the curve μ is a boundary of a one-dimensional analytic set A. But $\mu \subset \varphi_a$. By the local boundary uniqueness property of analytic sets, $A \subset \varphi_a$. Hence, μ is a boundary of a compact set $A \subset \varphi_a$. Therefore, μ is homologically trivial on φ_a, a contradiction.

Case 3. The curve μ is closed but not necessary simple. Let us call a closed curve $\lambda \in \pi_1(\varphi_a, a)$ nonseparable if the points of the covering manifold $\widetilde{M}$ represented by the curves γ_1 and $\gamma_1\lambda$ cannot be separated by holomorphic functions from $\mathcal{O}(\widetilde{M})$. Our goal is to prove that there is no such curves; but we supposed that they exist, and μ is one of them. We will bring this assumption to the contradiction.

Homotopy classes of nonseparable curves form a subgroup $\Gamma \subset \pi_1(\varphi_a, a)$. The difference $\pi_1(\varphi_a, a) \setminus \Gamma$ contains a countable number of elements; we denote the loops that represent them by $\lambda_1, \dots, \lambda_n, \dots$. Let $f_n \in \mathcal{O}(\widetilde{M})$ be the function that separates points $b_1 \in \widetilde{M}$ and $a_n \in \widetilde{M}$ represented by γ_1 and $\gamma_1\lambda_n$. The function

$$f = \Sigma_1^\infty c_n f_n,$$

with suitable rapidly decreasing coefficients separates the couples b_1, a_n for all n, we recall that b_1 is represented by γ_1.

Let's consider a map

$$\Pi : \widetilde{M} \to \mathbb{C}^{n+1}, \quad p \mapsto (\widetilde{\pi}(p), f(p)), \tag{2.4}$$

and denote by ψ the image $\Pi\varphi_{p_1}$. We recall that $p_1 = \pi b_1$. A projection $\pi_0 : \Pi(p) \to \widetilde{\pi}(p)$, $\psi \to \varphi_a$ is well defined. The fundamental group $\pi_1(\psi, \Pi(b_1))$ is isomorphic to Γ. It contains a class represented by a loop, say μ_1, which is simple, smooth and nonhomological to 0 on ψ, that is, μ_1 is nontrivial in the group $H_1(\psi, \mathbb{Z})$ with compact supports. The curve $\pi_0\mu_1 = \mu_0 \subset \varphi_a$ is nonseparable. On the other hand, repeating the construction of Case 2 with the polynomial p defined on $\mathbb{C}^{n+1}$ instead of $\mathbb{C}^n$, we prove that the curve μ_1 is separable. Hence μ_0 is separable too, a contradiction.

Case 4. The curve $\mu \subset \varphi_a$ is nonclosed and selfintersecting. We can replace this curve by another one, having all the selfintersection points in a. Let λ_1 and λ_2 be simple curves on φ_a that connect p_1 and p_2 respectively with a such that $\lambda_1^{-1}\lambda_2$ is simple. Then $\mu = \lambda_1^{-1}\gamma\lambda_2$ where $\gamma \in \pi_1(\varphi_a, a)$.

Let $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ be the covering on $\widetilde{M}$ of γ_1, γ_2 respectively, $\widetilde{\gamma}_j$ begins at p_j; the curve $\widetilde{\gamma}_j$ represents the point $b_j \in \widetilde{M}$.

Regarding Case 3 we proved that all the loops in $\pi_1(\varphi_a, a)$ are separable. Hence, for a suitable function $f \in \mathcal{O}(\widetilde{M})$ and the corresponding map Π, see (2.4), the curve

$$\mu_1 = (\Pi\widetilde{\gamma}_1)^{-1}(\Pi\widetilde{\gamma}_2),$$

is simple. Using the same arguments as in case 1, we find a function $g \in \mathcal{O}(\widetilde{M})$ that separates the points $b_1, b_2 \in \widetilde{M}$, a contradiction. □

Lemma 2.1 is proved. Together with the Remmert theorem [GR], [H], it implies that $\widetilde{M}$ is either a Stein manifold, or admits a holomorphic extension S which is a Stein manifold and a Riemann domain once more.

To prove Theorem 1, we need to prove that $S = \widetilde{M}$; in other words, we have to bring to the contradiction the hypothesis;

$$S \setminus \widetilde{M} \neq \emptyset. \tag{2.5}$$

This is done in the next three sections. Section 3 contains the Closure Lemma which becomes trivial as soon as we know that $S = \widetilde{M}$. But, on the contrary, we need the Closure Lemma to prove this equality.

Section 4 contains the Connection Lemma. This lemma is used in the proof of Theorem 1; in the same time, it has an independent interest.

3. Closure Lemma

Let $\mathcal{F}$ be a foliation with singularities of $\mathbb{C}^n$ by analytic curves, and $B \subset \mathbb{C}^n$ be a hypersurface transversal to the leaves. Let B be a Stein manifold, let φ_p be a leaf of $\mathcal{F}$ passing through p.

Let $\widetilde{M}$ be a covering manifold over a family $M = \bigcup_{p \in B} \varphi_p$.

Lemma 2.1 asserts that $\widetilde{M}$ is a Riemann domain. Let S be a Riemann domain which is a holomorphic envelope of $\widetilde{M}$ and a Stein manifold at the same time. Such an S exists by the Remmert Theorem.

We want to bring to a contradiction the hypothesis (2.5). Let's suppose that (2.5) holds.

Lemma 3.1 (Closure Lemma). *For any $p \in B$, the fiber $\widetilde{\varphi}_p$ is closed in S.*

Proof. First of all, let us describe the structures inherited by S from $\widetilde{M}$. The manifold S is a Riemann domain with the projection that extends $\widetilde{\pi}$; let us denote it by the same symbol. The projection $\widetilde{\pi}\widetilde{M}$ did not contain singular points of F; the projection $\widetilde{\pi}S$ may contain some (this possibility is not excluded until the equality $\widetilde{M} = S$ is proved).

On the other hand, projection $\pi : \widetilde{M} \to B$ may be extended to S. Indeed, B is a Stein manifold by assumption. Hence, it may be holomorphically embedded into some $\mathbb{C}^k$ as a closed submanifold; let $h_1, \dots, h_k$ be the corresponding functions, determined after the choice of coordinates in $\mathbb{C}^k, h = (h_1, \dots, h_k)$. The vector function $h \circ \pi \in \mathcal{O}(\widetilde{M})$ may be extended up to $\widetilde{h} \in \mathcal{O}(S)$. This vector-function maps S onto $h(B)$.

Indeed, $h(B) \subset \mathbb{C}^k$ may be given as a zero set of some ideal $\mathbf{I}$ of functions $F \in \mathcal{O}(\mathbb{C}^k)$. That is, for any $F \in \mathbf{I}$, $F \circ h = 0$. On the other hand, the statement $F(z) = 0$, for all $F \in \mathbf{I}$, implies $z \in h(B)$. For the S-extension $\widetilde{h}$ of h we have: $F \circ \widetilde{h} = 0$ for any $F \in \mathbf{I}$. Hence, $\widetilde{h}(S) = h(B)$. The map π is now extended to S

in the following way: for any $q \in S$, let $p = h^{-1}(\widetilde{h}(q))$, $\pi(q) = p$. The fibers of the map π on S are closed analytic sets, not necessarily one-dimensional.

Now we pass to the proof of Lemma 3.1. Let's suppose, the lemma is wrong. Let $\widetilde{\varphi}$ be a fiber of the skew cylinder in $\widetilde{M}$, which is nonclosed in S. Let $p = \pi\widetilde{\varphi}$, and $\Phi = \pi^{-1}p \subset S$. The set Φ is closed analytic. We decompose it to irreducible components and denote by ψ the component that contains $\widetilde{\varphi}$. As the intersection of ψ with the set $\widetilde{M}$ (which is open in S) is one-dimensional, the set ψ is one-dimensional itself. We shall prove that inequality $\widetilde{\varphi} \neq \psi$ contradicts the fact that the map $\widetilde{\pi} : \widetilde{\varphi} \to \varphi_p$ is the universal covering.

PROPOSITION 3.1. *The image of* (*Closure* $\widetilde{\varphi}$) $\cap \partial\widetilde{M}$ *under* $\widetilde{\pi}$ *belongs to the singular set* Σ *of the foliation* F.

PROOF. We suppose that the contrary holds. Let $b \in (\text{Closure}\,\widetilde{\varphi}) \cap \partial\widetilde{M}$ and $a = \widetilde{\pi}b \notin \Sigma$. Then there exist neighborhoods U of b and V of a in S and $\mathbb{C}^n$ such that $\widetilde{\pi} : U \to V$ is biholomorphic. Let $s : V \to U$ be the inverse map. Moreover, V may be chosen in such a way that the connected component of the intersection of any leaf with V is a disk. We denote by Δ that one of these disks that contains a.

On the other hand, the neighborhood U may be chosen in such a way that the intersection $\psi \cap U$ is a topological disk, because ψ is a closed one-dimensional analytic set. The projection $\widetilde{\pi}(\varphi \cap U)$ contains the point a and a domain on some of the leafs of $\mathcal{F}$. Hence, $\widetilde{\pi}(\psi \cap U) = \Delta$. The union of the disk $s\Delta$ and the fiber $\widetilde{\varphi}$ forms a covering over the leaf φ_p. This contradicts the fact that $\widetilde{\pi} : \widetilde{\varphi} \to \varphi_p$ is the universal covering. $\square$

Let us now return to the proof of Lemma 3.1. Proposition 3.1 implies that

$$\widetilde{\pi}(\psi \cap \partial\widetilde{M}) \subset \Sigma. \tag{3.1}$$

The set $\widetilde{\pi}^{-1}\Sigma$ is an analytic subset of S. Hence, the intersection $\psi \cap \widetilde{\pi}^{-1}\Sigma$ consists of isolated points. Let $b \in \partial\widetilde{M}$ be one of them. Some neighbourhood U_0 of b on any irreducible component of $\psi \cap U$ may be uniformized. In particular, there exists a disk K and a biholomorphic map $t : K \setminus 0 \to U_0 \setminus b$. By (3.1),

$$U_0 \setminus b \subset \widetilde{\varphi}.$$

The image under t of a small circle γ centered at 0 is a loop on $\widetilde{\varphi}$. It may be contracted on $\widetilde{\varphi}$, but not across b, because $b \notin \widetilde{\varphi}$. Let K' be the image of the contraction of $t(\gamma)$. Then the union $K' \cup U_0$ is homeomorphic to a Riemann sphere holomorphically embedded in the Stein manifold S. But any holomorphic map of a sphere into a Stein manifold is constant.

The contradiction proves Lemma 3.1. $\square$

4. Fibers Connection Lemma

Lemma 4.1 (Fibers Connection Lemma). *Let the total space M of the skew cylinder $C = (M, \pi, B)$ be embedded in $\mathbb{C}^n$. Let the holomorphic hull of M be a closed submanifold $S \subset \mathbb{C}^n$. Let the skew cylinder $C' = (K, \pi', B')$ belong to C in a sense that $K \subset M$, $B' \subset B$, $\pi|_K = \pi'$. Let the closure $\overline{K}$ of K be compact, and $\widehat{K}$ be the polynomial hull of $\overline{K}$. Let φ be an arbitrary fiber of the skew cylinder C such that the intersection $\varphi \cap \overline{K}$ is nonempty. Then the intersection $\varphi \cap \widehat{K}$ is connected.*

Remark. The intersection of K with any fiber of C is either empty, or coincides with the fiber of the skew cylinder C' and thus is connected and simply connected.

Proof. We suppose that the lemma is wrong. Let φ be a fiber of the skew cylinder C whose intersection with K is nonempty and the intersection with $\widehat{K}$ is disconnected. Let γ be a real analytic curve on φ that connects a point $a \in K \cap \varphi$ with a point b that belongs to the component of the intersection $\varphi \cap \overline{K}$ disjoint from K.

We shall construct a function F holomorphic on M, hence on S, and such that

$$|F(b)| > \max_{\overline{K}} |F|. \tag{4.1}$$

This function may be holomorphically extended to $\mathbb{C}^n$ from the Stein manifold S, and uniformly approximated by a polynomial on any ball. Hence, there exists a polynomial Q on $\mathbb{C}^n$ such that

$$|Q(b)| > \max_{\overline{K}} |Q|.$$

This contradicts the statement that $\overline{K}$ is a polynomially convex hull of K, and proves the lemma, modulo the existence of the function F with the property (4.1). The function itself will be constructed by the methods of Section 2.

Proposition 4.1. *Let $\widehat{K}$ be a polynomially convex compact set in C^n, φ be an analytic curve having a disconnected intersection with $\overline{K}$. Let γ be a real analytic curve on φ that connects two points on two different connected components of the intersection $\overline{K} \cap \varphi$. Then any continuous function on $\overline{K} \cup \varphi$ that may be uniformly approximated by a polynomial on K may be in the same time uniformly approximated by a polynomial on $\overline{K} \cup \varphi$.*

This proposition will be proved below. Now we deduce from it Lemma 4.1. By assumption, the set K is open, and $\overline{K}$ is compact. Hence there exists $c > 0$ such that any two points on any fiber of the skew cylinder C' may be connected by a curve of the length no greater than c.

Let ζ be the restriction of the coordinate function z_1 to γ. Let s be the arc length along γ. Without loss of generality, we may assume that $\partial\zeta/\partial s \neq 0$. Let ψ be a smooth function $\gamma \to [0,1]$ equal to 0 on $\widehat{K} \cap \gamma$ and equal to 1 on some arc γ_0. Let us define a continuous function f on $\widehat{K} \cup \gamma$ as follows:

$$f|_\gamma = \frac{\psi}{\partial\zeta/\partial s}, \quad f|_{\overline{K}} = 0.$$

By Proposition 4.1, this function may be uniformly approximated by a polynomial P on $\widehat{K} \cup \gamma$. Let P approximate f with the accuracy sufficient for what follows. Let

$$\omega = P\,dz_1, \quad F(q) = \int_{\gamma_q} P\,dz_1,$$

where γ_q is a curve on the fiber φ_p, $p = \pi q$ that starts at p and ends at q. The function $F \in \mathcal{O}(M)$ is well defined because the fibers φ_p are simply connected. For a polynomial P approximating f with good precision we have

$$F(b) = \int_\gamma P\,dz_1 \geq |\gamma_0|/2,$$

where $|\gamma_0|$ is the length of the curve γ_0. On the other hand,

$$|F||_{\overline{K}} \leq c \max_{\overline{K}} |P| < |\gamma_0|/2.$$

These inequalities prove (4.1). Hence, they prove Lemma 4.1, modulo Proposition 4.1. □

PROOF OF PROPOSITION 4.1. The proof is based on the following theorem. As before, the hat denotes the polynomially convex hull.

THEOREM (Stolzenberg Theorem ([S])). *Let K be a polynomially convex set in $\mathbb{C}^n$, γ a compact subset of $\mathbb{C}^n$, which is a finite union of smooth curves. Then*

(A) *The difference $(\widehat{K \cup \gamma}) \setminus (K \cup \gamma)$ is the one-dimensional analytic subset of $\mathbb{C}^n \setminus (K \cup \gamma)$ (it can be empty).*

(B) *Every continuous function on $K \cup \gamma$ that is uniformly approximable by polynomials on K can be uniformly approximated by rational functions on $K \cup \gamma$. If the set $K \cup \gamma$ is polynomially convex, then rational functions may be replaced by polynomials.*

We shall prove polynomial convexity of the set $\widehat{K} \cup \gamma$. After that assertion (B) of Stolzenberg Theorem yields Proposition 4.1.

Let's suppose that the union $\widehat{K} \cup \gamma$ isn't polynomially convex. Then, by assertion (A) of Stolzenberg Theorem, there exists a nonempty one-dimensional analytic subset $A \subset \mathbb{C}^n \setminus (\widehat{K} \cup \gamma)$. Its closure is compact. We say that the difference $\partial A :=$ Cl $A \setminus A$ is the boundary of A. Obviously, $\partial A \subset \widehat{K} \cup \gamma$. We note that ∂A cannot be contained in $\widehat{K}$ because in that case the relatively compact set A would be analytic in the complement $\mathbb{C}^n \setminus \widehat{K}$. But the distance to the polynomially convex set $\widehat{K}$ is a plurysubharmonic function (see, for example [H]), and its restriction to the analytic set A cannot have a maximum at any inner point of A. So the one-dimensional analytic set A contains in its boundary the real analytic curve γ lying on the complex analytic curve φ. Hence the curve φ contains the irreducible component A' of the set A such that $\gamma \subset \mathrm{Cl}\, A'$. It is the result of Aleksander Theorem about analytic continuation of a one-dimensional analytic set across a real analytic manifold [Ch] and uniqueness of the one-dimensional analytic set containing the real analytic curve. So we have the connected complex analytic curve φ, and the compact domain A' on it such that $\partial A'$ contains the unclosed curve γ. Hence the difference $\partial A' \setminus \gamma$ is connected. This contradicts to inclusion $\partial A' \subset \gamma \cup (\widehat{K} \cap \varphi)$: by our assumption the intersection $\widehat{K} \cap \varphi$ isn't connected. This proves Proposition 4.1. □

This completes the proof of the Fibers Connection Lemma.

5. Proof of the main theorem

Here we reduce the main theorem to the Closure and Fibers Connection Lemmas. Any skew cylinder may be exhausted by compact skew cylinders. Therefore, it is sufficient to prove the following

Proposition 5.1. *Let $C = (\widetilde{M}, \pi, B)$ be a skew cylinder from Theorem* 1.1, *B being a Stein manifold. Let (K, π, B') be a compact skew cylinder that belongs to $C = (\widetilde{M}, \pi, B)$. Then the $\mathcal{O}(\widetilde{M})$-holomorphic hull of $\overline{K}$ belongs to $\widetilde{M}$.*

Proof. We assume the converse and consider, as before, the holomorphic hull S of $\widetilde{M}$ as a closed analytic submanifold of $\mathbb{C}^m$. Then the $\mathcal{O}(\widetilde{M})$-holomorphic hull of $\overline{K}$ coincides with the polynomial convex hull of K in $\mathbb{C}^m$.

We supposed that $\widehat{K} \not\subset \widetilde{M}$. Then there exists a sequence $b_n \in \widehat{K} \cap \widetilde{M}$ such that

$$b := \lim b_n \in S \setminus \widetilde{M}.$$

No subsequence of this sequence belongs to a single fiber, by the Closure lemma. Then, passing to a subsequence, we may assume that the points b_n belong to different fibers of the skew cylinder C. Let

$$p_n = \pi b_n, \quad p = \lim p_n \in B.$$

The projection $\pi\widehat{K}$ of a compact set is a compact set itself. Hence, the above limit exists. Let

$$k_n = \widehat{K} \cap \widetilde{\varphi}_{p_n}.$$

By the Fibers Connection Lemma, k_n is connected. We consider an "upper topological limit" of the sequence k_n :

$$k = \{c \mid \text{there exist } c_n \in k_n \text{ such that } c = \lim c_n\}.$$

By the well known set theoretical fact, the upper topological limit of connected sets k_n is connected itself, provided that there exists a convergent sequence $b_n \in k_n$.

By definition of k, $k \ni b$ and $k \ni p$. We shall prove that this contradicts the Closure Lemma. Indeed, by this lemma, the fiber $\widetilde{\varphi}_p$ is closed in S. Hence, it is a connected component of the inverse image $\pi^{-1}p$ because $\varphi_p = \pi^{-1}p \cap \widetilde{M}$, and $\widetilde{M}$ is open in S. Hence, $k \subset \varphi_p$, because $k \subset \pi^{-1}p$, and k is connected. Therefore, $b \in \varphi_p \subset \widetilde{M}$, a contradiction. □

This contradiction proves Proposition 5.1, and the main theorem.

Conclusion

The algebraic version of the problems discussed here is solved in [B] and [G]. This version deals with the case when the foliation is given by the polynomial map $P : \mathbb{C}^n \to \mathbb{C}^{n-1}$, and the projective compactifications of the leaves are nonsingular Riemann surfaces. In this case the simultaneous uniformization problem is positively solved.

Let's note that for the case of leaves with singularities the same problem for the algebraic foliation is not yet solved. Its positive solution, together with the main theorem of the present paper, would imply the positive solution of the general simultaneous uniformization problem.

Recently Shcherbakov [Sh] proved that any skew cylinder with a Stein total space may be exhausted by a nested sequences of embedded compact skew cylinders which are strictly pseudoconvex and have a smooth boundary. The proof uses the corollary from the introduction to this paper.

On the other hand, a small perturbation of a complex structure on the standard compact strictly pseudoconvex skew cylinder with the smooth boundary transforms the cylinder to a new one, which is still uniformizable, that is, equivalent to a standard one. This theorem is proved in [IS].

Simultaneous uniformization for the generic foliations in $\mathbb{C}^n$ remains a challenging problem. We finally note that in Theorems 1.1, 1.2, $\mathbb{C}^n$ may be replaced by an arbitrary Stein manifold.

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