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ON JULIUSZ SCHAUDER'S PAPER ON LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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Dedicated, with admiration, to Jürgen Moser

In what follows by Ω we denote a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, $\partial\Omega \in C_{2,\alpha}$ and $0 < \alpha < 1$. We let

$$Lu(x) = \sum_{i,k=1}^{n} a_{i,k}(x) \cdot u_{x_i x_k}(x) + \sum_{i=1}^{n} a_i(x) \cdot u_{x_i}(x) + a(x) \cdot u(x) = f(x)$$

to be a linear elliptic differential equation of second order with coefficients a_{ik}, a_i, a, $f \in C_{0,\alpha}(\overline{\Omega})$ and $a(x) \leq 0$ for all $x \in \Omega$.

The paper of J. Schauder "Über lineare elliptische Differentialgleichungen zweiter Ordnung" is treating the solvability of Dirichlet's problem for the above equation. The main results of Schauder's work are given in the following theorems.

Theorem 1 (A priori estimate). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $\partial\Omega \in C_{2,\alpha}$, $0 < \alpha < 1$. There exists a constant $k > 0$, which depends on Ω, $\|a_{ik}\|_{0,\alpha,\Omega}$, $\|a_i\|_{0,\alpha,\Omega}$, $\|a\|_{0,\alpha,\Omega}$, so that for all $u \in C_{2,\alpha}(\overline{\Omega})$ is valid the inequality*[1]

$$\|u\|_{2,\alpha,\Omega} \leq k \cdot \{\|Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\}.$$

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[1] $C_{k,\alpha}(\overline{\Omega})$ is the Banach space, which is defined as the linear space of k times Hölder continuously differentiable functions in $\overline{\Omega}$ under the norm $\|u\|_{k,\alpha,\Omega} = \sup_{x\in\overline{\Omega}} |u(x)| + \ldots + \sum_{|\beta|=k}\{\sup_{x\in\Omega} |D^\beta u(x)| + H^\alpha_\Omega[D^\beta u]\}$.

Theorem 2 (Kellogg's Theorem)[2]. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $\partial\Omega \in C_{2,\alpha}$, $0 < \alpha < 1$. For each $(f, g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$ there exists precisely one solution $u \in C_{2,\alpha}(\overline{\Omega})$ to the boundary problem*

$$\Delta u(x) = \sum_{i=1}^{n} u_{x_i x_i}(x) = f(x), \quad u|_{\partial\Omega} = g.$$

Theorem 3 (Continuity Method). *For $0 \leq t \leq 1$ let*

$$L_t u(x) = (1 - t) \cdot \Delta u(x) + t \cdot Lu(x),$$

$$\begin{aligned} \Gamma = \{t \mid 0 \leq t \leq 1,\ &\text{for all } (f, g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega) \text{ the Dirichlet problem} \\ &L_t u(x) = f(x),\ u|_{\partial\Omega} = g \text{ possesses a solution } u(\cdot, t) \in C_{2,\alpha}(\overline{\Omega})\}. \end{aligned}$$

If Γ is not empty, then $\Gamma = [0, 1]$.

Theorem 4 (General Existence Theorem). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $\partial\Omega \in C_{2,\alpha}$, $0 < \alpha < 1$. For each $(f, g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$ there exists precisely one solution $u \in C_{2,\alpha}(\overline{\Omega})$ to the boundary problem*

$$Lu(x) = f(x), \quad u|_{\partial\Omega} = g.$$

In order to read the original work of J. Schauder [8] a strong background in potential theory is required so that the proofs in that paper could only be carried out by a few of the readers. In the literature, the results of above paper are used without going into the proofs. For this reason it is often overlooked that not only do Theorems 1, 3 and 4 stem from Schauder but Theorem 2 as well.

In what follows our aim is to show that Theorem 2 is in fact the principal result of the Schauder's paper. If one has Theorem 2 the Theorems 1, 3 and 4 follow with much less work than is necessary in giving a direct proof. This is a consequence of the following equivalence theorem, which is the main result of the present note:

Theorem (Equivalence Theorem). *The following statements are equivalent.*

(A) *For every bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C_{2,\alpha}$, $0 < \alpha < 1$, the following statement holds: if*

$$Lu(x) = \sum_{i,k=1}^{n} a_{i,k}(x) \cdot u_{x_i x_k}(x) + \sum_{i=1}^{n} a_i(x) \cdot u_{x_i}(x) + a(x) \cdot u(x)$$

is an arbitrary elliptic differential operator with coefficients a_{ik}, a_i, $a \in C_{0,\alpha}(\overline{\Omega})$ and $a(x) \leq 0$ for all $x \in \Omega$ and if $(f, g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$

[2] A partial case of Theorem 2 was proved by O. D. Kellogg in [3], the proof in full generality was given by J. Schauder in [8].

is arbitrary, then there exists a function $u \in C_{2,\alpha}(\overline{\Omega})$ *which solves the boundary value problem*

$$Lu(x) = f(x), \quad u|_{\partial\Omega} = g.$$

(B) *For every bounded domain* $\Omega \subset \mathbb{R}^n$ *with* $\partial\Omega \in C_{2,\alpha}$, $0 < \alpha < 1$, *the following statement holds: if* $(f, g) \in C_{0,\alpha}(\{\overline{\Omega}\}) \times C_{2,\alpha}(\partial\Omega)$ *is arbitrary, then there exists a function* $u \in C_{2,\alpha}(\{\overline{\Omega}\})$ *which solves the boundary value problem*

$$\Delta u(x) = f(x), \quad u|_{\partial\Omega} = g.$$

Proof. The proof that (A) implies (B) is evident. To prove the reverse implication we need the following inequality

$$\|u\|_{2,\alpha,\Omega} \leq k \cdot \{\|(1-t) \cdot \Delta u + t \cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\} \tag{1}$$

with a constant $k > 0$ independent of $t \in [0, 1]$ and all $u \in C_{2,\alpha}(\overline{\Omega})$. The proof of formula 1 is carried out in two steps.

Step 1. We first show that for each $t \in [0, 1]$ there exists a $k(t)$ so that for all $u \in C_{2,\alpha}(\overline{\Omega})$ the estimate

$$\|u\|_{2,\alpha,\Omega} \leq k(t) \cdot \{\|(1-t) \cdot \Delta u + t \cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\} \tag{2}$$

holds. Let $t \in [0, 1]$ be given. For $y \in \overline{\Omega}$ a family of elliptic operators

$$L_y\varphi = \sum_{i,k=1}^{n} \{(1-t) \cdot \delta_{ik}\varphi + t \cdot a_{ik}(y)\} \cdot \frac{\partial^2}{\partial x_i \partial x_k}\varphi$$

with constant coefficients is defined. By Lemma 1 of the Appendix it follows that for each $\widehat{x} \in \overline{\Omega}$ there exists a constant $c(\hat{x})$, so that for all $v \in C_{2,\alpha}(\overline{\Omega})$ the estimate

$$\|v\|_{2,\alpha,\Omega} \leq c(\widehat{x}) \cdot \{\|L_{\widehat{x}}v\|_{0,\alpha,\Omega} + \|v\|_{2,\alpha,\partial\Omega}\}$$

holds, where $c(\widehat{x})$ depends on the choice of t. Let

$$\begin{aligned} P(\widehat{x}) &= c(\widehat{x}) \cdot \sum_{i.k=1}^{n} \|a_{ik}\|_{0,\alpha,\Omega}, \\ O(\widehat{x}) &= \{x \mid x \in \mathbb{R}^n,\ \|x - \widehat{x}\|^\alpha < (2 \cdot P(\widehat{x}))^{-1}\}. \end{aligned}$$

The sets $\{O(\widehat{x})\}_{\widehat{x}\in\overline{\Omega}}$ form an open covering of $\overline{\Omega}$. By compactness, $\overline{\Omega}$ is covered by a finite family $\{O(x^j)\}_{j=1,\dots,M}$. Let now $\varphi_j \in C_0^\infty(\mathbb{R}^n)$ $(j = 1, \dots, M)$ with $\operatorname{supp} \varphi_j \subset O(x^j)$ be a subordinated partition of unity. Let $j \in \{1, \dots, M\}$ be given. From

$$L_{x^j}(\varphi_j \cdot u) = t \cdot \sum_{i,k=1}^{n} (a_{ik}(x^j) - a_{ik}(x)) \cdot (\varphi_j \cdot u)_{x_i x_k} + L_x(\varphi_j \cdot u)$$

we have the estimates

$$\|\varphi_j u\|_{2,\alpha,\Omega} \leq k(x^j) \cdot \left\{ \left\| t \cdot \sum_{i,k=1}^{n} (a_{ik}(x^j) - a_{ik}(x)) \cdot (\varphi_j u)_{x_i x_k} + L_x(\varphi_j u) \right\|_{0,\alpha,\Omega} + \|\varphi_j u\|_{2,\alpha,\partial\Omega} \right\},$$

and

$$\|\varphi_j u\|_{2,\alpha,\Omega} \leq k(x^j) \cdot \left\{ (2 \cdot P(x^j))^{-1} \cdot \sum_{i,k=1}^{n} \|a_{ik}\|_{0,\alpha,\Omega} \cdot \|\varphi_j u\|_{2,\alpha,\Omega} + 2 \cdot \sum_{i,k=1}^{n} \|a_{ik}\|_{0,\alpha,\Omega} \cdot \|\varphi_j u\|_{2,o,\Omega} + \|L_x(\varphi_j u)\|_{0,\alpha,\Omega} \|\varphi_j u\|_{2,\alpha,\partial\Omega} \right\}.$$

Next from the Corollary to Ehrling's Lemma (see Appendix, corollary to Lemma 2) we get for a given $\varepsilon > 0$ an estimate

$$\|w\|_{k,0,\Omega} \leq \varepsilon \cdot \|w\|_{k,\alpha,\Omega} + C(\varepsilon) \cdot \|w\|_{0,0,\Omega} \tag{3}$$

for all $w \in C_{k,\alpha}(\overline{\Omega})$. Furthermore, using the estimate

$$\|\varphi_j u\|_{0,0,\Omega} \leq \text{const } \cdot \{\|L_x(\varphi_j u)\|_{0,0,\Omega} + \|\varphi_j u\|_{0,0,\partial\Omega}\}, \tag{4}$$

which follows from the maximum-minimum principle, we find

$$\|\varphi_j u\|_{2,\alpha,\Omega} \leq \overline{c(x^j)} \cdot \{\|L_x(\varphi_j u)\|_{0,\alpha,\Omega} + \|\varphi_j u\|_{2,\alpha,\partial\Omega}\},$$

where $\overline{c(x^j)}$ is a suitable constant depending on x^j. Now summing up over j from 1 to M we obtain

$$\|u\|_{2,\alpha,\Omega} = \left\| \sum_{j=1}^{M} \varphi_j u \right\|_{2,\alpha,\Omega} \leq \sum_{j=1}^{M} \|\varphi_j u\|_{2,\alpha,\Omega} \leq \sum_{j=1}^{M} \overline{c(x^j)} \cdot \{\|L_x(\varphi_j u)\|_{0,\alpha,\Omega} + \|\varphi_j u\|_{2,\alpha,\partial\Omega}\}. \tag{5}$$

By differentiating each of $\varphi_j u$ and using triangle inequality, it follows from (5) that with suitable constants k_1 and k_2 we get the estimate which follows:

$$\|u\|_{2,\alpha,\Omega} \le k_1 \cdot \{\|(1-t)\cdot \Delta u + t\cdot Lu\|_{0,\alpha,\Omega}\} + k_2 \cdot \Bigg\{ \sum_{i,k=1}^{n} \|a_{ik}\|_{2,\alpha,\Omega} \cdot \|u\|_{2,0,\Omega} + \left(\sum_{i=1}^{n} \|a_i\|_{0,\alpha,\Omega} + \|a\|_{0,\alpha,\Omega}\right) \cdot \|u\|_{1,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\cdot \Bigg\}$$

From the last inequality in view of (3) and (4), the inequality (2) follows.

Step 2. Now we show the existence of a constant k independent of $t \in [0,1]$, so that for all $u \in C_{2,\alpha}(\overline{\Omega})$ the inequality

$$\|u\|_{2,\alpha,\Omega} \le k \cdot \{\|(1-t)\cdot \Delta u + t\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\}$$

is valid. To this end let $u \in C_{2,\alpha}(\overline{\Omega})$ be given. By Step 1 we know that for each $t_0 \in [0,1]$ there exists a constant $k(t_0)$, so that

$$\|u\|_{2,\alpha,\Omega} \le k(t_0) \cdot \{\|(1-t_0)\cdot \Delta u + t_0\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\}$$

holds. Consequently, for all t with

$$|t-t_0| \le \left[2\cdot k(t_0)\cdot \left\{ n + \sum_{i,k=1}^{n} \|a_{ik}\|_{0,\alpha,\Omega} + \sum_{i=1}^{n} \|a_i\|_{0,\alpha,\Omega} + \|a\|_{0,\alpha,\Omega} \right\}\right]^{-1},$$

we get the inequalities

$$\begin{aligned} \|u\|_{2,\alpha,\Omega} &\le k(t_0)\cdot\{|t-t_0|\cdot\|\Delta u - Lu\|_{0,\alpha,\Omega} + \|(1-t)\cdot\Delta u \\ &\qquad + t\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\} \\ \|u\|_{2,\alpha,\Omega} &\le \frac{1}{2}\cdot\|u\|_{2,\alpha,\Omega} + k(t_0)\cdot\{\|(1-t)\cdot\Delta u + t\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\} \end{aligned}$$

and therefore

$$\|u\|_{2,\alpha,\Omega} \le 2\cdot k(t_0)\cdot\{\|(1-t)\cdot\Delta u + t\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\}.$$

Now we observe that the sets

$$O(\tau) = \left\{ t \,\middle|\, |t-\tau| < \left[2\cdot k(\tau)\cdot\left\{ n + \sum_{i,k=1}^{n} \|a_{ik}\|_{0,\alpha,\Omega} + \sum_{i=1}^{n} \|a_i\|_{0,\alpha,\Omega} + \|a\|_{0,\alpha,\Omega}\right\}\right]^{-1} \right\},$$

where $\tau \in [0,1]$, form an open covering of the interval $[0,1]$, so by the compactness there exists a finite family $\{O(\tau_j)\}_{j\in\{1,\dots,s\}}$, which covers $[0,1]$.
Let $k = 2\cdot\max_{j\in\{1,\dots,s\}} k(\tau_j)$. Since for each $t \in [0,1]$, there is a τ_0 with $t \in O(\tau_0)$, we obtain

$$\|u\|_{2,\alpha,\Omega} \le 2\cdot k(\tau_0)\cdot\{\|(1-t)\cdot\Delta u + t\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\}.$$

Consequently, by definition of k, we get the desired inequality

$$\|u\|_{2,\alpha,\Omega} \le k\cdot\{\|(1-t)\cdot\Delta u + t\cdot Lu\|_{0,\alpha,\Omega} + \|u\|_{2,\alpha,\partial\Omega}\}.$$

Now to prove that statement (A) follows from statement (B), one must show that for each $(f,g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$ there exists a function $u \in C_{2,\alpha}(\overline{\Omega})$ with $Lu(x) = f(x)$, $u|_{\partial\Omega} = g$. This proof is carried out by means of the continuity method. For this we need the inequality (1).

Let $(f,g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$ be arbitrary but fixed. For $t \in [0,1]$ we define the following family of uniformly elliptic differential operators

$$L_t\varphi = (1-t)\cdot\Delta\varphi + t\cdot L\varphi.$$

Let

$$\begin{aligned}\Gamma = \{t \mid t \in [0,1] &\text{ and for all } (f,g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega) \text{ the problem}\\ &L_t u(x) = f(x),\ u|_{\partial\Omega} = g \ \text{ possesses a solution } u(\circ,t) \in C_{2,\alpha}(\overline{\Omega})\}.\end{aligned}$$

Observe that $t = 0 \in \Gamma$. Let $t_0 \in \Gamma$ and k be the constant derived from the estimate (1) and

$$\varepsilon = [2\cdot k\cdot n\cdot(n+3)\cdot \max_{i,k\in\{1,\dots,n\}}\{1, \|a_{ik}\|_{0,\alpha,\Omega}, \|a_i\|_{0,\alpha,\Omega}, \|a\|_{0,\alpha,\Omega}\}]^{-1}.$$

Then all $t \in [0,1]$ with $|t-t_0| < \varepsilon$ belong to Γ because for $t \in [0,1]$ and $|t-t_0| < \varepsilon$ and for each $u \in C_{2,\alpha}(\overline{\Omega})$ one has

$$\widehat{f}(\varphi,t) = (t-t_0)\cdot(\Delta u - Lu) + f \in C_{0,\alpha}(\overline{\Omega})$$

and the problem $L_{t_0}v = \widehat{f}(\circ,t)$, $v|_{\partial\Omega} = g$ has precisely one solution $v \in C_{2,\alpha}(\overline{\Omega})$ since t_0 belongs to Γ.

By $u \to v$ a mapping $T_{t_0} : C_{2,\alpha}(\overline{\Omega}) \to C_{2,\alpha}(\overline{\Omega})$ is defined. For arbitrary u_1 and u_2 in $C_{2,\alpha}(\overline{\Omega})$ the function $T_{t_0}u_1 - T_{t_0}u_2$ is a solution to the boundary value problem

$$\begin{aligned}L_{t_0}(T_{t_0}u_1 - T_{t_0}u_2) &= (t-t_0)\cdot[\Delta(u_1-u_2) - L(u_1-u_2)],\\ (T_{t_0}u_1 - T_{t_0}u_2)|_{\partial\Omega} &= 0.\end{aligned}$$

Thus, applying (1) to $T_{t_0}u_1 - T_{t_0}u_2$ we obtain the inequality

$$\begin{aligned}\|T_{t_0}u_1 - T_{t_0}u_2\|_{2,\alpha,\Omega} &\le k\cdot|t-t_0|\cdot\{\|\Delta(u_1-u_2)\|_{0,\alpha,\Omega} + \|L(u_1-u_2)\|_{0,\alpha,\Omega}\},\\ \|T_{t_0}u_1 - T_{t_0}u_2\|_{2,\alpha,\Omega} &\le \frac{1}{2}\cdot\|u_1-u_2\|_{2,\alpha,\Omega},\end{aligned}$$

i.e. the mapping T_{t_0} is contracting. Consequently, by the Banach Fixed Point Theorem, there exists a solution $u \in C_{2,\alpha}(\overline{\Omega})$ to the equation $T_{t_0}u = u$, i.e. t belongs to Γ. Since ε does not depend on t_0, it follows also that $[0,1] \subseteq \Gamma$. There exists, therefore for all $(f,g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$ a solution $u \in C_{2,\alpha}(\overline{\Omega})$ to the problem $Lu(x) = f(x)$, $u|_{\partial\Omega} = g$.

Remark

J. Schauder makes the following remark on page 281 of his paper [8][3]:

> Ich möchte hier noch folgende vielleicht interessante Bemerkung machen. Nehmen wir an, es wäre bereits gelungen, den Existenzbeweis der Lösung für elliptische Differentialgleichungen auf irgendeine Art und Weise zu erledigen. Weiter setzen wir voraus, daß die qualitative Eigenschaft, die die Hölderstetigkeit der zweiten Ableitungen D_2u bis auf den Rand des Gebietes behauptet, falls die Randwerte zweimal α-H-stetig differenzierbar sind, auch als richtig erkannt wurde. Dann bin ich imstande, daraus die qualitative Abschätzung (48) (Anmerkung: a priori Abschätzung) zu folgern. Denn zuerst beweise ich durch Anwendung eines Satzes aus der Theorie der Funktionaloperationen[4] die Existenz einer Abschätzungskonstante C. Weiter schließe ich durch Anwendung der sukzessiven Approximationen und Kontinuitätsbetrachtungen von der aus S. 277 vorkommenden Art, daß die Konstante C nur von M abhängt.

We see that with the help of the Equivalence Theorem, one can sharpen the remark of J. Schauder in that one does not have to demand the solvability of the Dirichlet problem for the general elliptic differential equation in the class $C_{2,\alpha}(\overline{\Omega})$ but only the solvability of the Laplace equation in the class $C_{2,\alpha}(\overline{\Omega})$.

The existence of a solution to the Dirichlet problem for the Laplace equation one can show by the method of O. Perron [5] in the class $C_{0,0}(\overline{\Omega}) \cap C_{2,0}(\Omega)$ or by potential theoretic methods, J. Schauder [7], N. M. Günther [1] in the class $C_{1,\alpha}(\overline{\Omega}) \cap C_{2,0}(\Omega)$.

Together with the paper of E. Hopf [2] it follows then that the solution is in $C_{0,0}(\overline{\Omega}) \cap C_{2,\alpha}(\Omega)$ or $C_{1,\alpha}(\overline{\Omega}) \cap C_{2,\alpha}(\Omega)$ as the case may be.

To show that the solution belongs to the class $C_{2,\alpha}(\overline{\Omega})$ it remains only to investigate the behaviour of the solution near the boundary. For that see the paper [4] of the author. In this way it is possible to obtain the results of J. Schauder with the Equivalenz Theorem essential simpler.

Comments

The results of Schauder's paper are significant not only for the Dirichlet problem for an elliptic differential equation, but together with subsequent works of J. Schauder and J. Leray [6], [9] on the existence of fixed points for completely continuous mappings in Banach spaces, they play a key role in existence proofs

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[3]At this point I would like to make the following remark. Suppose in some way we had been able to give an existence proof for the solution to an elliptic differential equation. Let us further require that the Hölder continuity of the second derivatives holds up to the boundary when the boundary values are twice differentiable and satisfy a Hölder condition. Then I am in the position to obtain the qualitative estimate (48) (Remark of the writer: a priori estimate). For I first prove the existence of a constant C for the estimate by applying a theorem from the theory of functional analysis. Then I conclude by the method of successive approximation and continuity considerations of the kind occuring on page 277 that the constant depends only on M.

[4]S. Banach, Studia Math. 1 (1929), 223–239, insbesondere Satz 7 und J. Schauder, Studia Math. 2 (1930), 1–8.

for solutions u to the Dirichlet problem for nonlinear elliptic differential equations. We elaborate this by means of an example for the quasi-linear elliptic differential equation:

$$\sum_{i,k=1}^{n} a_{ik}(x, u_x, u) \cdot u_{x_i x_k}(x) = f(x, u_x, u), \quad x \in \Omega.$$

By means of the Schauder fixed point theorem the existence of solutions to nonlinear problems can be proven in the following way:

1. Find a suitable Banach space B, which contains the desired solution u.
2. In this Banach space define a mapping $T : B \to B$ so that a fixed point of the mapping T is a solution to the nonlinear problem under consideration.
3. If one can prove that T satisfies the hypotheses of the Schauder fixed point theorem, the existence proof for the solution to the original problem will be complete.

How does one translate this program to solving the Dirichlet problem for a quasi-linear elliptic equation?

With the help of Theorem 4 we find by the following consideration a suitable Banach space B and a mapping $T : B \to B$. For that we begin by requiring that the functions a_{ik} and f are at least in $C_{0,\alpha}(\overline{\Omega} \times \mathbb{R}^n \times \mathbb{R})$. If we now take as Banach space $B = C_{2,\alpha}(\overline{\Omega})$, then for each $v \in C_{2,0}(\overline{\Omega}) \subset C_{2,\alpha}(\overline{\Omega})$ the linear Dirichlet problem

$$\begin{aligned} \sum_{i,k=1}^{n} a_{ik}(x, v_x, v) \cdot u_{x_i x_k}(x) &= f(x, v_x, v), \quad x \in \Omega, \\ u|_{\partial\Omega} &= g, \end{aligned}$$

has by Theorem 4 a solution $u \in C_{2,\alpha}(\overline{\Omega})$. Hence a mapping $\widehat{T} : C_{2,0}(\overline{\Omega}) \to C_{2,\alpha}(\overline{\Omega})$ is defined by $v \to u$. Since the natural embedding $I : C_{2,\alpha}(\overline{\Omega}) \to C_{2,0}(\overline{\Omega})$ is completely continuous the mapping $T = \widehat{T} \circ I : C_{2,\alpha}(\overline{\Omega}) \to C_{2,\alpha}(\overline{\Omega})$ is completely continuous. From the construction of T we see immediately that a fixed point of T is a solution to the original problem.

Thus, only the third step remains to be verified, that is the proof that T satisfies the hypotheses of the Leray–Schauder fixed point theorem. We have already verified the compactness requirement. To prove that a bounded, convex subset of the Banach space $C_{2,\alpha}(\overline{\Omega})$ is mapped into itself by T, one makes use of the a priori estimate

$$\|Tv\|_{2,\alpha,\Omega} \leq C(\Omega, \|a_{ik}\|_{0,\alpha,\Omega}) \cdot \{\|f(x, v_x, v)\|_{0,\alpha,\Omega} + \|g\|_{2,\alpha,\partial\Omega}\}$$

from Theorem 1. The work remaining consists now in finding suitable assumptions on the coefficients a_{ik} and on the function f so that for all $\lambda \in [0, 1]$ the norms $\|v\|_{0,0,\Omega}$, $\|v\|_{1,0,\Omega}$ and $\|v\|_{1,\alpha,\Omega}$ are uniformly bounded for all solutions v to the problem

$$\sum_{i,k=1}^{n} a_{ik}(x, \lambda \cdot v_x, \lambda \cdot v) \cdot v_{x_i x_k}(x) = f(x, \lambda \cdot v_x, \lambda \cdot v), \quad x \in \Omega \text{ and } 0 \leq \lambda \leq 1,$$
$$v|_{\partial\Omega} = g.$$

Appendix

Lemma 1. *Suppose statement* (B) *holds and let*

$$L_0 \varphi = \sum_{i,k=1}^{n} b_{ik} \cdot \frac{\partial^2}{\partial x_i \partial x_k} \varphi$$

be an elliptic differential operator in Ω whose coefficients b_{ik} are constant. For all $(f, g) \in C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$ the boundary value problem $L_0 u(x) = f(x)$, $u|_{\partial\Omega} = g$ possesses one and only one solution $u \in C_{2,\alpha}(\overline{\Omega})$. Furthermore the estimate

$$\|u\|_{2,\alpha,\Omega} \leq k \cdot \{\|f\|_{0,\alpha,\Omega} + \|g\|_{2,\alpha,\partial\Omega}\}$$

holds. The constant k depends upon Ω and the coefficients b_{ik}, but not on f and g.

Proof. Since the operator L_0 is elliptic, there exists a nonsingular linear mapping $Y : \mathbb{R}^n \to \mathbb{R}^n$, $Y(\Omega) = Q$, so that the boundary value problem $L_0 u = f$, $u|_{\partial\Omega} = g$ can be transformed into the boundary value problem $\Delta v = \widehat{f}$, $v|_{\partial Q} = \widehat{g}$. Since Q, however, is again a bounded domain with $\partial Q \in C_{2,\alpha}$, there exists by hypothesis and the maximum-minimum principle precisely one solution $v \in C_{2,\alpha}(\overline{\Omega})$. Thus there exists a unique solution $u \in C_{2,\alpha}(\overline{\Omega})$ to the original problem. By $(f, g) \to u$ a linear mapping $T : C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega) \to C_{2,\alpha}(\overline{\Omega})$ is defined. T is one-to-one. The inverse $T^{-1} : C_{2,\alpha}(\overline{\Omega}) \to C_{0,\alpha}(\overline{\Omega}) \times C_{2,\alpha}(\partial\Omega)$, defined by $u \to (L_0 u, u|_{\partial\Omega})$, is continuous. By the Theorem of Banach, T is then continuous. Since T is linear, there exists a constant k of the type stated in the assertion

Lemma 2 (Ehrling's Lemma). *Let*

1. *X_1, X_2, X_3 be Banach spaces with corresponding norm $\| \circ \|_j$, $j \in \{1, 2, 3\}$.*
2. *$K : X_1 \to X_2$ be a linear compact map.*
3. *$T : X_2 \to X_3$ be a linear continuous map which is one-to-one.*

Then for each $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that the inequality

$$\|Kx\|_2 \leq \varepsilon \cdot \|x\|_1 + c(\varepsilon) \cdot \|(T \circ K)x\|_3$$

holds for all $x \in X_1$.

This Lemma is well known, so that we do not prove it here.

COROLLARY. *Let l be an integer, $0 < \alpha < 1$, $X_1 = C_{l,\alpha}(\overline{\Omega})$, $X_2 = C_{l,0}(\overline{\Omega})$ and $X_3 = C_{0,0}(\overline{\Omega})$. Let the mappings $K : C_{l,\alpha}(\overline{\Omega}) \to C_{l,0}(\overline{\Omega})$ and $T : C_{l,0}(\overline{\Omega}) \to C_{0,0}(\overline{\Omega})$ be the natural embeddings, i.e. $Ku = u$ and $Tu = u$. Then for each $\varepsilon > 0$ and for all $u \in C_{l,\alpha}(\overline{\Omega})$ there exists a constant $c(\varepsilon)$ such that the inequality*

$$\|u\|_{l,0,\Omega} \leq \varepsilon \cdot \|u\|_{l,\alpha,\Omega} + c(\varepsilon) \cdot \|u\|_{0,0,\Omega}$$

holds.

PROOF. T and K are continuous, linear, one-to-one maps and K is compact. Therefore the claim follows from Lemma 2.

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