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ON THE EFFECT OF DOMAIN TOPOLOGY IN A SINGULAR PERTURBATION PROBLEM

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1. Introduction

In this paper, we study the following nonlinear elliptic equation

$$(1.1)\qquad \begin{cases} \varepsilon^2 \triangle u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $1 < p < (N+2)/(N-2)$ for $N \geq 3$, $1 < p < \infty$ for $N = 2$ and $\varepsilon > 0$ is a positive small parameter. Our interest in (1.1) arises from two aspects. First, (1.1) is a typical singular perturbation problem. Singular perturbation problems have received much attention lately due to their significances in applications such as chemotaxis (see [18] and [19]), population dynamics (see [1], [16]) and chemical reaction theory (see [1]), etc. Secondly, we are interested in the effect of the properties of the domain, such as geometry, topology on the solutions of nonlinear elliptic problems. Problem (1.1) can be a prototype.

Recently, the geometry of the domain on the solutions of (1.1) has been a subject of study. Beginning in [20], Ni and Wei studied the "*least-energy solutions*" of (1.1) and showed that for ε sufficiently small, the least-energy solution has only one local maximum point P_ε and P_ε must lie in the *most centered* part of Ω, namely, $d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega)$, where $d(P, \partial\Omega)$ is the distance from

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P to $\partial\Omega$. On the other hand, in [26], a kind of converse was proved. Namely, for each strictly local maximum point of the distance function $d(x, \partial\Omega)$, there is a solution of (1.1) with only one local maximum point near that point. This shows that the geometry of the domain plays a very important role in the multiplicity of solutions of (1.1). In [27], the effect of the geometry of Ω on single-peaked solutions has been studied. In particular, both necessary and sufficient conditions for the existence of single-peaked solutions are established. These conditions depend highly on the geometry of the domain. Some further studies in this direction are in [9], [13], [17], etc.

On the other hand, Benci and Cerami [5] and [6] studied the effect of the topology of Ω on solutions of (1.1). More precisely, they showed that there are at least $\mathrm{cat}(\Omega) + 1$ solutions for $\varepsilon \ll 1$. In fact, what they actually showed was there are at least cat $(\Omega) + 1$ *single-peaked* solutions (i.e., solutions with single maximum point), where cat (Ω) denotes the category of Ω.

In this paper, we will study the effect of domain topology on *multiple-peak* solutions (i.e., solutions with more than 1 local maximum points). Note that when Ω is a ball or some symmetric domains, there are no *multiple-peak* solutions, see [12]. Thus the existence and multiplicity of *multiple-peak* solutions are related to the geometry and topology of Ω.

To state our results, we introduce some notations. Let w be the unique solution of

$$\begin{cases} \triangle w - w + w^p = 0 \text{ in } \mathbb{R}^N, \\ w > 0 \text{ in } \mathbb{R}^N, \ w(0) = \ \max_{z \varepsilon \mathbb{R}^N} w(z), \\ w(z) \to 0 \text{ at } \infty. \end{cases}$$

Let $J(w) = (1/2)\int_{\mathbb{R}^N} |\nabla w|^2 + (1/2)\int_{\mathbb{R}^N} w^2 - (1/(p+1))\int_{\mathbb{R}^N} w^{p+1}$ be its "energy". Let $c_k = kJ(w)$. For any $u \in W_0^{1,2}(\Omega)$, we define an energy functional

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \frac{1}{p+1}\int_\Omega u^{p+1}.$$

We called a family of solutions of (1.1) *k-peak* if $\varepsilon^{-N} J_\varepsilon(u) \to c_k$. It is easy to see by blow up arguments that a k-peak solution u_ε, u_ε has only k local maximum points for ε small. (See the proof of Theorem 1.1 in [20]. Note that there they proved for single-peaked case but the arguments there can be easily modified to treat multiple peak case). More precisely we have

Lemma 1.1. *Let u_ε be a family of k-peaked solutions, then for ε sufficiently small, u_ε has only k local maximum points $P_\varepsilon^1, \ldots, P_\varepsilon^k \in \Omega$ and we have*

$$d(P_\varepsilon^j, \partial\Omega)/\varepsilon \to \infty, \ |P_\varepsilon^i - P_\varepsilon^j|/\varepsilon \to \infty, \quad i \neq j, i, j = 1, \ldots, k.$$

Moreover,

$$\left\| u - \sum_{j=1}^{k} w((x - P_j^{\varepsilon})/\varepsilon) \right\| \to 0$$

as $\varepsilon \to 0$*, where*

$$\|u\|^2 := \varepsilon^{-N}\left(\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega u^2\right).$$

Set

$$J_\varepsilon^{c_k+\eta} = \{u \in H_0^1(\Omega) \mid \varepsilon^{-N} J_\varepsilon(u) \le c_k + \eta\}$$

and

$$J_\varepsilon^{c_k-\eta} = \{u \in H_0^1(\Omega) \mid \varepsilon^{-N} J_\varepsilon(u) \le c_k - \eta\}$$

for $0 < \eta < I(w)$. In this paper, we study the case when $k = 2$. Our main result is

Theorem 1.1. *The contribution to the relative homology*

$$H_*(J_\varepsilon^{c_2+\eta}, J_\varepsilon^{c_2-\eta})$$

of 2*-peak positive solutions as* $\varepsilon \to 0$, $0 < \eta < I(w)$ *is equal to* $H_*(T)$*, where* T *is the quotient space of* $(\Omega \times \Omega, M(\Omega)) \times (D^2, S^1)$ *under a free* Z_2 *group action which comes since we can interchange the two maxima and* $M(\Omega) = \{x \in \Omega \times \Omega \mid x_1 = x_2\}$.

Remark. More precisely, we mean there is a neighborhood V_ε of the 2-peak solutions such that $H_*(J_\varepsilon^{c_2+\eta} \cap V_\varepsilon, J_\varepsilon^{c_2-\eta} \cap V_\varepsilon)$ is equal to $H_*(T)$.

An interesting corollary is

Corollary 1.2. *If the reduced homology* $H_*(\Omega, Z_2) \neq 0$ *is nontrivial, then for* ε *sufficiently small, there is a* 2*-peak solution for* (1.1).

Another by-product of the proof of the theorem is the following necessary conditions of the locations of the 2-peaks.

Theorem 1.2. *There is a* $\delta > 0$ *such that if* u_ε *is a* 2*-peak solution and let* P_1^ε, P_2^ε *be its only two local maximum points, then* $d(P_1^\varepsilon, \partial\Omega) \ge \delta > 0$, $d(P_2^\varepsilon, \partial\Omega) \ge \delta > 0$. *Moreover, if* $P_1^\varepsilon \to P_1$, $P_2^\varepsilon \to P_2$*, then* $|P_1 - P_2| \ge 2\min\,(d(P_1, \partial\Omega), d(P_2, \partial\Omega))$.

Remarks. For some rather symmetric domains, it is proved in [12] that there are no 2-peaked positive solutions. On the other hand, a number of authors have constructed 2-peak positive solutions on some contractible domains. Thus the complete answer when there are 2-peak positive solutions is complicated. Note also that in 2 and 3 dimensions, our assumption on Ω is equivalent to assuming Ω is not contractible. This follows from standard topology (see Rourke and Sanderson [23] for the more complicated 3 dimensional case). It seems likely that a similar result holds for much more general nonlinearities and that if Ω is complicated one can use the theorem to obtain multiple positive 2 peaked solutions.

Theorem 1.1 and Corollary 1.2 point out the importance of the topology of the domain on the multiplicity of solutions of (1.1). For example, when $\Omega = \Omega_1 \backslash \Omega_0$ where Ω_1, Ω_0 are contractible domains (e.g. Ω is an annulus), then $H_*(\Omega, Z_2) \neq 0$, hence (1.1) has a 2-peaked solutions. Note that in [9] and [13], rather strong local geometric conditions were placed on Ω in order to show the existence of 2-peaked solutions.

Theorem 1.1 was motivated by the results of [4], where they studied a nearly critical exponent problem and computed the effect of domain topology on the blow up solutions.

This paper is organized as follows. In Section 2, technical framework is set up and we make a preliminary analysis of problem (1.1) in Section 3. We prove Theorem 1.1 in Section 4 and Corollary 1.2 in Section 5.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notations $O(A)$, $o(a)$ always mean that $|O(A)| \leq C|A|$, $o(a)/a \to 0$ as $\varepsilon \to 0$, respectively.

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2. Technical framework

In this section, we introduce some notations and set up a technical framework. We shall follow [4] and [27]. First we define $P_\Omega w$ to be the projection of $w((x-P)/\varepsilon)$ into $H_0^1(\Omega)$, i.e. $P_\Omega w((x-P)/\varepsilon)$ is the unique solution of

$$\begin{cases} \varepsilon^2 \triangle u - u + w^p((x-P)/\varepsilon) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Sometimes we use Pw to denote $P_\Omega w((x-P)/\varepsilon)$ and Pw_i for $P_\Omega w((x-P_i)/\varepsilon)$ or $P_\Omega w((x-x_i)/\varepsilon)$.

By the Maximum Principle, $0 \leq P_\Omega w < w$. Let

$$\begin{gathered} x = \varepsilon y + P, \quad \varphi_{\varepsilon,P}(y) = w(y) - P_\Omega w((x-P)/\varepsilon), \quad \beta = 1/\varepsilon, \\ \psi_{\varepsilon,P}(x) = -\varepsilon \log \varphi_{\varepsilon,P}(y), \quad \beta = 1/\varepsilon, \\ V_{\varepsilon,P}(y) = e^{\beta\psi_{\varepsilon,P}(P)} \varphi_{\varepsilon,P}(y), \quad \psi_\varepsilon(P) = \psi_{\varepsilon,P}(P). \end{gathered}$$

It is easy to see that $\psi_{\varepsilon,P}(x)$ is the unique solution of

$$(2.2)\qquad \begin{cases} \varepsilon^2\Delta u - |\nabla u|^2 + 1 = 0 & \text{in } \Omega,\\ u(x) = -\varepsilon\log w((x-P)/\varepsilon) & \text{on } \partial\Omega.\end{cases}$$

The following properties are proved in [20].

Proposition 2.1.

(i) *There exist a constant C_1 such that*

$$\|\psi_{\varepsilon,P}(x)\|_{L^\infty(\Omega)} \le C_1.$$

(ii) $\psi_{\varepsilon,P}(x)\to\psi_P(x)$ *uniformly on Ω as $\varepsilon\to 0$, where $\psi_P(x)$ in the unique viscosity solution of the following Hamilton-Jacobi equation*

$$(2.3)\qquad \begin{cases} |\nabla u|^2 = 1 & \textit{in } \Omega,\\ u(x) = |x-P| & \textit{on } \partial\Omega.\end{cases}$$

Indeed, $\psi_P(x) = \inf_{z\in\partial\Omega}(|z-P|+L(x,z))$, where $L(x,z)$ is the infimum of T such that there exists $\xi(s)\in C^{0,1}([0,T],\overline{\Omega})$ with $\xi(0)=x$, $\xi(T)=z$ and $|d\xi/ds|\le 1$, a.e. in $[0,T]$. Furthermore, $\psi_P(P) = 2d(P,\Omega)$.

(iii) *For every sequence $\varepsilon_k\to 0$, there is a subsequence $\varepsilon_{k_l}\to 0$, such that $V_{\varepsilon_{k_l},P}\to V_P$ uniformly on every compact set of $\mathbb{R}^N$, where V_P is a positive solution of*

$$(2.4)\qquad \begin{cases} \Delta u - u = 0 & \textit{in } \mathbb{R}^N,\\ u(0)=1,\ u>0 & \textit{in } \mathbb{R}^N.\end{cases}$$

Furthermore, for any $\sigma_1>0$,

$$\sup_{y\in\overline{\Omega}_{\varepsilon_{k_l},P}} e^{-(1+\sigma_1)|y|}|V_{\varepsilon_{k_l},P}(y) - V_P(y)| \to 0 \quad \textit{as } \varepsilon_{k_l}\to 0.$$

For $a>0$, we define a subset of $H_0^1(\Omega)$

$$F_a = \left\{ Pw\left(\frac{x-x_1}{\varepsilon}\right) + Pw\left(\frac{x-x_2}{\varepsilon}\right) \,\middle|\, \frac{d(x_1,\partial\Omega)}{\varepsilon} > \frac{1}{a},\ \frac{d(x_2,\partial\Omega)}{\varepsilon} > \frac{1}{a},\ \frac{|x_1-x_2|}{\varepsilon} > \frac{1}{a} \right\}$$

Let

$$\Lambda_a = \left\{ (\alpha_1,\alpha_2,x_1,x_2)\in\mathbb{R}^2\times\Omega^2 \,\middle|\, |\alpha_i - 1| < 4a,\ \frac{d(x_i,\partial\Omega)}{\varepsilon} > \frac{1}{4a},\ i=1,2,\ \frac{|x_1-x_2|}{\varepsilon} > \frac{1}{4a} \right\}$$

Set

$$\langle u,v\rangle = \varepsilon^{-N}\bigg(\int_\Omega \varepsilon^2\nabla u\nabla v + uv\bigg), \quad \|u\|^2 = \langle u,v\rangle,$$
$$\Omega_\varepsilon := \{y \mid \varepsilon y + P_0 \in \Omega\},$$

where $P_0 \in \Omega$ is a fixed point.

$$E_Q = \{v \in H_0^1(\Omega) \mid \langle v, Pw_1\rangle = \langle v, Pw_2\rangle = \langle v, \partial_i Pw_1\rangle = \langle v, \partial_i Pw_2\rangle = 0, \quad i = 1,\ldots,N\},$$

where we use $Q = (P_1, P_2)$ and $\partial_i Pw_j$ to denote $\partial Pw_j/\partial P_{j,i}$. Note that $P_j = (P_{j,1},\ldots,P_{j,N})$, $j = 1,2$.

Then, as in [4] or [9], it is easy to prove

Lemma 2.1. *If a is small and $u \in W_0^{1,2}(\Omega)$ is such that $\inf_{h\varepsilon F_a}\|u-h\|$ is small then the minimizing problem*

$$\inf_{(\alpha,x)\in\Lambda_a}\bigg\|u - \sum_{i=1}^{2}\alpha_i Pw_i\bigg\|$$

has a unique solution. Moreover, u can be expressed as

$$u = \alpha_1 Pw_1 + \alpha_2 Pw_2 + v$$

where $v \in E_x$. The expression is unique modulo interchanging both (α_1, P_1) with (α_2, P_2).

Therefore, by Lemmas 1.1 and 2.1, there exists a diffeomorphism between a neighbourhood of the possible 2-peak solutions of (1.1) we are interested in and the quotient of the open set

$$M_\eta = \bigg\{(\alpha,x,v) \in R^2\times\Omega^2\times H_0^1(\Omega)\ \bigg|\ |\alpha_i - 1| < \eta, \quad \frac{d(x_i,\partial\Omega)}{\varepsilon} > \frac{1}{\eta},\ \frac{|x_1-x_2|}{\varepsilon} > \frac{1}{\eta},\ \|v\| < \eta\bigg\}$$

where we identify $(\alpha_1,\alpha_2,x_1,x_2,v)$ with $(\alpha_2,\alpha_1,x_2,x_1,v)$ and $\eta > 0$ is a some suitable constant. Note that the quotient map is smooth on M_η.

Let us define the functional

$$K_\varepsilon : M_\eta \to R, \quad m = (\alpha,x,v) \to \varepsilon^{-N}J_\varepsilon(\alpha_1 Pw_1 + \alpha_2 Pw_2 + v).$$

It also follows easily (see Proposition 1 of [4] or Proposition 2.2 of [9]) that

Proposition 2.2. *$m = (\alpha, x, v) \in M_\eta$ is a critical point of K_ε if and only if $u = \alpha_1 Pw_1 + \alpha_2 Pw_2 + v$ is a critical point of J_ε, i.e. if and only if there exists $(A, C) \in R^2 \times R^{2N}$ such that the following holds.*

$$
\text{(E)} \quad \begin{cases} (\mathrm{E}_{\alpha_i}) & \dfrac{\partial K_\varepsilon}{\partial \alpha_i} = 0, \quad \forall i = 1,2, \\ (\mathrm{E}_{x_i}) & \dfrac{\partial K_\varepsilon}{\partial x_{i,j}} = \displaystyle\sum_{k=1}^{N} C_{ik} \left\langle \frac{\partial^2 Pw_i}{\partial x_{i,j} \partial x_{i,k}}, v \right\rangle, \quad \forall i = 1,2,\ j = 1, \dots, N, \\ (\mathrm{E}_v) & \dfrac{\partial K_\varepsilon}{\partial v} = A_1 Pw_1 + A_2 Pw_2 + \displaystyle\sum_{i=1,2, j=1,\dots,N} C_{ij} \frac{\partial Pw_i}{\partial x_{i,j}} \end{cases}
$$

3. Preliminary analysis

In this section, we use equations (E) to derive a preliminary analysis of problem (1.1). More precisely, we shall prove the following

Theorem 3.1. *There is a $\delta > 0$ such that if u_ε is a 2-peak solution and let P_1^ε, P_2^ε be its only two local maximum points, then $d(P_1^\varepsilon, \partial\Omega) \geq \delta > 0$, $d(P_2^\varepsilon, \partial\Omega) \geq \delta > 0$. Moreover, if $P_1^\varepsilon \to P_1$, $P_2^\varepsilon \to P_2$, then $|P_1 - P_2| \geq 2\min(d(P_1, \partial\Omega), d(P_2, \partial\Omega))$.*

Set

$$
\begin{gathered} w_1 = w((x - P_1^\varepsilon)/\varepsilon), \quad w_2 = w((x - P_2^\varepsilon)/\varepsilon), \\ \delta_{\varepsilon, P_1, P_2} = \varphi_{\varepsilon, P_1}(P_1) + \varphi_{\varepsilon, P_2}(P_2) + w(|P_1 - P_2|/\varepsilon). \end{gathered}
$$

Recall that $\varphi_{\varepsilon,P}(x) = w((x - P)/\varepsilon) - Pw((x - P)/\varepsilon)$ and $\psi_\varepsilon(P) := \psi_{\varepsilon,P}(P)$.

We first state some useful lemmas.

Lemma 3.2. *Let $f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $g \in C(\mathbb{R}^N)$ be radially symmetric and satisfy for some $\alpha \geq 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$,*

$$
\begin{gathered} f(x) \exp(\alpha|x|)|x|^\beta \to \gamma \quad \text{as } |x| \to \infty, \\ \int_{\mathbb{R}^N} |g(x)| \exp(\alpha|x|)(1 + |x|^\beta) < \infty. \end{gathered}
$$

Then

$$
\left(\int_{\mathbb{R}^N} f(x+y) g(x)\, dx \right) \exp(\alpha|y|)|y|^\beta \to \gamma \int_{\mathbb{R}^N} g(x) \exp(-\alpha x_1)\, dx \quad \text{as } |y| \to \infty.
$$

For the proof, see Proposition 1.2 of [3].

We then have the following estimates.

Lemma 3.3.

(1)
$$\frac{1}{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)} \int_{\mathbb{R}^N} w_1^p w_2 \to \gamma_1 > 0.$$

(2)
$$\begin{aligned}&\frac{\varepsilon}{w(|P_1 - P_2|/\varepsilon)} \int_{\mathbb{R}^N} w_1^p \frac{\partial w_2}{\partial P_{2,j}} \\ &= \frac{1}{w(|P_1 - P_2|/\varepsilon)} \int_{\mathbb{R}^N} w_1^p w_2' \frac{P_{2,j} - x_j}{|\varepsilon y + P_1 - P_2|} \to -\frac{P_{2,j} - P_{1,j}}{|P_2 - P_1|}\gamma_2\end{aligned}$$

for some constants $\gamma_1 > 0$, $\gamma_2 > 0$.

Proof. Note that $|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon \to \infty$ as $\varepsilon \to 0$ by Lemma 1.1 and $w(y) \sim |y|^{(1-N)/2} e^{-|y|}$ as $|y| \to \infty$, by Lemma 3.2,

$$\bigg(\int_{\mathbb{R}^N} w_1^p w_2 \, dx\bigg) \exp(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)^{(N-1)/2} \to \gamma \int_{\mathbb{R}^N} w(y) e^{-\alpha y_1} \, dy.$$

Note that $\gamma > 0$. Hence

$$\frac{1}{w(|P_1 - P_2|/\varepsilon)} \int_{\mathbb{R}^N} w_1^p w_2 \to \gamma_1 > 0.$$

Similarly, we have (2). □

We first deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomena.

The proof of the following proposition is very similar to that of Lemma 4.2 in [26] and of Proposition 4 in [22, p. 15] and is thus omitted. Note that we do not have troubles close to the boundary because the region we are working with stays far enough from the boundary so that we do not have difficulties.

Proposition 3.4. *There exists a* $\varepsilon_0 > 0$, $\eta_0 > 0$ *such that if* $\varepsilon < \varepsilon_0$, $\|v\| < \eta_0$ *then there exists a smooth map which to any* (α, P, v) *such that* $(\alpha, P, 0) \in M_\eta$ *associates* $v_{\varepsilon,\alpha,P} \in E_P$, $\|v_{\varepsilon,\alpha,P}\| < \eta_0$ *such that* (E_v) *is satisfied for some* $(A, C) \in \mathbb{R} \times \mathbb{R}^{2N}$. *Such a* $v_{\varepsilon,\alpha,P}$ *is unique and minimizes* $K_\varepsilon(\alpha, P, v)$ *with respect to* v *in* $\{v \in E_P \mid \|v\| < \eta_0\}$ *and we have the estimates*

(3.1)
$$\|v_{\varepsilon,\alpha,P}\|^2 \le O(\varphi_{\varepsilon,P_1}^{1+2\sigma}(P_1) + \varphi_{\varepsilon,P_2}^{1+2\sigma}(P_2) + w^{1+2\sigma}(|P_1 - P_2|/\varepsilon))$$

where $2\sigma = \min(1, p-1)$.

Once $v_{\varepsilon,\alpha,P}$ is obtained, we can estimate A_1, A_2, C_{ij} in Proposition 2.2. In fact we have by Appendix C in [27] (set $\Gamma_1 := \int_{\mathbb{R}^N} w^{p+1}$, $\Gamma_2 := \int_{\mathbb{R}^N} pw^{p-1}(\partial w/\partial y_i)^2$)

$$\begin{aligned}
\langle Pw_i, Pw_i\rangle &= \Gamma_1 + O(\varphi_{\varepsilon,P_i}(P_i)), && i=1,2,\\
\left\langle Pw_i, \frac{\partial}{\partial P_{i,j}} Pw_i\right\rangle &= O\left(\frac{\varphi_{\varepsilon,P_i}(P_i)}{\varepsilon}\right), && i=1,2,\\
\left\langle \frac{\partial Pw_i}{\partial P_{i,j}}, \frac{\partial Pw_i}{\partial P_{i,k}}\right\rangle &= \frac{1}{\varepsilon^2}\Gamma_2\delta_{jk} + O\left(\frac{\varphi_{\varepsilon,P_i}(P_i)}{\varepsilon^2}\right), && i=1,2,\\
\left\langle \frac{\partial K_\varepsilon}{\partial v}, Pw_i\right\rangle &= \frac{\partial K_\varepsilon}{\partial \alpha_i},\\
\left\langle \frac{\partial K_\varepsilon}{\partial v}, \frac{\partial Pw_i}{\partial P_{i,j}}\right\rangle &= \frac{1}{\alpha_i}\frac{\partial K_\varepsilon}{\partial P_{i,j}}.
\end{aligned}$$

Explicit computations yield

$$\begin{aligned}
\frac{\partial K_\varepsilon}{\partial \alpha_1} &= \alpha_1\int_{\Omega_\varepsilon} w_1^p Pw_1 + \alpha_2\int_{\Omega_\varepsilon} w_2^p Pw_1 - \int_{\Omega_\varepsilon}(\alpha_1 Pw_1+\alpha_2 Pw_2)^p Pw_1\\
&= \int_{\mathbb{R}^N} w^{p+1}(\alpha_1-\alpha_1^p) + O(\varphi_{\varepsilon,P_1}(P_1) + w(|P_1-P_2|/\varepsilon)),\\
\frac{\partial K_\varepsilon}{\partial \alpha_2} &= \int_{\mathbb{R}^N} w^{p+1}(\alpha_2-\alpha_2^p) + O(\varphi_{\varepsilon,P_2}(P_2) + w(|P_1-P_2|/\varepsilon)),\\
\frac{\partial K_\varepsilon}{\partial P_{i,j}} &= \int_{\Omega_\varepsilon}(\alpha_1 w_1^p+\alpha_2 w_2^p)\left(\alpha_i\frac{\partial Pw_i}{\partial P_{i,j}}\right)\\
&\quad - \int_{\Omega_\varepsilon}(\alpha_1 Pw_1+\alpha_2 Pw_2+v)^p\left(\alpha_i\frac{\partial Pw_i}{\partial P_{i,j}}\right)\\
&= O((\varphi_{\varepsilon,P_1}(P_1)+\varphi_{\varepsilon,P_2}(P_2)+w(|P_1-P_2|/\varepsilon))/\varepsilon).
\end{aligned}$$

By using equation (E_v) and the previous estimates we obtain a system of equations.

$$\begin{aligned}
&A_1(\Gamma_1 + O(\varphi_{\varepsilon,P_1})(P_1)) + A_2\left(w\left(\frac{|P_1-P_2|}{\varepsilon}\right)\right) + C_{ij}O\left(\frac{\varphi_{\varepsilon,P_1}(P_i)}{\varepsilon}\right)\\
&\qquad = \left\langle \frac{\partial K_\varepsilon}{\partial v}, Pw_1\right\rangle = \frac{\partial K_\varepsilon}{\partial \alpha_1} = 0,\\
&A_1\left(w\left(\frac{|P_1-P_2|}{\varepsilon}\right)\right) + A_2(\Gamma_1 + O(\varphi_{\varepsilon,P_2}(P_2)) + C_{ij}O\left(\frac{\varphi_{\varepsilon,P_2}(P_2)}{\varepsilon}\right)\\
&\qquad = \left\langle \frac{\partial K_\varepsilon}{\partial v}, Pw_2\right\rangle = \frac{\partial K_\varepsilon}{\partial \alpha_2} = 0,\\
&A_1\left(\frac{\delta_{\varepsilon,P_1,P_2}}{\varepsilon}\right) + A_2\left(\frac{\delta_{\varepsilon,P_1,P_2}}{\varepsilon}\right) + C_{ij}\left(\frac{\Gamma_2\delta_{ij}}{\varepsilon^2} + O\left(\frac{\delta_{\varepsilon,P_1,P_2}}{\varepsilon}\right)\right)\\
&\qquad = \left\langle \frac{\partial K_\varepsilon}{\partial v}, \frac{\partial Pw_i}{\partial P_{i,j}}\right\rangle = \frac{1}{\alpha_i}\frac{\partial K_\varepsilon}{\partial P_{i,j}} = O\left(\frac{\delta_{\varepsilon,P_1,P_2}}{\varepsilon}\right).
\end{aligned}$$

Since $w((P_1-P_2)/\varepsilon)$, $\varphi_{\varepsilon,P_1}(P_1)$, $\varphi_{\varepsilon,P_2}(P_2)$ are small, we can think of this system for A_i, C_{ij}/ε as a small perturbation of an invertible diagonal system. Hence

$$\begin{aligned} A_i &= O(\varphi_{\varepsilon,P_1}(P_1)+\varphi_{\varepsilon,P_2}(P_2)+w(|P_1-P_2|/\varepsilon)), \quad i=1,2,\\ C_{ij} &= \varepsilon O(\varphi_{\varepsilon,P_1}(P_1)+\varphi_{\varepsilon,P_2}(P_2)+w(|P_1-P_2|/\varepsilon)). \end{aligned}$$

Therefore the equation $(\mathrm{E}_{P_{i,j}})$ becomes

$$\begin{aligned} \varepsilon\frac{\partial K_\varepsilon}{\partial P_{i,j}} &= \sum_{k=1}^{N} C_{ik}\left\langle \frac{\partial^2 Pw_i}{\partial P_{i,j}\partial P_{i,k}}, v\right\rangle \\ &= O(\varphi^{1+\sigma}_{\varepsilon,P_1}(P_1)+\varphi^{1+\sigma}_{\varepsilon,P_2}(P_2)+w^{1+\sigma}(|P_1-P_2|/\varepsilon)). \end{aligned}$$

But

$$\begin{aligned} \varepsilon\frac{\partial K_\varepsilon}{\partial P_{1,j}} &= \varepsilon\int_{\Omega_\varepsilon}(\alpha_1 w_1^p+\alpha_2 w_2^p)\frac{\partial Pw_i}{\partial P_{1,j}} - \varepsilon\int_{\Omega_\varepsilon}(\alpha_1 Pw_1+\alpha_2 Pw)^p\frac{\partial Pw_1}{\partial P_{1,j}} \\ &\quad + O(\varphi^{1+\sigma}_{\varepsilon,P_1}(P_1)+\varphi^{1+\sigma}_{\varepsilon,P_2}(P_2)+w^{1+\sigma}(|P_1-P_2|/\varepsilon)) \\ &= \varepsilon\int_{\Omega_\varepsilon}\left(w_1^p\frac{\partial w_1}{\partial P_{1,j}}-(Pw_1)^p\frac{\partial w_1}{\partial P_{1,j}}\right)+\varepsilon\int_{\Omega_\varepsilon}\left(w_2^{p-1}\frac{\partial w}{\partial P_{1,j}}w_2\right) \\ &\quad + O(\varphi^{1+\sigma}_{\varepsilon,P_1}(P_1)+\varphi^{1+\sigma}_{\varepsilon,P_2}(P_2)+w^{1+\sigma}(|P_1-P_2|/\varepsilon)). \end{aligned}$$

Hence we have

Lemma 3.5. *Equation* $(\mathrm{E}_{P_{i,j}})$ *is equivalent to*

$$(\mathrm{E}_{P_1})\qquad \begin{aligned} &\varepsilon\int_{\Omega_\varepsilon}\left(w_1^p\frac{\partial w_1}{\partial P_{1,j}}-(Pw_1)^p\frac{\partial w_1}{\partial P_{1,j}}\right)+\varepsilon\int_{\Omega_\varepsilon}\left(w_1^{p-1}\frac{\partial w}{\partial P_{1,j}}w_2\right)\\ &\qquad = O(\varphi^{1+\sigma/2}_{\varepsilon,P_1}(P_1)+\varphi^{1+\sigma}_{\varepsilon,P_2}(P_2)+w^{1+\sigma}(|P_1-P_2|/\varepsilon)), \end{aligned}$$

$$(\mathrm{E}_{P_2})\qquad \begin{aligned} &\varepsilon\int_{\Omega_\varepsilon}\left(w_2^p\frac{\partial w_2}{\partial P_{2,j}}-(Pw_2)^p\frac{\partial w_2}{\partial P_{2,j}}\right)+\varepsilon\int_{\Omega_\varepsilon}\left(w_2^{p-1}\frac{\partial w_2}{\partial P_{2,j}}w_1\right)\\ &\qquad = O(\varphi^{1+\sigma}_{\varepsilon,P_1}(P_1)+\varphi^{1+\sigma}_{\varepsilon,P_2}(P_2)+w^{1+\sigma}(|P_1-P_2|/\varepsilon)). \end{aligned}$$

We can now prove Theorem 3.1.

Proof. We first show that there exists $\delta>0$ such that for ε sufficiently small

$$\min(d(P_1^\varepsilon,\partial\Omega),d(P_2^\varepsilon,\partial\Omega))\geq\delta>0.$$

Suppose not. Suppose $d(P_2^\varepsilon,\partial\Omega)\to 0$. We shall discuss three cases.

Case 1. $\lim_{\varepsilon\to 0}\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)/\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)=0$.

In this case, if $\lim_{\varepsilon\to 0} w(|P_1^\varepsilon-P_2^\varepsilon|/\varepsilon)/\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)=0$, then we have by $(\mathrm{E}_{P_2^\varepsilon})$ (noting that the second integral in equation $(\mathrm{E}_{P_2^\varepsilon})$ is of order $w(|P_1^\varepsilon-P_2^\varepsilon|/\varepsilon)$ by Lemma 3.2), we have

$$\frac{1}{\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)}\varepsilon\int_{\Omega_\varepsilon}(w_2^p-(Pw_2)^p)\frac{\partial w_2}{\partial P_{2,j}^\varepsilon}\to 0.$$

By Lemma 5.1 of [27], this is impossible if $d(P_2^\varepsilon, \partial\Omega) \to 0$.

If $\lim_{\varepsilon\to 0} w((P_1^\varepsilon - P_2^\varepsilon)/\varepsilon)/\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon) = \infty$, then $\lim_{\varepsilon\to 0} \varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)/w((P_1^\varepsilon - P_2^\varepsilon)/\varepsilon) = 0$. By $(\mathrm{E}_{P_2^\varepsilon})$ we have

$$\frac{1}{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)}\varepsilon \int_{\Omega_\varepsilon} w_2^p \frac{\partial w_1}{\partial P_{1,j}^\varepsilon} \to 0$$

for all $j = 1, \ldots, N$, which is impossible by Lemma 3.2 since $P_1^\varepsilon \neq P_2^\varepsilon$. We are left with one case, i.e.

$$\lim_{\varepsilon\to 0} \frac{w((P_1^\varepsilon - P_2^\varepsilon)/\varepsilon)}{\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)} \to K > 0.$$

Since $d(P_2^\varepsilon, \partial\Omega) \to 0$, we conclude that

$$d(P_1^\varepsilon, \partial\Omega) \to 0, \quad d(P_2^\varepsilon, \partial\Omega) \to 0, \quad |P_1^\varepsilon - P_2^\varepsilon| \to 0.$$

Moreover, we have

$$\lim_{\varepsilon\to 0} \frac{\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)}{w((P_1^\varepsilon - P_2^\varepsilon)/\varepsilon)} = 0.$$

By equation $(\mathrm{E}_{P_1^\varepsilon})$, we have

$$\frac{\varepsilon}{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)} \int_{\Omega_\varepsilon} w_1^p \frac{\partial w_2}{\partial P_{2,j}^\varepsilon} \to 0,$$

a contradiction to Lemma 3.2. Hence case 1 is false.

Case 2. A similar argument shows that $\lim_{\varepsilon\to 0} \varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)/\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon) = 0$ is impossible.

Hence we now have

$$\lim_{\varepsilon\to 0} \frac{\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)}{\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)} = K_1 > 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)}{\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)} = K_2 > 0.$$

Adding equation $(\mathrm{E}_{P_1^\varepsilon})$ and $(\mathrm{E}_{P_2^\varepsilon})$ and noting that by Lemma 3.2,

$$\varepsilon \int_{\Omega_\varepsilon} w_1^{p-1} \frac{\partial w_1}{\partial P_{1,j}} w_2 + \varepsilon \int_{\Omega_\varepsilon} w_2^{p-1} \frac{\partial w_2}{\partial P_{2,j}} w_1 \bigg/ w\left(\frac{P_1^\varepsilon - P_2^\varepsilon}{\varepsilon}\right)$$
$$= -\gamma_2 \frac{P_1^\varepsilon - P_2^\varepsilon}{|P_1^\varepsilon - P_2^\varepsilon|}(1 + o(1)) - \gamma_2 \frac{P_2^\varepsilon - P_1^\varepsilon}{|P_1^\varepsilon - P_2^\varepsilon|}(1 + o(1)) \to 0.$$

We obtain, by Lemmas 3.5, 3.2 and 5.1 of [27]

$$\frac{\varepsilon}{\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)} \int_{\Omega_\varepsilon} ((Pw_1)^p - w_1^p) \frac{\partial w_1}{\partial P_{1,j}^\varepsilon}$$
$$+ \frac{\varepsilon}{\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)} \int_{\Omega_\varepsilon} ((Pw_2)^p - w_2^p) \frac{\partial w_2}{\partial P_{2,j}^\varepsilon} \to C\nu \neq 0,$$

where $C > 0$ is a positive constant and ν is the outer normal at P_0 where $P_1^\varepsilon, P_2^\varepsilon \to P_0 \in \partial\Omega$. A contradiction again! Hence $d(P_1^\varepsilon, \partial\Omega) \geq \sigma > 0$, $d(P_2^\varepsilon, \partial\Omega) \geq \sigma > 0$.

We next show that if $P_1^\varepsilon \to P_1, P_2^\varepsilon \to P_2$, then

$$|P_1 - P_2| \geq 2\min(d(P_1,\partial\Omega), d(P_2,\partial\Omega)).$$

Suppose not, then

$$|P_1 - P_2| < 2\min(d(P_1,\partial\Omega), d(P_2,\partial\Omega)).$$

Hence

$$\lim_{\varepsilon\to 0} \frac{\varphi_{\varepsilon,P_1^\varepsilon}(P_1^\varepsilon)}{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)} = 0, \qquad \lim_{\varepsilon\to 0} \frac{\varphi_{\varepsilon,P_2^\varepsilon}(P_2^\varepsilon)}{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)} = 0.$$

Thus by equation $(\mathrm{E}_{P_1^\varepsilon})$, we have

$$\frac{\varepsilon}{w(|P_1^\varepsilon - P_2^\varepsilon|/\varepsilon)} \int_{\Omega_\varepsilon} w_2 w_1^p \frac{\partial w_1}{\partial P_{1,j}} \to 0,$$

which is impossible by Lemma 3.2 again. Hence Theorem 3.1 is proved. □

4. Proof of Theorem 1.1

Let η be a fixed number such that $0 < \eta < I(w)$. Our aim in this section is to compute the contribution to the relative topology of $J_\varepsilon^{c_2+\eta}$ with respect to $J_\varepsilon^{c_2-\eta}$ of the 2-peak positive solutions to (1.1) that we studied before.

We first have a rough estimate of the energy.

Lemma 4.1. *There exist $c_0 > 0$, $0 < \sigma_0 < 0.01$, $d > 0$ such that for ε sufficiently small and any 2-peak positive solution u_ε, we have*

$$\varepsilon^{-N} J_\varepsilon(u_\varepsilon) \geq c_2 - c_0 e^{-\beta(2+3\sigma_0)d}.$$

Proof. We use the notation of Section 3. By Theorem 3.1 and Proposition 3.4, we have $\|v_{\varepsilon,\alpha,x}\| = O(e^{-(2+3\sigma_0)\beta d})$ for some $d > 0$, $0 < \sigma_0 < 0.01$, where $\beta = 1/\varepsilon$.

By equations (E_{α_1}), we have

$$\begin{aligned}
0 = \frac{\partial K_\varepsilon}{\partial \alpha_1} &= \int_{\Omega_\varepsilon} \alpha_1 w_1^p P w_1 + \alpha_2 \int_{\Omega_\varepsilon} w_2^p P w_1 - \int_{\Omega_\varepsilon} (\alpha_1 P w_1 + \alpha_2 P w_2 + v)^p P w_1 \\
&= (\alpha_1 - \alpha_1^p) \int_{\Omega_\varepsilon} (P w_1)^{p+1} + \alpha_1 \int_{\Omega_\varepsilon} [w_1^p P w_1 - (P w_1)^{p+1}] \\
&\quad + \alpha_2 \int_{\Omega_\varepsilon} w_2^p P w_1 + \alpha_1^p \int_{\Omega_\varepsilon} (P w_1)^{p+1} - \int_{\Omega_\varepsilon} (\alpha_1 P w_1 + \alpha_2 P w_2 + v)^p P w_1.
\end{aligned}$$

Hence $\alpha_1 = 1 + O(e^{-\beta(2+3\sigma_0)d})$. Similarly, $\alpha_2 = 1 + O(e^{-\beta(2+3\sigma_0)d})$. Thus

$$\begin{aligned}
\varepsilon^{-N} J_\varepsilon(u_\varepsilon) &= \varepsilon^{-N} J_\varepsilon(\alpha_1 P w_1 + \alpha_2 P w_2 + v_{\varepsilon,\alpha,x}) \\
&= 2I(w) + O(e^{-\beta(2+3\sigma_0)d}) \geq c_2 - c_0 e^{-\beta(2+3\sigma_0)d}.
\end{aligned}$$

Lemma 4.1 is thus proved. □

Let us now define

$$\max_{x,d(x,\partial\Omega)\geq d} \varphi_{\varepsilon,x}(x) = w(2d_\varepsilon/\varepsilon).$$

(Note that $d_\varepsilon = d + o(1)$.) By Lemma 4.1, we just need to compute the relative topology between the levels $c_2+\eta$ and $c_2 - C_1 e^{-2d_\varepsilon/\varepsilon}$ for some $C_1 > 0$. Namely, we have

$$(J_\varepsilon^{c_2+\eta}, J_\varepsilon^{c_2-\eta}) \cong (J_\varepsilon^{c_2+\eta}, J_\varepsilon^{c_2-C_1e^{-2d_\varepsilon/\varepsilon}})$$

for ε sufficently small and some $C_1 > 0$.

We now construct an open neighbourhood V_ε of the eventual 2-peak positive solutions to (1.1) such that on the boundary of V_ε, either $-J_\varepsilon^{'}$ is pointing inward V_ε or J_ε is less than $c_2 - C_1e^{-2d_\varepsilon/\varepsilon}$. We also show below that V_ε contains all positive critical points with energy near c_2 and it is easy to see that it contains no sign changing solutions.

Let C_2 be a sufficiently large number to be defined later. We use the letter C to denote various constants which depend on Ω only. Set

$$\begin{aligned} V_\varepsilon = \{&(\alpha,x,v) \mid |\alpha_i - 1| < \alpha_0 e^{-(1+2\sigma_0)d/2\varepsilon},\ d(x_i,\partial\Omega) > d,\ i=1,2,\\ &|x_1-x_2| > (2-\varepsilon\log C_2)d_\varepsilon,\ v\in E_x \text{ and } \|v - v_{\varepsilon,\alpha,x}\| < \nu_0 e^{-(1+2\sigma_0)d/\varepsilon}\}. \end{aligned}$$

Note that for $(\alpha,x,v)\in V_\varepsilon$ and d small,

$$w(|x_1-x_2|/\varepsilon) \leq C_2 \max_{d(x,\partial\Omega)\geq d} \varphi_{\varepsilon,x}(x),$$
$$\|v_{\varepsilon,\alpha,x}\| \leq Ce^{-(1+2\sigma_0)d/\varepsilon}, \quad \|v\| \leq Ce^{-(1+2\sigma_0)d/\varepsilon},$$

by Proposition 3.4 and since P_1, P_2 are not close to the boundary and not close together (and d is small). Note that the estimate holds on more than V_ε. This estimate shows that all the positive critical points with energy close to c_2 lie in V_ε.

Next we show that V_ε satisfies the above properties. We first consider the variable α. Note that

$$\begin{aligned} \frac{\partial K_\varepsilon}{\partial\alpha_1} &= \int_{\Omega_\varepsilon} \alpha_1 w_1^p Pw_1 + \alpha_2 \int_{\Omega_\varepsilon} w_2^p Pw_1 - \int_{\Omega_\varepsilon} (\alpha_1 Pw_1 + \alpha_2 Pw_2 + v)^p Pw_1\\ &= (\alpha_1 - \alpha_1^p)\int_{\Omega_\varepsilon} (Pw_1)^{p+1} + \alpha_1 \int_{\Omega_\varepsilon} (w_1^p Pw_1 - (Pw_1)^{p+1})\\ &\quad + \alpha_2 \int_{\Omega_\varepsilon} w_2^p Pw_1 + \alpha_1^p \int_{\Omega_\varepsilon} (Pw_1)^{p+1} - \int_{\Omega_\varepsilon} (\alpha_1 Pw_1 + \alpha_2 Pw_2 + v)^p Pw_1\\ &= I_1 + I_2 + I_3 \end{aligned}$$

where I_1, I_2 and I_3 will be defined in a moment.

Now

$$I_2 = \alpha_1 \int_{\Omega_\varepsilon} (w_1^p P w_1 - (P w_1)^{p+1}) = O(e^{-2d/\varepsilon})$$

and

$$\begin{aligned} I_3 = {} & \alpha_2 \int_{\Omega_\varepsilon} w_2^p P w_1 + \alpha_1^p \int_{\Omega_\varepsilon} (P w_1)^{p+1} - \int_{\Omega_\varepsilon} (\alpha_1 P w_1 + \alpha_2 P w_2 + v)^p P w_1 \\ = {} & \alpha_2 \int_{\Omega_\varepsilon} w_2^p P w_1 - \int_{\Omega_\varepsilon} (\alpha_2 P w_2)^p P w_1 \\ & + \int_{\Omega_\varepsilon} (\alpha_1^p (P w_1)^p + (\alpha_2 P w_2)^p + p(\alpha_1 P w_1)^{p-1} v \\ & - (\alpha_1 P w_1 + \alpha_2 P w_2 + v)^p)(P w_1) + p \alpha_1^p \int_{\Omega_\varepsilon} (P w_1)^p v = O(e^{-2d/\varepsilon}). \end{aligned}$$

Finally,

$$I_1 = (\alpha_1 - \alpha_1^p) \int_{\Omega_\varepsilon} (P w_1)^{p+1} = (\alpha_1 - \alpha_1^p) \int_{\mathbb{R}^N} w^{p+1} + O(e^{-2d/\varepsilon})$$

(note that $P w_1 - w_1 = O(e^{-2\beta d})$).

Hence on the boundary of $|\alpha_i - 1| \leq C e^{-(1+2\sigma_0)d/2\varepsilon}$, we have

$$\frac{\partial K_\varepsilon}{\partial \alpha_1} = (\alpha_1 - \alpha_1^p) \int_{\mathbb{R}^N} w^{p+1} + O(e^{-2d/\varepsilon}).$$

Similarly,

$$\frac{\partial K_\varepsilon}{\partial \alpha_2} = (\alpha_2 - \alpha_2^p) \int_{\mathbb{R}^N} w^{p+1} + O(e^{-2d/\varepsilon}).$$

Hence, for some $0 < \lambda < 1$, we have

$$\begin{aligned} K_\varepsilon(\alpha, x, v) &= K_\varepsilon(1, x, v) + \sum_{i=1}^2 \frac{\partial K_\varepsilon}{\partial \alpha_i}(\lambda \alpha + 1 - \lambda, x, v)(\alpha_i - 1) \\ &= K_\varepsilon(1, x, v) + \sum_{i=1}^2 (O(e^{-2d/\varepsilon}) + (\alpha_i - \alpha_i^p))(\alpha_i - 1) \\ &= K_\varepsilon(1, x, v) - C e^{-(2+2\sigma_0)d/\varepsilon}. \end{aligned}$$

But

$$\begin{aligned} K_\varepsilon(1, x, v) = {} & \frac{1}{2} \bigg\langle \sum_{i=1}^2 P w_i, \sum_{i=1}^2 P w_i \bigg\rangle + \frac{1}{2} \langle v, v \rangle - \frac{1}{p+1} \int_{\Omega_\varepsilon} \bigg(\sum_{i=1}^2 P w_i + v \bigg)^{p+1} \\ = {} & \frac{1}{2} \int_{\Omega_\varepsilon} (w_1^p + w_2^p)(P w_1 + P w_2) \\ & + \frac{1}{2} \langle v, v \rangle - \frac{1}{p+1} \int_{\Omega_\varepsilon} \bigg(\sum_{i=1}^2 P w_i + v \bigg)^{p+1} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\Omega_\varepsilon} w_1^p P w_1 + \frac{1}{2}\int_{\Omega_\varepsilon} w_2^p P w_1 + \frac{1}{2}\int_{\Omega_\varepsilon} w_1^p P w_2 \\
&\quad + \frac{1}{2}\int_{\Omega_\varepsilon} w_2^p P w_2 + \frac{1}{2}\langle v, v\rangle - \frac{1}{p+1}\int_{\Omega_\varepsilon}\bigg(\sum_{i=1}^{2} P w_i + v\bigg)^{p+1} \\
&= \bigg(\frac{1}{2} - \frac{1}{p+1}\bigg)\int_{\mathbb{R}^N} w^{p+1} + \bigg(\frac{1}{2} - \frac{1}{p+1}\bigg)\int_{\mathbb{R}^N} w^{p+1} \\
&\quad + (C + o(1))\varphi_{\varepsilon,x_1}(x_1) + (C + o(1))\varphi_{\varepsilon,x_2}(x_2) \\
&\quad + \int_{\Omega_\varepsilon} w_2^p w_1 - \frac{1}{p+1}\int_{\Omega_\varepsilon}((p+1)(w_1)^p w_2 + (p+1)(w_2)^p w_1) \\
&\quad + O(e^{-(2+\sigma_0)d/\varepsilon}) \\
&\le c_2 + (C + o(1))\max_{d(x,\partial\Omega)\ge d}\varphi_{\varepsilon,x}(x) - \int_{\Omega_\varepsilon} w_2^p w_1 \\
&\le c_2 - C(C_2 - C)e^{-2d_\varepsilon/\varepsilon} < c_2 - C_1 e^{-2d_\varepsilon/\varepsilon}
\end{aligned}
$$

if $C_1 < C(C_2 - C)$. Hence we obtain $K_\varepsilon(\alpha, x, v) < c_2 - C_1 e^{-2d_\varepsilon/\varepsilon}$.

Secondly, we consider the variable v. We claim that if ν_0 is large enough then we have for $(\alpha, x, v) \in V_\varepsilon$, $\|v - v_{\varepsilon,\alpha,x}\| = \nu_0\|v_{\varepsilon,\alpha,x}\|$ and ε small enough, we have

$$
(v - v_{\varepsilon,\alpha,x})\frac{\partial K_\varepsilon}{\partial v}(\alpha, x, v) > 0.
$$

In fact, let $v - v_{\varepsilon,\alpha,x} = \|v_{\varepsilon,\alpha,x}\|\varphi$. Then

$$
\begin{aligned}
(v - v_{\varepsilon,\alpha,x})\frac{\partial K_\varepsilon}{\partial v} &= \int_{\Omega_\varepsilon} \nabla v \nabla (v - v_{\varepsilon,\alpha,x}) + v(v - v_{\varepsilon,\alpha,x}) \\
&\quad - p\int_{\Omega_\varepsilon} (\alpha_1 P w_1 + \alpha_2 P w_2)^{p-1}(v - v_{\varepsilon,\alpha,x})v + O(\|v_{\varepsilon,\alpha,x}\|^{2+\delta_1}) \\
&= \|v_{\varepsilon,\alpha,x}\|^2\bigg(\int_{\Omega_\varepsilon} |\nabla\varphi|^2 + \varphi^2 - p\int_{\Omega_\varepsilon}(\alpha_1 P w_1 + \alpha_2 P w_2)^{p-1}\varphi^2\bigg) \\
&\quad - \|v_{\varepsilon,\alpha,x}\|\bigg(\int_{\Omega_\varepsilon} \nabla v_{\varepsilon,\alpha,x}\nabla\varphi + v_{\varepsilon,\alpha,x}\cdot\varphi \\
&\quad - p\int_{\Omega_\varepsilon}(\alpha_1 P w_1 + P w_2)^{p-1} v_{\varepsilon,\alpha,x}\cdot\varphi) + O(\|v_{\varepsilon,\alpha,x}\|^{2+\delta_1}) > 0
\end{aligned}
$$

if ν_0 is large enough, where $\delta_1 = \min(1, p-1)$.

Thirdly, we consider the variable $x_1 - x_2$. For $|x_1 - x_2| = (2 - \varepsilon\log C_2)d_\varepsilon$ small, we have

$$
\begin{aligned}
&K_\varepsilon(\alpha, x, v) \\
&\quad = \frac{1}{2}\bigg\langle \sum_{i=1}^{2}\alpha_i P w_i, \sum_{i=1}^{2}\alpha_i P w_i \bigg\rangle + \frac{1}{2}\langle v, v\rangle - \frac{1}{p+1}\int_{\Omega}\bigg(\sum_{i=1}^{2}\alpha_i P w_i + v\bigg)^{p+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\Omega_\varepsilon} (\alpha_1 w_1^p + \alpha_2 w_2^p)(\alpha_1 Pw_1 + \alpha_2 Pw_2) \\
&\quad + \frac{1}{2}\langle v, v\rangle - \frac{1}{p+1}\int_{\Omega_\varepsilon} \bigg(\sum_{i=1}^2 \alpha_i Pw_i + v\bigg)^{p+1} \\
&= \frac{1}{2}\alpha_1^2 \int_{\Omega_\varepsilon} w_1^p Pw_1 + \frac{1}{2}\alpha_1\alpha_2 \int_{\Omega_\varepsilon} w_2^p Pw_1 + \frac{1}{2}\alpha_1\alpha_2 \int_{\Omega_\varepsilon} w_1^p Pw_2 \\
&\quad + \frac{1}{2}\alpha_2^2 \int_{\Omega_\varepsilon} w_2^p Pw_2 + \frac{1}{2}\langle v, v\rangle - \frac{1}{p+1}\int_{\Omega_\varepsilon} \bigg(\sum_{i=1}^2 \alpha_i Pw_i + v\bigg)^{p+1} \\
&\le \bigg(\frac{\alpha_1^2}{2} - \frac{\alpha_1^{p+1}}{p+1}\bigg)\int_{\mathbb{R}^N} w^{p+1} + \bigg(\frac{\alpha_2^2}{2} - \frac{\alpha_2^{p+1}}{p+1}\bigg)\int_{\mathbb{R}^N} w^{p+1} \\
&\quad + \int_{\Omega_\varepsilon} w_2^p w_1 - \frac{1}{p+1}\int_{\Omega_\varepsilon} ((p+1)(w_1)^p w_2 + (p+1)(w_2)^p w_1) \\
&\quad + (C + o(1))\varphi_{\varepsilon,x_1}(x_1) + (C + o(1))\varphi_{\varepsilon,x_2}(x_2) + O(e^{-(2+\sigma_0)d/\varepsilon}) \\
&\le c_2 + O(e^{-(2+\sigma_0)d/\varepsilon}) + (C + o(1)) \max_{d(x,\partial\Omega)\ge d} \varphi_{\varepsilon,x}(x) - \int_{\Omega_\varepsilon} w_2^p w_1 \\
&\le c_2 - C(C_2 - C)e^{-2d_\varepsilon/\varepsilon} \le c_2 - C_1 e^{-2d_\varepsilon/\varepsilon}
\end{aligned}
$$

if we choose $C(C_2 - C) \ge C_1$.

Finally, we consider the variable x_1, x_2. If $d(x_i, \partial\Omega) = d$ for $i = 1$ or 2 (possibly both), we then have

$$
\begin{aligned}
\frac{\partial K_\varepsilon}{\partial x_1} &= \int_{\Omega_\varepsilon} (\alpha_1 w_1^p + \alpha_2 w_2^p)\alpha_1 \frac{\partial Pw_1}{\partial x_1} - \int_{\Omega_\varepsilon} (\alpha_1 Pw_1 + \alpha_2 Pw_2 + v)^p \frac{\partial Pw_1}{\partial x_1} \\
&= \int_{\Omega_\varepsilon} (\alpha_1 w_1^p + \alpha_2 w_2^p)\alpha_1 \frac{\partial Pw_1}{\partial x_1} - \int_{\Omega_\varepsilon} (\alpha_1 Pw_1 + \alpha_2 Pw_2)^p \alpha_1 \frac{\partial Pw_1}{\partial x_1} \\
&\quad + O(e^{-(1+\delta)\min(2d(x_1,\partial\Omega), 2d(x_2,\partial\Omega), |x_1 - x_2|)/\varepsilon}) \\
&= \int_{\Omega_\varepsilon} (\alpha_1 w_1^p + \alpha_2 w_2^p)\alpha_1 \frac{\partial Pw_1}{\partial x_1} \\
&\quad - \int_{\Omega_\varepsilon} ((\alpha_1 Pw_1)^p + p(\alpha Pw_1)^{p-1}\alpha_2 Pw_2 + \alpha_2 (Pw_2)^p)\alpha_1 \frac{\partial Pw_1}{\partial x_1} \\
&\quad + O(e^{-(1+\delta)\min(2d(x_1,\partial\Omega), 2d(x_2,\partial\Omega), |x_1 - x_2|)/\varepsilon}) \\
&= \int_{\Omega_\varepsilon} (w_1^p - (Pw_1)^p)\frac{\partial Pw_1}{\partial x_1} + \int_{\Omega_\varepsilon} (w_2^p - (Pw_2)^p)\frac{\partial Pw_1}{\partial x_1} \\
&\quad - p\int_{\Omega_\varepsilon} w_1^{p-1} w_2 \frac{\partial w_1}{\partial x_1} + O(e^{-(1+\delta)\min(2d(x_1,\partial\Omega), 2d(x_2,\partial\Omega), |x_1 - x_2|)/\varepsilon}) \\
&= \int_{\Omega_\varepsilon} ((w_1^p - (Pw_1)^p)\frac{\partial w_1}{\partial x_1} - p\int_{\Omega_\varepsilon} w_1^{p-1} w_2 \frac{\partial w_1}{\partial x_1} \\
&\quad + O(e^{-(1+\delta)\min(2d(x_1,\partial\Omega), 2d(x_2,\partial\Omega), |x_1 - x_2|)/\varepsilon}) = J_1 - J_2 + Er
\end{aligned}
$$

where J_1 and J_2 are defined at the last equality and Er is the error term.

For $x \in \partial\Omega_d := \{x \in \Omega : d(x, \partial\Omega) = d\}$, let ν_x be its outward normal and for d sufficiently small, let $\overline{x} \in \partial\Omega$ be such that $|x - \overline{x}| = d(x, \partial\Omega)$, then $\nu_x = \nu_{\overline{x}} + o(d)$. We first compute

$$\nu_x \varepsilon \int_{\Omega_\varepsilon} (w_1^p - (Pw_1)^p) \frac{\partial w_1}{\partial x_1}.$$

By Section 3 of [27],

$$\frac{1}{\varphi_{\varepsilon,x_1}(x_1)} \int_{\Omega_\varepsilon} \varepsilon (w_1^p - (Pw_1)^p) \frac{\partial w_1}{\partial x_1} \to C\nu_x,$$

for some $C > 0$. Hence

$$\nu_x \cdot \frac{\varepsilon}{\varphi_{\varepsilon,x_1}(x_1)} \int_{\Omega} (w_1^p - (Pw_1)^p) \frac{\partial w_1}{\partial x_1} \to C\nu_x \cdot \nu_x > 0.$$

We next compute J_2. Since $|x_1 - x_2| > (2 - \varepsilon \log C_2) d_\varepsilon$, we have that

$$w\left(\frac{x_1 - x_2}{\varepsilon}\right) \leq C_2 \max_{d(x,\partial\Omega) \geq d} \varphi_{\varepsilon,x}(x).$$

Note that when d is very small, we have $\max_{d(x,\partial\Omega)\geq d} \varphi_{\varepsilon,x}(x)$ is obtained at a point x' with $d(x', \partial\Omega) = d$ since $\nabla \psi_{\varepsilon,x} = -\nu_x + o(1)$ as $\varepsilon \to 0$ (see Section 3 of [27]). Hence

$$\max_{d(x,\partial\Omega)\geq d} \varphi_{\varepsilon,x}(x) = \varphi_{\varepsilon,x'}(x').$$

On the other hand, for $d(x', \partial\Omega) = d(x_1, \partial\Omega) = d$, we have $\psi_{\varepsilon,x'}(x') = \psi_{\varepsilon,x_1}(x_1) + O(\varepsilon)$. Hence

$$\max_{d(x,\partial\Omega)\geq d} \varphi_{\varepsilon,x}(x) = O(\varphi_{\varepsilon,x_1}(x_1)).$$

But we have

$$\frac{\varepsilon \int_{\Omega_\varepsilon} w_1^{p-1} w_2 \frac{\partial w_1}{\partial x_1}}{w(|x_1 - x_2|/\varepsilon)} = \frac{1}{w(|x_1 - x_2|/\varepsilon)} \int_{\Omega_\varepsilon} w_1^{p-1} w_1^{'} w_2 \frac{x_1 - x}{|x - x_1|}$$

$$\to \int_{\mathbb{R}^N} w^p \frac{w^{'}(r)}{|r|} \cdot \frac{(x_1 - x_2)}{|x_1 - x_2|} = \left(- \int_{R^N} w^{p+1} \frac{w^{'}(r)}{r} \right) \frac{(x_2 - x_1)}{|x_2 - x_1|}.$$

Note that when d is very small and x_2 is close to x_1 $(d(x_2, \partial\Omega) > d)$, we have

$$\lim_{d\to 0} \frac{x_2 - x_1}{|x_2 - x_1|} \cdot \nu_x \leq 0.$$

Hence $\lim_{d\to 0} -\nu_x \cdot J_2 / w(|x_1 - x_2|/\varepsilon) \geq 0$. Therefore, in any case, we have

$$\nu_{x_1} \varepsilon \frac{\partial K_\varepsilon}{\partial x_1} = \varphi_{\varepsilon,x_1}(x_1) C\nu_x \cdot \nu_x + o(\varphi_{\varepsilon,x_1}(x_1)) > 0.$$

Similarly, $\nu_{x_2} \varepsilon \partial K_\varepsilon / \partial x_2 > 0$. Thus, for $x \in \partial\Omega_d$, we have $\partial K_\varepsilon / \partial x$ is pointing outward to V_ε.

We now turn to the computation of the relative topology. The first step is concerned with the v-variable. We set

$$\widetilde{K}_\varepsilon(\alpha,x)=K_\varepsilon(\alpha,x,v_{\varepsilon,\alpha,x})\quad\text{for } (\alpha,x)\in\widetilde{V}_\varepsilon,$$

where

$$\widetilde{V}_\varepsilon:=\{(\alpha,x)\mid|\alpha_i-\alpha|<\alpha_0e^{-(1+2\sigma_0)d/2\varepsilon},\\ d(x_i,\partial\Omega)>d,\ |x_1-x_2|>(2-\varepsilon\log C_2)d\}$$

Then from Morse Theory we have, since $v_{\varepsilon,\alpha,x}$ is a strict nondegenerate minimizer of K_ε in a fixed neighborhood (uniform in the other variables) of $v=0$, that

$$K_\varepsilon^{c_2+\eta}\cap V_\varepsilon=V_\varepsilon=\{(\alpha,x,v)\mid(\alpha,x)\in\widetilde{V}_\varepsilon,\ v\in E_x\cap B_{\nu_0e^{-(1+2\sigma_0)d/\varepsilon}}(v_{\varepsilon,\alpha,x})\}$$

and

$$K_\varepsilon^{c_2-C_1e^{-2d_\varepsilon/\varepsilon}}\cap V_\varepsilon\\ =\{(\alpha,x)\in\widetilde{V}_\varepsilon)\mid(\alpha,x)\in\widetilde{V}_\varepsilon,\ \widetilde{K}_\varepsilon\le c_2-C_1e^{2d_\varepsilon/\varepsilon},\ v\in D(\alpha,x)\},$$

where $D(\alpha,x)$ is a subset of E_x topologically equivalent to a disk.

Set $\tau:=e^{-2d_\varepsilon/\varepsilon}$. Therefore

$$(K_\varepsilon^{c_2+\eta}\cap V_\varepsilon,K_\varepsilon^{c_2-C_1\tau}\cap V_\varepsilon)\cong(\widetilde{V}_\varepsilon,\widetilde{K}_\varepsilon^{c_2-C_1\tau}\cap\widetilde{V}_\varepsilon).$$

In the next step we define $\widetilde{\widetilde{K}}_\varepsilon=\widetilde{K}_\varepsilon(\overline{\alpha},x)$ for

$$x\in\widetilde{\widetilde{V}}_\varepsilon=\{x\mid d(x_i,\partial\Omega)\ge d,\ |x_1-x_2|>(2-\varepsilon\log C_2)d_\varepsilon\},$$

where $\overline{\alpha}=\overline{\alpha}(x)$ is such that $\frac{\partial\widetilde{K}_\varepsilon}{\partial\alpha}(\overline{\alpha},x)=0$. Such an $\overline{\alpha}=(\overline{\alpha}_1,\overline{\alpha}_2)$, $|\overline{\alpha}_i-1|<\eta_0$, $i=1,2$, is unique and corresponds to a strict and nondegenerate maximum (the proof is similar to that of [15]). Morse theory yields

$$\widetilde{K}_\varepsilon^{c_2-C_1\tau}\cap\widetilde{V}_\varepsilon=\{(\alpha,x)\in\widetilde{V}_\varepsilon\mid\widetilde{\widetilde{K}}_\varepsilon\le c_2-\widetilde{c}\tau\text{ and }\alpha\in D\}\\ \cup\{(\alpha,x)\in\widetilde{V}_\varepsilon\mid\widetilde{\widetilde{K}}(x)>c_2-\widetilde{c}\tau\text{ and }\alpha\in C(x)\},$$

where $\widetilde{c}<C_1$.

Here, D denotes that 2-square $[\alpha-1,\alpha+1]^2$, topologically equivalent to the unit disk D^2 of $\mathbb{R}^2$ and $C(x)$ is equal to D with a subset equivalent to a disk deleted, whose radius goes to zero as $\widetilde{\widetilde{K}}_\varepsilon(x)$ goes to $c_2-C_1\tau$. At the same time

$$\widetilde{V}_\varepsilon=\widetilde{\widetilde{V}}_\varepsilon\times D.$$

Then, we have a natural map

$$T:(\widetilde{\widetilde{V}}_\varepsilon,\widetilde{\widetilde{K}}_\varepsilon{}^{c_2-\widetilde{c}\tau}\cap\widetilde{\widetilde{V}}_\varepsilon)\times(D^2,S^1)\to(\widetilde{V}_\varepsilon,\widetilde{K}_\varepsilon{}^{c_2-C_1\tau}\cap\widetilde{V}_\varepsilon).$$

T is injective if we identify points (x_1, x_2) with (x_2, x_1). Since all the critical values for $\tilde{\tilde{K}}_\varepsilon$ on $\tilde{\tilde{V}}_\varepsilon$ are larger than $c_2 - \tilde{c}\tau$, T is also surjective and then is an isomorphism of the quotient of the left hand side by the Z_2 action and the right hand side.

Let us now compute $\tilde{\tilde{K}}(x)$. We first note that

$$\begin{aligned}
&J_\varepsilon(\alpha_1 Pw_1 + \alpha_2 Pw_2 + v_{\varepsilon,\alpha,x}) \\
&= \frac{1}{2}\int_{\Omega_\varepsilon} \nabla\Big(\sum_{i=1}^{2}\alpha_i Pw_i\Big)\nabla\Big(\sum_{i=1}^{2}\alpha_i Pw_i\Big) + \Big(\sum_{i=1}^{2}\alpha_i Pw_i\Big)\Big(\sum_{i=1}^{2}\alpha_i Pw_i\Big) \\
&\quad + \frac{1}{2}\langle v_{\varepsilon,\alpha,x}, v_{\varepsilon,\alpha,x}\rangle - \frac{1}{p+1}\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^{2}\alpha_i Pw_i + v_{\varepsilon,\alpha,x}\Big)^{p+1} \\
&= \frac{1}{2}\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^{2}\alpha_i w_i^p\Big)\Big(\sum_{i=1}^{2}\alpha_i Pw_i\Big) - \frac{1}{p+1}\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^{2}\alpha_i Pw_i\Big)^{p+1} + O(e^{2d/\varepsilon}) \\
&= \frac{1}{2}\int_{\Omega_\varepsilon}(\alpha_1^2 w_1^p Pw_1 + \alpha_1\alpha_2(w_1^p Pw_2 + w_2^p Pw_1) + \alpha_2^2 w_2^p Pw_2) \\
&\quad - \frac{1}{p+1}\int_{\Omega_\varepsilon}\Big(\sum_{i=1}^{2}\alpha_i^{p+1}(Pw_i)^{p+1} + (p+1)(\alpha_1 Pw_1)^p(\alpha_2 Pw_2) \\
&\quad + (p+1)(\alpha_2 Pw_2)^p(\alpha_1 Pw_1)\Big) + \text{high order term.}
\end{aligned}$$

Here the *high order term* means $o(e^{-2d_\varepsilon/\varepsilon})$.

We now compute $\overline{\alpha}_i$. In fact, from (E_α) we have

$$\alpha_1\int w_1^p Pw_1 + \alpha_2\int w_2^p Pw_1 - \int(\alpha_1 Pw_1 + \alpha_2 Pw_2 + v_{\varepsilon,\alpha,x})^p Pw_1 = 0.$$

Hence

$$\begin{aligned}
\alpha_1\Big(\int_{\Omega_\varepsilon} w_1^p Pw_1\Big) &- \alpha_1^p\int_{\Omega_\varepsilon}(Pw_1)^{p+1} - p\alpha_1^{p-1}\int_{\Omega_\varepsilon}(w_1)^p Pw_2 \\
&- \alpha_2^p\int_{\Omega_\varepsilon}(Pw_2)^p Pw_1 + \alpha_2\int_{\Omega_\varepsilon} w_2^p Pw_1 \\
&- \int_{\Omega_\varepsilon}((\alpha_1 Pw_1)^p + p(\alpha_1 Pw_1)^{p-1}Pw_2 + \alpha_2^p(Pw_2)^p \\
&- (\alpha_1 Pw_1 + \alpha_2 Pw_2 + v_{\varepsilon,\alpha,x})^p)Pw_1 = 0.
\end{aligned}$$

We then have the first rough estimates

$$\begin{aligned}
|\overline{\alpha_i} - 1| &= O\big(e^{-2\min(d(x_1,\partial\Omega), d(x_2,\partial\Omega), |x_1-x_2|/2)/\varepsilon}\big) \\
&= O(e^{-2\min(d_\varepsilon, d)/\varepsilon}), \quad i = 1, 2.
\end{aligned}$$

Similar to previous computations we have that

$$\begin{aligned}\widetilde{\widetilde{K}}_\varepsilon(x) &= \frac{1}{2}\bigg(\overline{\alpha}_1^2 \int_{\Omega_\varepsilon} w_1^p P w_1 + 2\overline{\alpha}_1\overline{\alpha}_2 \int_{\Omega_\varepsilon} w_1^p P w_2 + \overline{\alpha}_2^2 \int_{\Omega_\varepsilon} w_2^p P w_2\bigg) \\ &\quad - \frac{1}{p+1}\int_{\Omega_\varepsilon} (\overline{\alpha}_1 P w_1 + \overline{\alpha}_2 P w_2)^{p+1} + O(e^{-(2+\sigma_0)d/\varepsilon}) \\ &= \bigg(\frac{1}{2}\overline{\alpha}_1^2 - \frac{1}{p+1}\overline{\alpha}_1^{p+1}\bigg)\int_{\mathbb{R}^N} w^{p+1} \\ &\quad + \bigg(\frac{1}{2}\overline{\alpha}_2^2 - \frac{1}{p+1}\overline{\alpha}_2^{p+1}\bigg)\int_{\mathbb{R}^N} w^{p+1} - (2+o(1))\int_{\mathbb{R}^N} w_1^p w_2 \\ &\quad + (C+o(1))\varphi_{\varepsilon,x_1}(x_1) + (C+o(1))\varphi_{\varepsilon,x_2}(x_2) \\ &= c_2 - (C+o(1))w(|x_1-x_2|/\varepsilon) \\ &\quad + (C+o(1))\varphi_{\varepsilon,x_1}(x_1) + (C+o(1))\varphi_{\varepsilon,x_2}(x_2).\end{aligned}$$

Thus

$$\begin{aligned}\widetilde{\widetilde{K}}_\varepsilon{}^{c_2-\widetilde{c}\tau} \cap \widetilde{\widetilde{V}}_\varepsilon &= \{x \in \widetilde{\widetilde{V}}_\varepsilon \mid w(|x_1-x_2|/\varepsilon) > \widetilde{c}\tau/(C+o(1)) \\ &\qquad + C\varphi_{\varepsilon,x_1}(x_1) + C\varphi_{\varepsilon,x_2}(x_2)\} \\ &= \{x \in \widetilde{\widetilde{V}}_\varepsilon \mid |x_1-x_2| < (2-\varepsilon\log C_3)d_\varepsilon\}\end{aligned} \tag{4.2}$$

for some C_3. Now we choose C_2 sufficiently large so that $C_2 > C_3$. It is easy to see that for d small $\{x \in \widetilde{\widetilde{V}}_\varepsilon \mid |x_1-x_2| < (2-\varepsilon\log C_3)d_\varepsilon\}$ retracts by deformation onto $\{x \in \widetilde{\widetilde{V}}_\varepsilon \mid |x_1-x_2| < (2-\varepsilon\log C_2)d_\varepsilon\}$. Therefore

$$\begin{aligned}(\widetilde{\widetilde{V}}_\varepsilon, \widetilde{\widetilde{K}}_\varepsilon{}^{c_2-\widetilde{c}\tau} \cap \widetilde{\widetilde{V}}_\varepsilon) &\cong (\widetilde{\widetilde{V}}_\varepsilon, \{x \in \widetilde{\widetilde{V}}_\varepsilon \mid |x_1-x_2| < (2-\varepsilon\log C_3)d_\varepsilon\}) \\ &\cong (\{x \in \Omega_d \mid |x_1-x_2| > (2-\varepsilon\log C_2)d_\varepsilon\}, \\ &\qquad \{x \in \Omega_d| \mid (2-\varepsilon\log C_2)d_\varepsilon < |x_1-x_2| < (2-\varepsilon\log C_3)d_\varepsilon\}) \\ &\cong (\Omega\times\Omega, M(\Omega))\end{aligned}$$

for d small, which completes the proof of Theorem 1.1.

5. Proof of Corollary 1.2

In this section, we prove Corollary 1.2. From now on the homology will always denote reduced singular homology with coefficients in Z_2.

Firstly, note that the diagonal $M(\Omega) := \{(x_1,x_2) \in \Omega\times\Omega : x_1 = x_2\}$ is homeomorphic to Ω and hence $M(\Omega)$ and Ω have the same homology. Next, we prove that $H_*(\Omega\times\Omega, M(\Omega))$ is non-trivial. If not, the exactness of the homology sequence for the pair $(\Omega\times\Omega, M(\Omega))$ (as in [24], p. 184) implies that the natural inclusion of $M(\Omega)$ into $\Omega\times\Omega$ induces an isomorphism of $H_*(\Omega\times\Omega)$ and $H_*(M(\Omega)) = H_*(\Omega)$. To see that this is impossible, first note that, by [14, Proposition 8.3.3], $H_r(\Omega) = 0$ for $r > n$. Thus, by our assumption, there exists a

k such that $H_k(\Omega) \neq \{0\}$ while $H_r(\Omega) = \{0\}$ if $r > k$ (note that $k > 0$ since Ω is connected and $\widetilde{H}$ denotes the reduced homology). By the Kunneth formula for the homology of a product (see [24, p. 235]) it follows that $H_{2k}(\Omega \times \Omega) \neq \{0\}$. Note that we use that the coefficients are chosen to avoid torsion problems. Hence $\Omega \times \Omega$ and Ω have different homology and thus $H_*(\Omega \times \Omega, M(\Omega))$ is nontrivial. Hence by the Kunneth formula again, $(\Omega \times \Omega, M(\Omega)) \times (D^2, S^1)$ has non-trivial homology.

We have a Z_2 group action on $X = (\Omega \times \Omega) \times D^2$. Let $\widetilde{p}$ denote the natural mapping on the orbit space B and let $B_0 = \widetilde{p}((\Omega \times \Omega \times S^1) \cup M(\Omega) \times D^2)$. We apply Smith theory as on p. 143 of Bredon [8] with $p = 2$. In particular, we use 7.5 and 7.6 there and use that since $p = 2, \sigma = \tau$ and $\widetilde{\sigma} = \sigma$ (Note that σ and τ are defined on p. 122 there). We see from the exactness of the triangle that if $H^*(\widetilde{p}X, \widetilde{p}X_0) = H^*(B, B_0)$ is trivial, then $H^*(X, X_0)$ is trivial (where $X_0 = (\Omega \times \Omega \times S^1) \cup M(\Omega) \times D^2$). However, by what we have already proved and the universal coefficient theorem (as in [8, p. 247–248]) $H^*(X, X_0)$ is nontrivial. Thus $H^*(B, B_0)$ is nontrivial and hence by the universal coefficient theorem again, $H_*(B, B_0)$ is nontrivial, as required. This proves the corollary.

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