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MORSE THEORY FOR RIEMANNIAN GEODESICS WITHOUT NONDEGENERACY ASSUMPTIONS

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1. Introduction

Let $f \in C^2(\mathcal{M},\mathbb{R})$ be a functional defined on a Hilbert manifold $\mathcal{M}$. It is well known that if f is a Morse functional (i.e. every critical point of f is nondegenerate) and f satisfies the so called Palais–Smale condition, the Morse relations hold. More precisely, let $x \in \mathcal{M}$ be a critical point of f, and $m(x,f)$ denote the Morse index at x (i.e. the maximal dimension of the subspaces of $T_x\mathcal{M}$ where the hessian at x is negative definite). The polynomial defined by

$$m_\lambda(f) = \sum_{x\in K_f} \lambda^{m(x,f)},$$

(here K_f is the set of critical points f), is called *Morse polynomial* of f. Morse relations link the Morse polynomial of f with the Poincaré polynomial of $\mathcal{M}$ and, in particular, they state that

$$m_\lambda(f) = \mathcal{P}_\lambda(\mathcal{M}) + (1+\lambda)\mathcal{Q}_\lambda,$$

where $\mathcal{P}_\lambda(\mathcal{M})$ is the Poincaré polynomial of $\mathcal{M}$ and $\mathcal{Q}_\lambda$ is a formal series with coefficients in $\mathbb{N}\cup\{\infty\}$.

In this paper we present a Morse theory for functionals of class C^2 whose critical points are not necessarily nondegenerate. We shall see that also in this case

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we can write some "generalized" Morse relations, which in the nondegenerate case coincide with the classical ones.

This kind of problems has already been studied in [1], for a wide class of C^1 functionals, using an approximation technique: roughly speaking, the generalized Morse polynomial of f is the limit of the Morse polynomials of suitable Morse functionals which coincide with f far from its critical points. In the C^2 case, the approximating functionals can be chosen in a more natural class: for any approximating (in the C^2 norm) functional h_n and for any $k = 0, \ldots, n$ the number of critical points of h_n with the Morse index equal to k is required to be greater than or equal to the number of critical points of f with the Morse index equal to k. By a modification of the approximation technique developed in [7], we show that such a class is not empty (assuming that the linear form associated with the hessian is a Fredholm map). We shall take a kind of $\liminf$ on the class of such approximating functionals.

We wish to point out that, as in [1], the limit procedure used here allows us to get Morse relations between a topological invariant (the Poincaré polynomial) and a differential invariant (the generalized Morse index).

In the second part of the paper we shall apply the abstract theory to the case of geodesics on a Riemannian manifold. For any $x_0, x_1 \in \mathcal{M}$, we shall study the functional

$$f(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds,$$

defined on the Sobolev manifold

$$\Omega^1(x_0, x_1, \mathcal{M}) = \{x \in H^{1,2}([0,1], \mathbb{R}^N) \mid \forall s \in [0,1], \\ x(s) \in \mathcal{M},\ x(0) = x_0,\ x(1) = x_1\}.$$

In this case imposing that all the critical points of f are nondegenerate means that x_0 and x_1 are not conjugate along every geodesic from x_0 to x_1. In Section 5, we shall write Morse relations without the assumption that x_0, and x_1 are not conjugate.

Moreover, in Section 6, we shall give a geometrical interpretation of the generalized Morse index. In particular, we shall prove that the Morse index of any critical point x_ε of small perturbations of f used in the approximating scheme is equal to the number of conjugate points along x_ε. This makes it possible to generalize Morse relations to the degenerate case using the geometric index.

In [4] the ideas and the results developed here have been applied to obtain Morse relations for light rays on a Lorentzian manifold, without nondegeneration assumptions. They can be applied to the mathematical interpretation of the gravitational lens effect (see [5], [13], [14]).

2. Preliminaries

In this section we give some preliminary results. They will be useful in the sequel, in proving some perturbation theorems. We omit the proofs that can be found in [7] or in standard functional analysis texts.

In the sequel, we shall assume that $\mathcal{M}$ be a Riemannian manifold modelled on a Hilbert space H and $f \in C^2(\mathcal{M}, \mathbb{R})$ is a functional on $\mathcal{M}$. For $x \in \mathcal{M}$, $T_x\mathcal{M}$ denotes the tangent space at x and $df(x)$ the differential map of f at x. We recall that $x \in \mathcal{M}$ is a *critical point* of f if $df(x) = 0$ and $a \in \mathbb{R}$ is a *critical value* of f if there exists a critical point $x \in \mathcal{M}$ such that $f(x) = a$. Otherwise a is said *regular value*. To any critical point $x \in \mathcal{M}$ of f it is associated a bilinear form, the *Hessian* of f at x defined as

$$H^f(x)[\xi, \xi] = \frac{d^2 f(\gamma(r))}{dr^2}\bigg|_{r=0}$$

where $\gamma = \gamma(r) :]-\sigma, \sigma[\to \mathcal{M}$ is such that $\gamma(0) = x$, $\partial_r\gamma(0) = \xi$. $H^f(x)$ is well defined and it does not depend on the linear connection given on $\mathcal{M}$. For details see [8].

If X, Y are Banach spaces, we shall denote by $\mathcal{L}(X, Y)$ the space of linear continuous maps from X to Y. As $T_x\mathcal{M}$ is a Hilbert space, by the Riesz Representation Theorem, there exists a linear continuous operator $H_x \in \mathcal{L}(T_x\mathcal{M}, T_x\mathcal{M})$ such that, for all $v, v' \in T_x\mathcal{M}$

$$H^f(x)[v, v'] = \langle H_x v, v' \rangle_x \ ,$$

where $\langle \cdot\, , \cdot \rangle_x$ is the scalar product on $T_x\mathcal{M}$. From now on, we shall refer indifferently to properties of $H^f(x)$ as properties of the linear operator associated to it and we shall denote H_x by the same symbol used for the hessian.

Definition 2.1. A critical point $x \in \mathcal{M}$ of f is called *nondegenerate* if H_x is invertible, otherwise x is called *degenerate*.

Note that, as a consequence of the Local Inversion Theorem, every nondegenerate critical point of f is isolated.

Definition 2.2. Let x be a critical point of f. The *Morse index* of x, $m(x, f)$ is the maximal dimension of a subspace V of $T_x\mathcal{M}$ on which the restriction of $H^f(x)$ is negative definite. The *large Morse index* is

$$m^*(x, f) = m(x, f) + \dim\ker H^f(x).$$

We shall apply the Morse theory to C^2-functionals whose hessian at a critical point belongs to the class of the Fredholm maps of index 0 (see [2]).

Remark 2.3. It is well known that, if L is a selfadjoint, Fredholm map of index 0 on a Hilbert space $(H, \langle\cdot\,,\cdot\rangle)$, there exist H^-, H^0, H^+ orthogonal eigenspaces of L, such that

$$H = H^- \oplus H^0 \oplus H^+,$$

where $H^0 = \ker L$, L is negative definite on H^- and positive definite on H^+. If $\dim H^- < \infty$, it is possible to introduce a new scalar product $\langle\cdot\,,\cdot\rangle_1$ on H such that

$$\langle Lu, v\rangle = \langle L_1 u, v\rangle_1 \quad \forall u, v \in H, \tag{2.1}$$

where L_1 is a compact perturbation of the identity. Indeed, if $\overline{\lambda} \in \mathbb{R}^-$ is the smallest negative eigenvalue of L, we find that

$$\langle Lv^-, v^-\rangle \geq \overline{\lambda}\langle v^-, v^-\rangle \quad \forall v^- \in H^-.$$

Moreover, there exists $m > 0$, such that

$$\langle Lv^+, v^+\rangle \geq m\langle v^+, v^+\rangle \quad \forall v^+ \in H^+.$$

Taking $c = \min\{-\overline{\lambda}, m\}$ we can write $L = T + K$ where

$$T = L - 2\overline{\lambda} I_{|H^-} + cI_{|H^0}, \quad K = 2\overline{\lambda} I_{|H^-} - cI_{|H^0},$$

and $I_{|H^-}$ (respectively $I_{|H^0}$) is the linear selfadjoint operator on H equal to the identity on H^- (respectively on H^0) and equal 0 otherwise. T is a positive operator: indeed if we take take $v \in H$, $v = v^- + v^0 + v^+$, with $v^- \in H^-$, $v^0 \in H^0$, $v^+ \in H^+$, then

$$\begin{aligned}\langle Tv, v\rangle &= \langle Lv^-, v^-\rangle + \langle Lv^+, v^+\rangle - 2\overline{\lambda}\langle v^-, v^-\rangle + c\langle v^0, v^0\rangle \\ &\geq -\overline{\lambda}\langle v^-, v^-\rangle + m\langle v^+, v^+\rangle + c\langle v^0, v^0\rangle \geq c\langle v, v\rangle.\end{aligned}$$

Moreover, as $\dim H^- < \infty$, $\dim H^0 < \infty$, K is compact. Then, if we set

$$\langle\cdot\,,\cdot\rangle_1 = \langle T\cdot\,,\cdot\rangle,$$

we get (2.1) with $L_1 = I + T^{-1}K$.

For the sake of simplicity, some proofs in the paper will be supplied only considering the case where the hessian at the critical points is a compact perturbation of the identity. In this way we do not lose generality.

Definition 2.4. We say that $F \in C^1(X, Y)$ is $\mathcal{F}_0$ at $x \in X$ if $dF(x) : X \to Y$ is a Fredholm map of index 0.

If, for some $x \in \mathcal{M}$, $H^f(x)$ is a Fredholm map of index 0, this property is preserved when we compose f with a diffeomorphism. More precisely, we have the following

Lemma 2.5. *Let us assume that $f \in C^2(\mathcal{M}, \mathbb{R})$ and let Ω be an open subset of $\mathcal{M}$. Let W be an open subset of another Riemannian manifold $\mathcal{M}'$ and $\phi : W \to \Omega$ be a C^2 diffeomorphism. We take $x_1 \in W$ and suppose $\phi(x_1) = x_0 \in \Omega$. Then*

(i) *x_0 is a critical point of f if and only if x_1 is a critical point of $f \circ \phi$,*
(ii) *x_0 is a nondegenerate critical point of f if and only if x_1 is a nondegenerate critical point of $f \circ \phi$ and $m(x_0, f) = m(x_1, f \circ \phi)$,*
(iii) *df is $\mathcal{F}_0$ at x_0 if and only if $d(f \circ \phi)$ is $\mathcal{F}_0$ at x_1.*

Proof. Easy calculations show that

$$d(f\circ\phi)(x_1) = df(x_0)\circ d\phi(x_1) = 0 \quad \text{and} \quad H^{(f\circ\phi)}(x_1) = d\phi(x_1)^*\circ H^f(x_0)\circ d\phi(x_1),$$

where $d\phi(x_1)^*$ is the operator adjoint to $d\phi(x_1)$. Therefore, since $d\phi(x_1)^*$ and $d\phi(x_1)$ are isomorphisms, (i), (ii), (iii) are true. □

Now we introduce a condition useful in proving that small perturbations of functionals whose differential map is $\mathcal{F}_0$ in some open set Ω preserve the same property.

Definition 2.6. A linear operator $L \in \mathcal{L}(X, Y)$, where X and Y are Banach spaces is said to be *proper* in X if for all compact sets $K \subset Y$, $L^{-1}(K)$ is compact.

Definition 2.7. Let X, Y be Banach spaces. Let Ω be an open subset of X and $A \in C^1(\Omega, Y)$. We say that A satisfies *condition* (S) in Ω, if there exists $\varepsilon = \varepsilon(A, \Omega) > 0$ such that every $B \in C^1(\Omega, Y)$ with

$$\|dB(x) - dA(x)\| \leq \varepsilon \quad \forall x \in \Omega,$$

is $\mathcal{F}_0$ in Ω and proper in every closed subset of Ω.

Note that, if A satisfies condition (S), A is $\mathcal{F}_0$ in Ω and proper in every closed subset of Ω. A sufficient condition for condition (S) to be satisfied is given by the following Lemma whose proof can be found in [7].

Lemma 2.8. *Let X, Y be Banach spaces, Ω an open subset of X and $A \in C^2(\Omega, Y)$. Let $x_0 \in \Omega$ be such that $dA(x_0)$ is $\mathcal{F}_0$. Then there exists U, a neighbourhood of x_0, such that A satisfies condition* (S) *in U.*

Clearly, if $A, B \in C^1(\Omega, Y)$ are such that

(i) A satisfies condition (S) in $U \subset \Omega$;
(ii) there exists $\varepsilon \in]0, \varepsilon(A, U)[$ sufficiently small, such that for all $x \in U$

$$\|dA(x) - dB(x)\| \leq \varepsilon,$$

then also B satisfies condition (S) in U.

In critical points theorems, the well known Palais–Smale condition is used. Let $\mathcal{M}$ be a Riemannian manifold and $f \in C^2(\mathcal{M}, \mathbb{R})$. We set for $a, b \in \mathbb{R}$, $a < b$

$$\begin{aligned} f^a &= \{x \in \mathcal{M} \mid f(x) \leq a\}, \\ f_a^b &= \{x \in \mathcal{M} \mid a \leq f(x) \leq b\}, \\ K_f &= \{x \in \mathcal{M} \mid df(x) = 0\}. \end{aligned}$$

Definition 2.9. We say that f satisfies *Palais–Smale condition* (P.S.), if every sequence $(x_n)_{n\in\mathbb{N}}$ such that $(f(x_n))_{n\in\mathbb{N}}$ is bounded and $df(x_n) \to 0$ admits a converging subsequence to $x \in \mathcal{M}$.

It is easy to prove that, if f satisfies (P.S.), $K_f \cap f_b^a$ is compact for all $a, b \in \mathbb{R}$. Then, if f is bounded from below and f satisfies (P.S.), $K_f \cap f^a$ is compact. Moreover, if df is proper in $U \subset \mathcal{M}$, it is easy to show that f satisfies (P.S.) in U.

To end this section, we shall characterize the Morse index of a functional whose hessian at a critical point is a compact perturbation of the identity in terms of the negative eigenvalues of the Hessian. To this end, it is useful to state the well known Poincaré principle (for the proof see for example [3]) which will be also used in the next sections to study the properties of the generalized geometrical index.

Theorem 2.10. *Let H be a Hilbert space and $T \in \mathcal{L}(H, H)$ be compact and selfadjoint. Then the eigenvalues of T are a decreasing sequence $(\mu_n)_{n\in\mathbb{N}}$, such that*

$$\mu_n = \max_{\dim V = n} \{ \min_{v \in V,\, v \neq 0} \langle Tv, v\rangle / \|v\|^2 \}, \tag{2.2}$$

where V is a subspace of H and $\langle \cdot\,, \cdot \rangle$ is the scalar product of H.

Then, as a consequence of the previous Theorem, for any critical point x of a functional f such that $H^f(x) = I - K$, where I is the identity and K is a compact selfadjoint operator, the Morse index of x is equal to the number of negative eigenvalues of $I - K$.

3. Some perturbation results

In this section we prove some perturbation Lemmas which make possible the approximation of a functional f with a sequence of functionals without degenerate critical points. This technique has already been used in [7], but in our case we shall require something more: we choose a finite number of critical points of f and we construct an approximating functional for which these points are nondegenerate critical points and keep the same index. This property allows us to define a generalized Morse index for f and write the Morse relations in a different way which seems more natural for the C^2 case with respect to the approach used in [1] for the C^1 case. The most important step in doing this is the following Lemma. Let $(H, (\,\cdot\,,\,\cdot\,))$ be a Hilbert space. We set, for $R > 0$, $x_0 \in H$

$$B_R(x_0) = \{x \in H \mid \|x - x_0\| < R\},$$

and for $\delta > 0$, $A \subset H$

$$N_\delta(A) = \{x \in H \mid d(x, A) < \delta\},$$

where d is the distance of H.

LEMMA 3.1. *Let U be an open subset of H, $f \in C^2(U, \mathbb{R})$ and $x_0 \in U$. Let's suppose that*

(1) *there exists $R_2 > 0$ such that $\overline{B_{R_2}(x_0)} \subset U$ and df satisfies condition* (S) *in $\overline{B_{R_2}(x_0)}$,*
(2) *x_0 is a critical point of f with $H^f(x_0) = I - K$ where $K \in \mathcal{L}(H, H)$ is a compact selfadjoint operator.*

Let $R_1 < R_2$ and take $\varepsilon, \delta > 0$ sufficiently small. Then there exists $g \in C^2(U, \mathbb{R})$ with the following properties

(i) *$g(x) = f(x)$ if $x \in U \backslash B_{R_2}(x_0)$, $\|g - f\|_{C^2(U)} \leq \varepsilon$,*
(ii) *$K_g \subset N_\delta(K_f)$,*
(iii) *there exists $\rho > 0$, such that x_0 is the only critical point of g in $B_\rho(x_0)$, x_0 is nondegenerate and $m(x_0, g) = m(x_0, f)$,*
(iv) *dg satisfies condition* (S) *in $B_{R_2}(x_0)$.*

if f satisfies (P.S.) *condition, the same is true for g.*

PROOF. We can assume that $x_0 = 0$ and denote $B_{R_i} = B_{R_i}(0)$ for $i = 1, 2$. Let $\omega_1 \in C^\infty(U, \mathbb{R})$ be such that

$$\omega_1(x) = \begin{cases} 1 & x \in B_{R_1}, \\ 0 & x \in U \setminus B_{R_2}. \end{cases}$$

To construct ω_1, we can choose a function $\widetilde{\omega} \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\widetilde{\omega}(t) = \begin{cases} 1 & |t| < R_1, \\ 0 & |t| > R_2, \end{cases}$$

and set $\omega_1(x) = \widetilde{\omega}(\|x\|^2)$. Obviously ω_1 is C^∞ and has bounded derivatives. Defining for $\sigma > 0$, $x \in U$

$$g_\sigma(x) = f(x) + \sigma \|x\|^2 \omega_1(x)/2,$$

we have, for all $x \in U$

$$dg_\sigma(x) = df(x) + \sigma\omega_1(x)(x, \cdot\,) + \sigma\|x\|^2 d\omega_1(x)/2,$$

therefore

$$\|dg_\sigma(x)\| \geq \|df(x)\| - \sigma\left[\,\|x\||\omega_1(x)| + \|x\|^2\|d\omega_1(x)\|/2\,\right]. \tag{3.1}$$

By (1), df is proper in $\overline{B}_{R_2}$, then there exists $m > 0$ such that

$$\|df(x)\| \geq m \quad \forall x \in \overline{B}_{R_2} \setminus N_\delta(K_f). \tag{3.2}$$

As ω_1 has bounded derivatives, if σ is small

$$\sigma\left[\,\|x\||\omega_1(x)| + \|x\|^2\|d\omega_1(x)\|/2\,\right] \leq m/2, \tag{3.3}$$

then, from (3.3), (3.2), (3.1) we infer that

$$\|dg_\sigma(x)\| \geq m/2 \quad \forall x \in \overline{B}_{R_2} \setminus N_\delta(K_f). \tag{3.4}$$

Now observe that

$$g_\sigma(x) = f(x) \quad \forall x \in U \setminus B_{R_2}, \tag{3.5}$$

and, for $x \in B_{R_1}$

$$\begin{aligned} g_\sigma(x) &= f(x) + \sigma/2\,\|x\|^2, \\ dg_\sigma(x) &= df(x) + \sigma(x,\,\cdot\,), \\ H^{g_\sigma}(x) &= H^f(x) + \sigma I, \end{aligned} \tag{3.6}$$

from which we get that 0 is still a critical point of g_σ. Moreover, if we denote by μ an eigenvalue of $H^{g_\sigma}(0)$, from (3.6) we get $\mu = \lambda + \sigma$ where λ is an eigenvalue of $H^f(0)$. Therefore, we have

$$\lambda > 0 \Rightarrow \mu > 0, \quad \lambda = 0 \Rightarrow \mu > 0,$$

and if σ is sufficiently small, we get also $\lambda < 0 \Rightarrow \mu < 0$. Then, according to hypothesis 2., 0 is a nondegenerate critical point of g_σ and

$$m(0, g_\sigma) = m(0, f).$$

As 0 is nondegenerate, it is isolated. Then there exists $\rho > 0$ such that $\overline{B}_{2\rho} \subset B_{R_1}$ and 0 is the only critical point of g_σ in B_ρ. If σ is suitably chosen, g_σ verifies (i)–(iv). Indeed, as ω_1 has bounded derivatives, we can take σ so small that

$$\|g_\sigma - f\|_{C^2(U)} \leq \varepsilon,$$

and if ε is sufficiently small dg_σ is still $\mathcal{F}_0$ in $\overline{B}_{R_2}$ (as the set of the Fredholm maps is open). The other properties are obviuosly verified. It remains to prove that if f satisfies (P.S.), the same is true for g_σ. We take a sequence $(x_n)_{n\in\mathbb{N}} \subset U$ such that $g_\sigma(x_n)$ is bounded and $dg_\sigma(x_n) \to 0$. If there exists $\overline{n} \in \mathbb{N}$ such that for $n \geq \overline{n}$, $x_n \notin B_{R_2}$, then $g_\sigma(x_n) = f(x_n)$ and (P.S.) holds. If for all $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ with $k_n \to \infty$, such that $x_{k_n} \in B_{R_2}$, thanks to (iv), dg_σ is proper in B_{R_2}, then, also in this case, there exists a converging subsequence. □

Now we need to extend the previous Lemma to the case of Riemannian manifolds. To this end, we shall use the following Lemma whose proof can be found in [7].

Lemma 3.2. *Let $\mathcal{M}$ be a Riemannian manifold of class C^2 on a Banach space X and Z be a compact subset of $\mathcal{M}$. Then there exists a sequence $(W_\alpha, \psi_\alpha)_{\alpha\in\mathcal{A}}$, with a finite set $\mathcal{A}$, such that*

(i) *for all $\alpha \in \mathcal{A}$, (W_α, ψ_α) is a local chart,*
(ii) *$Z \subset \bigcup_{\alpha\in\mathcal{A}} W_\alpha$,*
(iii) *for all $\alpha, \beta \in \mathcal{A}$ with $W_\alpha \cap W_\beta \neq \emptyset$, $\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(W_\alpha \cap W_\beta) \to \psi_\alpha(W_\alpha \cap W_\beta)$ has bounded derivatives.*

Now we can define a perturbation of a functional f on a Riemannian manifold $\mathcal{M}$.

Theorem 3.3. *Let $f \in C^2(\mathcal{M}, \mathbb{R})$ be a functional bounded from below and satisfying* (P.S.) *condition. Let $x_1, \dots, x_m \in \mathcal{M}$ be m distinct critical points of f, such that $H^f(x_i) = I - K_i$, for all $i = 1, \dots, m$, where K_i is a compact selfadjoint operator. We take $\varepsilon, \delta > 0$, sufficiently small. Then there exist $A_1, \dots, A_m \subset \mathcal{M}$ open disjoint neighbourhoods of $x_1, \dots, x_m$ respectively, such that*

$$\bigcup_{i=1}^{m} \overline{A}_i \subset N_\delta(K_f),$$

and there exists $g \in C^2(\mathcal{M}, \mathbb{R})$ such that

(i) *$g(x) = f(x)$ for $d(x, K_f) \geq \delta$, $\|g - f\|_{C^2(U)} \leq \varepsilon$,*
(ii) *$K_g \subset N_\delta(K_f)$,*
(iii) *for all $i = 1, \dots, m$ x_i is the only critical point of g in A_i, it is nondegenerate and $m(x_i, g) = m(x_i, f)$,*
(iv) *g satisfies* (P.S.) *condition.*

Proof. We only give an outline of the proof which is similar to the one given in [7], thanks to Lemma 3.1. We take $a \in \mathbb{R}$ such that for all $i = 1, \dots, m$

$$x_i \in f^a \quad \text{and} \quad d(x_i, \partial f^a) > \delta.$$

We shall modify f in f^a. Since (P.S.) holds, and f is bounded from below, $f^a \cap K_f$ is compact. Hence, applying Lemma 3.2, we find that there exists a finite sequence of charts $(W_\alpha, \psi_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$K_f \cap f^a \subset \bigcup_{\alpha\in\mathcal{A}} W_\alpha,$$

and the other properties of that Lemma hold. Then for all $i = 1, \dots, m$ there exists $\alpha_i \in \mathcal{A}$ with $x_i \in W_{\alpha_i}$. As x_i are distinct, it is possible to choose m open disjoint neighborhoods $U_1, \ldots, U_m$ of $x_1, \ldots, x_m$ in $\mathcal{M}$ such that, $U_i \subset W_{\alpha_i}$ and

$$\bigcup_{i=1}^{m} U_i \subset N_\delta(K_f). \tag{3.7}$$

We define now $\varphi_i = \psi_{\alpha_i|U_i}$, $V_i = \varphi_i(U_i)$ and $y_i = \varphi_i(x_i)$ where V_i is an open subset of H and consider $\widetilde{f}_1 = f \circ \varphi_1^{-1} : V_1 \to \mathbb{R}$. By Lemma 2.5, y_1 is a critical point of $\widetilde{f}_1$ and $d\widetilde{f}_1(y_1)$ is $\mathcal{F}_0$, then, by Lemma 2.8, there exists $R_1'' > 0$ such that $B_{R_1''}(y_1) \subset V_1$ and $d\widetilde{f}_1$ satisfies condition (S) in $B_{R_1''}(y_1)$. We take $R_1' > 0$, with $R_1' < R_1''$, $\varepsilon_1 < \varepsilon/m$ and δ_1 so small that

$$\varphi_1^{-1}(N_{\delta_1}(K_{\widetilde{f}_1})) \subset N_\delta(K_f). \tag{3.8}$$

Now we can apply Lemma 3.1 to $\widetilde{f}_1$, V_1, $B_{R_1''}(y_1)$, $B_{R_1'}(y_1)$ and ε_1, δ_1. We denote by $\widetilde{g}_1$ the perturbation of $\widetilde{f}_1$ given by that Lemma and define

$$f_1(x) = \begin{cases} (\widetilde{g}_1 \circ \varphi_1)(x) & x \in U_1, \\ f(x) & x \notin U_1. \end{cases}$$

With the same technique we can modify f_1 in U_2 and get a functional f_2 with the previous properties. We repeat this operation m times. At the end we shall get a functional $f_m = g$ that satisfies (i)–(iv). For any $i = 1, \ldots, m$, if ρ_i is as in (iii) of Lemma 3.1, the set A_i is given by $A_i = \varphi_i^{-1}(B_{\rho_i}(y_i))$. Obviously, we set $g(x) = f(x)$ in $x \notin f^a$. $\square$

Remark 3.4. Notice that, since f is bounded from below and (i) of the previous theorem holds, also g is bounded from below.

Using Theorem 3.3, we get a small perturbation g of our functional which has a fixed number of critical points of f as nondegenerate critical points keeping the same Morse index. But, at this point, other critical points of g may be degenerate. As our aim is to apply the classical Morse relations to the perturbed functional, we need a Morse functional. Then we shall use the method used by Marino and Prodi in [7]. In that paper the following Lemma was proved.

Lemma 3.5. *Let V, V', V'' be open subsets of a Riemannian manifold $\mathcal{M}$ on a Hilbert space H such that (V, φ) is a local chart with bounded derivatives. We set $\varphi(V) = U$, $\varphi(V') = U'$, $\varphi(V'') = U''$ and suppose that U', U'' are open balls in H with the same centre and $\overline{U}'' \subset U$, $\overline{U}' \subset U''$. We take $f \in C^2(\mathcal{M}, \mathbb{R})$ such that $d(f \circ \varphi^{-1})$ verifies condition (S) in U and $\varepsilon, \delta > 0$ sufficiently small. Then, for all $h \in C^2(\mathcal{M}, \mathbb{R})$ such that*

$$\|h - f\|_{C^2} \le \varepsilon, \tag{3.9}$$

and $K_h \subset N_\delta(K_f)$, there exists $l \in C^2(\mathcal{M}, \mathbb{R})$ such that

(i) *$l(x) = h(x)$ for $x \in \mathcal{M} \setminus U''$, $\|l - h\|_{C^2(U)} \le \varepsilon$,*

(ii) *the critical points of* l *in* $\overline{V'}$ *are nondegenerate and their number is finite,*
(iii) $K_l \subset N_\delta(K_f)$,
(iv) *let* A *be a subset of* $\mathcal{M}$ *such that in* $\overline{A}$ *all the critical points of* h *are nondegenerate and their number is finite, then the same property holds for* l,
(v) *if* h *satisfies* (P.S.) *condition, the same holds for* l.

REMARK 3.6. Differently from Marino–Prodi in [7], we do not suppose that our functional f is such that K_f is compact, but only that $K_f \cap f^a$ is compact, for all $a \in \mathbb{R}$. This fact obliges us to work in a sublevel or in a strip to define the perturbed functional.

THEOREM 3.7. *Let* $f \in C^2(\mathcal{M}, \mathbb{R})$, $x_1, \dots x_m$ *as in Theorem* 3.3. *We fix* $\varepsilon, \delta > 0$. *Then there exists* $h \in C^2(\mathcal{M}, \mathbb{R})$ *such that*

(1) $h(x) = f(x)$ *for* $d(x, K_f) \geq \delta$, $\|h - f\|_{C^2(U)} \leq \varepsilon$,
(2) *for all* $i = 1, \dots, m$, x_i *is a nondegenerate critical point of* h *and*
$$m(x_i, h) = m(x_i, f),$$
(3) *all the critical points of* h *are nondegenerate,*
(4) h *satisfies* (P.S.) condition.

PROOF. We consider a, A_i, $x_1, \dots, x_m$ as in Theorem 3.3 and the functional g given by the same Theorem. Thanks to Remark 3.4, the set
$$(K_g \cap g^a) \setminus \bigcup_{i=1}^m A_i$$
is compact. Hence, using again Lemma 3.2 and seeing that dg is $\mathcal{F}_0$ in K_g, with the same method as used in Theorem 3.3, we can construct a finite sequence of open subsets of $\mathcal{M}$
$$V_1, \dots, V_k \quad V'_1, \dots, V'_k \quad V''_1, \dots, V''_k,$$
such that

(a) V_i is the domain of a chart $\varphi_i : V_i \to \varphi_i(V_i) = U_i$ and $\varphi_i(V'_i) = U'_i$, $\varphi_i(V''_i) = U''_i$ where U'_i, U''_i are open balls in H with the same centre and $\overline{U}'_i \subset U''_i$, $\overline{U}''_i \subset U_i$,
(b) $\varphi_i \circ \varphi_j^{-1}$ has bounded derivatives,
(c) $x_i \notin \bigcup_{i=1}^k V_i$ for all $i = 1, \dots, m$,
$$(K_g \cap g^a) \setminus \bigcup_{i=1}^m A_i \subset \bigcup_{i=1}^k V'_i, \qquad \bigcup_{i=1}^k V''_i \subset N_\delta(K_f),$$
(d) if $f_i = f \circ \varphi_i^{-1}$, df_i verifies condition (S) in U_i.

We can apply Lemma 3.5 k times to V_i, V_i', V_i'' with $\varepsilon_i, \delta_i > 0$ suitably chosen. Notice that, as in each $\overline{V}_i$ the functional to be modified has only nondegenerate critical points, from (iv) of Lemma 3.5, the same property holds for the new functional. We denote by h the last modification which is defined in $g^{a+\eta}$ for some $\eta > 0$. As the critical points of each perturbation belong to $\bigcup_{i=1}^k V_i'$, h is a Morse functional and, as an easy consequence of Lemma 3.5, h satisfies (1)–(4). The last step is to define h on $\mathcal{M}$. To this end we take a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{n\to\infty} a_n = \infty$ and $a_0 = a$. We consider the set

$$g_{a_0+\eta}^{a_1} \cap K_g.$$

As (P.S.) condition holds for g, this set is compact. Then we can apply Lemma 3.5 a finite number of times modifying g also in $g_{a_0+\eta}^{a_1}$ and obtaining a functional defined on $g_{a_0+\eta}^{a_1+\eta_1}$, for some $\eta_1 > 0$, still satisfying (1)–(4). We can use these arguments in every $g_{a_{n-1}+\eta_{n-1}}^{a_n}$ getting a functional defined on $\mathcal{M}$. □

4. The generalized Morse relations

The results proved in previous sections allow us to define a generalized Morse index and write generalized Morse relations for a C^2 functional whose critical points are not necessarily nondegenerate. Before doing this, let's recall some well known facts of the classical theory.

Deffinition 4.1. Let X be a topological space, A a subspace of X and F a field. For any $k \in \mathbb{N}$ we denote by $H_k(X, A; F)$ the *singular homology groups* of the couple (X, A). The polynomial defined by

$$\mathcal{P}_\lambda(X, A, F) = \sum_{k=0}^{\infty} \dim H_k(X, A; F)\lambda^k,$$

is called *Poincaré polynomial* of the couple (X, A).

It is well known that the homology groups are topological invariants. The classical Morse relations link the Poincaré polynomial to the Morse polynomial of the critical points of a smooth functional. Indeed, the following result holds:

Theorem 4.2. *Let's consider a functional $f \in C^2(\mathcal{M}, R)$ where $\mathcal{M}$ is a complete Riemannian manifold. We take $a < b$, two regular values for f, and assume that f satisfies* (P.S.) *condition and every critical point of f in f_a^b is nondegenerate. Then the following relation holds:*

$$\sum_{p\in K_f\cap f_a^b} \lambda^{m(x,f)} = \mathcal{P}_\lambda(f^b, f^a; F) + (1+\lambda)\mathcal{Q}_\lambda, \tag{4.1}$$

where $\mathcal{Q}$ is a polynomial with positive integer coefficients.

Under the same hypothesis of Theorem 4.2 and if f is bounded from below, it is possible to send b to ∞ and obtain

$$\sum_{p\in K_f} \lambda^{m(x,f)} = \mathcal{P}_\lambda(\mathcal{M};F) + (1+\lambda)\mathcal{Q}_\lambda, \tag{4.2}$$

where $\mathcal{P}_\lambda(\mathcal{M};F) = \mathcal{P}_\lambda(\mathcal{M},\emptyset;F)$ and $\mathcal{Q}_\lambda$ is a formal series. For the proofs of (4.1) and (4.2) see for example [9].

Remark 4.3. The left hand side of (4.1) and (4.2) is the *Morse polynomial* of f and will be denoted by $m_\lambda(f)$. It can be written in the following form:

$$m_\lambda(f) = \sum_{k=0}^{\infty} a_k(f)\lambda^k, \tag{4.3}$$

where $a_k(f)$ denotes the number of critical points of f having Morse index equal to k.

Remark 4.4. As it is known, we can not expect to find relation (4.2) in the degenerate case. For example, we take $\mathcal{M} = S^1\times S^1$ and for $P=(a,b,x,y)\in\mathcal{M}$ set

$$f(P) = (1-a)y + (1-x)b.$$

Easy calculations show that f has the following critical points:

$$A=(1,0,1,0),\quad B=\Big(-\frac12,\frac{\sqrt3}{2},-\frac12,\frac{\sqrt3}{2}\Big),\quad C=\Big(-\frac12,-\frac{\sqrt3}{2},-\frac12,-\frac{\sqrt3}{2}\Big),$$

and A is degenerate. Moreover,

$$\mathcal{P}_\lambda(\mathcal{M}) = \mathcal{P}_\lambda(S^1)\mathcal{P}_\lambda(S^1) = (1+\lambda)(1+\lambda) = \lambda^2+2\lambda+1,$$

then, if (4.2) holds, we should obtain

$$a_0(f)\ge 1,\quad a_1(f)\ge 2,\quad a_2(f)\ge 1,$$

a contradiction. However, a limit process, based on the results of Section 3, allows us to generalize (4.2) to the degenerate case in order to have, once again, a relation between the differential structure and the topological structure of the manifold.

We take a functional $f\in C^2(\mathcal{M},R)$ that satisfies (P.S.) condition, bounded from below and whose hessian at a critical point is a Fredholm map of index 0. Then we take $n\in\mathbb{N}$ and for $k=0,\dots n$ consider $a_k(f)$. If $a_k(f)<\infty$, we fix all the critical points of f having Morse index k. As we do not suppose that the critical points of f are nondegenerate it could happen that $a_k(f)=\infty$. In this case, we can arbitrarily choose n critical points with index k. In this way we consider a finite number of critical points of f, which we can denote by

$x_1, \dots, x_m$. Using the perturbation Theorem 3.7 and taking $\varepsilon = \delta = 1/n$, for n sufficiently large, we find that there exists a C^2 functional h such that

(i) $h(x) = f(x)$ for $d(x, K_f) \geq 1/n$,
(ii) $\|h - f\|_{C^2} \leq 1/n$,
(iii) every critical point of h is nondegenerate,
(iv) h satisfies (P.S.) condition,
(v) for $k = 1, \dots, n$

$$a_k(f) < \infty \Rightarrow a_k(h) \geq a_k(f), \quad a_k(f) = \infty \Rightarrow a_k(h) \geq n.$$

Remark 4.5. Property (v) is a consequence of (2) of Theorem 3.7 and means that, even if f is not a Morse functional in the classical sense, it can be approximated by Morse functionals whose critical points are "as close and minimal as possible" to that of f.

Now, for all $n \in \mathbb{N}$ we define the class $\mathcal{F}_n$ of the C^2 functionals on a Riemannian manifold $\mathcal{M}$, satisfying (i)–(v). By Theorem 3.7, for all $n \in \mathbb{N}$, $\mathcal{F}_n \neq \emptyset$. Using the class $\mathcal{F}_n$ we can define a generalized Morse index. To this end it is useful to recall some definitions about formal series. Let's denote by S the family of the formal series in one variable λ with coefficients in $\mathbb{N} \cup \{\infty\}$. For $\mathcal{P} \in S$ we set

$$c_k(\mathcal{P}) = a_k \Leftrightarrow \mathcal{P} = \sum_{k=0}^{\infty} a_k \lambda^k.$$

We can consider the following total order relation on S:

$$\sum_{k=0}^{\infty} a_k \lambda^k < \sum_{k=0}^{\infty} b_k \lambda^k \Leftrightarrow \exists n \in \mathbb{N} \textit{ such that } a_k = b_k \textit{ for } k \leq n-1 \textit{ and } a_n < b_n.$$

Moreover, on S the notion of limit is defined in the following way:

$$\mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n \Leftrightarrow \lim_{n\to\infty} c_k(\mathcal{P}_n) = c_k(\mathcal{P}) \quad \forall k \in \mathbb{N}.$$

With the topology induced by the above convergence, S is compact (see [1]). If $A \subset S$, we shall denote by $\overline{A}$ the closure of A i.e.

$$\overline{A} = \left\{ \mathcal{P} \in S \;\middle|\; \exists (\mathcal{P}_n)_{n\in\mathbb{N}} \subset A : \mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n \right\}.$$

Now we can define the infimum as follows:

Definition 4.6. If $A \subset S$, we set $\mathcal{R} = \inf A$ if and only if $\mathcal{R} = \min \overline{A}$.

About the existence of the infimum you can see [1], Theorem 1.11. We just point out that, if we set, by induction

$$b_0 = \min\{c_0(\mathcal{P}) \mid \mathcal{P} \in \overline{A}\}, \qquad B_0 = \{\mathcal{P} \in \overline{A} \mid c_0(\mathcal{P}) = b_0\},$$
$$b_n = \min\{c_n(\mathcal{P}) \mid \mathcal{P} \in B_{n-1}\}, \qquad B_n = \{\mathcal{P} \in B_{n-1} \mid c_n(\mathcal{P}) = b_n\},$$

we find that

$$\bigcap_{n=0}^{\infty} B_n \neq \emptyset \quad \text{and} \quad \mathcal{R} \in \bigcap_{n=0}^{\infty} B_n,$$

is $\min \overline{A}$. Now we can give the following notion of the generalized Morse index.

Definition 4.7. Let $f \in C^2(\mathcal{M}, \mathbb{R})$ be bounded from below, satisfying the (P.S.) condition and for any $x \in K_f$, $H^f(x)$ be a Fredholm map of index 0. We define

$$\mathcal{A}_f = \left\{ \mathcal{P} \in S \,\middle|\, \exists h_n \in \mathcal{F}_n \ \forall n \in \mathbb{N} : \lim_{n\to\infty} m_\lambda(h_n) = \mathcal{P} \right\}.$$

The formal series defined by $i_\lambda(f) = \inf \mathcal{A}_f$ is called the *generalized Morse index* of f.

Remark 4.8. If f is a Morse functional (i.e. all its critical points are non-degenerate), taking into account Definition 4.7, it is not difficult to prove that the generalized Morse index coincides with the Morse polynomial (4.2).

Of course, i_λ has been defined in such a way that the Morse relations are still valid. Indeed the following theorem holds:

Theorem 4.9. *Let $f \in C^2(\mathcal{M}, \mathbb{R})$ be a functional on a complete Riemannian manifold $\mathcal{M}$, satisfying* (P.S.) *condition, whose hessian at each critical point is a Fredholm map of index* 0 *and is bounded from below. Then there exists a formal series $\mathcal{Q}_\lambda$ with integer positive coefficients such that*

$$i_\lambda(f) = \mathcal{P}_\lambda(\mathcal{M}) + (1+\lambda)\mathcal{Q}_\lambda. \tag{4.4}$$

Proof. From the definition of $i_\lambda(f)$, there exists a sequence $h_n \in \mathcal{F}_n$ such that

$$\lim_{n\to\infty} m_\lambda(h_n) = i_\lambda(f). \tag{4.5}$$

For all $n \in \mathbb{N}$, h_n satisfies the classical Morse relation, i.e.

$$m_\lambda(h_n) = \mathcal{P}_\lambda(\mathcal{M}) + (1+\lambda)\mathcal{Q}_\lambda^n, \tag{4.6}$$

where $\mathcal{Q}_\lambda^n$ is a formal series. As the first term of the right hand side of (4.6) is constant with respect to n, by (4.5), there exists a formal series Q_λ such that

$$\lim_{n\to\infty} \mathcal{Q}_\lambda^n = \mathcal{Q}_\lambda.$$

Then, taking the limit in (4.6), we complete our proof. □

5. Application to geodesics

Now we shall study the application of the theory developed in the previous sections to the case of geodesics with fixed extreme points. In particular, we shall investigate a geometrical meaning of the Morse index and we shall write the generalized Morse relations. At first, we recall some basic notions of Riemannian geometry. Let $(\mathcal{M}, \langle \cdot\,, \cdot \rangle)$ be a finite dimensional, complete Riemannian manifold of class C^∞. By virtue of the well known Nash theorem, $\mathcal{M}$ can be embedded in the Euclidean space $\mathbb{R}^N$, for sufficiently large $N > 0$. The Riemannian structure at $x \in \mathcal{M}$ is given by the restriction of the Euclidean scalar product of $\mathbb{R}^N$ to $T_x\mathcal{M}$.

Definition 5.1. We say that a smooth curve $x \in C^2([0,1], \mathcal{M})$ is a *geodesic* if x satisfies the equation

$$D_s\dot{x}(s) = 0,$$

where by D_s we denote the covariant derivative of $\dot{x}$ with respect to the Riemannian structure of $\mathcal{M}$.

Remark 5.2. Thanks to the embedding of $\mathcal{M}$ in R^N, the vector field $D_s\dot{x}(s)$ can be seen as the projection of $\ddot{x}(s)$ on the tangent space $T_{x(s)}\mathcal{M}$.

The problem of finding geodesics with fixed extreme points admits a variational formulation. More precisely, we consider $x_0, x_1 \in \mathcal{M}$ and define the space

$$\Omega^1(x_0, x_1, \mathcal{M}) = \{x \in H^{1,2}([0,1], \mathbb{R}^N) \mid \forall s \in [0,1],\\ x(s) \in \mathcal{M},\ x(0) = x_0,\ x(1) = x_1\}.$$

$\Omega^1(x_0, x_1, \mathcal{M})$ is a Hilbert manifold modelled on $H_0^1([0,1], \mathbb{R}^N)$, whose tangent space at $x \in \Omega^1(x_0, x_1, \mathcal{M})$ is given by

$$T_x\Omega^1(x_0, x_1, \mathcal{M}) = \{\xi \in H^{1,2}([0,1], \mathbb{R}^N) \mid \xi(s) \in T_{x(s)}\mathcal{M}\ \ \xi(0) = 0 = \xi(1)\}.$$

Later on, we shall need to know how local charts can be defined in

$$\Omega^1(x_0, x_1, \mathcal{M}).$$

To this end, we recall that for $x \in \mathcal{M}$ we can define the exponential map $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ in the following way $\exp_x v = \gamma(1)$, where $\gamma : [0,1] \to \mathcal{M}$ is the geodesic such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Since $\mathcal{M}$ is smooth, also $\exp_x v$ is smooth and is a local diffeomorphism. Then, we fix $\sigma \in \Omega^1(x_0, x_1, \mathcal{M})$ and for U, a suitably chosen neighbourhood of σ in $\Omega^1(x_0, x_1, \mathcal{M})$, set $\varphi_\sigma : U \to T_\sigma\Omega^1(x_0, x_1, \mathcal{M})$ such that

$$\varphi_\sigma(\lambda)(t) = \exp^{-1}_{\sigma(t)}\lambda(t). \tag{5.1}$$

It can be proved that (see e.g. [12]) the map φ_σ is smooth and is a local chart.

For $x \in \Omega^1(x_0, x_1, \mathcal{M})$ we consider a functional

$$f(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle\, ds. \tag{5.2}$$

It is well known that f is a smooth functional and for $x \in \Omega^1(x_0, x_1, \mathcal{M})$ and $\xi \in T_x\Omega^1(x_0, x_1, \mathcal{M})$

$$df(x)[\xi] = \int_0^1 \langle \dot{x}, D_s\xi\rangle\, ds. \tag{5.3}$$

Using (5.3), we get that every critical point of f is smooth and integrating by parts in (5.3), we find that x is a critical point of f if and only if x is a geodesic joining x_0 and x_1. For the proofs see e.g. [8], [10].

We denote by $H^f(x)$ the hessian of f at x, i.e. for any $x \in \Omega^1(x_0, x_1, \mathcal{M})$ and $\xi \in T_x\Omega^1(x_0, x_1, \mathcal{M})$

$$H^f(x)[\xi, \xi'] = \frac{d^2 f(\alpha(r, \cdot\,))}{dr^2}\bigg|_{r=0},$$

where $\alpha(r, s) :]{-\sigma}, \sigma[\, \times [0, 1] \to \mathcal{M}$ is a smooth map satisfying

$$\alpha(0, s) = x(s), \qquad \partial_r\alpha(0, s) = \xi(s), \qquad \alpha(r, 0) = x_0, \qquad \alpha(r, 1) = x_1.$$

Let x be a geodesic joining x_0 and x_1. For all $\xi, \xi' \in T_x\Omega^1(x_0, x_1, \mathcal{M})$ we have

$$H^f(x)[\xi, \xi'] = \int_0^1 [\langle D_s\xi, D_s\xi'\rangle - \langle R(\xi, \dot{x})\dot{x}, \xi'\rangle]\, ds, \tag{5.4}$$

where R is the Riemann curvature tensor of $\mathcal{M}$, (see e.g. [8] and also (6.16)). Moreover, by regularity results, $\xi \in \ker H^f(x)$ if and only if ξ satisfies the ordinary differential equation

$$\begin{cases} D_s^2\xi + R(\xi, \dot{x})\dot{x} = 0, \\ \xi(0) = 0 = \xi(1). \end{cases} \tag{5.5}$$

Every ξ satisfying (5.5) is called *Jacobi field.*

REMARK 5.3. By the definition of the tangent space, the scalar product on $T_x\Omega^1(x_0, x_1, \mathcal{M})$ is given by

$$\langle \xi, \xi'\rangle_1 = \int \langle \xi, \xi'\rangle\, ds + \int_0^1 \langle \dot{\xi}, \dot{\xi}'\rangle\, ds.$$

It can be proved (see [8]) that

$$\langle \xi, \xi'\rangle_0 = \int_0^1 \langle D_s\xi, D_s\xi'\rangle\, ds,$$

is a scalar product on $T_x\Omega^1(x_0, x_1, \mathcal{M})$ equivalent to $\langle\, \cdot\,, \cdot\,\rangle_1$.

As a consequence of the previous Remark and the Sobolev embedding theorems, the hessian defined by (5.4) is given by

$$H^f(x)[\xi,\xi] = \langle (I - K(x))\xi, \xi\rangle_0, \tag{5.6}$$

where $K(x)$ is a compact perturbation of the identity. Moreover, f satisfies (P.S.) condition (see [8], Proposition 2.11.4). Then, as f verifies all the hypothesis of Theorem 4.9, we can write the Morse relations.

Theorem 5.4. *Let $(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a complete, finite dimensional, smooth Riemannian manifold, $x_0, x_1 \in \mathcal{M}$ and f the functional defined by* (5.2). *Then there exists a formal series $\mathcal{Q}_\lambda$ such that*

$$i_\lambda(f) = \mathcal{P}_\lambda(\Omega^1(x_0, x_1, \mathcal{M})) + (1+\lambda)\mathcal{Q}_\lambda, \tag{5.7}$$

where $i_\lambda(f)$ is the generalized Morse index of f.

6. The Index Theorem

In this section we shall study the geometrical meaning of the Morse index. We shall see that the Morse polynomial can be written using an index defined by means of the hessian of the perturbed functional. Moreover, we shall prove an extension to the perturbed functional of the Index Theorem that is valid for f. Before doing that, it is necessary to state some properties of the small perturbations of f defined in Section 3. Let $x_0, x_1 \in \mathcal{M}$ and f be defined as in (5.2). Considering a functional $h_\varepsilon \in C^2(\Omega^1(x_0, x_1, \mathcal{M}), \mathbb{R})$ given by Theorem 3.7, we have

$$h_\varepsilon(x) = f(x) + g_\varepsilon(x), \tag{6.1}$$

with

$$\|g_\varepsilon\|_{C^2} \le \varepsilon, \tag{6.2}$$

for some $\varepsilon > 0$ as small as we want. To prove the Index Theorem, we need some regularity results about the critical points of h_ε. From the definition of local charts in $\Omega^1(x_0, x_1, \mathcal{M})$, taking into account the construction of h_ε we made Section 3, and fixing an open sublevel

$$f^b = \{x \in \Omega^1(x_0, x_1, \mathcal{M}) \mid f(x) < b\},$$

we can write h_ε in the following form:

$$\begin{aligned} h_\varepsilon(x) = f(x) &+ \omega_1\bigg(\int_0^1 \langle \dot{\overbrace{E_0(s, x(s))}}, \dot{\overbrace{E_0(s, x(s))}}\rangle\, ds\bigg) \\ &+ \sum_{i=1}^p \omega_{2,i}\bigg(\int_0^1 \langle \dot{\overbrace{E_i(s, x(s))}}, \dot{\overbrace{E_i(s, x(s))}}\rangle\, ds\bigg) \int_0^1 \langle \dot{\overbrace{E_i(s, x(s))}}, \dot\zeta_i(s)\rangle\, ds, \end{aligned} \tag{6.3}$$

where $p \in \mathbb{N}$ depends only on b and

$$E_i(s, x(s)) = \exp^{-1}_{\sigma_i(s)} x(s) \quad \forall i = 0, \dots, p,$$

for some $\sigma_i \in \Omega^1(x_0, x_1, \mathcal{M})$, where ζ_i are smooth curves in $H^1_0([0,1], \mathbb{R}^N)$ and $\omega_1, \omega_{2,i} \in C^\infty(\mathbb{R}, \mathbb{R})$. To avoid further technical complications, we can study the case $p = 1$. Evaluating the derivatives of E_i, we can write h_ε as follows

$$\begin{aligned} h_\varepsilon(x) = f(x) &+ \omega_{1,\varepsilon}(g_1(x) + g_2(x) + g_3(x)) \\ &\cdot \omega_{2,\varepsilon}(l_1(x) + l_2(x) + l_3(x))(I_1(x) + I_2(x)), \end{aligned} \tag{6.4}$$

where

$$\begin{aligned}
g_1(x) &= \int_0^1 \phi_1(s, x)\, ds, & g_2(x) &= \int_0^1 \langle \phi_2(s, x), \dot{x} \rangle\, ds, \\
g_3(x) &= \int_0^1 \langle \phi_3(s, x)\dot{x}, \dot{x} \rangle\, ds, \\
l_1(x) &= \int_0^1 \varphi_1(s, x)\, ds, & l_2(x) &= \int_0^1 \langle \varphi_2(s, x), \dot{x} \rangle\, ds, \\
l_3(x) &= \int_0^1 \langle \varphi_3(s, x)\dot{x}, \dot{x} \rangle\, ds, \\
I_1(x) &= \int_0^1 \psi_1(s, x)\, ds, & I_2(x) &= \int_0^1 \langle \psi_2(s, x), \dot{x} \rangle\, ds,
\end{aligned}$$

where $\phi_i, \varphi_i, \psi_i$ are smooth functions or vector fields on $\mathcal{M}$. Now we can state the following regularity result.

Theorem 6.1. *If $x_\varepsilon \in \Omega^1(x_0, x_1, \mathcal{M})$ is a critical point of h_ε, x_ε is a smooth curve.*

Proof. As x_ε is a critical point of h_ε, for all $\xi \in T_{x_\varepsilon}\Omega^1(x_0, x_1, \mathcal{M})$ we have

$$dh_\varepsilon(x_\varepsilon)[\xi] = 0.$$

Then taking a smooth map $\alpha = \alpha(r, s) : [0,1] \times [0,1] \to \mathcal{M}$ such that

$$\alpha(0, s) = x_\varepsilon(s), \quad \partial_r \alpha(0, s) = \xi(s), \quad \alpha(r, 0) = x_0, \quad \alpha(r, 1) = x_1, \tag{6.5}$$

we find that

$$\left. \frac{dh_\varepsilon(\alpha(r, \cdot))}{dr} \right|_{r=0} = 0. \tag{6.6}$$

Taking into account (6.4) and evaluating (6.6) we find that x_ε solves the following equation

$$\begin{aligned} \int_0^1 \langle \dot{x}_\varepsilon, D_s\xi \rangle\, ds &+ \varepsilon_1 \int_0^1 \langle A(s, x_\varepsilon), \xi \rangle\, ds + \varepsilon_2 \int_0^1 \langle B(s, x_\varepsilon)\dot{x}_\varepsilon, \xi \rangle\, ds \\ &+ \varepsilon_3 \int_0^1 \langle C(s, x_\varepsilon), D_s\xi \rangle\, ds + \varepsilon_4 \int_0^1 \langle F(s, x_\varepsilon)\dot{x}_\varepsilon, D_s\xi \rangle\, ds = 0, \end{aligned} \tag{6.7}$$

for all $\xi \in T_{x_\varepsilon}\Omega^1(x_0, x_1, \mathcal{M})$. We notice that A, B, C, F are smooth functions or vector fields on $\mathcal{M}$ and, thanks to (6.2), the constants ε_i are as small as we want and they depend only on x_ε. Now we can complete the proof using a boot-strap argument. Taking $v \in C_0^\infty([0,1], \mathbb{R}^N)$, we can write

$$v = P(x_\varepsilon)v + Q(x_\varepsilon)v, \tag{6.8}$$

where $P(x)$ is the projection on $T_x\mathcal{M}$ and $Q(x)$ is the projection on $T_x\mathcal{M}^\perp$. Setting $\xi = P(x_\varepsilon)v \in T_{x_\varepsilon}\Omega^1(x_0, x_1, \mathcal{M})$ and differentiating (6.8) we find that

$$D_s\xi = D_s v - P(x_\varepsilon) \circ dQ(x_\varepsilon)[\dot{x}_\varepsilon]v = D_s v - L(x_\varepsilon, \dot{x}_\varepsilon)v. \tag{6.9}$$

Then, from (6.8) and (6.9), substituting in (6.7) we get

$$\begin{aligned}\int_0^1 \langle (I + \varepsilon_4 F(s, x_\varepsilon))\dot{x}_\varepsilon + \varepsilon_3 C(s, x_\varepsilon), \dot{v}\rangle\, ds \\ - \int_0^1 \langle (I + \varepsilon_4 F(s, x_\varepsilon))\dot{x}_\varepsilon + \varepsilon_3 C(s, x_\varepsilon), L(x_\varepsilon, \dot{x}_\varepsilon)v\rangle\, ds \\ + \varepsilon_1 \int_0^1 \langle A(s, x_\varepsilon), v\rangle\, ds + \varepsilon_2 \int_0^1 \langle B(s, x_\varepsilon)\dot{x}_\varepsilon, v\rangle\, ds = 0.\end{aligned}$$

Hence we get

$$\int_0^1 \langle (I + \varepsilon_4 F(s, x_\varepsilon))\dot{x}_\varepsilon + \varepsilon_3 C(s, x_\varepsilon), \dot{v}\rangle\, ds = \int_0^1 \langle h, v\rangle\, ds, \tag{6.10}$$

where

$$\begin{aligned}h = {} & L(x_\varepsilon, \dot{x}_\varepsilon)^* \circ (I + \varepsilon_4 F(s, x_\varepsilon))\dot{x}_\varepsilon - \varepsilon_1 A(s, x_\varepsilon) \\ & - \varepsilon_2 B(s, x_\varepsilon)\dot{x}_\varepsilon + \varepsilon_3 L(x_\varepsilon, \dot{x}_\varepsilon)^* \circ C(s, x_\varepsilon).\end{aligned}$$

Now we see that $h \in L^1([0,1], \mathbb{R}^N)$, hence

$$H(s) = -\int_0^s h(\tau)\, d\tau,$$

is absolutely continuous and $H' = -h$. Integrating the right hand side of (6.10) by parts, we get

$$\int_0^1 \langle (I + \varepsilon_4 F(s, x_\varepsilon))\dot{x}_\varepsilon + \varepsilon_3 C(s, x_\varepsilon), \dot{v}\rangle\, ds = \int_0^1 \langle H, \dot{v}\rangle\, ds,$$

then, as v is arbitrary, there exists $c \in \mathbb{R}^N$ such that

$$(I + \varepsilon_4 F(s, x_\varepsilon))\dot{x}_\varepsilon + \varepsilon_3 C(s, x_\varepsilon) = H + c. \tag{6.11}$$

Because operator $I + \varepsilon_4 F(s, x_\varepsilon)$ is invertible (it is a small perturbation of the identity), then from (6.11)

$$\dot{x}_\varepsilon = (I + \varepsilon_4 F(s, x_\varepsilon))^{-1} H_1, \tag{6.12}$$

where H_1 is a continuous function. From the definition of $I + \varepsilon_4 F(s, x_\varepsilon)$ and (6.12), also $\dot{x}_\varepsilon$ is continuous, that is x_ε is C^1. Then h is continuous and H is C^1. Also H_1 is C^1, then from (6.12) x_ε is C^2. Repeating this argument, we get that x_ε is smooth. $\square$

Now we need to study the hessian of h_ε at a critical point. Let

$$x_\varepsilon \in \Omega^1(x_0, x_1, \mathcal{M})$$

be a critical point of h_ε. It is well known that if $\xi \in T_{x_\varepsilon}\Omega^1(x_0, x_1, \mathcal{M})$

$$H^{h_\varepsilon}(x_\varepsilon)[\xi, \xi] = \left.\frac{d^2 h_\varepsilon(\alpha(r, \cdot))}{dr^2}\right|_{r=0},$$

where $\alpha = \alpha(r, s) : [0,1] \times [0,1] \to \mathcal{M}$ is a smooth map satisfying (6.5). Moreover, $H^{h_\varepsilon}(x_\varepsilon)[\xi, \xi]$ does not depend on the choice of α. Hence, we can take α solving the following Cauchy problem, for all $s \in [0,1]$

$$\begin{cases} D_r \partial_r \alpha(r, s) = 0, \\ \alpha(0, s) = x_\varepsilon(s), \\ \partial_r \alpha(0, s) = \xi(s). \end{cases} \tag{6.13}$$

It can be proved (see [11]) that

$$D_r \partial_s \alpha(r, s) = D_s \partial_r \alpha(r, s), \tag{6.14}$$

and if Y is a vector field along x_ε

$$D_r D_s Y - D_s D_r Y = R(\partial_r \alpha, \partial_s \alpha) Y. \tag{6.15}$$

Then using (6.13)–(6.15) we can evaluate

$$\frac{df(\alpha(r, \cdot))}{dr} = \int_0^1 \langle D_r \partial_s \alpha, \partial_s \alpha \rangle \, ds = \int_0^1 \langle D_s \partial_r \alpha, \partial_s \alpha \rangle \, ds$$

and

$$\begin{aligned} \frac{d^2 f(\alpha(r, \cdot))}{dr^2} &= \int_0^1 \langle D_r D_s \partial_r \alpha, \partial_s \alpha \rangle \, ds + \int_0^1 \langle D_s \partial_r \alpha, D_r \partial_s \alpha \rangle \, ds \\ &= \int_0^1 \langle D_s D_r \partial_r \alpha, \partial_s \alpha \rangle \, ds + \int_0^1 \langle R(\partial_r \alpha, \partial_s \alpha) \partial_r \alpha, \partial_s \alpha \rangle \\ &\quad + \int_0^1 \langle D_s \partial_r \alpha, D_s \partial_r \alpha \rangle \, ds \\ &= \int_0^1 \langle R(\partial_r \alpha, \partial_s \alpha) \partial_r \alpha, \partial_s \alpha \rangle \, ds + \int_0^1 \langle D_s \partial_r \alpha, D_s \partial_r \alpha \rangle \, ds, \end{aligned}$$

from which we get, using the well known properties of R

$$\left.\frac{d^2 f(\alpha(r, \cdot))}{dr^2}\right|_{r=0} = \int_0^1 \langle D_s \xi, D_s \xi \rangle - \langle R(\xi, \dot{x}_\varepsilon) \dot{x}_\varepsilon, \xi \rangle \, ds. \tag{6.16}$$

As x_ε is a critical point of h_ε, also $d^2 h_\varepsilon(\alpha(r,\cdot))/dr^2|_{r=0}$ is well defined so, from (6.1), $d^2 g_\varepsilon(\alpha(r,\cdot))/dr^2|_{r=0} = H^{g_\varepsilon}(x_\varepsilon)[\xi,\xi]$ is well defined. Then, choosing the map α as in (6.13), we can write for all $\xi,\xi' \in T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M})$

$$H^{h_\varepsilon}(x_\varepsilon)[\xi,\xi'] = \int_0^1 \langle D_s\xi, D_s\xi'\rangle - \langle R(\xi,\dot{x}_\varepsilon)\dot{x}_\varepsilon,\xi'\rangle\,ds + H^{g_\varepsilon}(x_\varepsilon)[\xi,\xi']. \tag{6.17}$$

Using the Riesz Representation Theorem, as $H^{g_\varepsilon}(x_\varepsilon)$ is a bilinear form on $T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M})$, there exists $A(x_\varepsilon) \in \mathcal{L}(T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M}), T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M}))$ such that

$$H^{g_\varepsilon}(x_\varepsilon)[\xi,\xi'] = \langle A(x_\varepsilon)\xi,\xi'\rangle_0,$$

then

$$\begin{aligned} H^{h_\varepsilon}(x_\varepsilon)[\xi,\xi'] \\ = \int_0^1 [\langle D_s\xi, D_s\xi'\rangle - \langle R(\xi,\dot{x}_\varepsilon)\dot{x}_\varepsilon,\xi'\rangle + \langle D_s(A(x_\varepsilon)\xi), D_s\xi'\rangle]\,ds, \end{aligned} \tag{6.18}$$

and from (6.2)

$$\|A(x_\varepsilon)\| \le \varepsilon. \tag{6.19}$$

Taking into account the expression of h_ε given by (6.3) with $p=1$ and integrating by parts where it is possible we can write the hessian at a critical point x_ε of h_ε. For any $\xi,\xi' \in T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M})$, we have

$$\begin{aligned} H^{h_\varepsilon}(x_\varepsilon)[\xi,\xi'] &= \int_0^1 (\langle D_s\xi, D_s\xi'\rangle - \langle R(\xi,\dot{x}_\varepsilon)\dot{x}_\varepsilon,\xi'\rangle)\,ds \\ &+ \eta_1\int_0^1 \langle A_1(s)\xi,\xi'\rangle\,ds + \eta_2\int_0^1\langle A_2(s)\xi, D_s\xi'\rangle\,ds \\ &+ \eta_3\int_0^1 \langle A_3(s)D_s\xi,\xi'\rangle\,ds + \eta_4\int_0^1\langle A_4(s)D_s\xi, D_s\xi'\rangle\,ds \\ &+ \sum_{i=1}^3 \varepsilon_i \int_0^1 \langle F_i(s),\xi\rangle\,ds \int_0^1 \langle G_i(s),\xi'\rangle\,ds \\ &+ \sum_{i=1}^3 \varepsilon_i \int_0^1 \langle F_i(s),\xi'\rangle\,ds \int_0^1 \langle G_i(s),\xi\rangle\,ds, \end{aligned} \tag{6.20}$$

where η_i and ε_i are positive constants less or equal ε, depending only on x_ε, and A_i, B_i, C_i, F_i, G_i are smooth functions.

Remark 6.2. Changing the scalar product on $T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M})$ we find that

$$H^{h_\varepsilon}(x_\varepsilon) = I - K_\varepsilon,$$

where K_ε is a compact selfadjoint operator. Then, $m(x_\varepsilon,h_\varepsilon)$ is given by the number of negative eigenvalues of $I-K_\varepsilon$. Indeed, thanks to the previous calculations and (5.6) we see that

$$H^{h_\varepsilon}(x_\varepsilon)[\xi,\xi] = \langle (I-K(x_\varepsilon))\xi,\xi\rangle_0 + \langle A(x_\varepsilon)\xi,\xi\rangle_0.$$

By (6.19) and the continuity of $A(x_\varepsilon)$,

$$\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon = \langle (I+A(x_\varepsilon))\,\cdot\,,\,\cdot\,\rangle_0$$

is a scalar product equivalent to $\langle\,\cdot\,,\,\cdot\,\rangle_0$ and $(I+A(x_\varepsilon))$ is a selfadjoint invertible operator. With respect to the new scalar product we find that

$$H^{h_\varepsilon}(x_\varepsilon) = I - (I+A(x_\varepsilon))^{-1}K(x_\varepsilon).$$

Easy calculations show that $K_\varepsilon = (I+A(x_\varepsilon))^{-1}K(x_\varepsilon)$ is compact and selfadjoint.

Now we can introduce the definition of conjugate points as a generalization of that usually given for geodesics. We take $x \in \Omega^1(x_0,x_1,\mathcal{M})$ and $\sigma \in\,]0,1]$. Let us define the subspace of $T_x\Omega^1(x_0,x_1,\mathcal{M})$ given by

$$T_x\Omega^1_\sigma = \{\xi \in T_x\Omega^1(x_0,x_1,\mathcal{M}) \mid \xi(s)=0 \ \ \forall s\in[\sigma,1]\}.$$

Definition 6.3. We consider $h \in C^2(\Omega^1(x_0,x_1,\mathcal{M}),\mathbb{R})$ and choose x a critical point of h. Take $\sigma \in\,]0,1]$. We say that $x(\sigma)$ is *conjugate* to x_0 along x if $H^h(x)_{|T_x\Omega^1_\sigma}$ is degenerate, that is

$$M_\sigma = \ker H^h(x)_{|T_x\Omega^1_\sigma} \neq \{0\}.$$

The *multiplicity* of $x(\sigma)$ is the dimension of M_σ.

Remark 6.4. If $H^h(x)$ is a Fredholm map of index 0 and $x(\sigma)$ is conjugate to x_0, $x(\sigma)$ has finite multiplicity.

Definition 6.5. Let x and h be the same as in Definition (6.3). The *geometric index* of x, $\mu(x,h)$ is the number of conjugate points along x, counted with their multiplicity.

Remark 6.6. If x is a nondegenerate critical point of h, $x_1 = x(1)$ is not conjugate to x_0 along x.

Remark 6.7. Let h_ε be a perturbation of f defined in (6.1), given by Theorem 3.7. Then x_1 is not conjugate to x_0 along any x_ε critical point of h_ε.

We take x_ε such that $H^{h_\varepsilon}(x_\varepsilon) = I - K_\varepsilon$, where I is the identity and K_ε is the compact selfadjoint operator of Remark 6.2. Then

$$H^{h_\varepsilon}(x_\varepsilon)|_{T_{x_\varepsilon}\Omega^1_\sigma} = I - K^\sigma_\varepsilon,$$

where $K^\sigma_\varepsilon = K_{\varepsilon|T_{x_\varepsilon}\Omega^1_\sigma}$ is still a compact selfadjoint operator. From the theory of this kind of operators, the eigenvalues of K^σ_ε have finite multiplicity and are a decreasing sequence that we shall denote by

$$\lambda^\varepsilon_1(\sigma) \geq \ldots \geq \lambda^\varepsilon_k(\sigma) \geq \ldots \tag{6.21}$$

Each eigenvalue is repeated according to its multiplicity. It is obvious that $x_\varepsilon(\sigma)$ is conjugate to x_0 along x_ε with multiplicity m if and only if there exists $k \in \mathbb{N}$ such that

$$\lambda^\varepsilon_{k+1}(\sigma) = \ldots = \lambda^\varepsilon_{k+m}(\sigma) = 1.$$

Remark 6.8. We see that $\xi \in \ker(\lambda^\varepsilon_k(\sigma)I - K^\sigma_\varepsilon)$ for some k, if and only if ξ satisfies the following equation

$$\lambda^\varepsilon_k(\sigma)\int_0^\sigma \langle D_s(\xi + A(x_\varepsilon)\xi), D_s\xi'\rangle\, ds - \int_0^\sigma \langle R(\xi, \dot{x}_\varepsilon)\dot{x}_\varepsilon, \xi'\rangle\, ds, \tag{6.22}$$

for all $\xi' \in T_{x_\varepsilon}\Omega^1_\sigma$.

Remark 6.9. In the following Theorem, the regularity of ξ can be proved using a general theorem about regularity (see [6], Theorem 1.1'), since in one variable the situation is simpler. However, we prefer to give a direct proof.

We find the following

Theorem 6.10. *If $\xi \in \ker(\lambda^\varepsilon_k(\sigma)I - K^\sigma_\varepsilon)$ for some k, ξ is smooth.*

Proof. Taking into account (6.22) and (6.20) (where, for the sake of simplicity we suppose that $i = 1$), for all $\xi' \in T_{x_\varepsilon}\Omega^1_\sigma$, we find that ξ satisfies the following equation

$$\begin{aligned}
\lambda^\varepsilon_k(\sigma)\bigg[\int_0^\sigma &\langle \dot{\xi}, D_s\xi'\rangle\, ds + \eta_1\int_0^\sigma \langle A_1(s)\xi, \xi'\rangle\, ds \\
&+ \eta_2\int_0^\sigma \langle A_2(s)\xi, D_s\xi'\rangle\, ds + \eta_3\int_0^\sigma \langle A_3(s)D_s\xi, \xi'\rangle\, ds \\
&+ \eta_4\int_0^\sigma \langle A_4(s)D_s\xi, D_s\xi'\rangle\, ds + \varepsilon_1\int_0^\sigma \langle F(s), \xi\rangle\, ds\int_0^\sigma \langle G(s), \xi'\rangle\, ds \\
&+ \varepsilon_1\int_0^\sigma \langle G(s), \xi\rangle\, ds\int_0^\sigma \langle F(s), \xi'\rangle\, ds\bigg] - \int_0^\sigma \langle R(\xi, \dot{x}_\varepsilon)\dot{x}_\varepsilon, \xi'\rangle\, ds = 0,
\end{aligned} \tag{6.23}$$

where F and G are smooth. For $x \in \mathcal{M}$, we denote by $P(x)$ and $Q(x)$ the projection on $T_x\mathcal{M}$ and $T_x\mathcal{M}^\perp$ respectively and see that, as $\xi(s) \in T_{x_\varepsilon(s)}\mathcal{M}$ for all $s \in [0, \sigma]$,

$$Q(x_\varepsilon)\xi = 0. \tag{6.24}$$

Differentiating (6.24), we get $dQ(x_\varepsilon)[\dot{x}_\varepsilon]\xi + Q(x_\varepsilon)\dot{\xi} = 0$, then

$$\dot{\xi} = D_s\xi + Q(x_\varepsilon)\dot{\xi} = D_s\xi - dQ(x_\varepsilon)[\dot{x}_\varepsilon]\xi. \tag{6.25}$$

By (6.25), it is sufficient to prove that $\xi_0 = D_s\xi$ is smooth. We can write (6.23) as

$$\begin{aligned}
\lambda^\varepsilon_k(\sigma)&\int_0^\sigma \langle (I + \eta_4 A_4(s))\xi_0, \dot{\xi}'\rangle\, ds \\
&+ \eta_2\,\lambda^\varepsilon_k(\sigma)\int_0^\sigma \langle A_2(s)\xi, \dot{\xi}'\rangle\, ds + \int_0^\sigma \langle T(s)\xi, \xi'\rangle\, ds = 0,
\end{aligned} \tag{6.26}$$

where

$$\begin{aligned} T(s)\xi = \lambda_k^\varepsilon(\sigma)\bigg[&\eta_1 A_1(s)\xi + \eta_3 A_3(s) D_s\xi + \varepsilon_1 \int_0^\sigma \langle F(s),\xi\rangle\, ds\, G \\ &+ \varepsilon_1 \int_0^\sigma \langle G(s),\xi\rangle\, ds\, F\bigg] - R(\xi,\dot x_\varepsilon)\dot x_\varepsilon. \end{aligned}$$

Now we take $v \in C_0^\infty([0,1],\mathbb{R}^N)$. As $v = P(x_\varepsilon)v + Q(x_\varepsilon)v$, we set

$$\xi' = P(x_\varepsilon)v = v - Q(x_\varepsilon)v \in T_{x_\varepsilon}\Omega^1(x_0,x_1,\mathcal{M}),$$

then

$$\dot\xi' = \dot v - dQ(x_\varepsilon)[\dot x_\varepsilon]v - Q(x_\varepsilon)\dot v.$$

Now, substituting ξ' and $\dot\xi'$ in (6.26), using a boot-strap argument and (6.25), we infer that the proof is completely analogous to that of Theorem 6.1. $\square$

In proving the Index Theorem we need the following two Lemmas.

LEMMA 6.11. *We take $\varepsilon > 0$ small and $a \in \mathbb{R}$. Let h_ε be the perturbation of f defined in* (6.1) *and x_ε be a critical point of h_ε such that*

$$h_\varepsilon(x_\varepsilon) \le a.$$

For $\sigma > 0$, let ξ_ε be an eigenfunction of K_ε^σ (for some eigenvalue $\lambda_k^\varepsilon(\sigma)$ of the sequence (6.21)*) such that*

$$\|\xi_\varepsilon\|_{H_0^1([0,1],\mathbb{R}^N)} = 1. \tag{6.27}$$

Then there exists $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, a geodesic $x \in \Omega^1(x_0,x_1,\mathcal{M})$ and $\xi \in T_x\Omega_\sigma^1$ with

$$\xi \in \ker(\lambda(\sigma)I - K^\sigma(x)), \tag{6.28}$$

where $K^\sigma(x) = K(x)_{|T_x\Omega_\sigma^1}$ and $\lambda(\sigma)$ is an eigenvalue of K^σ, such that

$$\lim_{n\to\infty} x_{\varepsilon_n} = x \quad \text{in } C^2([0,1],\mathbb{R}^N) \quad \text{and} \qquad \lim_{n\to\infty} \xi_{\varepsilon_n} = \xi \quad \text{in } C^2([0,\sigma],\mathbb{R}^N).$$

PROOF. Thanks to Theorem 6.1 and Theorem 6.10, we see that x_ε and ξ_ε are smooth. From the definition of h_ε, and (6.2) we get that $f(x_\varepsilon)$ is bounded. Moreover, as x_ε is a critical point of h_ε,

$$\|df(x_\varepsilon)\| = \|dg_\varepsilon(x_\varepsilon)\| \le \varepsilon.$$

Then as the (P.S.) condition holds for f, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 and a geodesic $x \in \Omega^1(x_0,x_1,\mathcal{M})$ such that

$$\lim_{n\to\infty} x_{\varepsilon_n} = x,$$

in $H_0^1([0,1],\mathbb{R}^N)$. Thanks to the regularity of x_{ε_n}, taking the limit in the equation satisfied by x_{ε_n} as a critical point of h_{ε_n}, we have the C^2 convergence. We consider ξ_{ε_n} now. From (6.27), up to a subsequence, there exists $\xi \in H_0^1([0,1],\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \xi_{\varepsilon_n} = \xi,$$

weakly in $H_0^1([0,1],\mathbb{R}^N)$ and uniformly. As for all $s \in [0,1]$, $\xi_{\varepsilon_n}(s) \in T_{x_{\varepsilon_n}(s)}\mathcal{M}$, $\xi(s) \in T_{x(s)}\mathcal{M}$, then $\xi \in T_x\Omega^1_\sigma$. For all $n \in \mathbb{N}$, ξ_{ε_n} satisfies the following differential equation in $[0,\sigma]$:

$$\begin{aligned}\lambda_{\varepsilon_n}(\sigma)\bigg[&D_s^2\xi_{\varepsilon_n} + \eta_{1,n}A_1(s)\xi_{\varepsilon_n} + \eta_{2,n}A_2(s)D_s\xi_{\varepsilon_n} \\ &+ \eta_{3,n}A_3(s)D_s^2\xi_{\varepsilon_n} + \sum_{i=1}^3 \varepsilon_{i,n}\int_0^\sigma \langle F_i(s),\xi_{\varepsilon_n}\rangle\,ds G_i \\ &+ \sum_{i=1}^3 \varepsilon_{i,n}\int_0^\sigma \langle G_i(s),\xi_{\varepsilon_n}\rangle\,ds\, F_i\bigg] + R(\xi_{\varepsilon_n},\dot x_{\varepsilon_n})\dot x_{\varepsilon_n} = 0,\end{aligned} \tag{6.29}$$

where $\lim_{n\to\infty}\eta_{i,n} = 0$ and $\lim_{n\to\infty}\varepsilon_{i,n} = 0$. Multiplying (6.29) by ξ_{ε_n} and integrating we get that $\lambda^k_{\varepsilon_n}(\sigma)$ is bounded with respect to n, then up to a subsequence, there exists $\lambda(\sigma) \in \mathbb{R}$ such that

$$\lim_{n\to\infty}\lambda^k_{\varepsilon_n}(\sigma) = \lambda(\sigma).$$

Then taking the limit in (6.29), we get that ξ_{ε_n} converges in $C^2([0,\sigma],\mathbb{R}^N)$ and ξ satisfies the equation

$$\lambda(\sigma)D_s^2\xi + R(\xi,\dot x)\dot x = 0, \tag{6.30}$$

that is, (6.28) holds true. □

Lemma 6.12. *There exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in\,]0,\varepsilon_0[$, if x_ε is a critical point of h_ε, $\sigma,\tau \in [0,1]$ with $\sigma < \tau$ and $\xi_\varepsilon \in T_{x_\varepsilon}\Omega^1_\sigma$ with*

$$\|\xi_\varepsilon\|_{H_0^1([0,1],\mathbb{R}^N)} = 1, \tag{6.31}$$

then $\xi_\varepsilon \notin \ker(\lambda^\varepsilon_k(\tau)I - K^\tau_\varepsilon)$, for all $k \in \mathbb{N}$.

Proof. We notice that $\sigma < \tau$ implies $T_{x_\varepsilon}\Omega^1_\sigma \subset T_{x_\varepsilon}\Omega^1_\tau$, then $\xi_\varepsilon \in T_{x_\varepsilon}\Omega^1_\tau$. If, by contradiction, $\xi_\varepsilon \in \ker(\lambda^\varepsilon_k(\tau)I - K^\tau_\varepsilon)$, for some $k \in \mathbb{N}$, as $\xi_\varepsilon \in T_{x_\varepsilon}\Omega^1_\sigma$ and from the regularity of ξ_ε in $[0,\tau]$, we get

$$\xi_\varepsilon(\sigma) = 0 \qquad \dot\xi_\varepsilon(\sigma) = 0. \tag{6.32}$$

But, as we stated in Lemma 6.11, x_ε and ξ_ε converge respectively to a geodesic x and to $\xi \in T_x\Omega^1_\tau$. Then, from (6.32),

$$\xi(\sigma) = 0 \quad\text{and}\quad \dot\xi(\sigma) = 0, \tag{6.33}$$

so we get $\xi = 0$, recalling the regularity of the solutions of the equation (6.30). This is in contradiction with (6.31), using the uniform convergence. $\square$

Now we can state the Index Theorem. We adapt to our case the proofs used in [9] and [8] respectively for Hamiltonian systems and for geodesics on Lorentzian manifolds.

Theorem 6.13. *We take $h_\varepsilon \in C^2(\mathcal{M}, \mathbb{R})$, and the perturbation of f defined in* (6.1). *Let x_ε be a critical point of h_ε. Then we have*

$$m(x_\varepsilon, h_\varepsilon) = \mu(x_\varepsilon, h_\varepsilon).$$

Proof. We divide the proof into several steps.

Step 1. *The eigenvalues $\lambda_k^\varepsilon(\sigma)$ defined in* (6.21) *are continuous functions of σ.*

It is an easy consequence of the variational characterization of eigenvalues given by Theorem 2.10, which holds for $\lambda_k^\varepsilon(\sigma)$.

Step 2. *If σ is small, $H^{h_\varepsilon}(x_\varepsilon)_{|T_{x_\varepsilon}\Omega^1_\sigma}$ is positive definite.*

We see that, from the definition of $T_{x_\varepsilon}\Omega^1_\sigma$ and (6.18), we have for all $\xi \in T_{x_\varepsilon}\Omega^1_\sigma$

$$H^{h_\varepsilon}(x_\varepsilon)[\xi,\xi] = \int_0^\sigma [\langle D_s\xi, D_s\xi\rangle - \langle R(\xi,\dot x_\varepsilon)\dot x_\varepsilon, \xi\rangle\, ds + \langle D_s(A(x_\varepsilon)\xi), D_s\xi\rangle]\, ds. \tag{6.34}$$

From the continuity of R it follows that, there exists $c > 0$ such that

$$\int_0^\sigma \langle R(\xi,\dot x_\varepsilon)\dot x_\varepsilon, \xi\rangle\, ds \le c \int_0^\sigma \langle \xi,\xi\rangle\, ds. \tag{6.35}$$

We take $s \in [0,\sigma]$ and, using Hölder inequality, we get

$$\begin{aligned}\langle \xi(s), \xi(s)\rangle &= \int_0^s \frac{d}{d\tau}\langle \xi,\xi\rangle\, d\tau = 2\int_0^s \langle D_\tau\xi, \xi\rangle\, d\tau \le 2\int_0^s |D_\tau\xi||\xi|\, d\tau \\ &\le 2\left(\int_0^\sigma \langle D_\tau\xi, D_\tau\xi\rangle\, d\tau\right)^{1/2}\left(\int_0^\sigma \langle \xi,\xi\rangle\, d\tau\right)^{1/2}.\end{aligned} \tag{6.36}$$

Integrating (6.36) on $[0,\sigma]$ we find that

$$\int_0^\sigma \langle \xi,\xi\rangle\, ds \le 2\sigma\left(\int_0^\sigma \langle D_s\xi, D_s\xi\rangle\, ds\right)^{1/2}\left(\int_0^\sigma \langle \xi,\xi\rangle\, ds\right)^{1/2},$$

from which

$$\int_0^\sigma \langle \xi,\xi\rangle\, ds \le 4\sigma^2 \int_0^\sigma \langle D_s\xi, D_s\xi\rangle\, ds. \tag{6.37}$$

Moreover, from (6.19) we get

$$\left|\int_0^\sigma \langle D_s(A(x_\varepsilon)\xi), D_s\xi\rangle\, ds\right| \le \varepsilon \int_0^\sigma \langle D_s\xi, D_s\xi\rangle\, ds. \tag{6.38}$$

Hence, from (6.35), (6.37), (6.38), substituting in (6.34), we conclude that there exists $M > 0$ such that, for all $\xi \in T_{x_\varepsilon}\Omega^1_\sigma$

$$H^{h_\varepsilon}(x_\varepsilon)[\xi,\xi] \geq (1 - M\sigma^2 - \varepsilon)\int_0^\sigma \langle D_s\xi, D_s\xi\rangle\, ds.$$

Now, taking ε and σ sufficiently small we complete the proof.

Step 3. *The eigenvalues of $H^{h_\varepsilon}(x_\varepsilon)_{|T_{x_\varepsilon}\Omega^1_\sigma}$ are decreasing with respect to σ.*

It is sufficient to prove that $\lambda^\varepsilon_k(\sigma)$ is increasing with respect to σ. We fix $\sigma, \tau \in\,]0,1]$, with $\sigma \leq \tau$ and notice that $T_{x_\varepsilon}\Omega^1_\sigma \subset T_{x_\varepsilon}\Omega^1_\tau$. As the variational characterization holds, there exists a subspace $V \subset T_{x_\varepsilon}\Omega^1_\sigma$, $\dim V = k$, such that

$$\lambda^\varepsilon_k(\sigma) = \min_{\substack{\xi\in V\\ \|\xi\|_\varepsilon=1}} \langle K^\sigma_\varepsilon\xi,\xi\rangle_\varepsilon = \min_{\substack{\xi\in V\\ \|\xi\|_\varepsilon=1}} \langle K^\tau_\varepsilon\xi,\xi\rangle_\varepsilon \leq \lambda^\varepsilon_k(\tau).$$

Step 4. *The eigenvalues of $H^{h_\varepsilon}(x_\varepsilon)_{|T_{x_\varepsilon}\Omega^1_\sigma}$ are strictly decreasing with respect to σ.*

As in the previous step we can prove that $\lambda^\varepsilon_k(\sigma)$ is strictly increasing with respect to σ. We fix $\sigma, \tau \in\,]0,1]$, with $\sigma < \tau$. As in Step 3, suppose that

$$\lambda^\varepsilon_k(\sigma) = \min_{\substack{\xi\in V\\ \|\xi\|_\varepsilon=1}} \langle K^\sigma_\varepsilon\xi,\xi\rangle_\varepsilon,$$

where V is a subspace of $T_{x_\varepsilon}\Omega^1_\sigma$ with $\dim V = k$. From the previous step we know that

$$\lambda^\varepsilon_k(\sigma) \leq \lambda^\varepsilon_k(\tau), \tag{6.39}$$

then, if $\lambda^\varepsilon_k(\sigma)$ is not an eigenvalue of K^τ_ε the proof is complete. Now we suppose now that $\lambda^\varepsilon_k(\sigma) = \lambda$ is an eigenvalue of K^τ_ε. Then, from the spectral properties of K^τ_ε, we conclude that

$$T_{x_\varepsilon}\Omega^1_\tau = H_1 \oplus H_2 \oplus H_3, \tag{6.40}$$

where H_1 is the maximal subspace where $\lambda I - K^\tau_\varepsilon$ is positive definite, $H_2 = \ker(\lambda I - K^\tau_\varepsilon)$, H_3 is the maximal subspace where $\lambda I - K^\tau_\varepsilon$ is negative definite. Let us prove that

$$V \cap (H_1 \oplus H_2) = \{0\}. \tag{6.41}$$

We take $\xi \in V \cap (H_1 \oplus H_2)$, $\xi = \xi_1 + \xi_2$, $\xi_1 \in H_1$, $\xi_2 \in H_2$. If, by contradiction, $\xi_1 \neq 0$, then also $\xi \neq 0$, and

$$\begin{aligned}\left\langle K^\sigma_\varepsilon \frac{\xi}{\|\xi\|_\varepsilon}, \frac{\xi}{\|\xi\|_\varepsilon}\right\rangle_\varepsilon &= \left\langle K^\tau_\varepsilon \frac{\xi}{\|\xi\|_\varepsilon}, \frac{\xi}{\|\xi\|_\varepsilon}\right\rangle_\varepsilon \\ &= \left\langle K^\tau_\varepsilon \frac{\xi_1}{\|\xi\|_\varepsilon}, \frac{\xi_1}{\|\xi\|_\varepsilon}\right\rangle_\varepsilon + \left\langle K^\tau_\varepsilon \frac{\xi_2}{\|\xi\|_\varepsilon}, \frac{\xi_2}{\|\xi\|_\varepsilon}\right\rangle_\varepsilon\end{aligned}$$

$$
\begin{aligned}
&= \left\langle K_\varepsilon^\tau \frac{\xi_1}{\|\xi\|_\varepsilon}, \frac{\xi_1}{\|\xi\|_\varepsilon} \right\rangle_\varepsilon + \lambda \left\langle \frac{\xi_2}{\|\xi\|_\varepsilon}, \frac{\xi_2}{\|\xi\|_\varepsilon} \right\rangle_\varepsilon \\
&< \lambda \left\langle \frac{\xi_1}{\|\xi\|_\varepsilon}, \frac{\xi_1}{\|\xi\|_\varepsilon} \right\rangle_\varepsilon + \lambda \left\langle \frac{\xi_2}{\|\xi\|_\varepsilon}, \frac{\xi_2}{\|\xi\|_\varepsilon} \right\rangle_\varepsilon = \lambda = \lambda_k^\varepsilon(\sigma),
\end{aligned}
$$

from which $\lambda_k^\varepsilon(\sigma) < \lambda_k^\varepsilon(\sigma)$, a contradiction. Then $\xi_1 = 0$ and $\xi = \xi_2 \in \ker(\lambda I - K_\varepsilon^\tau)$. We consider now $\eta = \xi/\|\xi\|_{H_0^1}$. We have $\eta \in V \subset T_{x_\varepsilon}\Omega_\sigma^1$ and $\eta \in \ker(\lambda I - K_\varepsilon^\tau)$, in contradiction with Lemma 6.12. Then $\xi = 0$. By (6.40) and (6.41), we have $V \subset H_3$, then $k = \dim V \le \dim H_3$. So, let's take a subspace H of H_3, with $\dim H = k$, for example V. We have

$$
\lambda_k^\varepsilon(\sigma) = \lambda < \min_{\substack{\xi \in H \\ \|\xi\|_\varepsilon = 1}} \langle K_\varepsilon^\tau \xi, \xi \rangle_\varepsilon \le \lambda_k^\varepsilon(\tau).
$$

Step 5. Let's denote by $m(\sigma)$ the maximal dimension of a subspace of $T_{x_\varepsilon}\Omega_\sigma^1$ where $H^{h_\varepsilon}(x_\varepsilon)_{|T_{x_\varepsilon}\Omega_\sigma^1}$ is negative definite. As $m(\sigma)$ is given by the number of negative eigenvalues of $I - K_\varepsilon^\sigma$, by Step 2, if σ is small, $m(\sigma) = 0$, that is, all the eigenvalues are positive. By Steps 1, 3, 4, when σ becames greater, if $x(\sigma)$ is conjugate with multiplicity m for some σ, there are m eigenvalues equal 0 that became negative. Hence, $m(x_\varepsilon, h_\varepsilon) = m(1)$ is exactly the number of conjugate points counted with their multiplicity, that is $\mu(x_\varepsilon, h_\varepsilon)$. □

Remark 6.14. Theorem 6.13 shows that for all $n \in \mathbb{N}$, there exists a not empty class $\mathcal{F}_n' \subset \mathcal{F}_n$ where the Index Theorem holds. From now on we shall only consider functionals in $\mathcal{F}_n'$.

As an immediate consequence of the Index Theorem we have the following.

Corollary 6.15. *Let h_ε be defined as in Theorem* 6.13 *and x_ε be a critical point of h_ε. Then $\mu(x_\varepsilon, h_\varepsilon)$ is finite.*

Proof. From Theorem 6.14, $m(x_\varepsilon, h_\varepsilon) = \mu(x_\varepsilon, h_\varepsilon)$. We see that $m(x_\varepsilon, h_\varepsilon)$ is finite as $H^{h_\varepsilon}(x_\varepsilon) = I - K_\varepsilon$, where K_ε is a compact operator. Then there are only a finite number of negative eigenvalues of $H^{h_\varepsilon}(x_\varepsilon)$. □

Another Corollary of the Index Theorem is a result already known for geodesics. Indeed, it has been proved that conjugate points along a geodesic are isolated. The same holds for the small perturbations of f we are considering.

Corollary 6.16. *Let h_ε be defined as in Theorem* 6.13 *and x_ε be a critical point of h_ε. Then every conjugate point $x_\varepsilon(\sigma)$ to x_0 along x_ε is isolated.*

Proof. Take a conjugate point $x_\varepsilon(\sigma)$. If, by contradiction, there exists a sequence $(\sigma_n)_{n\in\mathbb{N}} \subset [0,1]$ such that

$$
\lim_{n\to\infty} \sigma_n = \sigma,
$$

and $x_\varepsilon(\sigma_n)$ is conjugate to x_0 along x_ε we should have $\mu(x_\varepsilon, h_\varepsilon) = \infty$, in contradiction with Corollary 6.16. □

The following follows immediately from Corollary 6.16.

Corollary 6.17. *Let h_ε be defined as in Theorem* 6.13 *and x_ε be a critical point of h_ε. Then the number of conjugate points along x_ε is finite.*

Using Theorem 6.13, we can write the generalized Morse relations using the geometric index. Let f be the functional defined in (5.2). Let $\mathcal{F}'_n$ be the class of functionals defined in Remark 6.14. For $h \in \mathcal{F}'_n$ we can consider the following polynomial

$$\mu_\lambda(h) = \sum_{x \in K_h} \lambda^{\mu(x,h)}.$$

From the Index Theorem

$$\mu_\lambda(h) = m_\lambda(h), \tag{6.42}$$

where $m_\lambda(h)$ is the Morse polynomial of h. Then we can formulate the following

Definition 6.18. Let f be the functional defined in (5.2) and

$$\mathcal{A}'_f = \{\mathcal{P} \in S \mid \exists h_n \in \mathcal{F}'_n \ \forall n \in \mathbb{N} \ \lim_{n\to\infty} \mu_\lambda(h_n) = \mathcal{P}\}.$$

The formal series $i_\lambda(f) = \inf \mathcal{A}'_f$ is called the *generalized geometric Morse index* of f.

As in Section 4, the generalized Morse index is well defined and obviously Morse relations are still valid. Indeed the following theorem holds.

Theorem 6.19. *Under the same assumptions as for Theorem* 4.9*, there exists a polynomial $\mathcal{Q}_\lambda$ with integer positive coefficients such that*

$$i_\lambda(f) = \mathcal{P}_\lambda(\Omega^1(x_0, x_1, \mathcal{M})) + (1+\lambda)\mathcal{Q}_\lambda.$$

Proof. In exactly the same way as for the proof of Theorem 4.9, seeing that for every perturbation h in the class $\mathcal{F}'_n$ classical Morse relations hold, then using (6.42) we can write

$$\mu_\lambda(h) = \mathcal{P}_\lambda(\Omega^1(x_0, x_1, \mathcal{M})) + (1+\lambda)\mathcal{Q}^h_\lambda,$$

where $\mathcal{Q}^h_\lambda$ is a formal series. □

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