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ON r-NEIGHBOURLY SUBMANIFOLDS IN $\mathbb{R}^N$

Victor A. Vassiliev

(*Submitted by A. Granas*)

To Jürgen Moser with very best wishes

1. Introduction

Let M be a k-dimensional manifold, and r a natural number.

Definition. A smooth embedding $M \to \mathbb{R}^N$ is r-*neighbourly* if for any r points in M there exists an affine hyperplane in $\mathbb{R}^N$, supporting M and touching it at exactly these r points.

We denote by $\delta(M, r)$ the minimal dimension N of an Euclidean space, such that there exists an r-neighbourly embedding $M \to \mathbb{R}^N$, and by $\delta(k, r)$ the maximum of numbers $\delta(M, r)$ over all k-dimensional manifolds M.

The problem of estimating the numbers $\delta(k, r)$ for all k and r was posed by M. Perles in the 1970-ies by analogy with similar problems of combinatorics, and was discussed at the Oberwolfach Combinatorics meetings in 1982 and 1986. Nontrivial examples of r-neighbourly manifolds were constructed in [5]; as G. Kalai communicated to me, in their non-published work with A. Wigderson a polynomial upper estimate $\delta(k, r)$ is proved. However, no nontrivial general lower estimates of these numbers are known.

We consider a similar problem concerning a slightly stronger condition.

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DEFINITION. A smooth embedding $M \to \mathbb{R}^N$ is *stably r-neighbourly*, if it is r-neighbourly, and any other embedding, sufficiently close to it in the C^2-topology, also is. The corresponding analogues of numbers $\delta(M,r)$ and $\delta(k,r)$ are denoted by $\Delta(M,r)$ and $\Delta(k,r)$, respectively.

It is more or less obvious that

$$\Delta(k,r) \geq (k+1)r. \tag{1}$$

Indeed, if $N < (k+1)r$, then for any generic submanifold $M^k \subset \mathbb{R}^N$ and almost any set of r points in M^k the minimal affine plane in $\mathbb{R}^N$, touching M^k at all points of this set, coincides with entire $\mathbb{R}^N$. If $k=1$, then this estimate is sharp: the moment embedding $S^1 \to \mathbb{R}^{2r}$, sending the point α to $(\sin\alpha, \cos\alpha, \ldots, \sin r\alpha, \cos r\alpha)$, is stably r-neighbourly.

We prove the following lower estimates of numbers $\Delta(M,r)$. We denote by $d(r)$ the number of ones in the binary representation of r.

THEOREM 1. *If k is a power of 2, then*

$$\Delta(\mathbb{R}^k, r) > (k+1)r + (k-1)(r-d(r)) - 1. \tag{2}$$

In particular, if r also is a power of 2, then $\Delta(\mathbb{R}^k, r) > 2kr - k$.

Of course, the same estimate of the number $\Delta(M,r)$ is true for any k-dimensional manifold M.

For an *arbitrary fixed* r and growing k we have a slightly better estimate of $\Delta(k,r)$.

THEOREM 2. *For any $r \leq k = 2^j$,*

$$\Delta(\mathbb{R}P^k, r) > (k+1)r + kr - r(r+1)/2 - 1. \tag{3}$$

The estimate of Theorem 2 can be improved if $r=1$.

PROPOSITION 1. *For any compact M, $\Delta(M,1) \leq N$ if and only if M can be smoothly embedded into S^{N-1}. In particular, for $k = 2^j$, $\Delta(\mathbb{R}P^k, 1) = 2k+1$.*

CONJECTURE. In both theorems 1 and 2, Δ can be replaced by δ.

In fact, we prove this conjecture in a much more general situation than it follows from Theorems 1 and 2.

THEOREMS 1!, 2!. *All r-neighbourly embeddings $\mathbb{R}^k \to \mathbb{R}^N$ (respectively, $\mathbb{R}P^k \to \mathbb{R}^N$) with N equal to the right-hand side of the inequality* (2) *(respectively,* (3)*), belong to a subset $\Sigma\Sigma$ of infinite codimension in the space $C^2(\mathbb{R}^k, \mathbb{R}^N)$ (respectively, $C^2(\mathbb{R}P^k, \mathbb{R}^N)$).*

I believe that in fact such embeddings do not exist.

Of course, Theorems 1 and 2 follow from these ones and even from similar statements with "infinite codimension" replaced by "positive codimension". The set $\Sigma\Sigma$ will be described in Section 3.

I thank Professor Gil Kalai for stating the problem.

2. A topological lemma

Convention and notation. Using the parallel translations in $\mathbb{R}^N$, we will identify the space $\mathbb{R}^N$ with its tangent spaces at all points $y \in \mathbb{R}^N$, and use the canonical projection $\pi : T\mathbb{R}^N \to \mathbb{R}^N$, sending any vector $v \in T_y\mathbb{R}^N$ to the point $y + v$. In particular, any vector subspace $\Xi \subset T_y\mathbb{R}^N$ defines the affine subspace $\pi\Xi \subset \mathbb{R}^N$. The sign $\oplus$ denotes the direct (Whitney) sum of two vector bundles. $w(\xi)$ is the total Stiefel–Whitney class of the vector bundle ξ, see [7], and $\overline{w}(\xi)$ is the total Stiefel–Whitney class of any vector bundle η such that the bundle $\xi \oplus \eta$ is trivial. In particular, $w(\xi)\overline{w}(\xi) = \mathbf{1} \in H^0(\text{the base of } \xi, \mathbb{Z}_2)$.

Let L be a l-dimensional compact manifold without boundary (maybe not connected), and ξ a t-dimensional vector bundle over L ($t < N - l$).

Definition. A *smooth homomorphism* $i : \xi \to T\mathbb{R}^N$ is a pair, consisting of a smooth map $\iota : L \to \mathbb{R}^N$ and, for any $x \in L$, a homomorphism $\xi_x \to T_{\iota(x)}\mathbb{R}^N$ smoothly depending on x. i is a *monomorphism* if all these homomorphisms $\xi_x \to T_{\iota(x)}\mathbb{R}^N$ are injective.

Let $i : \xi \to T\mathbb{R}^N$ be a smooth monomorphism such that the corresponding map $\iota : L \to \mathbb{R}^N$ is a smooth embedding and for any $x \in L$ two subspaces $i(\xi_x)$ and $T_{\iota(x)}(\iota(L)) \equiv \iota_*(T_xL)$ of the tangent space $T_{\iota(x)}\mathbb{R}^N$ have no common nonzero vectors. This monomorphism i induces the proper map $i' \equiv \pi \circ i : E \to \mathbb{R}^N$ of the total space E of ξ into $\mathbb{R}^N$, sending any vector $a \in \xi_x$ to the point $\iota(x) + i(a)$. Suppose that this map i' is transversal to $\iota(L)$ everywhere in $E \setminus L$ (where $L \subset E$ denotes the zero section of ξ). Then this intersection set $\iota(L) \cap i'(E \setminus L)$ defines a $\mathbb{Z}_2$-cycle in $\iota(L) \sim L$.

Lemma 1. *The class in $H^*(L, \mathbb{Z}_2)$, Poincaré dual to this cycle $\iota^{-1}(\iota(L) \cap i'(E \setminus L))$, is equal to*

$$\overline{w}_{N-l-t}(TL \oplus \xi), \tag{4}$$

the $(N - l - t)$-dimensional homogeneous component of $\overline{w}(TL \oplus \xi)$.

In particular, if this class (4) *is nontrivial, then the set $\iota(L) \cap i'(E \setminus L)$ is not empty. Moreover, the latter is true even if i' is not transversal to $\iota(L)$ in $E \setminus L$.*

Example. If $t = 0$, we get the known obstruction to the existence of an embedding $L^l \to \mathbb{R}^N$ (see [7]). Indeed, if $\overline{w}_{N-l}(TL) \neq 0$, then the *selfintersection set* of any such *embedding* is non-empty.

Proof of Lemma 1. There exists a tubular ε-neighbourhood U_ε of the submanifold $\iota(L)$ in $\mathbb{R}^N$ such that the containing L connected component of $(i')^{-1}(U_\varepsilon)$ is a tubular neighbourhood of L in E, and the restriction of i' on the latter neighbourhood (which we denote by W_ε) is a diffeomorphism onto its image. Let us deform $\iota(L)$ in $\mathbb{R}^N$ by a sufficiently C^1-small generic diffeomorphism v of $\mathbb{R}^N$ so that

(a) $|v(y) - y| < \varepsilon/2$ for any $y \in \iota(L)$,
(b) the image $v\iota(L)$ of $\iota(L)$ under this shift is a smooth manifold in U_ε transversal to the manifold $i'(W_\varepsilon)$,
(c) the map i' remains transversal to $v\iota(L)$ everywhere in $E \setminus W_\varepsilon$.

The variety $v\iota(L) \cap i'(E)$ consists of two disjoint parts. The first, $v\iota(L) \cap i'(E \setminus W_\varepsilon)$, defines in the group $H_*(v\iota(L), \mathbb{Z}_2) \simeq H_*(L, \mathbb{Z}_2)$ the same homology class as $\iota(L) \cap i'(E \setminus W_\varepsilon)$, which we are going to calculate. The second, $v\iota(L) \cap i'(W_\varepsilon)$, is Poincaré dual to the first homological obstruction to the existence of a continuous non-zero section of the quotient bundle $\mathbb{R}^N/(TL \oplus \xi)$ over L, and thus is equal to $\overline{w}_{N-l-t}(TL \oplus \xi)$, see [7].

Finally, the homology class in $H_*(v\iota(L), \mathbb{Z}_2)$ of the sum of these two cycles in $v\iota(L)$ is Poincaré dual to the restriction to $v\iota(L)$ of the cohomology class $[E] \in H^{N-t-l}(\mathbb{R}^N, \mathbb{Z}_2)$ Poincaré dual to the direct image of the fundamental cycle of the closed manifold E under the proper map i'. The latter class belongs to a trivial group, hence our two cycles in $v\iota(L)$ are homologous mod 2. $\square$

3. The general topological estimate

Let M be a k-dimensional manifold, and $B(M, r)$ the r-th configuration space of M, i.e. the topological space, whose points are subsets of cardinality r in M. Let ψ (respectively, $\widetilde{\psi}$) be the canonical r-dimensional (respectively, $(r-1)$-dimensional) vector bundle over $B(M, r)$, whose fibre over the point $\mathbf{x} = \{x_1, \dots, x_r\} \in B(M, r)$ is the space of all functions $f : \{x_1, \dots, x_r\} \to \mathbb{R}$ (respectively, of all such functions with $\sum f(x_i) = 0$). Obviously $\psi \simeq \widetilde{\psi} \oplus \mathbb{R}^1$.

Any embedding $I : M \to \mathbb{R}^N$ defines the map $I^r : B(M, r) \to \mathbb{R}^N$, sending any point $\mathbf{x} = \{x_1, \dots, x_r\} \in B(M, r)$ to the mass center

$$\frac{1}{r} \sum I(x_i), \tag{5}$$

of points $I(x_1), \dots, I(x_r)$. The image of the tangent space $T_{\mathbf{x}}B(M, r)$ under the derivative map I^r_* of this map at the point $\mathbf{x}$ is equal to the linear span of tangent spaces $T_{I(x_i)}I(M)$, $i = 1, \dots, r$, translated to the point (5). Also there exists the natural homomorphism $\chi : \widetilde{\psi} \to T\mathbb{R}^N$, sending any function $f : (x_1, \dots, x_r) \to \mathbb{R}$ to the vector $\sum_{i=1}^r f(x_i) I(x_i) \in T_{I^r(\mathbf{x})}\mathbb{R}^N$. The corresponding subset $\pi \circ \chi(\widetilde{\psi}|_{\mathbf{x}}) \subset \mathbb{R}^N$ is the minimal affine subspace in $\mathbb{R}^N$ containing all points $I(x_i)$. Thus for any $\mathbf{x} \in B(M,r)$ we have two important subspaces in $T_{I^r(\mathbf{x})}\mathbb{R}^N$: the image of $T_{\mathbf{x}}B(M,r)$ under the derivative map I^r_* and $\chi(\widetilde{\psi}|_{\mathbf{x}})$. Let $\tau(\mathbf{x}) \subset T_{I^r(\mathbf{x})}\mathbb{R}^N$ be the linear span of these two subspaces.

Lemma 2. *If we suppose that the embedding I is r-neighbourly, then for any $\mathbf{x} = (x_1, \dots, x_r) \in B(M,r)$ the affine plane $\pi(\tau(\mathbf{x})) \subset \mathbb{R}^N$ intersects the set $I(B(M,r))$ only at the point $I(\mathbf{x})$.*

Proof. The affine hyperplane in $\mathbb{R}^N$, touching M at the points $x_1, \dots, x_r$ and participating in the definition of a r-neighbourly submanifold, contains this plane $\pi(\tau(\mathbf{x}))$ but cannot contain any point of $I^r(B(M,r))$ other than $\mathbf{x}$. $\square$

Definition. The set $\Omega(I) \subset B(M,r)$ is the set of all points $\mathbf{x}$ such that $\dim \tau(\mathbf{x}) < kr + r - 1$. For $r > 1$ and $N \geq (k+1)r - 1$, the set $\mathrm{Reg}(r) \subset C^\infty(M, \mathbb{R}^N)$ consists of all maps $I : M \to \mathbb{R}^N$ such that the topological codimension of $\Omega(I)$ in $B(M,r)$ is greater than $N-(k+1)r+1$ in the following exact sense: any compact $(N-(k+1)r+1)$-dimensional submanifold in $B(M,r)$ is isotopic to the one not intersecting $\Omega(I)$. For $r = 1$, set $\mathrm{Reg}(1) = C^\infty(M, \mathbb{R}^N)$.

In particular, for any such submanifold L, not intersecting $\Omega(I)$, the restriction on L of the bundle $\tau(\mathbf{x})$ is isomorphic to $TB(M,r) \oplus \widetilde{\psi}$, and the restriction of the map I on L is an immersion into $\mathbb{R}^N$ (and even an embedding if $r > 1$).

Lemma 3. *For any $r > 1$, the set $\Sigma\Sigma(r) \equiv C^\infty(M, \mathbb{R}^N) \setminus \mathrm{Reg}(r)$ is a subset of infinite codimension in $C^\infty(M, \mathbb{R}^N)$.*

Proof. Any map $I : M \to \mathbb{R}^N$ defines its multijet extension $I^1_r : B(M,r) \to J^1_r(M, \mathbb{R}^N)$, sending any point $(x_1, \dots, x_r) \in B(M,r)$ to the collection of 1-jets of I at these points, see e.g. [4]. The set $\Omega(I)$ can be described as the pre-image under this map of a certain algebraic subset $\Sigma \subset J^1_r(M, \mathbb{R}^N)$, whose codimension is equal to $N-(k+1)r+2$. If the map I is of class $\Sigma\Sigma$, then this map I^1_r is non-transversal to Σ at infinitely many points, and our lemma follows from the Thom's multijet transversality theorem (see [4], [9]). $\square$

Theorem 3. *Let us suppose that $N = (k+1)r + l - 1$ and there is a l-dimensional compact submanifold $L \subset B(M,r)$ such that*

$$\langle [L], \overline{w}_l(TB(M,r) \oplus \widetilde{\psi}) \rangle \neq 0, \tag{6}$$

where $[L]$ is the $\mathbb{Z}_2$-fundamental class of L. Then there are no r-neighbourly embeddings $M \to \mathbb{R}^N$ of the class $\mathrm{Reg}(r)$. In particular, $\Delta(M,r) \geq (k+1)r + l$.

Proof. Let's suppose that I is a r-neighbourly embedding $M \to \mathbb{R}^N$ of class $\mathrm{Reg}(r)$, $r > 1$. We denote by ν the $(kr-l)$-dimensional vector subbundle in the restriction of $TB(M,r)$ to L, orthogonal to the tangent bundle TL; let be $\xi = \nu \oplus \widetilde{\psi}_L$, so that $TL \oplus \xi = T|_L B(M,r) \oplus \widetilde{\psi}_L$. Since $I \in \mathrm{Reg}(r)$, we can assume that L does not meet $\Omega(I)$, in particular the homomorphism $I_*^r \oplus \chi : \xi \to T\mathbb{R}^N$ is a monomorphism satisfying conditions of Lemma 1. Then conclusions of Lemmas 1 and 2 contradict one another. Finally, in the case $r = 1$ our condition (6) prevents the existence of *any* embedding $M \to \mathbb{R}^N$ (see [7] or Example after Lemma 1), and Theorem 3 is completely proved. □

4. Proof of main theorems

4.1. Proof of Proposition 1. Any embedding $M^k \to S^{N-1}$ obviously is a stable 1-neighbourly embedding $M \to \mathbb{R}^N$. Conversely, suppose that M is a 1-neighbourly submanifold in $\mathbb{R}^N$. The Gauss map establishes a natural homeomorphism between S^{N-1} and the set of all supporting hyperplanes of M. For any point $x \in M$, we consider the set $H(x)$ of all oriented hyperplanes, supporting M only at this point. This is a convex semialgebraic subset of the sphere $S^{N-1-k} \subset S^{N-1}$, consisting of all oriented hyperplanes parallel to T_xM. If M is generic, then the set of interior points of $H(x)$ in S^{N-1-k} is non-empty, and the union of such sets forms a smooth fiber bundle over M with fibers homeomorphic to $\mathbb{R}^{N-1-k}$. Any smooth section of this bundle is the desired embedding.

Remark. The same considerations prove that if M is a (non-stably) 1-neighbourly submanifold of $\mathbb{R}^N$, then it is *homeomorphic* to a subset of S^{N-1}. However, the example of the curve $t \to (t, t^3, t^4)$ shows that for non-generic embeddings the set $H(t)$ can consist of a unique point, and our fiber bundle $\{H(x) \to x\}$ can be not smooth and even not locally trivial.

4.2. Proof of Theorem 1!. First suppose that $r = 2^s$. The corresponding submanifold $L(r) \subset B(\mathbb{R}^k, r)$, satisfying the conditions of Theorem 3, is constructed as follows (cf. [3], [9]).

Let us fix some $\varepsilon \in (0, 1/2]$. We consider in $\mathbb{R}^k$ a sphere of radius 1 and mark on it some two opposite points A_1, A_2. Then we consider two spheres of radius ε with centers in these two points and mark on both of them two opposite points: A_{11}, A_{12} on the first and A_{21}, A_{22} on the second. We consider four spheres of radius ε^2 with centers at all these four points and mark on all of them two opposite points: $A_{111}, \ldots, A_{222}$, etc. After the s-th step we obtain $2^s = r$ pairwise different points in $\mathbb{R}^k$, i.e. a point of the space $B(\mathbb{R}^k, r)$. $L(r)$ is defined as the union of all points of the latter space, which can be obtained in this way; this is a $(k-1)(r-1)$-dimensional smooth compact manifold.

Proposition 2.

1. *For any k and r, the tangent bundle $TB(\mathbb{R}^k, r)$ is isomorphic to the direct sum of k copies of the bundle ψ.*

2. *If k is a power of 2, then the k-th power of the total Stiefel–Whitney class of the bundle $\widetilde{\psi}$ (or ψ) over $B(\mathbb{R}^k, r)$ is equal to $\mathbf{1} \in H^0(B(\mathbb{R}^k, r), \mathbb{Z}_2)$. In particular, $\overline{w}(\widetilde{\psi}) = (w(\widetilde{\psi}))^{k-1}$.*
3. *If both k and r are powers of 2, then the $(k-1)(r-1)$-dimensional homogeneous component $\overline{w}_{(k-1)(r-1)}(\widetilde{\psi}) \in H^{(k-1)(r-1)}(B(\mathbb{R}^k, r), \mathbb{Z}_2)$ of the total inverse Stiefel–Whitney class $\overline{w}_*(\widetilde{\psi})$ is nontrivial and its value on the fundamental cycle of the manifold $L(r)$ is equal to 1.*

Proof. Statement 1 is obvious (and remains true if we replace $\mathbb{R}^k$ by any parallelizable manifold). Statement 2 (and, moreover, a similar assertion concerning any class of the form $\mathbf{1} + \{\text{terms of positive dimension}\} \in H^*(B(\mathbb{R}^k, r), \mathbb{Z}_2))$ is proved in [6]. Statement 3 is proved in [10], § I.3.7 (and no doubt was known to the author of [6]). Thus the submanifold $L(r) \subset B(\mathbb{R}^k, r)$, $k = 2^j$, satisfies the conditions of Theorem 3, and Theorem 1! is proved in the case when r is a power of 2. For an arbitrary r the similar submanifold $L(r)$ is constructed as follows. Suppose that $r = 2^{t_1} + \ldots + 2^{t_d}$, $t_1, \ldots, t_d \in \mathbb{N}$, $d = d(r)$. Then the manifold $L(r)$ consists of all collections of r points in $\mathbb{R}^k$, such that first 2^{t_1} of them belong to the manifold $L(2^{t_1})$, the next 2^{t_2} are obtained from some collection $\mathbf{x}_2 \in L(2^{t_2})$ by the translation along the vector $(5, 0, \ldots, 0)$, the next 2^{t_3} are obtained from some collection $\mathbf{x}_3 \in L(2^{t_3})$ by the translation along the vector $(10, 0, \ldots, 0)$, etc. In particular, $L(r) \sim L(2^{t_1}) \times \ldots \times L(2^{t_d})$. (If $t_d = 1$, then we set $L(t_d) = \{\text{the point } 0\}$.) In restriction to $L(r)$, the bundle $\widetilde{\psi}$ is obviously isomorphic to the direct sum of d bundles, induced from similar $(2^{d_i} - 1)$-dimensional bundles over the factors $L(2^{t_i})$, and the $(r-1)$-dimensional trivial bundle. Thus the manifold $L(r)$ for an arbitrary r also satisfies conditions of Theorem 3 (with $l = (k-1)(r-d)$). □

4.3 Proof of Theorem 2!. Let's suppose that $k = 2^j$ and define the submanifold $L(k, r) \subset B(\mathbb{R}P^k, r)$ as the set of all unordered collections of r pairwise orthogonal points in $\mathbb{R}P^k$ (with respect to any Euclidean metrics in $\mathbb{R}^{k+1}$) lying in some fixed subspace $\mathbb{R}P^{k-1} \subset \mathbb{R}P^k$. This is a smooth $\left(kr - \binom{r}{2}\right)$-dimensional manifold.

We consider also the manifold $\Lambda(k, r)$, consisting of similar ordered collections; it is a submanifold of the space $(\mathbb{R}P^k)^r$ and also the space of a $r!$-fold covering $\theta : \Lambda(k, r) \to L(k, r)$.

Proposition 3.

1. *There is a ring isomorphism*

$$H^*(\Lambda(k, r), \mathbb{Z}_2) \simeq H^*(\mathbb{R}P^{k-1}, \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^{k-2}, \mathbb{Z}_2) \otimes \ldots \otimes H^*(\mathbb{R}P^{k-r}, \mathbb{Z}_2). \tag{7}$$

2. *The vector bundle over $\Lambda(k, r)$, induced by the map θ from the bundle ψ or $\widetilde{\psi}$ on $L(k, r)$, is equivalent to the trivial one.*

3. *The vector bundle over* $\Lambda(k,r)$*, induced by the map* θ *from the tangent bundle* $TB(\mathbb{R}P^k, r)$*, coincides with the restriction on* $\Lambda(k,r)$ *of the tangent bundle of the manifold* $(\mathbb{R}P^k)^r$*.*
4. *The inverse Stiefel–Whitney class* $\overline{w}_* \equiv (w_*)^{-1}$ *of the latter tangent bundle satisfies the inequality* $\langle [\Lambda(k,r)], \overline{w}_{kr-\binom{r}{2}}(T(\mathbb{R}P^k)^r)\rangle \neq 0$*.*

PROOF. Statement 1 is a standard exercise on homology of fiber bundles, see e.g. [2]. Namely, let's consider the fiber bundle $p : \Lambda(k,r) \to \Lambda(k, r-1)$, sending any ordered collection $(x_1, \dots, x_r)$ to $(x_1, \dots, x_{r-1})$. Its fiber $\mathcal{F}$ is equal to $\mathbb{R}P^{k-r}$, hence the fundamental group of the base acts trivially on $H^*(\mathcal{F}, \mathbb{Z}_2)$, and the term $E_2^{p,q}$ of the $\mathbb{Z}_2$-spectral sequence of this bundle is naturally isomorphic to $H^p(\Lambda(k, r-1), \mathbb{Z}_2) \otimes H^q(\mathcal{F}, \mathbb{Z}_2)$. All further differentials d^i, $i \geq 2$, of the spectral sequence act trivially on all elements of the column $E^{0,*} \simeq H^*(\mathcal{F}, \mathbb{Z}_2)$, because the embedding homomorphism $H^*(\Lambda(k,r), \mathbb{Z}_2) \to H^*(\mathcal{F}, \mathbb{Z}_2)$ is epimorphic: indeed, its composition with the map $H^*(\mathbb{R}P^k, \mathbb{Z}_2) \to H^*(\Lambda(k,r), \mathbb{Z}_2)$, defined by the projection of the space $\Lambda(k,r) \subset (\mathbb{R}P^k)^r$ onto the r-th copy of $\mathbb{R}P^k$, is just the epimorphism $H^*(\mathbb{R}P^k, \mathbb{Z}_2) \to H^*(\mathbb{R}P^{k-r}, \mathbb{Z}_2)$, induced by the embedding. Since the spectral sequence is multiplicative, it degenerates at the term E_2.

Statements 2 and 3 of Proposition 3 are obvious, and statement 4 follows from the induction conjecture for $\Lambda(k, r-1)$ and the fact that for $k = 2^j$ the inverse Stiefel–Whitney class $\overline{w}_*(T\mathbb{R}P^k)$ is equal to $1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1}$, where α is the multiplicative generator of the ring $H^*(\mathbb{R}P^k, \mathbb{Z}_2)$ (see e.g. [7]).

Using the naturality of Stiefel–Whitney classes and the Whitney multiplication formula for these classes of a Whitney sum of bundles, we obtain from this proposition that class $\overline{w}_{kr-\binom{r}{2}}(TB(\mathbb{R}P^k, r) \oplus \widetilde{\psi})$ is a non-trivial element of $H^{kr-\binom{r}{2}}(L(k,r), \mathbb{Z}_2)$. Theorem 2! is now reduced to Theorem 3. $\square$

REMARK. The number $\Delta(M,k)$ is a measure of the "topological complexity" of the configuration space $B(M,r)$. This complexity appears from two issues: the obvious free action of the symmetric group $S(r)$ and the topological complexity of the manifold M itself. In Theorem 1! we essentially exploit only the first issue, and in Theorem 2! only the second. The simultaneous consideration of these two interacting components should give us more precise estimates of $\Delta(M,r)$.

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VICTOR A. VASSILIEV
Russian Academy of Sciences
Steklov Mathematics Institute
Moscow 117901, RUSSIA

E-mail address: vassil@vassil.mccme.rssi.ru