Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 10, 1997, 327–338

ON ASYMPTOTICALLY AUTONOMOUS DIFFERENTIAL EQUATIONS IN THE PLANE

Boris S. Klebanov

1. Introduction

In this paper, we study qualitative behaviour of trajectories of solutions of perturbed autonomous differential equations in the plane. We work in the framework of an axiomatic theory of solution spaces of ordinary differential equations suggested by V. V. Filippov (see the survey [7] and the references therein). This theory provides a unified approach to the study of solutions of ordinary differential equations, including equations with singularities, as well as of differential inclusions. The theory sets a series of axioms which reflect fundamental properties of solution sets of ordinary differential equations and deals with sets of functions satisfying one or another set of these axioms. Topological structures introduced make it possible to deal with sets of solutions as with elements of a topological space.

It is well known that many results in the classical qualitative theory of ordinary differential equations extend to dynamical systems. The theory suggested by Filippov allows one to develop such results in another direction and to extend them, in particular, to differential equations with singularities of various types. There are also many situations where the methods developed lead to new results in the classical realms, even for equations $y' = f(t, y)$ $(y' = dy/dt)$ with

1991 *Mathematics Subject Classification.* 34C05, 34C37.

Key words and phrases. Asymptotically autonomous differential equation, ω-limit set, stationary point, Markus' theorem.

continuous functions f. The scope of the axiomatic theory naturally embraces differential inclusions, and many results concerning them can be generalized in the framework of the theory.

The present paper is based on previous studies by Filippov on autonomous and perturbed autonomous differential equations, in particular, on planar ones, in the framework of the axiomatic theory. Our results and proofs are formulated in the terminology of the axiomatic theory. In the next section we present the notions from the axiomatic theory that are used in the sequel and briefly comment on them (we refer the unfamiliar reader to [7] for a broad account).

2. Notions from the axiomatic theory and notation

2.1. For a function z, we denote by $\pi(z)$ its domain of definition, and by $\mathrm{Im}(z)$ the set of its values, also called the *trajectory* of z.

2.2. Let U be an open subset of $\mathbb{R} \times \mathbb{R}^n$. Consider the set of all continuous mappings of all finite closed intervals $[a, b]$, $a \leq b$, of $\mathbb{R}$ into $\mathbb{R}^n$ whose graphs lie in U. This set equipped with the Hausdorff distance between their graphs is a metric space denoted by $C_s(U)$.

Denote by $R(U)$ the set of all subspaces Z of $C_s(U)$ satisfying the following two conditions (axioms).

(1) If $z \in Z$ and the closed interval I (possibly degenerate) lies in $\pi(z)$, then the restriction $z|_I$ belongs to Z.

(2) If the domains of definition of $z_1, z_2 \in Z$ intersect and these functions coincide on $\pi(z_1) \cap \pi(z_2)$, then the function z defined on $\pi(z_1) \cup \pi(z_2)$ by the formula $z(t) = z_i(t)$ if $t \in \pi(z_i)$ $(i = 1, 2)$ also belongs to Z.

Denote by $R_{ce}(U)$ the set of all $Z \in R(U)$ satisfying the following conditions (c) and (e).

(c) For any compact set $K \subseteq U$ the set of all elements of Z whose graphs lie in K is compact.

(e) For any point $(t_0, y_0) \in U$ there exists a function $z \in Z$ defined on an interval containing t_0 in its interior such that $z(t_0) = y_0$.

2.3. Given a differential equation $y' = f(t, y)$ (or a differential inclusion $y' \in F(t, y)$) in the set U, we define its solutions to be generalized absolutely continuous functions [12] defined on arbitrary finite closed intervals $[a, b]$, $a \leq b$, that satisfy it almost everywhere. These functions form a subspace of $C_s(U)$ called the *space of solutions* of this equation (respectively, inclusion) and denoted by $D(f)$ (respectively, $D(F)$). We remark that under the hypotheses of the classical Peano and Carathéodory theorems the above definition of solution agrees with the standard definitions [8, Chapter VII].

The spaces of solutions defined above belong to $R(U)$. The space $D(f)$ belongs to $R_{ce}(U)$ if f satisfies the hypotheses of the Peano and Carathéodory theorems. Moreover, $R_{ce}(U)$ contains $D(F)$ if the multivalued function F satisfies the hypotheses of Davy's theorem [4].

If f is continuous in U everywhere except at points of a closed, at most denumerable set, then $D(f) \in R_{ce}(U)$ [5], [7]. This is an example of a space of solutions $D(f)$ which belongs to $R_{ce}(U)$, under non-classical assumptions on f.

2.4. Let V be an open subset of $\mathbb{R}^n$. A space $Z \subseteq C_s(\mathbb{R} \times V)$ is said to be *autonomous* if it is closed with respect to translations along $\mathbb{R}$ (that is, for any $z \in Z$ with $\pi(z) = [a, b]$ and any real number τ, the function w defined on $[a - \tau, b - \tau]$ by the formula $w(t) = z(t + \tau)$ also belongs to Z). Denote by $A(V)$ the set of all autonomous spaces $Z \subseteq C_s(\mathbb{R} \times V)$, and let

$$A_{ce}(V) = A(V) \cap R_{ce}(\mathbb{R} \times V).$$

If $Z \in A_{ce}(V)$ satisfies a condition (k) defined in [7, §2] that corresponds to the connectedness property described by Kneser's theorem [9, Theorem II.4.1], we write $Z \in A_{cek}(V)$.

Clearly the spaces of solutions of autonomous differential equations (inclusions) defined in $\mathbb{R} \times V$ belong to $A(V)$. If a multivalued function $F : V \to \mathbb{R}^n$ is upper semicontinuous and for every $y \in V$ the set $F(y)$ is nonempty, compact, and convex, then $D(F) \in A_{cek}(V)$ (this can be derived, e.g., from results in [6, §7]; see also [7, §2]).

2.5. The concept presented below follows a similar one given in [7, §11].

Let $Z \subseteq C_s(\mathbb{R} \times V)$ and $Z_\infty \in A(V)$. We will write $\langle Z, Z_\infty \rangle \in \beta(V)$ if for any compact set $K \subseteq V$ and any positive number c the following condition is satisfied: if a sequence $\{z_n : n \in \mathbb{N}\} \subseteq Z$ with $\pi(z_n) = [a_n, b_n]$ is such that $\mathrm{Im}(z_n) \subseteq K$, $b_n - a_n \le c$ for any $n \in \mathbb{N}$ and $a_n \to \infty$ as $n \to \infty$, and if z_n^* is a function with $\pi(z_n^*) = [0, b_n - a_n]$ defined by the formula

$$z_n^*(t) = z_n(t + a_n),$$

then the sequence $\{z_n^* : n \in \mathbb{N}\}$ has a subsequence converging to a function from Z_∞.

If $\langle Z, Z_\infty \rangle \in \beta(V)$, the space Z will be called *asymptotically autonomous* (as $t \to \infty$).

Let an autonomous equation $y' = f(y)$ be defined for $y \in V$, and a perturbed autonomous equation

$$y' = f(y) + g(t, y)$$

be defined for $y \in V$ and $t \in (a, \infty)$, $a \in \mathbb{R} \cup \{-\infty\}$. It can be shown that if the second equation is asymptotically autonomous in the sense of Artstein [1], [2] and the first equation is its limiting equation (see the definition in [1]–[3]), then

$$\langle D(f+g), D(f)\rangle \in \beta(V).$$

This means that in order to verify that $D(f+g)$ is asymptotically autonomous, one can use the corresponding convergence conditions in [1]–[3]. An example of an asymptotically autonomous space $D(f+g)$ with continuous functions f and g, not covered by the conditions in [1]–[3], is given in [7, Example 11.2].

The space $D(f+g)$ is asymptotically autonomous (and the equation $y' = f(y)+g(t,y)$ has $y' = f(y)$ as its limiting equation) if, for instance, the following hypotheses hold: f is continuous, g satisfies the Carathéodory conditions locally, and for any compact set $B \subseteq V$ there exists a real-valued function $\varphi(t)$ locally Lebesgue integrable on (a,∞) such that

$$\|g(t,y)\| \le \varphi(t) \text{ for all } t \in (a,\infty) \text{ and } y \in B,$$

and

$$\lim_{s\to\infty} \int_s^{s+c} \varphi(t)\,dt = 0 \text{ for any } c > 0.$$

2.6. For $Z \in R(U)$, we denote by Z^+ (respectively, by Z^-) the set of all continuous functions $z : [a,b) \to \mathbb{R}^n$, $-\infty < a < b \le \infty$ (respectively, $z : (a,b] \to \mathbb{R}^n$, $-\infty \le a < b < \infty$) such that $z|_I \in Z$ for any finite closed interval $I \subseteq \pi(z)$ and no function in Z extends z. Denote by Z^{-+} the set of all continuous functions $z : (a,b) \to \mathbb{R}^n$, $-\infty \le a < b \le \infty$, such that for some $t \in (a,b)$ we have $z|_{(a,t]} \in Z^-$ and $z|_{[t,b)} \in Z^+$.

It is clear that the sets Z^+, Z^-, and Z^{-+} consist of analogues of maximally forward extended, backward extended, and both forward and backward extended solutions, respectively. For Z being the space of solutions of a differential equation or inclusion, the elements of Z^{-+} will be called its *full solutions*. A full solution z with $\pi(z) = \mathbb{R}$ is called *periodic* if $z(t) = z(t+p)$ for all $t \in \mathbb{R}$ and some $p > 0$.

2.7. A point $y \in V$ is called a *stationary point* of a space $Z \in R(\mathbb{R} \times V)$ if there exists $z \in Z^{-+}$ such that $\mathrm{Im}(z) = \{y\}$.

2.8. For any function $z \in Z^+ \cup Z^{-+}$, its *ω-limit set* $\Omega(z)$ is defined by the formula

$$\Omega(z) = \bigcap \{\overline{\mathrm{Im}(z|_{[t,\infty)\cap\pi(z)})} : t \in \pi(z)\}$$

(the bar signifies closure), and the *α-limit set* $A(z)$ of a function $z \in Z^- \cup Z^{-+}$ is defined by the formula

$$\mathrm{A}(z) = \bigcap \{\overline{\mathrm{Im}(z|_{(-\infty,t]\cap\pi(z)})} : t \in \pi(z)\}.$$

We now proceed to the presentation of our results.

3. Two theorems on ω-limit sets

Consider the following hypotheses:

(3.1) V is an open set in the plane, $a \in \mathbb{R} \cup \{-\infty\}$, $U = (a, \infty) \times V$;

(3.2) $$Z \in R_{ce}(U);$$

(3.4) $$Z_\infty \in A_{cek}(V) \text{ and } \langle Z, Z_\infty \rangle \in \beta(V).$$

The next theorems are generalizations of Theorems VII.4.3 and VII.4.4 of [9].

Theorem 1. *Suppose that* (3.1)–(3.3) *hold. Let $z \in Z^+$, and let $\Omega(z)$ be a nonempty compact subset of V. Then either $\Omega(z)$ contains a stationary point of the space Z_∞ or there exists a function $z_0 \in Z_\infty$ with $\pi(z_0) = [\alpha, \beta]$ such that the curve $y = z_0(t)$, $t \in [\alpha, \beta]$, is a Jordan curve* (*i.e., a topological image of a circle*) *lying in $\Omega(z)$.*

Proof. 1. Suppose that $\Omega(z)$ does not contain a stationary point of Z_∞. Then by Theorem IX.6.10 of [5] there exists $u \in Z_\infty^+$ such that $\operatorname{Im}(u) \subseteq \Omega(z)$. Since $\Omega(z)$ is nonempty and compact, the last inclusion implies that $\Omega(u) \subseteq \Omega(z)$ is also nonempty and compact.

We distinguish two cases.

Case 1. $u(b) = u(c)$ for some distinct $b, c \in \pi(u)$,

Case 2. $u(b) \neq u(c)$ for all distinct $b, c \in \pi(u)$.

2. Consider first Case 1. Let $b < c$. We claim that there is a $\delta > 0$ such that $u(p) \neq u(q)$ for all distinct points $p, q \in [b, c]$ with $|p - q| < \delta$.

Assume the contrary. Then for any $n \in \mathbb{N}$ there are $p_n, q_n \in [b, c]$, $p_n < q_n$, such that $u(p_n) = u(q_n)$ and $q_n - p_n < 1/n$. By passing to a subsequence, we may assume without loss of generality that $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ converge to some t_0. Since $\Omega(z)$ is assumed to contain no stationary points of Z_∞, the point $u(t_0) \in \Omega(z)$ is not a stationary point of Z_∞. As noted in [7, p. 129], this implies that there is a neighbourhood O of $u(t_0)$ and a $T > 0$ such that the trajectory of any function $v \in Z_\infty$ cannot lie entirely in O if the length of $\pi(v)$ exceeds T.

Denote by v_n the restriction of u to $[p_n, q_n]$. Then $v_n \in Z_\infty$ for any $n \in \mathbb{N}$. The continuity of u implies that $\operatorname{Im}(v_n) \subseteq O$ for all n large enough, say $n \geq n_0$. Since $v_{n_0}(p_{n_0}) = v_{n_0}(q_{n_0})$, we can extend v_{n_0} as a continuous periodic function over arbitrarily long closed intervals; the functions thus obtained belong to Z_∞ because the space $Z_\infty \in R(\mathbb{R} \times V)$ is autonomous. Hence Z_∞ must contain functions with domain of definition longer than T, whose trajectories lie in O. This contradicts the choice of T. Thus the claim is proved.

3. Since $u(b) = u(c)$, the above claim implies that there is a closed interval $[\alpha, \beta] \subseteq [b, c]$ such that the function $z_0 \in Z_\infty$, defined as the restriction of u to $[\alpha, \beta]$, has the required properties: the curve $y = z_0(t)$, $t \in [\alpha, \beta]$, is a Jordan curve and $z_0(t) = u(t) \in \Omega(z)$ for all $t \in [\alpha, \beta]$.

4. Consider now Case 2. Since the values $u(t)$ are different for distinct $t \in \pi(u)$, and $\langle Z_\infty, Z_\infty \rangle \in \beta(V)$ (this is obvious, since $Z_\infty \in A_{ce}(V)$), we can apply to u an analogue of the Poincaré–Bendixson theorem proved by Filippov [7, Theorem 11.2]. This gives us a $z_0 \in Z_\infty$ with $\operatorname{Im}(z_0) \subseteq \Omega(u) \subseteq \Omega(z)$ such that the curve $y = z_0(t)$, $t \in \pi(z_0)$, is a Jordan curve. Thus in both Cases 1 and 2 we have found a function z_0 as required. The theorem is proved. □

THEOREM 2. *Suppose that* (3.1)–(3.3) *hold. If in addition V is simply connected and does not contain stationary points of Z_∞, then no function from Z^+ remains in a compact subset of V.*

PROOF. Suppose the contrary, that is, there exists a $z \in Z^+$ whose trajectory is contained in some compact set $K \subseteq V$. Then $\Omega(z)$ is a nonempty compact subset of K. By Theorem 1, there is a function in Z^+_∞ whose trajectory is a Jordan curve lying in $\Omega(z)$. By [7, Theorem 11.1], the closure of the domain bounded by this curve contains a stationary point y_0 of Z_∞. Since V is simply connected, we have $y_0 \in V$. This contradicts the assumption that V contains no stationary points of Z_∞. The proof is complete. □

4. On Markus' theorem

Our next theorem is concerned with one of the most important properties of asymptotically autonomous differential equations in the plane, presented by Markus in [10].

Consider two planar differential equations

$$y' = f(y) \tag{4.1}$$

and

$$y' = h(t, y), \tag{4.2}$$

where the functions $f(y)$ and $h(t, y)$ satisfy a local Lipschitz condition in y and $h(t, y)$ is continuous in (t, y) for $t \in (a, \infty)$ and $y \in V$ ($a \in \mathbb{R} \cup \{-\infty\}$, V is an open subset of $\mathbb{R}^2$). Suppose that $h(t, y) \to f(y)$ as $t \to \infty$, uniformly in y on compact sets in V. Markus presented in [10] the following remarkable Poincaré–Bendixson type theorem.

THEOREM (Markus). *Let z be a maximally forward extended solution of* (4.2) *whose ω-limit set $\Omega(z)$ is a nonempty compact set in V. If $\Omega(z)$ contains no stationary points of* (4.1)*, then it is the union of periodic trajectories of* (4.1).

We remark that Markus' theorem was proved under more general assumptions by Artstein in [1, Theorem 9.1] where the assumptions on the right-hand sides were relaxed and a weaker type of convergence of h to f was considered (we also note that a version of Markus' theorem was given by Opial in [11]). The theorem of Markus was extended by Thieme [13, Theorem 1.4] to a case where $\Omega(z)$ may contain stationary points of (4.1). Thieme's paper [13] also contains a list of references on the use and impact of Markus' theorem.

In theorems presented below we extend Markus' theorem in the same direction as Thieme did, that is, to a case where the ω-limit set of an asymptotically autonomous equation may contain stationary points of its limiting autonomous equation. (We remark that our results and Theorem 1.4 of [13] do not cover each other.)

The key result of this section is Theorem 3. It is concerned with a pair $\langle Z, Z_\infty \rangle \in \beta(V)$, where Z_∞ has a special type: it is the space of solutions $D(F)$ of a differential inclusion

$$y' \in F(y) \tag{4.3}$$

such that the multivalued mapping $F : V \to \mathbb{R}^n$ satisfies the following conditions:

(4.4) F is upper semicontinuous;

(4.5) $F(y)$ is nonempty, compact, and convex for any $y \in V$.

If F satisfies (4.4) and (4.5), then $D(F) \in A_{ce}(V)$ (see Section 2.4).

In the sequel we use the following concepts of one-sided uniqueness of solutions.

DEFINITION. Solutions of an initial value problem $y' \in F(y)$, $y(t_0) = y_0$, $(t_0, y_0) \in \mathbb{R} \times V$, are said to be *forward* (respectively, *backward*) *unique* if for all $z_1, z_2 \in D(F)$ with $t_0 \in \pi(z_1) \cap \pi(z_2)$ and $y_0 = z_1(t_0) = z_2(t_0)$ we have $z_1(t) = z_2(t)$ for any $t \in \pi(z_1) \cap \pi(z_2)$ such that $t > t_0$ (respectively, $t < t_0$).

We remark that inclusions (4.3) for which solutions of the initial value problems satisfy the above definition form an important class of differential inclusions considered, for example, in [6].

To present our results we also need the concept of arc.

DEFINITION. A curve $u = u(t)$ such that $u(t_1) \neq u(t_2)$ for all distinct $t_1, t_2 \in \pi(u)$ will be called an *arc*.

THEOREM 3. *Suppose that the space Z and inclusion* (4.3) *satisfy* (3.1), (3.2), (4.4) *and* (4.5). *Let $\langle Z, D(F) \rangle \in \beta(V)$. Suppose that solutions of all initial value problems for* (4.3) *are forward* (*respectively, backward*) *unique. Let $z \in Z^+$ and let $\Omega(z)$ be a nonempty compact subset of V. Then any point of $\Omega(z)$ either*

(i) *is a stationary point of* (4.3), *or*

(ii) *lies on the trajectory of a periodic solution of* (4.3) *contained in* $\Omega(z)$, *or*

(iii) *lies on an arc contained in* $\Omega(z)$ *which is the trajectory of a maximally forward* (*respectively, backward*) *extended solution* φ *of* (4.3)*, and* $\mathrm{A}(\varphi)$ (*respectively,* $\Omega(\varphi)$) *consists only of stationary points of* (4.3)*.*

Proof. We first consider the case where all initial value problems for (4.3) are assumed to have forward unique solutions.

1. Let $y \in \Omega(z)$. By Theorem IX.6.10 of [5] there exists $\varphi \in D(F)^{-}$ with $\pi(\varphi) = (-\infty, 0]$ such that $y = \varphi(0)$ and $\mathrm{Im}(\varphi) \subseteq \Omega(z)$. As $\mathrm{Im}(\varphi)$ lies in the nonempty compact set $\Omega(z) \subseteq V$, the set $\mathrm{A}(\varphi) \subseteq \Omega(z)$ is also a nonempty compact subset of V. Denote by T the trajectory of φ.

Assume that T is not an arc. Then there exist distinct points $s_1, s_2 \in \pi(\varphi)$ and a point $y^* \in V$ such that $\varphi(s_1) = \varphi(s_2) = y^*$. Using the facts that all solutions of (4.3) are forward unique and $D(F)$ is invariant with respect to translations along the t-axis, it is not difficult to verify that either (i) or (ii) holds, depending on whether y^* is a stationary point of (4.3) or not ($y = y^*$ in the former case).

2. Suppose now that T is an arc. If y is a stationary point of (4.3), we are clearly done. So we assume henceforth that y is a non-stationary point of (4.3). Then T contains no stationary points of (4.3) due to the assumption of forward uniqueness. Our aim is to prove that $\mathrm{A}(\varphi)$ contains only stationary points of (4.3).

Assume, on the contrary, that there exists a point $b \in \mathrm{A}(\varphi)$ which is not a stationary point of (4.3). By the results similar to those in [6, §13] (we reformulate for the α-limit set the assertions given there for the ω-limit set), there exists a transversal line segment L through b such that

(a) T intersects L in points $b_i = \varphi(t_i)$, $i \in \mathbb{N}$, with $0 > t_1 > t_2 > \dots$, $t_i \to -\infty$ as $i \to \infty$, such that $b_i \to b$ along L strictly monotonicly (that is, b_{i+1} is between b_i and b for all $i \in \mathbb{N}$);
(b) $L \cap \mathrm{A}(\varphi) = \{b\}$;
(c) the trajectories of all solutions of (4.3) intersecting L, traverse it from one side, common to all of them, to the other.

Note that

(d) T and $\mathrm{A}(\varphi)$ are disjoint.

Indeed, their intersection cannot contain a stationary point of (4.3), because T contains no such points; nor can it contain non-stationary points of (4.3), since this would contradict Theorem 2 from §13 in [6].

Fix an $i \in \mathbb{N}$. Let T_i be the subarc of the trajectory T with endpoints b_i and b_{i+1}, and let L_i be the interval of the transversal line segment L with the same endpoints. Denote $T_i \cup L_i$ by J_i. The curve J_i is a Jordan curve (this follows from (a) and the fact that T is an arc).

3. It easily follows from (a), (b) and (d) that J_i and $\mathrm{A}(\varphi)$ are disjoint. Since $t_n < 0$ for all $n \in \mathbb{N}$ and T is an arc, the point $y = \varphi(0)$ does not belong to J_i. Thus, both $\mathrm{A}(\varphi)$ and y lie in $\mathbb{R}^2 \setminus J_i$. By the Jordan curve theorem, the set $\mathbb{R}^2 \setminus J_i$ consists of two components, G_i and H_i, with J_i being their boundary. For definiteness, let $y \in G_i$. Then $\varphi(t) \in H_i$ for all $t < t_{i+1}$, since T is an arc satisfying (c). Therefore $\mathrm{A}(\varphi) \subseteq H_i$, and since $b \in \mathrm{A}(\varphi)$, it follows that $b \in H_i$.

Thus, y and b lie in different components of $\mathbb{R}^2 \setminus J_i$. Hence, using the fact that both y and b belong to $\Omega(z)$ and lie on different sides of J_i, one can find sequences $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ lying in $\pi(z)$ such that for all $n \in \mathbb{N}$ we have $p_n < q_n < p_{n+1}$, $z(p_n) \in J_i$, $z(q_n) \to b$ as $n \to \infty$, and $z(t) \in H_i$ for any $t \in (p_n,\ q_n)$. Since J_i is compact, one may assume without loss of generality that $\{z(p_n) : n \in \mathbb{N}\}$ converges to some point $c \in J_i$.

We distinguish two cases: the sequence $\{q_n - p_n : n \in \mathbb{N}\}$ is either bounded (Case 1) or unbounded (Case 2).

4. Consider Case 1. Let z_n be defined on $[0, q_n - p_n]$ by the formula

$$z_n(t) = z(t + p_n).$$

Since the trajectory of z is bounded (this is equivalent to $\Omega(z)$ being nonempty and compact), one can use the hypothesis $\langle Z, D(F)\rangle \in \beta(V)$ to find a subsequence of $\{z_n : n \in \mathbb{N}\}$ converging to a function $\psi \in D(F)$ with domain $[0, r]$ such that $\psi(0) = c$, $\psi(r) = b$ and $\mathrm{Im}(\psi) \subseteq \overline{H}_i$.

The point c cannot belong to L_i (the contrary would contradict (c)). Neither can it belong to T_i, since otherwise the forward uniqueness of solutions of (4.3) would imply that $\mathrm{Im}(\psi) \subseteq T$. This is impossible because $b \in \mathrm{Im}(\psi) \cap \mathrm{A}(\varphi)$, but $T \cap \mathrm{A}(\varphi) = \emptyset$ (see (d)). Since $c \in J_i = T_i \cup L_i$, we obtain a contradiction.

Thus, Case 1 cannot occur.

5. Let us now consider Case 2. In this case we can apply Lemma IX.6.8 of [5] to the sequence $\{z_n : n \in \mathbb{N}\}$ to find $\psi \in D(F)^+$ with $\pi(\psi) = [0, \infty)$ such that $\psi(0) = c$ and the trajectory of ψ lies entirely in the compact set $\overline{H}_i \cap \Omega(z)$.

The point c cannot belong to L_i for the same reason as in Case 1. It cannot belong to T_i either. For otherwise, if there were some $\tau \in (t_i, t_{i+1})$ with $\varphi(\tau) = c = \psi(0)$, the forward uniqueness of solutions of (4.3) would imply that $\psi(t-\tau) = \varphi(t)$ for all $t \in [\tau, 0]$. Then $\psi(-\tau) = \varphi(0) = y \in G_i$. Since $\mathrm{Im}(\psi) \subseteq \overline{H}_i$ and $G_i \cap \overline{H}_i = \emptyset$, this is impossible. So again we have a contradiction to the fact that $c \in J_i$. Hence Case 2 is also impossible.

Since both Cases 1 and 2 have appeared to be impossible, the assumption that $\mathrm{A}(\varphi)$ contains a non-stationary point of (4.3) is false.

Thus, the theorem is proved under the assumption of forward uniqueness.

6. The proof under the assumption of backward uniqueness is similar. We just indicate the key changes.

(I) In part 1, we now find $\varphi \in D(F)^+$ with $\pi(\varphi) = [0,\infty)$ satisfying the other properties, and deal throughout the proof with $\Omega(\varphi)$ instead of $\mathrm{A}(\varphi)$.

(II) In part 2, the points $t_i \in (0,\infty)$ tend monotonicly to ∞, and not to $-\infty$ as previously.

(III) In part 3, the sequences $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ are chosen to satisfy $q_n < p_n < q_{n+1}$.

(IV) In part 4, the domain of $\psi \in D(F)$ will be $[r,0]$, $r < 0$.

(V) In part 5, the domain of $\psi \in D(F)^-$ will be $(-\infty,0]$.

The proof is complete. □

Before stating our next result we recall the notion of a totally disconnected space.

DEFINITION. A topological space is *totally disconnected* if the connected component of each of its points reduces to this point.

THEOREM 4. *Suppose that the space* Z *and inclusion* (4.3) *satisfy* (3.1), (3.2), (4.4) *and* (4.5). *Let* $\langle Z, D(F)\rangle \in \beta(V)$. *Suppose that solutions of all initial value problems for* (4.3) *are forward and backward unique. Let* $z \in Z^+$ *and let* $\Omega(z)$ *be a nonempty compact subset of* V. *Assume that the* (*possibly empty*) *set* E *of stationary points of* (4.3) *that lie in* $\Omega(z)$ *is totally disconnected. Then any point of* $\Omega(z)$ *belongs either to* E, *or to the trajectory of a periodic solution of* (4.3) *contained in* $\Omega(z)$, *or to an arc contained in* $\Omega(z)$ *which is the trajectory of a full solution* φ *of* (4.3) *such that* $\pi(\varphi) = \mathbb{R}$ *and the limits* $\lim_{t\to\infty}\varphi(t)$ *and* $\lim_{t\to-\infty}\varphi(t)$ *exist and are in* E.

PROOF. Let $y \in \Omega(z)$. Suppose that the first two possibilities stated in the theorem do not hold for y. By Theorem 3 there exist functions $\varphi_1 \in D(F)^-$ and $\varphi_2 \in D(F)^+$ such that

$$\mathrm{Im}(\varphi_1) \cup \mathrm{Im}(\varphi_2) \subseteq \Omega(z) \quad \text{and} \quad \mathrm{A}(\varphi_1) \cup \Omega(\varphi_2) \subseteq E.$$

Since inclusion (4.3) is autonomous, we may assume without loss of generality that

$$0 = \max\{t : t \in \pi(\varphi_1)\} = \min\{t : t \in \pi(\varphi_2)\} \quad \text{and} \quad \varphi_1(0) = \varphi_2(0) = y.$$

As the trajectories of φ_1 and φ_2 lie in the compact set $\Omega(z)$, $\pi(\varphi_1)$ and $\pi(\varphi_2)$ are unbounded (see, e.g., [7, Theorem 2.8]), so that $\pi(\varphi_1) = (-\infty,0]$ and $\pi(\varphi_2) = [0,\infty)$. Define $\varphi \in D(F)^{-+}$ on $\mathbb{R}$ by $\varphi(t) = \varphi_1(t)$ if $t \le 0$, and $\varphi(t) = \varphi_2(t)$ if $t > 0$.

Clearly $A(\varphi) = A(\varphi_1)$ and $\Omega(\varphi) = \Omega(\varphi_2)$. Since $\mathrm{Im}(\varphi) \subseteq \Omega(z)$, the sets $A(\varphi)$ and $\Omega(\varphi)$ are nonempty compact subsets of the totally disconnected set E. These sets are connected [6, §12, Section 4], so they are singletons. Therefore (see [6, §12, Section 4]) $\varphi(t)$ tends to $\Omega(z)$ (respectively, to $A(z)$) as $t \to \infty$ (respectively, as $t \to -\infty$). The theorem is proved. □

Let (3.1) hold. Consider the differential equations

$$y' = f(y), \tag{4.6}$$

$$y' = f(y) + g(t, y), \tag{4.7}$$

where $f : V \to \mathbb{R}^2$ is locally Lipschitz continuous and $g : U \to \mathbb{R}^2$ is continuous (or, more generally, g locally satisfies the Carathéodory conditions). Suppose that $\|g(t, y)\| \to 0$ as $t \to \infty$, uniformly in y on compact sets in V. More generally, one may assume that $g(t, y)$ converges to zero as $t \to \infty$ in the sense of the definition introduced by Artstein (see [1, Sections 4 and 5], [2, Section 9]), i.e., equation (4.7) is asymptotically autonomous in the sense of Artstein with (4.6) being its limiting equation.

A representation of ω-limit sets of forward extended solutions of (4.7) as unions of trajectories of solutions of (4.6) is described in Theorem 5 below. This theorem, which is an immediate corollary to Theorem 4, gives a positive answer to a problem of Thieme [13].

Theorem 5. *Let z be a maximally forward extended solution of* (4.7) *such that $\Omega(z)$ is a nonempty compact subset of V. Assume that the set of stationary points of* (4.6) *that lie in $\Omega(z)$ is totally disconnected* (*in particular, empty or finite*). *Then any point of $\Omega(z)$ either is a stationary point of* (4.6), *or lies on the trajectory of a periodic solution of* (4.6) *contained in $\Omega(z)$, or lies on an arc contained in $\Omega(z)$ which is the trajectory of a full solution of* (4.6) *whose α- and ω-limit sets are stationary points of* (4.6).

References

[1] Z. Artstein, *Limiting equations and stability of nonautonomous ordinary differential equations*, Appendix A in: J. P. LaSalle, *The Stability of Dynamical Systems*, CBMS Regional Conf. Ser. in Appl. Math., vol. **25**, SIAM, Philadelphia, 1976.

[2] ———, *The limiting equations of nonautonomous ordinary differential equations*, J. Differential Equations **25** (1977), 184–202.

[3] ———, *Uniform asymptotic stability via the limiting equations*, J. Differential Equations **27** (1978), 172–189.

[4] J. L. Davy, *Properties of the solution set of a generalized differential equation*, Bull. Austral. Math. Soc. **6** (1972), 379–398.

[5] V. V. Fedorchuk and V. V. Filippov, *General Topology. Basic Constructions*, Moscow University Press, Moscow, 1988. (Russian)

[6] A. F. FILIPPOV, *Differential Equations with Discontinuous Righthand Sides*, Math. Appl. (Soviet Ser.) **18** (1988), Kluwer Academic Publ. Dordrecht.

[7] V. V. FILIPPOV, *The topological structure of solution spaces of ordinary differential equations*, Russian Math. Surveys **48** (1993), no. 1, 101–154.

[8] ———, *Spaces of Solutions of Ordinary Differential Equations*, Moscow University Press, Moscow, 1993. (Russian)

[9] P. HARTMAN, *Ordinary Differential Equations*, Wiley, New York, 1964.

[10] L. MARKUS, *Asymptotically autonomous differential systems*, Contributions to the Theory of Nonlinear Oscillations, vol. III, Annals of Math. Stud. vol. **36**, Princeton University Press, Princeton, N.J., 1956, pp. 17–29.

[11] Z. OPIAL, *Sur la dépendance des solutions d'un système d'équations différentielles de leurs seconds membres. Application aux systèmes presque autonomes*, Ann. Polon. Math. **8** (1960), 75–89.

[12] S. SAKS, *Theory of the Integral*, PWN, Warszawa, 1937.

[13] H. R. THIEME, *Convergence results and a Poincaré–Bendixson trichotomy for asymptotically autonomous differential equations*, J. Math. Biol. **30** (1992), 755–763.

Manuscript received July 15, 1994

BORIS S. KLEBANOV
School of Mathematics and Statistics
The University of Birmingham
Birmingham B15 2TT, UNITED KINGDOM

Current address: The Moscow Institute for Teacher Development
Moscow, RUSSIA

E-mail address: klebanov@int.glas.apc.org