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MULTIPLICITY OF FORCED OSCILLATIONS FOR THE SPHERICAL PENDULUM

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1. Introduction

In this paper we deal with the forced oscillations of a particle, of mass m, constrained to a two dimensional sphere S, with radius r, and acted on by the sum of three forces: a vector field depending only on the position, a possible friction and a T-periodic forcing term. More precisely, we are concerned with the following second order differential equation on S:

$$m\,\ddot{x} = -\frac{m\,|\dot{x}|^2}{r^2}\,x + h(x) - \eta\dot{x} + \lambda\varphi(t,x,\dot{x}), \quad \lambda \geq 0 \tag{1}$$

where $h : S \to \mathbb{R}^3$ is C^1 and tangent to S, $\eta \geq 0$ and $\varphi : \mathbb{R} \times TS \to \mathbb{R}^3$ is continuous, T-periodic in t, and such that $\varphi(t,q,v) \in T_qS$ for any $(t,q,v) \in \mathbb{R} \times TS$.

In the case when φ does not depend on $\dot{x}$, the problem of the existence of T-periodic solutions of (1), for any value of λ, has been positively solved in [2] and [3], and extended to the case of even dimensional spheres in [5]. In this paper, in order to get multiplicity results for forced oscillations of (1), we combine the methods used in [3] and [5] with a result of [7] about the set of harmonic solutions of periodically perturbed first order autonomous ODE's.

A physically relevant example is when h is the tangential component of the gravitational force. In this case we prove that the forced gravitational spherical

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pendulum has at least two essentially different T-periodic solutions for λ small enough. This multiplicity result for small perturbations has a topological nature and cannot be proved via implicit function theorem. In fact, this is obtained as a consequence of a general result about multiplicity of harmonic solutions for periodically perturbed autonomous differential equations on a complete manifold (Theorem 3.4 below).

2. Preliminaries

Let U be an open subset of a (boundaryless) differentiable manifold $M \subset \mathbb{R}^k$, and $v : M \to \mathbb{R}^k$ be a continuous tangent vector field such that the set $v^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (v, U) an integer, often called the Euler characteristic (or Hopf index or rotation) of v in U, which, roughly speaking, counts (algebraically) the number of zeros of v in U (see e.g. [8]–[11], and references therein), and which, for reasons that will become clear in the sequel, we will call degree of the vector field v and denote by $\deg(v, U)$. If $v^{-1}(0) \cap U$ is a finite set, then $\deg(v, U)$ is simply the sum of the indices at the zeros of v, i.e.

$$\deg(v, U) = \sum_{z \in v^{-1}(0) \cap U} \mathrm{i}(v, z).$$

In the general admissible case, i.e. when $v^{-1}(0) \cap U$ is a compact set, $\deg(v, U)$ is defined by taking a convenient smooth approximation of v having finitely many zeros (provided that these zeros are sufficiently close to $v^{-1}(0) \cap U$). We stress that no orientability on M is necessary in order to define the degree of a tangent vector field. Given a compact relatively open subset Z of $v^{-1}(0)$, it is convenient to introduce the index $\mathrm{i}(v, Z)$ of v at Z as $\mathrm{i}(v, Z) = \deg(v, U)$, where U is any open neighbourhood of Z such that $Z = v^{-1}(0) \cap U$.

In the flat case, namely if U is an open subset of $\mathbb{R}^k$, $\deg(v, U)$ is just the Brouwer degree (with respect to zero) of v in U (i.e. in any bounded open set V containing $v^{-1}(0)$ and such that $\overline{V} \subset U$).

Using the equivalent definition of degree given in [1], one can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds.

In what follows, the inner product of two vectors v, w in $\mathbb{R}^k$ is denoted by $\langle v, w \rangle$, the vector product (when $k = 3$) by $v \times w$, and $|v|$ will stand for the Euclidean norm in $\mathbb{R}^k$ (i.e. $|v| = \sqrt{\langle v, v \rangle}$). Moreover, if $A : E \to E$ is an endomorphism of the vector space E, we will denote by $\sigma(A)$ the spectrum of A.

If M is a differentiable manifold embedded in some $\mathbb{R}^k$, we will denote by $C^n_T(M)$, $n \in \mathbb{N} \cup \{0\}$, the metric subspace of the Banach space $C^n_T(\mathbb{R}^k)$ of all the T-periodic C^n maps $x : \mathbb{R} \to M$ with the C^n norm given by $\|x\|_n =$

$\sum_{i=0}^{n} \max_{t\in\mathbb{R}} |x^{(i)}(t)|$ (in the case $n = 0$ we will simply write $C_T(M)$). Observe that $C_T^n(M)$ is not complete, unless M is complete (i.e. closed in $\mathbb{R}^k$). Nevertheless, since M is locally compact, $C_T^n(M)$ is always locally complete.

In the sequel we will make use of the following version of Ascoli's theorem.

Theorem 2.1. *Let X be a subset of $\mathbb{R}^k$ and B a bounded equicontinuous subset of $C([a,b], X)$. Then B is totally bounded in $C([a,b], X)$. In particular, if X is closed, B is relatively compact.*

3. Multiplicity for first order equations

We deal with the following parametrized differential equation

$$\dot{x} = g(x) + \lambda f(t,x), \quad \lambda \in [0,\infty), \tag{2}$$

where, throughout this section, $g : M \to \mathbb{R}^k$ and $f : \mathbb{R} \times M \to \mathbb{R}^k$ are continuous tangent vector fields on a boundaryless manifold $M \subset \mathbb{R}^k$ and f is T-periodic in the first variable.

We say that $(\lambda, x) \in [0,\infty) \times C_T(M)$ is a *T-pair* (or *solution pair*) if $x(\,\cdot\,)$ is a solution of (2) corresponding to λ. If $\lambda = 0$ and x is constant, then (λ, x) is said to be *trivial*. Clearly one may have nontrivial solutions even when $\lambda = 0$.

Denote by X the subset of $[0,\infty) \times C_T(M)$ of all the T-pairs. Known properties of the set of solutions of differential equations imply that X is closed, hence it is locally complete as a closed subset of a locally complete space.

Lemma 3.1. *Assume M is a complete manifold. Then any bounded subset of X is actually totally bounded. As a consequence, closed and bounded sets of T-pairs are compact.*

Proof. Since M is complete, its bounded subsets are relatively compact (in M). Thus, given $A \subset X$ bounded, the set

$$\{(\lambda, x(t)) \in [0,\infty) \times M : (\lambda, x) \in A, \quad t \in [0,T]\}$$

is contained in a compact set $K \subset [0,\infty) \times M$. Hence, there exists a constant $c \geq 0$ such that $|g(y) + \lambda f(t,y)| \leq c$ for all $t \in [0,T]$ and all $(\lambda, y) \in K$. This implies that $|\dot{x}(t)| \leq c$ for all $(\lambda, x) \in A$. Thus A can be regarded as an equibounded set of equicontinuous functions from $[0,T]$ into $[0,\infty) \times M$. Ascoli's Theorem implies that A is totally bounded.

If A is assumed to be closed in X then, as a closed subset of a complete metric space, A is complete. Thus, being totally bounded and complete, it is compact. $\square$

The proof of the above lemma shows that, even when M is not complete, X is always locally totally bounded. Thus, being locally complete, X is actually locally compact.

For the sake of simplicity, according to [6], we make some conventions. We will regard every space as its image in the following diagram of natural inclusions

$$\begin{tikzcd}
[0,\infty)\times M \arrow[r] & [0,\infty)\times C_T(M) \\
M \arrow[u] \arrow[r] & C_T(M). \arrow[u]
\end{tikzcd}$$

In particular, we will identify M with its image in $C_T(M)$ under the embedding which associates to any $p \in M$ the map $\widehat{p} \in C_T(M)$ constantly equal to p. Moreover, we will regard M as the slice $\{0\}\times M \subset [0,\infty)\times M$ and, analogously, $C_T(M)$ as $\{0\}\times C_T(M)$. We point out that the images of the above inclusions are closed.

According to these identifications, if Ω is an open subset of $[0,\infty)\times C_T(M)$, by $\Omega\cap M$ we mean the open subset of M given by all $p\in M$ such that the pair $(0,\widehat{p})$ belongs to Ω. If U is an open subset of $[0,\infty)\times M$, then $U\cap M$ represents the open set $\{p\in M : (0,p)\in U\}$. With the above conventions, $g^{-1}(0)$ can be viewed as the set of trivial T-pairs.

In [7] we proved the following result about the structure of the set X of T-pairs of (2).

Theorem 3.2. *Let $f:\mathbb{R}\times M\to\mathbb{R}^k$ and $g:M\to\mathbb{R}^k$ be continuous tangent vector fields defined on a (boundaryless) differentiable manifold $M\subset\mathbb{R}^k$, with f T-periodic in the first variable. Let Ω be an open subset of $[0,\infty)\times C_T(M)$, and assume that $\deg(g,\Omega\cap M)$ is well defined and nonzero. Then there exists a connected set Γ of nontrivial T-pairs of (2) in Ω whose closure in $[0,\infty)\times C_T(M)$ meets $g^{-1}(0)\cap\Omega$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbb{R}^k$ and $\Omega=[0,\infty)\times C_T(M)$, then Γ is unbounded.*

We will say that a point $p\in g^{-1}(0)$ is T-resonant for (2) if g is C^1 in a neighbourhood of p and if the linearized problem (on T_pM)

$$\begin{cases} \dot{x} = g'(p)x, \\ x(0)=x(T), \end{cases}$$

which corresponds to $\lambda=0$, admits nontrivial solutions.

Notice that a point $p\in g^{-1}(0)$ is not T-resonant if $g'(p)$ (which maps T_pM into itself, see e.g. [10]) has no eigenvalues of the form $2\pi li/T$ with $l\in\mathbb{Z}$. Thus, in particular, p is an isolated zero of g.

In what follows, given a subset Y of M, by $\mathrm{Fr}(Y)$ we denote the boundary of Y in M.

Lemma 3.3. *Let $g:M\to\mathbb{R}^k$ be a C^1 tangent vector field. If $p\in g^{-1}(0)$ is not T-resonant, then for any sufficiently small neighbourhood V of p in $C_T(M)$*

there exists a real number $\delta_V > 0$ such that $[0, \delta_V] \times \mathrm{Fr}(V)$ does not contain any T-pair of (2).

Proof. Since the set X of the T-pairs of (2) is locally compact, there exists an open neighbourhood W of p in $C_T(M)$ and a number $\mu > 0$ such that $X \cap ([0, \mu] \times \overline{W})$ is compact. Let us prove first that if W is sufficiently small, then $\{0\} \times \overline{W}$ does not contain any T-pair of (2) different from $(0, p)$.

Assume, by contradiction, that there exists a sequence of nontrivial T-pairs of the form $\{(0, x_n)\}_{n \in \mathbb{N}}$ converging to p (recall that, according to our convention, p can be seen as an element of $[0, \infty) \times C_T(M)$). By definition, we have $|x_n(t) - p| \to 0$ as $n \to \infty$ uniformly in t (recall that M is embedded in $\mathbb{R}^k$) and, in particular, $\lim_{n\to\infty} x_n(0) = p$. Put

$$p_n = x_n(0) \quad \text{and} \quad u_n = \frac{p_n - p}{|p_n - p|}.$$

Without loss of generality we can assume that $u_n \to u \in T_pM$. Denote by $P_t : M \to M$ the Poincaré t-translation operator associated with the equation $\dot{x} = g(x)$. It is well known, since g is C^1, that the map $P_t(\,\cdot\,)$ is differentiable. Define $\Phi : M \to \mathbb{R}^k$ by $\Phi(\xi) = \xi - P_T(\xi)$. Obviously Φ is differentiable and $\Phi(p_n) = \Phi(p) = 0$. Thus,

$$\Phi'(p)u = \lim_{n\to\infty} \frac{\Phi(p_n) - \Phi(p)}{|p_n - p|} = 0.$$

On the other hand $\Phi'(p) : T_pM \to \mathbb{R}^k$ operates as follows

$$\Phi'(p)v = v - P_T'(p)v, \quad \forall v \in T_pM.$$

It is well known that the map $\alpha : t \mapsto P_t'(p)u$ satisfies the Cauchy problem

$$\begin{cases} \dot{\alpha}(t) = g'(p)\alpha(t), \\ \alpha(0) = u. \end{cases}$$

Since p is not T-resonant, $\Phi'(p)u = \alpha(0) - \alpha(T) \neq 0$. A contradiction.

Let us now complete the proof. Take W satisfying the above properties and let $V \subset W$. Assume by contradiction that there are no numbers δ_V as in the assertion. Then, there exists a sequence $\{(\lambda_n, x_n)\}_{n \in \mathbb{N}} \subset ([0, \mu] \times \mathrm{Fr}(V))$ of T-pairs such that $\lambda_n \to 0$ and $x_n \to \overline{x}$ for $n \to \infty$. Thus $(0, \overline{x}) \in \{0\} \times \mathrm{Fr}(V) \subset \{0\} \times \overline{W}$ is a T-pair different from $(0, p)$, contradicting the choice of W. □

We are now in a position to prove a multiplicity result for first order equations.

Theorem 3.4. *Let $M \subset \mathbb{R}^k$ be a complete differentiable manifold and let $g : M \to \mathbb{R}^k$ be a continuous tangent vector field on M with $g^{-1}(0)$ compact. Assume that at least a zero p of g is not T-resonant and such that* $\deg(g, M) \neq$

$\mathrm{i}(p,g)$. *Suppose that the connected sets of T-periodic solutions of $\dot{x} = g(x)$ are bounded in $C_T(M)$. Then, given f as in* (2), *there exists $\lambda_f > 0$ such that, for any $\lambda \in [0, \lambda_f]$,* (2) *has at least two T-periodic solutions with different images.*

Proof. Put $Y = g^{-1}(0)\backslash\{p\}$ and define

$$\Omega_Y = ([0,\infty) \times C_T(M)) \setminus \{p\}, \quad \Omega_p = ([0,\infty) \times C_T(M)) \setminus Y.$$

Since p is not T-resonant, then $\deg(g, \Omega_p \cap M) = \deg(g, M\backslash Y) = \mathrm{i}(p,g) = \pm 1$. From the additivity property of the degree and the assumption $\deg(g,M) \neq \mathrm{i}(p,g)$, it follows $\deg(g, \Omega_Y \cap M) = \deg(g, M\backslash\{p\}) \neq 0$. Applying Theorem 3.2 to Ω_Y and Ω_p, we get two connected sets $\Gamma^Y \subset \Omega_Y$ and $\Gamma^p \subset \Omega_p$ of nontrivial T-pairs, whose closures $\overline{\Gamma^Y}$ and $\overline{\Gamma^p}$ in $[0,\infty) \times C_T(M)$ meet respectively Y and p, and are not contained in any compact subset of Ω_Y and Ω_p.

Let X be the set of the T-pairs of (2). Denote by $B(p,\varepsilon)$ the open ball in $C_T(M)$ centered at p with radius ε, and by $S(p,\varepsilon)$ its boundary. By Lemma 3.3, given ε sufficiently small, there exists $\lambda_\varepsilon > 0$ such that $X \cap ([0,\lambda_\varepsilon] \times S(p,\varepsilon)) = \emptyset$. By Lemma 3.1, $\overline{\Gamma^p}$ is not contained in $[0,\lambda_\varepsilon] \times \overline{B(p,\varepsilon)}$. Thus, $\overline{\Gamma^p}$ being connected, one has $\overline{\Gamma^p} \cap (\{\lambda\} \times B(p,\varepsilon)) \neq \emptyset$ for all $\lambda \in [0,\lambda_\varepsilon]$.

Let G^Y be the connected component of the set of T-pairs which contains $\overline{\Gamma^Y}$. Since G^Y is a connected component and $\overline{\Gamma^p}$ is connected, either $\overline{\Gamma^p} \subset G^Y$ or $\overline{\Gamma^p} \cap G^Y = \emptyset$.

Consider first the case when $\overline{\Gamma^p} \subset G^Y$. By the connectedness of G^Y and by the fact that $G^Y \cap Y \neq \emptyset$, it follows that for any $\lambda \in [0,\lambda_\varepsilon]$ there exists a T-pair in

$$G^Y \cap (\{\lambda\} \times (C_T(M)\backslash B(p,\varepsilon))).$$

Thus, in this case, for any $\lambda \in [0,\lambda_\varepsilon]$ we get at least two T-periodic solutions, one in $B(p,\varepsilon)$ and one in $C_T(M)\backslash B(p,\varepsilon)$. In particular these solutions have different images.

Assume now $\overline{\Gamma^p} \cap G^Y = \emptyset$. In this case G^Y is contained in Ω_Y and unbounded. In fact, if G^Y was bounded, then, by Lemma 3.1 it would be a compact subset of Ω_Y containing $\overline{\Gamma^Y}$, which is impossible. Since G^Y is unbounded, it cannot be contained in $\{0\} \times C_T(M)$ by assumption. Hence the projection $\pi_1(G^Y)$ on $[0,\infty)$ of G^Y is a nontrivial interval, that is, there exists $\delta_Y > 0$ such that $[0,\delta_Y] \subset \pi_1(G^Y)$. Choose $\lambda_f = \min\{\delta_Y, \lambda_\varepsilon\}$. The connectedness of G^Y implies that for any $\lambda \in [0,\lambda_f]$ there exists at least a T-pair in $\{\lambda\} \times (C_T(M)\backslash B(p,\varepsilon))$. Otherwise it is easy to see that G^Y would be disconnected by a closed set of the form $\{\lambda\} \times (C_T(M)\backslash B(p,\varepsilon))$. Thus, as before, for any $\lambda \in [0,\lambda_f]$ there is a solution in $B(p,\varepsilon)$ and one in $C_T(M)\backslash B(p,\varepsilon)$. $\square$

The following example shows that in the above theorem the non-resonance assumption at p cannot be merely replaced by the hypothesis that p is an isolated zero of g with nonzero index $\mathrm{i}(p,g) \neq \deg(g,M)$.

Example 3.1. On the plane $M = \{(q_1, q_2, q_3) \in \mathbb{R}^3 : q_3 = 1\} \cong \mathbb{R}^2$ consider the (tangent) vector fields $g(q) = (q_2, -q_1, 0)$ and $f(t, q) = (0, \sin t, 0)$. Following a classic procedure due to Poincaré, we associate to them the tangent vector fields on the unit sphere $\widetilde{g} : S^2 \to \mathbb{R}^3$ and $\widetilde{f} : \mathbb{R} \times S^2 \to \mathbb{R}^3$ as follows. Let $S^2_+ = \{(q_1, q_2, q_3) \in S^2 : q_3 > 0\}$ and $S^2_- = \{(q_1, q_2, q_3) \in S^2 : q_3 < 0\}$ be the "upper" and the "lower" hemispheres of S^2. Define $h_+ : S^2_+ \to M$ as the diffeomorphism which associates to any $q \in S^2_+$ the intersection of M with the straight line through the origin and q. The map $h_- : S^2_- \to M$ is defined analogously. We define $\widetilde{g}$ and $\widetilde{f}$ on S^2_+ (S^2_-) as the tangent vector fields which correspond to g and f under h_+ (h_-). Straightforward computations show that

$$\widetilde{g}(q) = (q_2, -q_1, 0), \qquad \widetilde{f}(t, q) = q_3 \sin t(-q_1 q_2, 1 - q_2^2, -q_2 q_3).$$

Thus $\widetilde{g}$ and $\widetilde{f}$ can be smoothly extended to the whole sphere. Obviously $\widetilde{g}$ has only two zeros, the North and the South poles, both with index one.

The following facts can be easily verified.

1. If $\lambda > 0$, any solution of

$$\dot{z} = \widetilde{g}(z) + \lambda \widetilde{f}(t, z) \tag{3}$$

   starting either from S^2_+ or S^2_- corresponds to a solution of $\dot{x} = g(x) + \lambda f(t, x)$ on M. Thus, it cannot be periodic.
2. Any solution of (3) starting from a point of the equator is 2π-periodic.
3. The image of any solution on the equator is the equator itself.

Thus (3) admits "essentially" only one 2π-periodic solution for any $\lambda > 0$.

4. The spherical pendulum

We deal with the problem of the motion of a particle of mass m constrained on a sphere and subjected to an active force consisting of three parts: a vector field depending only on the position (e.g. the gravity), a possible friction, and a T-periodic forcing term. Actually we deal with the second order differential equation on a sphere $S = \{q \in \mathbb{R}^3 : |q| = r\}$,

$$m\ddot{x} = -\frac{m|\dot{x}|^2}{r^2} x - \eta \dot{x} + h(x) + \lambda \varphi(t, x, \dot{x}), \tag{4}$$

where, as throughout the rest of this section, $h : S \to \mathbb{R}^3$ is a C^1 tangent vector field on the compact 2-dimensional smooth manifold S, η is a non-negative real number and $\varphi : \mathbb{R} \times TS \to \mathbb{R}^3$ is a continuous *T-periodic active force on S*; that is, a continuous map such that $\varphi(t; q, v) = \varphi(t + T; q, v) \in T_q S$ for any $(t; q, v) \in \mathbb{R} \times TS$. Here TS denotes the *tangent bundle* to the sphere S; i.e. the set

$$TS = \{(q, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : q \in S,\ \langle q, v \rangle = 0\}.$$

It is convenient to regard S as the *null section* of the tangent bundle, i.e. as the set $S\times\{0\}\subset\mathbb{R}^3\times\mathbb{R}^3$. In this way, given an open subset U of TS, the expression $U\cap S$ will denote the open subset of S given by $\{q\in S : (q,0)\in U\}$. The term $-m(|\dot x(t)|/r)^2x(t)$ on the right hand side of (4) is the reactive force due to the constraint S.

An important special case of equation (4) is when h is the tangential component of the gravitational force $(0,0,-mg)$, that is

$$h_g(q)=\frac{mg}{r^2}(q_3q_1,q_3q_2,-(r^2-q_3^2)),$$

$q=(q_1,q_2,q_3)$. In what follows, when $h=h_g$, the equation (4) and its equivalent form (9) below is called the *forced gravitational pendulum equation.*

In order to apply Theorem 3.4 we will show that in $C^1_T(S)$ there are no unbounded connected sets of solutions of $m\ddot x=-(m|\dot x|^2/r^2)x+h(x)-\eta\dot x$. The proof, for $\eta=0$, is based on [3]. We point out that if h is conservative (e.g. when $h=h_g$) the case $\eta>0$ is trivial. In this situation, in fact, the only possible T-periodic solutions are constant.

Lemma 4.1. *Let $h:S\to\mathbb{R}^3$ be a continuous tangent vector field on S. Any connected set of solutions in $C^1_T(S)$ of the second order equation*

$$m\ddot x=-\frac{m|\dot x|^2}{r^2}x+h(x)-\eta\dot x, \tag{5}$$

where $\eta\ge 0$, is bounded. Moreover, if $\eta>0$, then the set of solutions in $C^1_T(S)$ of (5) *is a priori bounded.*

Proof. Throughout this proof we let $H=\max\{|h(q)| : q\in S\}$.

Let us consider first the case $\eta>0$. If $x:\mathbb{R}\to S$ is a T-periodic solution of (5), then take $u(t)=|\dot x(t)|^2$ and let t_0 be a point in $[0,T]$ such that $u(t_0)=\max_{t\in[0,T]}u(t)$. Since u is differentiable, one gets

$$0=\dot u(t_0)=2\langle\dot x(t_0),\ddot x(t_0)\rangle=\frac{2}{m}\langle\dot x(t_0),h(x(t_0))-\eta\dot x(t_0)\rangle.$$

Hence,

$$\eta|\dot x(t_0)|^2=\langle\dot x(t_0),h(x(t_0))\rangle\le|\dot x(t_0)|H$$

In other words, if $x:\mathbb{R}\to S$ is a T-periodic solution of (5), then $\|x\|_1\le r+H/\eta$. Thus the set of T-periodic solutions of (5) is bounded.

Assume now $\eta=0$. In order to deal with this more delicate case, we need the notion of admissible curve (on a sphere) and of index of an admissible curve (see [3]). Let $y\in C^1_T(S)$ be a curve such that $\dot y(t)\ne0$ for any $t\in\mathbb{R}$. Fix $\tau\in\mathbb{R}$ and let α_τ be the straight line through the origin spanned and oriented by the vector product $y(\tau)\times\dot y(\tau)$. If, for any $\tau,t\in\mathbb{R}$, the distance $\rho_\tau(t)$ between $y(t)$ and α_τ is positive, then we say that y is *admissible.* It is readily verified that the set of admissible curves is an open subset of $C^1_T(S)$. To any admissible curve it is associated, in a continuous way, an index $\mathrm{ind}(y)$ which, roughly speaking, counts the number of turns the curve makes around any α_τ axis associated with y. One can show that the admissibility of y ensures that this number is independent of $\tau \in \mathbb{R}$.

Let now $x : \mathbb{R} \to S$ be a T-periodic solution of (5) such that $\|x\|_1 \geq 2HT/m + r$. In [3] it is proved that x is admissible and that, for any $\tau \in \mathbb{R}$,

$$\mathrm{ind}(x) \geq \frac{mT|\dot{x}(\tau)| - HT^2}{2\pi mr}.$$

Thus, since S is bounded, there exist $a, b > 0$ such that

$$\mathrm{ind}(x) \geq a\|x\|_1 - b. \tag{6}$$

Let Σ be a connected set of solutions for (5), which, without loss of generality, we may assume to be closed. By the Tietze extension theorem there exists a continuous function $\omega : \Sigma \to \mathbb{R}$ such that $\omega(x) = \mathrm{ind}(x)$ if $\|x\|_1 \geq 2HT/m + r$. The inequality (6) shows that the image of ω is unbounded. Moreover, since Σ is connected, this image must actually be an unbounded interval. This is impossible, because ω takes integer values outside the set

$$\Sigma_1 = \{x \in \Sigma : \|x\|_1 \leq 2HT/m + r\},$$

which, by Ascoli's theorem, is compact. □

Since Theorem 3.4 is stated for first order differential equations, it is convenient to write (4) as an equivalent first order equation on TS (the phase space). Notice that, while the configuration space S is compact, this is not the case for TS.

Given h and φ as in (4), we associate to them the tangent (to TS) vector fields $\widehat{h} : TS \to \mathbb{R}^3 \times \mathbb{R}^3$ and $\widehat{\varphi} : \mathbb{R} \times TS \to \mathbb{R}^3 \times \mathbb{R}^3$, given by

$$\widehat{h}(q, v) = \left(v, -\frac{|v|^2}{r^2} q + \frac{h(q)}{m}\right), \tag{7}$$

$$\widehat{\varphi}(t, q, v) = \left(0, \frac{1}{m}\varphi(t, q, v)\right). \tag{8}$$

The equation (4) is equivalent to the following first order differential equation on TS:

$$\dot{\xi} = \widehat{h}(\xi) - \eta\kappa(\xi) + \lambda\widehat{\varphi}(t; \xi), \quad \xi = (q, v), \tag{9}$$

where $\xi = (q, v)$ and $\kappa(q, v) = (0, v/m)$. Lemma 4.1 implies that any connected set of solutions in $C_T(TS)$ of the first order equation $\dot{\xi} = \widehat{h}(\xi) - \eta\kappa(\xi)$ is bounded.

We say that a point $p \in h^{-1}(0)$ is T-resonant for the second order equation (4) if the (linearized) equation $m\ddot{x} = h'(p)x - \eta\dot{x}$ on T_pS admits nontrivial T-periodic solutions. Obviously p is T-resonant if and only if so is $(p, 0)$ for (9). Standard computations show that p is not T-resonant for (4) if and only if $-m(2l\pi/T)^2 + \eta 2l\pi i/T \notin \sigma(h'(p))$ for any $l \in \mathbb{Z}$.

Theorem 4.2. *Let $h : S \to \mathbb{R}^3$ be a C^1 tangent vector field on S, with a not T-resonant zero for* (4). *Then, given a continuous T-periodic active force $\varphi : \mathbb{R} \times TS \to \mathbb{R}^3$ on the sphere S, there exists $\lambda_\varphi > 0$ such that, for $\lambda \in [0, \lambda_\varphi]$,* (4) *has at least two T-periodic solutions with different images.*

Proof. Let p be a zero of h which is not T-resonant. Since $0 \notin \sigma(h'(p))$, p is a nondegenerate zero of h, which implies $\mathrm{i}(h,p) = \pm 1$. Moreover, by the Poincaré–Hopf Theorem, $2 = \chi(S) = \deg(h,S)$.

We need to relate the degree of h with that of $\widehat{h} - \eta\kappa$. Define $G : TS \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}^3$ by $G(\xi,\mu) = \widehat{h}(\xi) - \mu\eta\kappa(\xi)$. Observe that G is an admissible homotopy between $\widehat{h}$ and $\widehat{h} - \eta\kappa$ in any open neighbourhood V of p such that $V \cap h^{-1}(0) = \{p\}$. Hence, by the homotopy property of the degree,

$$\mathrm{i}(\widehat{h},p) = \mathrm{i}(\widehat{h} - \eta\kappa, p).$$

The same homotopy shows that $\deg(\widehat{h},S) = \deg(\widehat{h} - \eta\kappa, S)$. From Lemma 1.3 of [4] it follows that $\mathrm{i}(\widehat{h},p) = -\mathrm{i}(h,p)$, and $\deg(\widehat{h},S) = -\deg(h,S)$. Thus, finally,

$$\mathrm{i}(\widehat{h} - \eta\kappa, p) = -\mathrm{i}(h,p) = \pm 1 \neq -2 = -\deg(h,S) = \deg(\widehat{h} - \eta\kappa, S).$$

Lemma 4.1 ensures that the connected sets of T-periodic solutions of (9) are bounded. Thus, by Theorem 3.4, for λ small enough (9) has at least two T-periodic solutions with different images. Thus the same assertion holds true for the equivalent equation (4). □

In the case when $h = h_g$, Theorem 4.2, has the following physical meaning.

The T-periodically perturbed gravitational pendulum for small perturbations admits at least two essentially different T-periodic solutions.

More precisely, we have the following

Corollary 4.3. *Given a T-periodic active force $\varphi : \mathbb{R} \times TS \to \mathbb{R}^3$ on the sphere S, there exists $\lambda_\varphi > 0$ such that, for any $\lambda \in [0, \lambda_\varphi]$, the perturbed gravitational pendulum equation*

$$m\ddot{x} = -\frac{m|\dot{x}|^2}{r^2}\,x + h_g(x) - \eta\dot{x} + \lambda\varphi(t,x,\dot{x}),$$

where $\eta \geq 0$, has (*at least*) *two T-periodic solutions with different images.*

Proof. The linearized equation in the north pole $\mathrm{N} = (0,0,1)$ is

$$\ddot{x} = (mg/r)x - \eta\dot{x}, \quad x \in T_{\mathrm{N}}S \cong \mathbb{R}^2.$$

Thus N is not T-resonant for any $T > 0$. □

We point out that when the south pole $\mathrm{S} = (0,0,1)$ is T-resonant for the gravitational pendulum equation, the multiplicity result in Corollary 4.3 cannot be deduced via the implicit function theorem.

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