Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 11, 1998, 1–18

MULTIPLICITY RESULTS OF AN ELLIPTIC EQUATION WITH NON–HOMOGENEOUS BOUNDARY CONDITIONS

A. M. Candela — A. Salvatore

1. Introduction

In this paper we study the nonlinear elliptic problem

$$\begin{cases} \Delta u \ + \ |u|^{p-2} \ u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open smooth bounded subset of $\mathbb{R}^N$, $N \geq 2$, $g : \partial\Omega \to \mathbb{R}$ is a given continuous function and $p > 2$ is fixed.

If $g \equiv 0$, it is well known that (1.1) has infinitely many distinct solutions for $2 < p < 2N/(N-2)$ if $N \geq 3$ or $p > 2$ if $N = 2$. Such results have been proved by using variational methods also for more general odd nonlinearities at the beginning of 70's (see e.g. [2], [3], [6], [9], [11] and references therein). In all these papers a fundamental role is played by the fact that the energy functional is even in a Banach space, hence it is possible to use a modified version of the classical Ljusternik–Schnirelman theory and the properties of the genus for symmetric sets.

On the contrary, if $g \not\equiv 0$ the more general boundary value problem (1.1) loses its symmetry and the previous recalled arguments do not hold. In fact, it is well

1991 *Mathematics Subject Classification.* 35J20, 35J65.

Key words and phrases. Non-homogeneous boundary data, non-symmetric variational principle, perturbative methods.

Supported by M.U.R.S.T. (research funds 40 % and 60%) and E.E.C., Program Human Capital Mobility (Contract ERBCHRXCT 940494).

©1998 Juliusz Schauder Center for Nonlinear Studies

known that the solutions of (1.1) are critical points of the energy functional

$$I^*(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p\,dx \tag{1.2}$$

in $E = \{u \in H^1(\Omega) : u = g \text{ on } \partial\Omega\}$, and I^* is not invariant under a group of symmetries in such a set.

However, we prove that it is possible to apply the perturbation results developed by Bahri and Berestycki (cf. [4]), Rabinowitz (cf. [10]), Struwe (cf. [13]); then, for $p > 2$ but not too larger, the existence of infinitely many solutions of (1.1) with higher and higher energy will be stated.

Indeed the following theorem holds:

THEOREM 1.1. *If g is continuous in $\partial\Omega$ and*

$$2 < p < 2(N+1)/N, \tag{1.3}$$

then the elliptic problem (1.1) *has infinitely many classical solutions* $(u_n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} I^*(u_n) = +\infty.$$

The idea of using perturbation methods for solving nonlinear boundary value problems was introduced by Struwe (cf. [12]), while Ekeland, Ghoussoub and Teherani use perturbative methods in order to prove the existence of infinitely many solutions of a second order Hamiltonian system joining two given points (cf. [8]; see also [5] for a generalization).

REMARK 1.2. The result in Theorem 1.1 holds for all $N \geq 2$ under the hypothesis (1.3) which arises from the pertubative methods used in the proof. In any case such condition seems a natural extension of the hypothesis $2 < p < 4$ which is introduced in [8] for solving the problem corresponding to (1.1) in the particular case $N = 1$.

2. Variational setting and perturbed functional

Let

$$\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}, \quad |u|_p = \left(\int_\Omega |u|^p\,dx\right)^{1/p} \quad \text{and} \quad |u|_\infty = \sup_{x\in\Omega}|u(x)|$$

be the standard norms of $H^1_0(\Omega)$, respectively $L^p(\Omega)$, $C(\Omega)$.

Since our aim is to give a suitable variational approach to the problem (1.1), we state the following results.

PROPOSITION 2.1. *For any continuous function $g : \partial\Omega \to \mathbb{R}$ there exists a unique function $\phi : \overline{\Omega} \to \mathbb{R}$ such that $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$ and*

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega, \\ \phi = g & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

PROOF. By a generalization of Weierstrass Theorem (see [13, Theorem 1.2]) the functional

$$L(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx, \quad u \in E,$$

attains its infimum at a point $\phi \in E$. Classical results imply that ϕ is the only smooth solution of (2.1). □

From now on, fixed $g \in C(\partial\Omega)$, let ϕ be a smooth solution of the corresponding problem (2.1). It is easy to see that the following lemma holds.

LEMMA 2.2. *The following items are equivalent:*

(i) *the function $u = u(x)$ is a classical solution of the problem* (1.1),
(ii) *the function $v = v(x)$ is a classical solution of the following Dirichlet problem:*

$$\begin{cases} \Delta v + |v + \phi|^{p-2}(v + \phi) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where it is $u(x) = v(x) + \phi(x)$ for all $x \in \overline{\Omega}$.

By Lemma 2.2 we are interested in classical solutions of the Dirichlet problem (2.2), then, by standard regularity arguments, it is enough to prove the existence of infinitely many critical points of the functional

$$I(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{1}{p} \int_\Omega |v + \phi|^p \, dx \tag{2.3}$$

in the Sobolev space $H_0^1(\Omega)$.

Since the functional (2.3) is not even, the classical Symmetric Mountain Pass Theorem, or some of its generalizations, can not apply (see e.g. [11]). Hence, arguing as in [8] or [10], it is necessary to introduce a modified functional whose critical levels are related to those ones of (2.3).

Let $\Phi \in C^\infty(\mathbb{R}, [0, 1])$ be a decreasing cut–function such that

$$\Phi(s) = \begin{cases} 1 & \text{if } s \le 1, \\ 0 & \text{if } s \ge 2, \end{cases} \tag{2.4}$$

and $\Phi'(s) \in]-2, 0[$ for all $s \in]1, 2[$. Let $A = A(p, \phi) > 0$ be a suitable constant (for more details, see Remark 2.11). Define the modified functional

$$J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{1}{p} \int_\Omega |v|^p \, dx - \frac{\Psi(v)}{p} \int_\Omega (|v + \phi|^p - |v|^p) \, dx, \tag{2.5}$$

$v \in H_0^1(\Omega)$, where it is

$$\Psi(v) = \Phi(\mathcal{H}(v)), \qquad \mathcal{H}(v) = \frac{1}{Q(v)} \int_\Omega |v+\phi|^p \, dx, \qquad Q(v) = 2pA(I^2(v)+1)^{1/2}. \tag{2.6}$$

It can be easily proved that J is a C^1-functional in the Sobolev space $H_0^1(\Omega)$.

Remark 2.3. By definition (2.4) it follows that

$$v \in H_0^1(\Omega),\ \mathcal{H}(v) < 1 \quad \Rightarrow \quad J(v) = I(v),\ J'(v) = I'(v).$$

Before proving the propositions which justify the introduction of (2.5), let us give the following technical lemma.

Lemma 2.4. *Let a, $b \geq 0$ be fixed and consider $q > 1$. Then for every $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that $\beta(\varepsilon) \to +\infty$ if $\varepsilon \to 0$ and*

$$a^{q-1} b \leq \varepsilon a^q + \beta(\varepsilon) b^q. \tag{2.7}$$

Proof. Let $\varepsilon > 0$ be fixed. By the well known Young inequality there results

$$a^{q-1} b = (\varepsilon\mu)^{1/\mu} a^{q-1} \frac{b}{(\varepsilon\mu)^{1/\mu}} \leq \varepsilon a^{\mu(q-1)} + \frac{1}{\nu}\left(\frac{b}{(\varepsilon\mu)^{1/\mu}}\right)^\nu,$$

for any μ, $\nu > 1$ such that $1/\mu + 1/\nu = 1$. Then by choosing $\mu = q/(q-1)$ it is $\nu = q$ and (2.7) follows if we assume

$$\beta(\varepsilon) = \frac{1}{q}\left(\frac{q-1}{\varepsilon q}\right)^{q-1}. \tag{2.8}$$

□

Proposition 2.5. *If the constant A is large enough, then every critical point of I is also a critical point of the modified functional J.*

Proof. By Remark 2.3 we have just to prove that, for A large enough, if v is a critical point of I in $H_0^1(\Omega)$ then $\mathcal{H}(v) \leq 1/2$, that is

$$\int_\Omega |v+\phi|^p dx \leq pA(I^2(v)+1)^{1/2}.$$

Let $v \in H_0^1(\Omega)$ be such that $I'(v) = 0$. Taken any $\varepsilon > 0$, by Lemma 2.4 there results

$$\begin{aligned} I(v) &= I(v) - \frac{1}{2}\, I'(v)[v] \\ &= \frac{p-2}{2p} \int_\Omega |v+\phi|^p \, dx - \frac{1}{2} \int_\Omega |v+\phi|^{p-2}(v+\phi)\phi \, dx \\ &\geq \frac{p-2}{2p} \int_\Omega |v+\phi|^p \, dx - \frac{1}{2} \int_\Omega |v+\phi|^{p-1}|\phi| \, dx \\ &\geq \frac{p-2}{2p} \int_\Omega |v+\phi|^p \, dx - \frac{\varepsilon}{2} \int_\Omega |v+\phi|^p \, dx - \frac{1}{2}\beta(\varepsilon) \int_\Omega |\phi|^p \, dx. \end{aligned}$$

Choosing $\overline{\varepsilon} = (p-2)/2p$, then the inequality

$$I(v) \geq \frac{p-2}{4p} \int_\Omega |v+\phi|^p\,dx - \frac{1}{2}\,\beta(\overline{\varepsilon}) \int_\Omega |\phi|^p\,dx$$

implies

$$\int_\Omega |v+\phi|^p\,dx \leq \frac{4p}{p-2}\biggl(I(v) + \frac{1}{2}\beta(\overline{\varepsilon})|\phi|_p^p\biggr).$$

Setting

$$\gamma_0 = \max\biggl\{1, \frac{1}{2}\beta(\overline{\varepsilon})|\phi|_p^p\biggr\}, \tag{2.9}$$

it is

$$\int_\Omega |v+\phi|^p\,dx \leq \frac{4p}{p-2}\gamma_0(|I(v)|+1) \leq p\,\frac{4\sqrt{2}}{p-2}\gamma_0(I^2(v)+1)^{1/2}.$$

Hence, the condition

$$A \geq \frac{4\sqrt{2}}{p-2}\gamma_0 \tag{2.10}$$

concludes the proof. □

Remark 2.6. By (2.8) and (2.9) Proposition 2.5 holds if A verifies (2.10) with

$$\gamma_0 = \max\biggl\{1, \frac{1}{2p}\biggl(2\frac{p-1}{p-2}\biggr)^{p-1}|\phi|_p^p\biggr\}. \tag{2.11}$$

Proposition 2.7. *If the constant A is large enough, then there exists $M_0 > 0$ such that if $v \in H_0^1(\Omega)$ is a critical point of J and $J(v) \geq M_0$, then v is a critical point of I and $I(v) = J(v)$.*

For the proof of Proposition 2.7 the following lemmas need.

Lemma 2.8. *There exist two positive constants c_1 and c_2 such that*

$$\biggl|\int_\Omega (|v+\phi|^p - |v|^p)\,dx\biggr| \leq c_1|I(v)|^{(p-1)/p} + c_2 \quad \textit{for all } v \in \operatorname{supp}\Psi.$$

Proof. By Lagrange Theorem and some simple inequalities for any $v \in H_0^1(\Omega)$ there exists $\theta \in [0,1]$ such that

$$\begin{aligned}
\biggl|\int_\Omega (|v+\phi|^p - |v|^p)\,dx\biggr| &= p\biggl|\int_\Omega |v+\theta\phi|^{p-2}(v+\theta\phi)\phi\,dx\biggr| \\
&\leq p\int_\Omega |v+\theta\phi|^{p-1}|\phi|\,dx \\
&\leq 2^{p-2}p\int_\Omega |v+\phi|^{p-1}|\phi|\,dx + 2^{p-2}p\int_\Omega |\phi|^p\,dx \\
&\leq 2^{p-2}p|\phi|_p\biggl(\int_\Omega |v+\phi|^p\,dx\biggr)^{(p-1)/p} + 2^{p-2}p|\phi|_p^p.
\end{aligned}$$

If $v \in \mathrm{supp}\Psi$, that is $\mathcal{H}(v) \leq 2$, then

$$\int_\Omega |v+\phi|^p\,dx \leq 4pA(I^2(v)+1)^{1/2} \leq 4pA(|I(v)|+1); \tag{2.12}$$

moreover, it is

$$(|I(v)|+1)^{(p-1)/p} \leq 2^{(p-1)/p}(|I(v)|^{(p-1)/p}+1).$$

Hence the conclusion follows by the previous inequalities. □

Lemma 2.9. *There exist $M_1 > 0$ and $\rho > 0$ such that for any $M \geq M_1$ there results*

$$v \in \mathrm{supp}\Psi,\ J(v) \geq M \ \Rightarrow\ I(v) \geq \rho M.$$

Proof. Let $v \in \mathrm{supp}\Psi$. By

$$J(v) = I(v) + \frac{1}{p}(1-\Psi(v))\int_\Omega (|v+\phi|^p - |v|^p)\,dx \tag{2.13}$$

and $0 \leq \Psi(v) \leq 1$, Lemma 2.8 implies

$$\begin{aligned} I(v) &= J(v) - \frac{1}{p}(1-\Psi(v))\int_\Omega (|v+\phi|^p - |v|^p)\,dx \\ &\geq J(v) - \left|\int_\Omega (|v+\phi|^p - |v|^p)\,dx\right| \geq J(v) - c_1|I(v)|^{(p-1)/p} - c_2. \end{aligned}$$

Hence

$$I(v) + c_1|I(v)|^{(p-1)/p} \geq J(v) - c_2. \tag{2.14}$$

Moreover, as $c_2 > 0$ is fixed, there exists $M_1 > 0$ such that if $M \geq M_1$ and $J(v) \geq M$ it is

$$J(v) - c_2 \geq M/2. \tag{2.15}$$

Then (2.14) and (2.15) imply

$$I(v) + c_1|I(v)|^{(p-1)/p} \geq M/2. \tag{2.16}$$

If $I(v) \leq 0$ then (2.16) gives

$$c_1|I(v)|^{(p-1)/p} \geq M/2 + |I(v)|;$$

on the other hand, by Young inequality there results

$$c_1|I(v)|^{(p-1)/p} \leq \frac{c_1^p}{p} + \frac{p-1}{p}|I(v)|;$$

whence

$$c_1^p - pM/2 \geq |I(v)|. \tag{2.17}$$

Since we can choose M_1 so large that $c_1^p - pM_1/2 < 0$, then (2.17) is impossible and it is $I(v) > 0$, hence (2.16) becomes

$$I(v) + c_1 I^{(p-1)/p}(v) \geq M/2. \tag{2.18}$$

By (2.18) it follows that

$$I(v) \geq \frac{M}{2(1+c_1)} \quad \text{or} \quad I^{(p-1)/p}(v) \geq \frac{M}{2(1+c_1)};$$

so if $M_1 > 1$ there results

$$I(v) \geq \rho M, \quad \rho = \left(\frac{1}{2(1+c_1)}\right)^{p/p-1}. \qquad \square$$

In order to study the critical points of J we examine the expression of $J'(v)$. By definitions (2.5), (2.6) and simple calculations it follows

$$\begin{aligned} J'(v)[v] = & \int_\Omega |\nabla v|^2\,dx - \int_\Omega |v|^p\,dx \\ & + 4pA^2 \frac{\Phi'(\mathcal{H}(v))\mathcal{H}(v)I(v)}{Q^2(v)} I'(v)[v] \int_\Omega (|v+\phi|^p - |v|^p)\,dx \\ & - \frac{\Phi'(\mathcal{H}(v))}{Q(v)} \int_\Omega |v+\phi|^{p-2}(v+\phi)v\,dx \int_\Omega (|v+\phi|^p - |v|^p)\,dx \\ & - \Psi(v) \int_\Omega (|v+\phi|^{p-2}(v+\phi)v - |v|^p)dx. \end{aligned}$$

Assuming

$$\begin{aligned} T_1(v) &= 4pA^2 \frac{\Phi'(\mathcal{H}(v))\mathcal{H}(v)I(v)}{Q^2(v)} \int_\Omega (|v+\phi|^p - |v|^p)\,dx, \\ T_2(v) &= \frac{\Phi'(\mathcal{H}(v))}{Q(v)} \int_\Omega (|v+\phi|^p - |v|^p)\,dx + T_1(v), \end{aligned}$$

by

$$I'(v)[v] = \int_\Omega |\nabla v|^2\,dx - \int_\Omega |v+\phi|^{p-2}(v+\phi)v\,dx$$

there results

$$\begin{aligned} J'(v)[v] = & (1+T_1(v)) \int_\Omega |\nabla v|^2\,dx - (1-\Psi(v)) \int_\Omega |v|^p\,dx \\ & - (T_2(v) + \Psi(v)) \int_\Omega |v+\phi|^{p-2}(v+\phi)v\,dx. \end{aligned} \tag{2.19}$$

Lemma 2.10. *If $v \in H_0^1(\Omega)$ is such that $J(v) \geq M$ and $M \to +\infty$ then $T_1(v)$ and $T_2(v)$ go to 0, that is for every $\delta > 0$ there exists $M > 0$ large enough such that*

$$v \in H_0^1(\Omega),\ J(v) \geq M \ \Rightarrow\ |T_1(v)| \leq \delta,\ |T_2(v)| \leq \delta.$$

Proof. Let M_1 be as in Lemma 2.9. Let $v \in H_0^1(\Omega)$ be such that $J(v) \geq M$ and $M \geq M_1$. If $v \notin \operatorname{supp}\Psi$, the proof is trivial.

Let $v \in \operatorname{supp}\Psi$. Then $0 \leq \mathcal{H}(v) \leq 2$ and $|\Phi'(\mathcal{H}(v))| \leq 2$; moreover, by Lemma 2.9, it is $I(v) > 0$. Lemma 2.8 and (2.6) imply

$$\begin{aligned}|T_1(v)| &= 4pA^2 \frac{|\Phi'(\mathcal{H}(v))|\mathcal{H}(v) I(v)}{Q^2(v)} \left| \int_\Omega (|v+\phi|^p - |v|^p)\, dx \right| \\ &\leq 16pA^2 \frac{I(v)}{Q^2(v)} (c_1\ I^{(p-1)/p}(v) + c_2) = \frac{4I(v)}{p\ (I^2(v)+1)} (c_1\ I^{(p-1)/p}(v) + c_2).\end{aligned}$$

Hence

$$|T_1(v)| \leq 4(c_1\ I^{-1/p}(v) + c_2 I^{-1}(v)). \tag{2.20}$$

In a similar way there results

$$\frac{|\Phi'(\mathcal{H}(v))|}{Q(v)} \left| \int_\Omega (|v+\phi|^p - |v|^p)\, dx \right| \leq \frac{1}{pA} (c_1\ I^{-1/p}(v) + c_2 I^{-1}(v)). \tag{2.21}$$

Thus Lemma 2.9, (2.20) and (2.21) imply the proof. □

Proof of Proposition 2.7. Let $v \in H_0^1(\Omega)$ be such that $J'(v) = 0$. By (2.19) and simple calculations it is

$$\begin{aligned}I(v) &= I(v) - \frac{J'(v)[v]}{2(1+T_1(v))} \\ &= -\frac{1}{p} \int_\Omega |v+\phi|^p\, dx + \frac{1-\Psi(v)}{2(1+T_1(v))} \int_\Omega |v|^p\, dx \\ &\quad + \frac{T_2(v)+\Psi(v)}{2(1+T_1(v))} \int_\Omega |v+\phi|^{p-2}(v+\phi)v\, dx \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \frac{T_2(v)+\Psi(v)}{1+T_1(v)} \int_\Omega |v+\phi|^p\, dx - \frac{T_1(v)-T_2(v)}{2(1+T_1(v))} \int_\Omega |v|^p\, dx \\ &\quad - \frac{T_2(v)+\Psi(v)}{2(1+T_1(v))} \int_\Omega |v+\phi|^{p-2}(v+\phi)\phi\, dx \\ &\quad + \frac{1}{p}\left(\frac{T_2(v)+\Psi(v)}{1+T_1(v)} - 1\right) \int_\Omega (|v+\phi|^p - |v|^p)\, dx \\ &\quad - \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{T_2(v)+\Psi(v)}{1+T_1(v)} - 1\right) \int_\Omega |v|^p\, dx.\end{aligned}$$

By Lagrange Theorem there exists $\theta \in [0,1]$ such that

$$\begin{aligned}I(v) &= \left(\frac{1}{2} - \frac{1}{p}\right) \frac{T_2(v)+\Psi(v)}{1+T_1(v)} \int_\Omega |v+\phi|^p\, dx - \frac{T_1(v)-T_2(v)}{2(1+T_1(v))} \int_\Omega |v|^p\, dx \\ &\quad - \frac{T_2(v)+\Psi(v)}{2(1+T_1(v))} \int_\Omega |v+\phi|^{p-2}(v+\phi)\phi\, dx \\ &\quad + \left(\frac{T_2(v)+\Psi(v)}{1+T_1(v)} - 1\right) \int_\Omega |v+\theta\phi|^{p-2}(v+\theta\phi)\phi\, dx\end{aligned}$$

$$
\begin{aligned}
&-\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\frac{T_2(v)+\Psi(v)}{1+T_1(v)}-1\Big)\int_\Omega |v+\phi|^p\,dx\\
&+p\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\frac{T_2(v)+\Psi(v)}{1+T_1(v)}-1\Big)\int_\Omega |v+\theta\phi|^{p-2}(v+\theta\phi)\phi\,dx\\
=&\Big(\frac{1}{2}-\frac{1}{p}\Big)\int_\Omega |v+\phi|^p\,dx-\frac{T_1(v)-T_2(v)}{2(1+T_1(v))}\int_\Omega |v|^p\,dx\\
&-\frac{T_2(v)+\Psi(v)}{2(1+T_1(v))}\int_\Omega |v+\phi|^{p-2}(v+\phi)\phi\,dx\\
&+\frac{p}{2}\Big(\frac{T_2(v)+\Psi(v)}{1+T_1(v)}-1\Big)\int_\Omega |v+\theta\phi|^{p-2}(v+\theta\phi)\phi\,dx.
\end{aligned}
$$

By Lemma 2.10 there exists $M_2 \geq M_1$ such that if $J(v) \geq M_2$ then

$$
\left|\frac{T_2(v)+\Psi(v)}{1+T_1(v)}\right| \leq 2, \quad \left|\frac{T_2(v)+\Psi(v)}{1+T_1(v)}-1\right| \leq 2.
$$

Moreover, $p>2$ implies

$$
|v+\theta\phi|^{p-1} \leq 2^{p-2}(|v+\phi|^{p-1}+|\phi|^{p-1}), \quad |v|^p \leq 2^{p-1}(|v+\phi|^p+|\phi|^p).
$$

Therefore, by Lemma 2.4, for any $\varepsilon>0$ there results

$$
\begin{aligned}
I(v) \geq& \frac{p-2}{2p}\int_\Omega |v+\phi|^p\,dx-2^{p-1}\left|\frac{T_1(v)-T_2(v)}{2(1+T_1(v))}\right|\int_\Omega (|v+\phi|^p+|\phi|^p)\,dx\\
&-\int_\Omega |v+\phi|^{p-1}|\phi|\,dx-p2^{p-2}\int_\Omega (|v+\phi|^{p-1}+|\phi|^{p-1})|\phi|\,dx\\
=&\Big(\frac{p-2}{2p}-2^{p-2}\left|\frac{T_1(v)-T_2(v)}{1+T_1(v)}\right|\Big)\int_\Omega |v+\phi|^p\,dx\\
&-(1+p2^{p-2})\int_\Omega |v+\phi|^{p-1}|\phi|\,dx\\
&-2^{p-2}\Big(\left|\frac{T_1(v)-T_2(v)}{1+T_1(v)}\right|+p\Big)\int_\Omega |\phi|^p\,dx\\
\geq&\Big(\frac{p-2}{2p}-2^{p-2}\left|\frac{T_1(v)-T_2(v)}{1+T_1(v)}\right|\Big)\int_\Omega |v+\phi|^p\,dx\\
&-(1+p2^{p-2})\int_\Omega (\varepsilon|v+\phi|^p+\beta(\varepsilon)|\phi|^p)\,dx\\
&-2^{p-2}\Big(\left|\frac{T_1(v)-T_2(v)}{1+T_1(v)}\right|+p\Big)|\phi|_p^p\\
=&\Big(\frac{p-2}{2p}-2^{p-2}\left|\frac{T_1(v)-T_2(v)}{1+T_1(v)}\right|-\varepsilon(1+p2^{p-2})\Big)\int_\Omega |v+\phi|^p\,dx\\
&-\Big((1+p2^{p-2})\beta(\varepsilon)+2^{p-2}\Big(\left|\frac{T_1(v)-T_2(v)}{1+T_1(v)}\right|+p\Big)\Big)|\phi|_p^p.
\end{aligned}
$$

Let $\varepsilon^* = (p-2)/8p(1+p2^{p-2})$. By Lemma 2.10 there exists $M_3 \geq M_2$ such that if $J(v) \geq M_3$ then

$$2^{p-2}\left|\frac{T_1(v) - T_2(v)}{1 + T_1(v)}\right| \leq \frac{p-2}{8p},$$

hence

$$I(v) \geq \frac{p-2}{4p}\int_\Omega |v+\phi|^p\,dx - \left((1+p2^{p-2})\beta(\varepsilon^*) + \frac{p-2}{8p} + p2^{p-2}\right)|\phi|_p^p$$

which implies

$$\int_\Omega |v+\phi|^p\,dx \leq \frac{4p}{p-2}\left[I(v) + \left((1+p2^{p-2})\beta(\varepsilon^*) + \frac{p-2}{8p} + p2^{p-2}\right)|\phi|_p^p\right].$$

Whence, if

$$\gamma_1 = \max\left\{1, \left((1+p2^{p-2})\beta(\varepsilon^*) + \frac{p-2}{8p} + p2^{p-2}\right)|\phi|_p^p\right\}, \tag{2.22}$$

it is

$$\int_\Omega |v+\phi|^p\,dx \leq p\frac{4\sqrt{2}}{p-2}\gamma_1(I^2(v)+1)^{1/2}.$$

So, if A satisfies

$$A \geq \frac{4\sqrt{2}}{p-2}\gamma_1, \tag{2.23}$$

it follows that if v is such that $J'(v) = 0$ and $J(v) \geq M_3$, then $\mathcal{H}(v) \leq 1/2$ and therefore v is a critical point of I. $\square$

Remark 2.11. By (2.8) and (2.12) it is

$$\gamma_1 = \max\left\{1, \left(\frac{1}{p}\left(8\,\frac{p-1}{p-2}\right)^{p-1}(1+p2^{p-2})^p + \frac{p-2}{8p} + p2^{p-2}\right)|\phi|_p^p\right\}.$$

Since by (2.11) there results $\gamma_1 \geq \gamma_0$, it follows that (2.23) implies (2.10), then from now on the constant A introduced in (2.6) is choosen such to satisfy (2.23).

Remark 2.12. The choice of the homogeneous term $|v|^{p-2}v$ needs only in the proof of Proposition 2.7. On the contrary it can be proved that all the other results hold also for more general odd superlinear functions.

3. Proof of Theorem 1.1

In order to find infinitely many critical levels of the not-even functional J, for several times we will apply a non-symmetric variational principle which was introduced by Rabinowitz in [10]. For completeness here we recall this theorem in the version due to Struwe (see [13, Ch. II, Theorem 7.1]).

THEOREM 3.1. *Let H be a Hilbert space endowed with the norm $\|\cdot\|$. Suppose $J \in C^1(H)$ satisfies the Palais–Smale condition, that is any sequence $(v_n)_{n\in\mathbb{N}} \subset H$ such that $(J(v_n))_{n\in\mathbb{N}}$ is bounded and $J'(v_n) \to 0$ has a converging subsequence. Let $V \subset H$ be a finite-dimensional subspace of H and $v^* \in H\backslash V$; moreover, define*

$$V^* = V \oplus \operatorname{span}\{v^*\}, \quad V^*_+ = \{v + tv^* : v \in V,\ t \geq 0\}.$$

Suppose

1. $J(0) \leq 0$,
2. *there exists $R > 0$ such that for any $v \in V$, $\|v\| \geq R$ implies $J(v) \leq J(0)$,*
3. *there exists $R^* \geq R$ such that for any $v \in V^*$, $\|v\| \geq R^*$ implies $J(v) \leq J(0)$,*

and let

$$\Gamma = \{h \in C(H,H) : \ h \text{ is odd}, \ h(v) = v \text{ if } \ \max\{J(v), J(-v)\} \leq 0\}.$$

Then, if

$$\alpha^* = \inf_{h\in\Gamma} \sup_{v\in V^*_+} J(h(v)) > \alpha = \inf_{h\in\Gamma} \sup_{v\in V} J(h(v)) \geq 0,$$

the functional J possesses a critical value greater than α^*.

Since we have to prove that J satisfies the Palais–Smale condition (at least at high levels) and the geometrical hypotheses of Theorem 3.1 are satisfied, we state the following lemmas.

LEMMA 3.2. *There exist two positive constants $\overline{c}_1$ and $\overline{c}_2$ such that*

$$\left| \int_\Omega (|v+\phi|^p - |v|^p)\,dx \right| \leq \overline{c}_1 \ |J(v)|^{(p-1)/p} + \overline{c}_2 \quad \text{for all } v \in \operatorname{supp}\Psi.$$

PROOF. By (2.13) and simple inequalities it is

$$|I(v)|^{(p-1)/p} \leq 2^{(p-1)/p}\bigg(|J(v)|^{(p-1)/p} + \left| \int_\Omega (|v+\phi|^p - |v|^p)\,dx \right|^{(p-1)/p}\bigg),$$

hence by Lemma 2.8 it follows

$$\begin{aligned}\left| \int_\Omega (|v+\phi|^p - |v|^p)\,dx \right| \leq{}& 2^{(p-1)/p} c_1 |J(v)|^{(p-1)/p} \\ &+ \ 2^{(p-1)/p} c_1 \left| \int_\Omega (|v+\phi|^p - |v|^p)\,dx \right|^{(p-1)/p} + c_2.\end{aligned}$$

By a suitable version of Lemma 2.4 for $\varepsilon = 1/2$ there exists a constant $c_3 > 0$ such that

$$2^{(p-1)/p} c_1 \left| \int_\Omega (|v+\phi|^p - |v|^p)\,dx \right|^{(p-1)/p} \leq \frac{1}{2} \left| \int_\Omega (|v+\phi|^p - |v|^p)\,dx \right| + c_3,$$

then the conclusion holds. □

Lemma 3.3. *There exists $c^* > 0$ such that*

$$|J(v) - J(-v)| \le c^*(|J(v)|^{(p-1)/p} + 1) \quad \textit{for all } v \in \operatorname{supp}\Psi.$$

Proof. Taken $v \in \operatorname{supp}\Psi$, by Lagrange Theorem and some calculations it follows

$$\left| \int_\Omega (|v-\phi|^p - |v|^p)\, dx \right| \le 2^{p-2} p \int_\Omega |v+\phi|^{p-1}|\phi|\, dx + 2^{2p-3} p \int_\Omega |\phi|^p\, dx,$$

then, by (2.12) and working as in the proofs of Lemmas 2.8 and 3.2, there exist $\overline{c}_3$, $\overline{c}_4 > 0$ such that

$$\left| \int_\Omega (|v-\phi|^p - |v|^p)\, dx \right| \le \overline{c}_3\ |J(v)|^{(p-1)/p} + \overline{c}_4. \tag{3.1}$$

Hence the proof follows by the inequality

$$|J(v) - J(-v)| \le \frac{\Psi(v)}{p}\left| \int_\Omega (|v+\phi|^p - |v|^p)\, dx \right| + \frac{\Psi(-v)}{p}\left| \int_\Omega (|v-\phi|^p - |v|^p)\, dx \right|,$$

Lemma 3.2 and (3.1). □

Proposition 3.4. *There exists $\eta > 0$ such that J satisfies the Palais–Smale condition in $J^{-1}([\eta,\infty[)$.*

Proof. Let $M_1 > 0$ be as in Lemma 2.9 and take $\eta \ge M_1$. Let $(v_n)_{n\in\mathbb{N}}$ be such that

$$\eta \le J(v_n) \le k \ \text{ for every } n \in \mathbb{N}, \quad \lim_{n\to\infty} J'(v_n) = 0, \tag{3.2}$$

for some $k > \eta$. Let us assume that, up to subsequences, it is $v_n \in \operatorname{supp}\Psi$ for every $n \in \mathbb{N}$ (otherwise it is $J(v_n) = I^*(v_n)$, $J'(v_n) = (I^*)'(v_n)$, where I^* is defined in (1.2), and I^* satisfies the Palais–Smale condition). First of all let us remark that (3.2) implies that there exists $k_1 > 0$ such that

$$I(v_n) \le k_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

In fact, taken $n \in \mathbb{N}$ by (2.13) and Lemma 3.2 it follows

$$\begin{aligned} I(v_n) &\le J(v_n) + \left| \int_\Omega (|v_n+\phi|^p - |v_n|^p)\, dx \right| \\ &\le J(v_n) + \overline{c}_1 |J(v_n)|^{(p-1)/p} + \overline{c}_2 \le k + \overline{c}_1\ k^{(p-1)/p} + \overline{c}_2. \end{aligned}$$

On the other hand (3.2) and Lemma 2.9 imply

$$I(v_n) \ge \rho\eta \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

By (2.19) and some calculations it follows

$$pJ(v_n) - J'(v_n)[v_n] = \left(\frac{p}{2} - 1 - T_1(v_n)\right) \int_\Omega |\nabla v_n|^2 \, dx + T_2(v_n) \int_\Omega |v_n + \phi|^p \, dx - (T_2(v_n) + \Psi(v_n)) \int_\Omega |v_n + \phi|^{p-2}(v_n + \phi)\phi \, dx,$$

while by (2.3) it is

$$\int_\Omega |v_n + \phi|^p \, dx = \frac{p}{2} \int_\Omega |\nabla v_n|^2 \, dx - pI(v_n), \tag{3.5}$$

hence there results

$$\begin{aligned} pJ(v_n) &- J'(v_n)[v_n] \\ &= \left(\frac{p}{2} - 1 - T_1(v_n) + \frac{p}{2}\, T_2(v_n)\right) \int_\Omega |\nabla v_n|^2 \, dx \\ &\quad - (T_2(v_n) + \Psi(v_n)) \int_\Omega |v_n + \phi|^{p-2}(v_n + \phi)\phi \, dx - pT_2(v_n)I(v_n) \\ &\geq \left(\frac{p}{2} - 1 - T_1(v_n) + \frac{p}{2} T_2(v_n)\right) \int_\Omega |\nabla v_n|^2 \, dx \\ &\quad - |T_2(v_n) + \Psi(v_n)||\phi|_\infty \int_\Omega |v_n + \phi|^{p-1} \, dx - pT_2(v_n)I(v_n). \end{aligned}$$

By (3.4), (3.5) and simple calculations there exist some positive constants c_4, c_5 and c_6 such that

$$\int_\Omega |v_n + \phi|^{p-1} \, dx \leq c_4 \left(\int_\Omega |v_n + \phi|^p \, dx\right)^{(p-1)/p} \leq c_5 \left(\int_\Omega |\nabla v_n|^2 \, dx\right)^{(p-1)/p} + c_6;$$

whence

$$\begin{aligned} pJ(v_n) - J'(v_n)[v_n] \geq& \left(\frac{p}{2} - 1 - T_1(v_n) + \frac{p}{2}\, T_2(v_n)\right) \int_\Omega |\nabla v_n|^2 \, dx \\ &- c_5 |T_2(v_n) + \Psi(v_n)||\phi|_\infty \left(\int_\Omega |\nabla v_n|^2 \, dx\right)^{(p-1)/p} \\ &- c_6 |T_2(v_n) + \Psi(v_n)||\phi|_\infty - p|T_2(v_n)|I(v_n). \end{aligned}$$

By Lemma 2.10, choosen $\delta > 0$ such that $\delta < (p-2)/(p+2)$ and taken η large enough, for all $n \in \mathbb{N}$ there results

$$|T_1(v_n)| \leq \delta, \quad |T_2(v_n)| \leq \delta, \quad c_7 = p/2 - 1 - (1 + p/2)\delta > 0. \tag{3.6}$$

Moreover, by (3.3) it is $p|T_2(v_n)|I(v_n) \leq p\delta k_1$, then there exist c_8, $c_9 > 0$ such that

$$pJ(v_n) - J'(v_n)[v_n] \geq c_7 \|v_n\|^2 - c_8 \|v_n\|^{2(p-1)/p} - c_9. \tag{3.7}$$

Whence (3.2) and (3.7) imply that $(v_n)_{n\in\mathbb{N}}$ has to be bounded in $H_0^1(\Omega)$. Since by (1.3) it follows $p < 2N/(N-2)$ if $N \geq 3$, then a well known embedding theorem implies that there exists $\overline{v} \in H_0^1(\Omega)$ such that, up to a subsequence, $v_n \rightharpoonup \overline{v}$ weakly in $H_0^1(\Omega)$ and $v_n \to \overline{v}$ strongly in $L^p(\Omega)$. Moreover, by

$$\begin{aligned} J'(v_n) = & -(1+T_1(v_n))\Delta v_n - (1-\Psi(v_n))|v_n|^{p-2}v_n \\ & -(T_2(v_n)+\Psi(v_n))|v_n+\phi|^{p-2}(v_n+\phi), \end{aligned}$$

(3.2), (3.6) and standard compacteness arguments imply that $v_n \to \overline{v}$ strongly in $H_0^1(\Omega)$. □

Now, suitable finite-dimensional subspaces of $H_0^1(\Omega)$ have to be introduced. Let λ_k be the kth eigenvalue (counting multiplicities) of the linear operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$. Let $(\varphi_k)_{k\ge 1}$ denote an orthonormalized set of eigenfunctions of $H_0^1(\Omega)$ such that φ_k is associated with λ_k, that is $-\Delta\varphi_k = \lambda_k\varphi_k$.

Let us recall that by the formula of the asymptotic behaviour of λ_k it is

$$\lambda_k \sim Ck^{2/N} \quad \text{as } k \to \infty \tag{3.8}$$

(see [1] or [7]). For any $m \ge 1$, define

$$V_m = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}, \quad V_m^+ = \{v + t\varphi_{m+1} : v \in V_m,\ t \ge 0\}.$$

Let Γ be as in Theorem 3.1, that is

$$\Gamma = \{h \in C(H_0^1(\Omega), H_0^1(\Omega)) : \text{ h is odd },h(v) = v \text{ if } \max\{J(v), J(-v)\} \le 0\}.$$

Lemma 3.5. *For every $m \in \mathbb{N}$ there exists $R(m) > 0$ such that for any $v \in V_m$ with $\|v\| \ge R(m)$ it results $J(v) \le J(0) \le 0$.*

Proof. Let $v \in V_m$. By definition (2.5), Lagrange Theorem and some calculations it follows

$$J(v) \le \frac{1}{2}\|v\|^2 - \frac{1}{p}|v|_p^p + 2^{p-2}\int_\Omega |v|^{p-1}|\phi|\,dx + 2^{p-2}|\phi|_p^p,$$

then, applying Lemma 2.4, for any $\varepsilon > 0$ it is

$$J(v) \le \frac{1}{2}\|v\|^2 - \left(\frac{1}{p} - \varepsilon 2^{p-2}\right)|v|_p^p + 2^{p-2}(\beta(\varepsilon)+1)|\phi|_p^p.$$

By choosing $\widetilde{\varepsilon} = 1/p2^{p-1}$ and since in V_m there exists $d = d(m) > 0$ such that $|v|_p \ge d\|v\|$, there results

$$J(v) \le \frac{1}{2}\|v\|^2 - \frac{d^p}{2p}\|v\|^p + 2^{p-2}(\beta(\widetilde{\varepsilon})+1)|\phi|_p^p$$

which implies that $J(v) \to -\infty$ as $v \in V_m$, $\|v\| \to +\infty$. □

Remark 3.6. Since $V_m \subset V_{m+1}$ it is possible to choose $R(m) < R(m+1)$.

Arguing as in [14], the following result can be proved.

PROPOSITION 3.7. *Let $\overline{k}_1 > 0$ be given and assume*

$$K(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \overline{k}_1 \int_\Omega |v|^p\,dx, \quad v \in H^1_0(\Omega).$$

If $N \geq 3$ there exists $C_0 > 0$ such that

$$\inf_{h\in\Gamma} \sup_{v\in V_m} K(h(v)) \geq C_0\ m^{[p/(p-2)][2/N]},$$

while if $N = 2$ for every $\sigma > 0$ there exists $C_{0\sigma} > 0$ such that

$$\inf_{h\in\Gamma} \sup_{v\in V_m} K(h(v)) \geq C_{0\sigma} m^{p/(p-2)-\sigma}.$$

PROOF OF THEOREM 1.1. For any $m \in \mathbb{N}$, $m \geq 1$, define

$$\alpha_m = \inf_{h\in\Gamma} \sup_{v\in V_m} J(h(v)), \quad \alpha_m^+ = \inf_{h\in\Gamma} \sup_{v\in V_m^+} J(h(v)). \tag{3.9}$$

Since Proposition 3.4, Lemma 3.5 and Remark 3.6 hold, in order to apply Theorem 3.1 we have just to prove that for some m it is

$$\alpha_m^+ > \alpha_m \geq \eta, \tag{3.10}$$

where η is as in Proposition 3.4. Let us remark that by Lagrange Theorem and Lemma 2.4 there exist two positive constants $\overline{k}_1$ and $\overline{k}_2$ such that

$$J(v) \geq \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \overline{k}_1 \int_\Omega |v|^p\,dx - \overline{k}_2, \quad v \in H^1_0(\Omega). \tag{3.11}$$

Then (3.11) and Proposition 3.7 imply that, if $N \geq 3$, there exist $\overline{C}_0 > 0$ and $m_0 \in \mathbb{N}$ such that

$$\alpha_m \geq \overline{C}_0\ m^{[p/(p-2)][2/N]} \geq \eta \quad \text{for all } m \geq m_0, \tag{3.12}$$

while if $N = 2$ for every $\sigma > 0$ there exists $\overline{C}_{0\sigma} > 0$ and $m_{0\sigma} \in \mathbb{N}$ such that

$$\alpha_m \geq \overline{C}_{0\sigma} m^{p/(p-2)-\sigma} \geq \eta \quad \text{for all } m \geq m_{0\sigma}. \tag{3.13}$$

Now, let $N \geq 3$ and fix $m \geq m_0$. Obviously $V_m \subset V_m^+$ implies $\alpha_m^+ \geq \alpha_m$, then (3.10) holds if $\alpha_m^+ \neq \alpha_m$. Assume

$$\alpha_m^+ = \alpha_m. \tag{3.14}$$

By (3.9) and (3.14) for any $\varepsilon > 0$ there exists $h_\varepsilon \in \Gamma$ such that

$$\sup_{v\in V_m^+} J(h_\varepsilon(v)) < \alpha_m + \varepsilon. \tag{3.15}$$

By Lemma 3.5, the Weierstrass Theorem implies that there exists $v^\varepsilon_{m+1} \in V_{m+1}$ such that $\|v^\varepsilon_{m+1}\| \leq R(m+1)$ and

$$J(h_\varepsilon(v^\varepsilon_{m+1})) = \sup_{v\in V_{m+1}} J(h_\varepsilon(v)).$$

Hence there results

$$\alpha_{m+1} \leq J(h_\varepsilon(v^\varepsilon_{m+1})), \tag{3.16}$$

moreover, $V_{m+1} = V^+_m \cup (-V^+_m)$ implies $v^\varepsilon_{m+1} \in V^+_m$ or $-v^\varepsilon_{m+1} \in V^+_m$.

If $v^\varepsilon_{m+1} \in V^+_m$ by (3.15) and (3.16) it follows

$$\alpha_{m+1} < \alpha_m + \varepsilon. \tag{3.17}$$

On the contrary, if $-v^\varepsilon_{m+1} \in V^+_m$, (3.15) implies

$$\begin{aligned} J(h_\varepsilon(v^\varepsilon_{m+1})) &\leq \sup_{v\in V^+_m} J(h_\varepsilon(v)) + |J(h_\varepsilon(v^\varepsilon_{m+1})) - J(h_\varepsilon(-v^\varepsilon_{m+1}))| \\ &< \alpha_m + \varepsilon + |J(h_\varepsilon(v^\varepsilon_{m+1})) - J(h_\varepsilon(-v^\varepsilon_{m+1}))|. \end{aligned} \tag{3.18}$$

We claim that there exist $m_1 \geq m_0$ and two positive constants c^*_1 and c^*_2 such that for all $m \geq m_1$ and $\varepsilon > 0$ there results

$$|J(h_\varepsilon(v^\varepsilon_{m+1})) - J(h_\varepsilon(-v^\varepsilon_{m+1}))| \leq c^*_1 |J(h_\varepsilon(-v^\varepsilon_{m+1}))|^{(p-1)/p} + c^*_2. \tag{3.19}$$

The proof of (3.19) is trivial if both $h_\varepsilon(v^\varepsilon_{m+1})$ and $h_\varepsilon(-v^\varepsilon_{m+1})$ are not in $\operatorname{supp}\Psi$. On the contrary it follows by $h_\varepsilon(v^\varepsilon_{m+1}) = -h_\varepsilon(-v^\varepsilon_{m+1})$ and Lemma 3.3 if $h_\varepsilon(-v^\varepsilon_{m+1}) \in \operatorname{supp}\Psi$. Let

$$h_\varepsilon(-v^\varepsilon_{m+1}) \notin \operatorname{supp}\Psi \quad \text{and} \quad h_\varepsilon(v^\varepsilon_{m+1}) \in \operatorname{supp}\Psi. \tag{3.20}$$

By Lemma 3.3 it follows

$$|J(h_\varepsilon(v^\varepsilon_{m+1})) - J(h_\varepsilon(-v^\varepsilon_{m+1}))| \leq c^*(|J(h_\varepsilon(v^\varepsilon_{m+1}))|^{(p-1)/p} + 1). \tag{3.21}$$

Obviously (3.21) implies (3.19) if we prove that there exist $m_1 \geq m_0$ and $c^*_3, c^*_4 > 0$ such that

$$|J(h_\varepsilon(v^\varepsilon_{m+1}))| \leq c^*_3 |J(h_\varepsilon(-v^\varepsilon_{m+1}))| + c^*_4 \quad \text{for all } m \geq m_1, \varepsilon > 0. \tag{3.22}$$

By (2.5), (3.20) and Lemma 3.2 it follows

$$\begin{aligned} J(h_\varepsilon(v^\varepsilon_{m+1})) &\leq J(h_\varepsilon(-v^\varepsilon_{m+1})) + \left| \int_\Omega (|h_\varepsilon(v^\varepsilon_{m+1}) + \phi|^p - |h_\varepsilon(v^\varepsilon_{m+1})|^p)\, dx \right| \\ &\leq J(h_\varepsilon(-v^\varepsilon_{m+1})) + \bar{c}_1 |J(h_\varepsilon(v^\varepsilon_{m+1}))|^{(p-1)/p} + \bar{c}_2, \end{aligned}$$

then there results

$$J(h_\varepsilon(v^\varepsilon_{m+1})) - \bar{c}_1 |J(h_\varepsilon(v^\varepsilon_{m+1}))|^{(p-1)/p} \leq J(h_\varepsilon(-v^\varepsilon_{m+1})) + \bar{c}_2. \tag{3.23}$$

By (3.12) and (3.16) it follows that there exists $m_1 \geq m_0$ such that for all $m \geq m_1$ and $\varepsilon > 0$ it is $\bar{c}_1 J^{-1/p}(h_\varepsilon(v^\varepsilon_{m+1})) < 1/2$; hence

$$J(h_\varepsilon(v^\varepsilon_{m+1}))/2 \leq J(h_\varepsilon(v^\varepsilon_{m+1})) - \bar{c}_1 J^{(p-1)/p}(h_\varepsilon(v^\varepsilon_{m+1})). \tag{3.24}$$

Whence (3.22) follows by (3.23) and (3.24). Now, we prove that there exists $m_2 \geq m_1$ such that

$$J(h_\varepsilon(-v^\varepsilon_{m+1})) > 0 \quad \text{for all } m \geq m_2, \varepsilon > 0. \tag{3.25}$$

Indeed, by (3.19) it is

$$J(h_\varepsilon(-v^\varepsilon_{m+1})) + c^*_1 |J(h_\varepsilon(-v^\varepsilon_{m+1}))|^{(p-1)/p} \geq J(h_\varepsilon(v^\varepsilon_{m+1})) - c^*_2;$$

then (3.12) and (3.16) imply (3.25).

Let $m \geq m_2$. By (3.15), (3.18), (3.19) and (3.25) it follows that the inequalities

$$\begin{aligned} J(h_\varepsilon(v^\varepsilon_{m+1})) &< \alpha_m + \varepsilon + c^*_1 \Big(\sup_{v \in V^+_m} J(h_\varepsilon(v)) \Big)^{(p-1)/p} + c^*_2 \\ &< \alpha_m + \varepsilon + c^*_1(\alpha_m + \varepsilon)^{(p-1)/p} + c^*_2 \end{aligned}$$

hold for all $\varepsilon > 0$; whence by (3.16) there results

$$\alpha_{m+1} \leq \alpha_m + c^*_1 \alpha_m^{(p-1)/p} + c^*_2. \tag{3.26}$$

Obviously by (3.17) this inequality holds even if $v^\varepsilon_{m+1} \in V^+_m$. Then there exist $m_3 \geq m_2$ and $c^*_3 > 0$ such that by (3.12) and (3.26) it follows

$$m \geq m_3,\ \alpha_m = \alpha^+_m \quad \text{implies} \quad \alpha_{m+1} \leq \alpha_m + c^*_3 \alpha_m^{(p-1)/p}. \tag{3.27}$$

At last, we are ready to prove that for all $\overline{m} \geq m_3$ there exists $m \geq \overline{m}$ such that it is $\alpha_m \neq \alpha^+_m$, $\alpha_m \geq M_0$ (M_0 as in Proposition 2.7). Arguing by contradiction, there exists $\overline{m} \geq m_3$ such that for all $m \geq \overline{m}$ (3.27) holds, then by [4, Lemma 5.3] there exists $c^*_4 > 0$ such that

$$\alpha_m \leq c^*_4 m^p \quad \text{for all } m \geq 1. \tag{3.28}$$

Hence the contradiction follows by (1.3), (3.12) and (3.28).

Let us remark that if $N = 2$ the same arguments hold; moreover, (1.3) means $2 < p < 3$. Then there exists $\sigma > 0$ such that $p < p/(p-2) - \sigma$, whence (3.13) and (3.28) imply the proof. □

References

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand Company Inc., Princeton N.J., 1965.

[2] A. Ambrosetti, *On the existence of multiple solutions for a class of nonlinear boundary value problems*, Rend. Sem. Mat. Univ. Padova **49** (1973), 195–204.

[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[4] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc. **267** (1981), 1–32.

[5] P. Bolle, *An extension of a result of Ekeland–Ghoussoub–Tehrani for the Bolza problem*, J. Differential Equations (to appear).

[6] C. V. Coffman, *A minimum–maximum principle for a class of nonlinear integral equations*, J. Analyse Math. **22** (1969), 391–419.

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.

[8] I. Ekeland, N. Ghoussoub and H. Tehrani, *Multiple solutions for a classical problem in the Calculus of Variations*, J. Differential Equations **131** (1996), 229–243.

[9] J. A. Hempel, *Multiple solutions for a class of nonlinear elliptic boundary value problems*, Indiana Univ. Math. J. **20** (1971), 983–996.

[10] P. H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc. **272** (1982), 753–769.

[11] ______, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series Math. **65** (1986), Amer. Math. Soc., Providence.

[12] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math. **32** (1980), 335–364.

[13] ______, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Berlin/New York, 1990.

[14] K. Tanaka, *Morse indices at critical points related to the Symmetric Mountain Pass Theorem and applications*, Comm. Partial Differential Equations **14** (1989), 99–128.

Manuscript received February 17, 1998

A. M. Candela and A. Salvatore
Dipartimento Interuniversitario di Matematica
via E. Orabona 4
70125 Bari, ITALY

E-mail address: candela@pascal.dm.uniba.it, salvator@pascal.dm.uniba.it