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HESSIAN MEASURES I

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Dedicated to Olga Ladyzhenskaya

1. Introduction

Let Ω be a domain in Euclidean n-space $\mathbb{R}^n$. For $k = 1, \ldots, n$ and $u \in C^2(\Omega)$ the k-Hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)), \tag{1.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix of second derivatives D^2u, and S_k is the k-th elementary symmetric function on $\mathbb{R}^n$, given by

$$S_k(\lambda) = \sum_{i_1 < \ldots < \lambda_k} \lambda_{i_1} \ldots \lambda_{i_k}. \tag{1.2}$$

Alternatively we may write

$$F_k[u] = [D^2u]_k, \tag{1.3}$$

where $[\mathcal{A}]_k$ denotes the sum of the $k \times k$ principal minors of an $n \times n$ matrix $\mathcal{A}$. Our purpose in this paper is to extend the definition of the F_k to corresponding classes of continuous functions so that $F_k[u]$ is a Borel measure and to consider the Dirichlet problem in this setting. A function $u \in C^2(\Omega)$ is called *k-convex* (*uniformly k-convex*) in Ω if $F_j[u] \geq 0$ (> 0) for $j = 1, \ldots, k$. The operator F_k

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is degenerate elliptic (elliptic) with respect to k-convex (uniformly k-convex) functions. When $k = 1$, we have $F_1[u] = \Delta u$ and 1-convex functions are subharmonic. When $k = n$, $F_k[u] = \det D^2 u$, the Monge–Ampère operator, and n-convex functions are convex. To extend these notions to continuous functions, we call a function $u \in C^0(\Omega)$, *k-convex*, if there exists a sequence $\{u_m\} \subset C^2(\Omega)$ such that in any subdomain $\Omega' \Subset \Omega$, u_m is k-convex for sufficiently large m and converges uniformly to u. It is easily seen that $u \in C^0(\Omega)$ is k-convex if and only if $F_k[u] \geq 0$ in the viscosity sense ([11], [16]), that is, whenever there exists a point $y \in \Omega$ and function $v \in C^2(\Omega)$ satisfying $u(y) = v(y)$, $u \leq v$ in Ω, we must have $F_k[v](y) \geq 0$. As above a function $u \in C^0(\Omega)$ is 1-convex if and only if it is subharmonic and n-convex if and only if it is convex. In each of these cases, it is well known that the operator F_k can be defined as a Borel measure μ_k. For $k = 1$, μ_1 is the positive distribution given by

$$\mu_1(\varphi) = \int_\Omega u \Delta \varphi \tag{1.4}$$

for $\varphi \in C_0^\infty(\Omega)$, while for $k = n$,

$$\mu_n(e) = |\chi_u(e)| \tag{1.5}$$

for any Borel set $e \subset \Omega$, where χ_u is the normal (subgradient) mapping of the convex function u ([1], [4]). Let $\Phi^k(\Omega)$ denote the class of k-convex functions in $C^0(\Omega)$. In this paper we shall prove that $F_k[u]$ may be extended to $\Phi^k(\Omega)$ as a Borel measure μ_k, for all $k = 1, \dots, n$, and that the corresponding mapping $u \to \mu_k[u]$ is weakly continuous on $C^0(\Omega)$. The resultant measure $\mu_k[u]$ will be called the *k-Hessian measure* generated by u.

Theorem 1.1. *For any $u \in \Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ such that*

(i) $$\mu_k[u](e) = \int_e F_k[u]$$

for any Borel set $e \subset \Omega$, if $u \in C^2(\Omega)$, and

(ii) *if $u_m \to u$ locally uniformly in Ω, then the corresponding measures $\mu_k[u_m] \to \mu_k[u]$ weakly, that is,*

$$\int_\Omega g \, d\mu_k[u_m] \to \int_\Omega g \, d\mu_k[u], \tag{1.6}$$

for all $g \in C^0(\Omega)$ with compact support.

Theorem 1.1 is proved in Section 2 of this paper as a consequence of various integral inequalities for the operators F_k.

In Section 3 we consider the corresponding Dirichlet problem,

$$\begin{cases} \mu_k[u] = \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

in the class of k-convex functions. Under the hypotheses that the domain Ω is uniformly $(k-1)$-convex, that is, $\partial\Omega \in C^2$ and $H_j[\partial\Omega] > 0$, $j = 1, \ldots, k-1$, where $H_j[\partial\Omega]$ denotes the j-mean curvature of the boundary $\partial\Omega$ (see [17], [18]), and that the Borel measure μ can be decomposed as a sum

$$\mu = \mu_1 + \mu_2, \tag{1.8}$$

where $\mu_1 \in L^1(\Omega)$ and μ_2 has compact support in Ω, we prove the following existence and uniqueness theorem.

Theorem 1.2. *For any $\varphi \in C^0(\overline{\Omega})$, there exists a unique $u \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$ satisfying* (1.7)*, provided $k > n/2$.*

Theorem 1.2 extends the case, $p = 1$, in [20], where an equivalent formulation of the Dirichlet problem (1.7) is treated for inhomogeneous terms in L^p spaces.

In Section 4, we consider the extension of the measures μ_k as signed measures on more general classes of functions including semi-convex functions (as in [10]) and admissible functions, for which the operators F_k are degenerate elliptic. Finally, in Section 5, we apply Theorem 1.1 to extend Hessian integrals, (as defined in [7], [19], [24]), to continuous k-convex functions. In particular we derive a convergence theorem, Theorem 5.1, monotonicity results, Lemma 5.2, Corollary 5.3, and a variational formula, Theorem 5.4.

In an ensuing paper [23], we consider the extension of Theorem 1.1 to convergence in measure, with applications to the cases $k \leq n/2$ in Theorem 1.2.

2. Integral inequalities

In this section we develop some basic integral properties for the operators F_k which lead to Theorem 1.1. First we establish a monotonicity property.

Lemma 2.1. *Let $u, v \in \Phi^k(\Omega) \cap C^2(\overline{\Omega})$ satisfy $u = v$ on $\partial\Omega$, $u \geq v$ in Ω. Then*

$$\int_\Omega F_k[u] \leq \int_\Omega F_k[v]. \tag{2.1}$$

Proof. By approximation of the functions u and v and use of Sard's theorem, we may assume $\partial\Omega \in C^2$. Setting, for a symmetric matrix r with eigenvalues $\lambda = (\lambda_1, \cdots, \lambda_n)$,

$$S_k^{ij}(r) = \frac{\partial}{\partial r_{ij}} S_k(\lambda(r)), \tag{2.2}$$

and, using the identity [15],

$$D_i S_k^{ij}(D^2u) = 0, \tag{2.3}$$

we then obtain, by the divergence theorem,

$$\begin{aligned}\int_\Omega \big(F_k[v] - F_k[u]\big) &= \int_0^1 \int_\Omega S_k^{ij}(sD^2u + (1-s)D^2v)D_{ij}(v-u) \\ &= \int_0^1 \int_{\partial\Omega} S_k^{ij}(sD^2u + (1-s)D^2v)\gamma_i D_j(v-u)\end{aligned} \tag{2.4}$$

where γ denotes the unit outer normal to $\partial\Omega$. Letting ∂ denote the tangential gradient in $\partial\Omega$, given by

$$\partial = D - \gamma(\gamma \cdot D), \tag{2.5}$$

we can write the integrand in (2.4) as

$$\begin{aligned}&S_k^{ij}(sD^2u + (1-s)D^2v)\gamma_i D_j(v-u) \\ &\qquad = S_k^{ij}(sD^2u + (1-s)D^2v)\gamma \cdot D(v-u)\gamma_i\gamma_j \geq 0\end{aligned} \tag{2.6}$$

since $\partial u = \partial v$ on $\partial\Omega$, $\gamma \cdot Dv \geq \gamma \cdot Du$ on $\partial\Omega$, and the function $su + (1-s)v$ will be k-convex for all $s \in [0,1]$ (see Lemma 2.3 below). □

Next we note that a global control on F_k is provided, for example, by Reilly's formula, [15] (see also [17]),

$$\int_\Omega F_k[u] = \frac{1}{k}\int_\Omega (\gamma \cdot Du)^k H_{k-1}[\partial\Omega], \tag{2.7}$$

when u vanishes on $\partial\Omega$. Our next estimate shows that we can control the integral of F_k *locally* in terms of the oscillation of u.

Lemma 2.2. *Let $u \in \Phi^k(\Omega) \cap C^2(\Omega)$. Then for any subdomain $\Omega' \Subset \Omega$, we have*

$$\int_{\Omega'} F_k[u] \leq C(\operatorname{osc}_\Omega u)^k, \tag{2.8}$$

where C is a constant depending on Ω and Ω'.

To prove Lemma 2.2, we need a further property of k-convex functions.

Lemma 2.3. *Let $u_1, \ldots, u_m \in \Phi^k(\Omega)$ and f be a convex, non-decreasing function in $\mathbb{R}^m$. Then the composite function $w = f(u_1, \ldots, u_m)$ is also k-convex.*

Proof. As a special case of Lemma 2.3, we see that linear combinations of k-convex functions with non-negative coefficients are also k-convex. This follows immediately from the convexity of the cones

$$\begin{aligned}\Gamma_k &= \{r \in \$^n \mid S_j(\lambda(r)) > 0,\ j = 1, \ldots, k\}, \\ \overline{\Gamma}_k &= \{r \in \$^n \mid S_j(\lambda(r)) \geq 0,\ j = 1, \ldots, k\}\end{aligned} \tag{2.9}$$

in $\mathbb{S}^n$, the space of real, $n \times n$, symmetric matrices. For the general case, it suffices to assume $u_1, \dots, u_m \in \Phi^k(\Omega) \cap C^2(\Omega)$ with $f \in C^2(\mathbb{R}^m)$. Then we have by calculation,

$$D_{ij}w = \frac{\partial f}{\partial u_p} D_{ij}u_p + \frac{\partial^2 f}{\partial u_p \partial u_q} D_i u_p D_j u_q,$$

so that $D^2 w \in \overline{\Gamma}_k$, since

$$\frac{\partial f}{\partial u_p} \geq 0, \quad p = 1, \dots, m, \quad \left[\frac{\partial^2 f}{\partial u_p \partial u_q}\right] \geq 0$$

and $\overline{\Gamma}_k$ is convex. □

PROOF OF LEMMA 2.2. Let $B = B_R(y)$ be a ball of radius R and centre y, lying in Ω and for $0 < \sigma < 1$, let $B_{\sigma R}$ denote the concentric ball of radius σR. Without loss of generality we may assume $y = 0$ and, by subtraction of a suitable constant, $u < -\varepsilon$ in B for some given positive constant ε. Setting

$$\begin{aligned} \psi(x) &= \frac{m_0}{1-\sigma^2}\left(1 - \frac{|x|^2}{R^2}\right), \quad m_0 = \inf_B u, \\ w(x) &= \max\{u, \psi\} \end{aligned} \tag{2.10}$$

it follows from Lemma 2.3, that w is k-convex in B and $w \leq u$ in $B_{\sigma R}$, $w = \psi$ on ∂B. Our desired result follows by applying Lemma 2.1 to the function w and ψ. To overcome the lack of smoothness of w, we replace it by

$$w_h = f_h(u, \psi),$$

where f_h, for $h > 0$, is the mollification,

$$f_h(x) = \int_{\mathbb{R}^2} \rho\left(\frac{x-y}{h}\right) \max(y_1, y_2)\, dy \tag{2.11}$$

and $\rho \geq 0$, in $C_0^\infty(\mathbb{R}^2)$, with $\int \rho = 1$, is the usual mollifier. With h sufficiently small, we obtain from Lemma 2.1,

$$\begin{aligned} \int_{B_{\sigma R}} F_k[u] &\leq \int_B F_k[\psi] = \binom{n}{k} \omega_n \left(\frac{2m_0}{1-\sigma^2}\right)^k R^{n-2k} \\ &= \binom{n}{k} \omega_n \left(\frac{2}{1-\sigma^2}\right)^k R^{n-2k} \left(\operatorname{osc}_B u\right)^k \end{aligned} \tag{2.12}$$

as $\varepsilon \to 0$. By covering Ω' with balls we conclude (2.8). □

We are now ready to prove Theorem 1.1. Let $u \in \Phi^k(\Omega)$ and suppose $\{u_m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ converges to u in $C^0(\Omega)$. By Lemma 2.2, the integrals

$$\int_{\Omega'} F_k[u_m]$$

are uniformly bounded, for any subdomain $\Omega' \Subset \Omega$ and hence a subsequence $\{F_k[u_{m_p}]\}$ converges weakly [2] (in the sense of measures) to a Borel measure

$\mu_k[u]$ on Ω. It remains to show that the measure $\mu_k[u]$ is determined uniquely by the function u. Accordingly suppose that $\{u_m\}$, $\{v_m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ are two sequences converging in $C^0(\Omega)$ to u and that the corresponding sequence of functions $\{F_k[u_m]\}$, $\{F_k[v_m]\}$ converge weakly to Borel measures ν_1 and ν_2 respectively. Let $B = B_R(y) \Subset \Omega$ and fix some $\sigma \in (0,1)$. Let $\eta \in C^2(\overline{B})$ be a convex function satisfying $\eta = 0$ in $B_{\sigma R}$, $\eta = 1$ on $\partial\Omega$. For fixed $\varepsilon > 0$, it then follows from the uniform convergence of $\{u_m\}$, $\{v_m\}$, that

$$u_m \le v_m + \varepsilon\eta \tag{2.13}$$

on ∂B, for sufficiently large m. Let

$$G_m = \{x \in B \mid u_m > v_m + \varepsilon\eta\}. \tag{2.14}$$

Without loss of generality we may assume that ∂G_m is sufficiently smooth so that from Lemma 2.1 we have

$$\int_{G_m} F_k[u_m] \le \int_{G_m} F_k[v_m + \varepsilon\eta]. \tag{2.15}$$

By adding $\varepsilon/2$ to u_m, we may also assume that $G_m \supset B_{\sigma R}$, so that from (2.15), we have

$$\begin{aligned} \int_{B_{\sigma R}} F_k[u_m] &\le \int_B F_k[v_m + \varepsilon\eta] \le \int_B [D^2 v_m + C\varepsilon I]_k \\ &\le \int_B F_k[v_m] + C \sum_{j=0}^{k-1} \varepsilon^{k-j} \int_B F_j[v_m], \end{aligned} \tag{2.16}$$

where C is a constant depending on η. Using the estimate (2.11) and sending $m \to \infty$, $\varepsilon \to 0$, $\sigma \to 1$, we then obtain

$$\nu_1(B) \le \nu_2(\overline{B}). \tag{2.17}$$

By replacing B by a sequence of balls $B_{\sigma_m R}$, with $\sigma_m \to 1$, satisfying

$$\nu_2(B_{\sigma_m R}) = \nu_2(\overline{B}_{\sigma_m R}),$$

we deduce $\nu_1(B) \le \nu_2(B)$ and subsequently by interchanging $\{u_m\}$ and $\{v_m\}$, we have $\nu_1(B) = \nu_2(B)$, whence $\nu_1 = \nu_2$. This completes the proof of Theorem 1.1, as the above argument shows that $\mu_k[u]$ is well defined as the weak limit of $F_k[u_m]$ for any sequence $\{u_m\}$ converging to u in $C^0(\Omega)$ and the mapping, $\mu_k : C^0(\Omega) \to M(\Omega)$, the space of locally finite Borel measures in Ω is weakly continuous. □

Using Theorem 1.1, our previous inequalities may be extended to functions in $\Phi^k(\Omega)$. In particular we have the following extensions of Lemmas 2.1 and 2.2.

COROLLARY 2.4. *Let $u, v \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$ satisfying $u = v$ on $\partial\Omega$, $u \geq v$ in Ω. Then the corresponding measures μ_k satisfy*

$$\mu_k[u](\Omega) \leq \mu_k[v](\Omega). \tag{2.18}$$

COROLLARY 2.5. *Let $u \in \Phi^k(\Omega)$. Then for any solution $\Omega' \Subset \Omega$, we have*

$$\mu_k[u](\Omega') \leq C(\operatorname{osc}_\Omega u)^k, \tag{2.19}$$

where C is a constant depending on Ω and Ω'.

3. The Dirichlet problem

In the paper [20], existence and uniqueness results are obtained for the Dirichlet problem for weak solutions of the equation

$$F_k[u] = \psi \tag{3.1}$$

for inhomogeneous term $\psi \in L^p(\Omega)$ for $p \geq 1$. The classical case had been previously treated in [5] (see also [18]). A function u was called a *weak solution* of equation (3.1) in Ω if there existed a sequence $\{u_m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ converging in $C^0(\Omega)$ to u with the corresponding sequence $\{F_k[u_m]\}$ converging in $L^1_{\mathrm{loc}}(\Omega)$ to ψ. From Theorem 1.1, we have immediately, $\mu_k[u] = \psi$, so that the notion in (1.7) is more general. (Note that when a Borel measure μ is absolutely continuous and representable by a locally integrable function ψ we identify μ with ψ.) A comparison principle for weak solutions is proved in [20] using estimates from [19]. From Corollary 2.4 we obtain a more general result as follows.

THEOREM 3.1. *Let $u, v \in C^0(\overline{\Omega}) \cap \Phi^k(\Omega)$ satisfy*

$$\begin{cases} \mu_k[u] \geq \mu_k[v] & \text{in } \Omega, \\ u \leq v & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Then $u \leq v$ in Ω.

PROOF. Assume $\{o\} \in \Omega$ and set

$$\overline{u}(x) = u(x) + \varepsilon(|x|^2 - d^2)$$

for some $\varepsilon > 0$, where $d = \operatorname{diam} \Omega$. Clearly, we have

$$\mu_k[\overline{u}] \geq \mu_k[u] + \binom{n}{k}(2\varepsilon)^k,$$

and $\overline{u} \leq u \leq v$ on $\partial\Omega$. Accordingly, setting

$$\Omega_\varepsilon = \{x \in \Omega \mid \overline{u}(x) > v(x)\},$$

and assuming Ω_ε is non-empty, we have, by Corollary 2.4,

$$\mu_k[u](\Omega_\varepsilon) < \mu_k[\overline{u}](\Omega_\varepsilon) \leq \mu_k[v](\Omega_\varepsilon),$$

which contradicts our hypothesis. Consequently, letting $\varepsilon \to 0$, we infer $u \le v$ in Ω. □

Note that Corollary 2.4 and Theorem 3.1, were proved by completely different methods, (using the normal mapping), in the case $k = n$ ([1], [4], [6]). The uniqueness assertion in Theorem 1.2 follows immediately from Theorem 3.1. We may obtain the existence part by approximation from the case $\nu_2 = 0$, ([20, Theorem 1.1]), using the Hölder estimate there to guarantee the local equicontinuity of the approximating solutions. However, this estimate may be bypassed as k-convex functions are automatically Hölder continuous if $k > n/2$. To see this we fix a ball $B = B_R(y) \subset \Omega$ and observe that the function w given by

$$w(x) = C|x-y|^{2-n/k}, \tag{3.3}$$

where C is a positive constant, satisfies

$$F_k[w] = 0, \quad \text{for } x \ne y. \tag{3.4}$$

Consequently, if $u \in \Phi^k(\Omega) \cap C^2(\Omega)$, we obtain, from the classical comparison principle in the punctured ball, $B_R(y) - \{y\}$,

$$u(x) - u(y) \le \operatorname{osc}_{B_R(y)} u \left(\frac{|x-y|}{R}\right)^{2-n/k}, \tag{3.5}$$

provided $k > n/2$. It follows then that $\Phi^k(\Omega) \subset C^{0,\alpha}(\Omega)$ for $\alpha = 2 - n/k > 0$ and moreover, for any $x, y \in \Omega$, $x \ne y$.

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \le \frac{\operatorname{osc} u}{d_{x,y}^\alpha}, \tag{3.6}$$

where $d_{x,y} = \min\{\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega)\}$. For $k > n/2$, the function w will be k-convex in any domain and from [19], (see, in particular, (3.15), (3.16) in [19]), we have

$$F_k[w] = \left[C\left(2 - \frac{n}{k}\right)\right]^k \binom{n}{k} \omega_n \delta_y, \tag{3.7}$$

where δ_y denotes the Dirac delta measure at y.

To complete the proof of Theorem 1.2, we let $\{\psi_m\}$ be a sequence of non-negative functions in $C_0^\infty(\Omega)$, converging weakly as measures to ν_2, with support lying in some subdomain $\Omega' \Subset \Omega$. By virtue of the case $p = 1$, ([20, Theorem 1.1]), there exists a sequence $\{u_m\} \subset C^0(\overline{\Omega}) \cap \Phi^k(\Omega)$ of weak solutions of the Dirichlet problems

$$\begin{cases} \mu_k[u_m] = \nu_1 + \psi_m & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

From the L^∞ estimates in [19], [20], the sequence $\{u_m\}$ is uniformly bounded in $L^\infty(\Omega)$ and hence, from (3.6), (see also [20, Theorem 4.1]), equicontinuous in Ω', so that a subsequence converges uniformly in Ω'. Relabelling the subsequence as $\{u_m\}$, we fix $\varepsilon > 0$, so that for sufficiently large m, l, we have,

$$|u_m - u_l| \le \varepsilon \quad \text{on } \Omega'. \tag{3.9}$$

Using the comparison principle, Theorem 3.1 (or [20, Theorem 2.2]), in the domain $\Omega - \overline{\Omega'}$, we then obtain (3.9) on the whole domain Ω and Theorem 1.2 follows from Theorem 1.1. □

We remark that the necessary L^∞ estimates for the above proof (and also that of Theorem 1.1 in [20]), also follow readily from the Sobolev inequality in [19], [24], and moreover, (in the case $k > n/2$), can be derived simply from comparison with the functions (3.3), [21].

As an example of Theorem 1.2, we see that for any uniformly $(k-1)$-convex domain Ω, and point $y \in \Omega$, there exists a bounded Greens function G_y, given by the solution of the Dirichlet problem,

$$\begin{cases} \mu_k[G_y] = \delta_y & \text{in } \Omega, \\ G_y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Furthermore, it is readily shown that $G_y \in C^{0,\alpha}(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega} - \{y\})$, where $\alpha = 2 - n/k$, and, in accordance with (3.7) (see also [19]), for $\Omega = B_R(y)$, we have

$$G_y(x) = \left[\frac{1}{\binom{n}{k}\omega_n}\right]^{1/k} \frac{1}{2 - n/k}\left(|x-y|^{2-n/k} - R^{2-n/k}\right). \tag{3.11}$$

The Greens function is used to sharpen maximum principles in [21].

When $k \le n/2$, we cannot expect to obtain a *continuous* k-convex solution of the Dirichlet problem (1.7) without further restrictions on ν, for example, $\nu \in L^p(\Omega)$ for $p > n/2k$, as in [20]. In order to embrace this case, we extend our notion of k-convexity to upper semi-continuous functions analogously to the general notion of subharmonic functions in the case $k = 1$. Accordingly, an upper semi-continuous function $u : \Omega \to [-\infty, \infty)$ is called k-convex if $F_k[u] \ge 0$ in the viscosity sense, that is, whenever there exists a point $y \in \Omega$ and function $v \in C^2(\Omega)$ satisfying $u(y) = v(y)$, $u \le v$ in Ω, we must have $F_k[v](y) \ge 0$. Because our comparison argument above automatically extends to upper semi-continuous k-convex functions, we infer again the estimate (3.5) and (3.6) when $k > n/2$ so that there is no gain in generality in this case. However, the functions (3.3) will be k-convex for all $k = 1, \dots, n$ and corresponding Greens functions arise by solving (3.10) in an appropriate sense. The general case is treated in our ensuing paper [23], together with further local properties of k-convex functions.

Finally, we note that Theorem 1.2 extends to embrace more general boundary data in the presence of barriers and that the Perron process [12] is also applicable.

In particular the condition $\nu_1 \in L^1(\Omega)$ may be replaced by

$$\nu_1 \leq \nu[\mathrm{dist}(x, \partial\Omega)]^{\beta-k-1} \tag{3.12}$$

for positive constants ν and β, as in the case $k = n$, (see [4], [20]).

4. Semi-convex and admissible functions

The theory in Section 2 extends to larger classes of functions. Analogously to the notion of semi-convexity, we may call a function $u \in C^0(\Omega)$, *k-semi-convex* if the function v given by

$$v(x) = u(x) + A|x|^2/2, \tag{4.1}$$

is k-convex for some fixed positive constant A. From the expansion

$$F_k[u] = \sum_{j=0}^{k} c(j,k,n)(-A)^j F_{k-j}[v], \tag{4.2}$$

where $c(j,k,n) = \binom{n}{k}\binom{k}{j}/\binom{n}{k-j}$, we can then define μ_k as a *signed* Borel measure in Ω, by

$$\mu_k[u] = \sum_{j=0}^{k} c(j,k,n)(-A)^j \mu_{k-j}[v]. \tag{4.3}$$

If $\{u_m\}$ is a sequence of k-semi-convex functions, with the same constant A, converging in $C^0(\Omega)$ to a k-semi-convex function u, the corresponding sequence of measures $\mu_k[u_m]$ will converge weakly to $\mu_k[u]$. It follows that the definition (4.3) is independent of the expansion (4.2).

Following usual terminology ([16], [18]), we call a function $u \in C^2(\Omega)$, *admissible* with respect to the operator F_k (or simply k-admissible) if

$$S_k(D^2u + \eta) \geq S_k(D^2u) \tag{4.4}$$

for all non-negative matrices $\eta \in \mathbb{R}^n$. Condition (4.4) implies that the operator F_k is degenerate elliptic with respect to u, that is,

$$[S_k^{ij}(D^2u)] \geq 0, \tag{4.5}$$

and is weaker that k-convexity, although the two conditions coincide in the convex case $k = n$. A function $u \in C^0(\Omega)$ is called *k-admissible* if there exists a sequence $\{u_m\} \subset C^2$ of k-admissible functions converging to u in $C^0(\Omega)$. If additionally the sequence $\{u_m\}$ satisfies

$$F_k[u_m] \geq -\binom{n}{k}A^k \tag{4.6}$$

for a positive constant A, then the function u will be k-semi-convex, (with the same constant A). To see this, we set

$$v_m = u_m + A|x|^2/2$$

and expand

$$F_k[v_m] = \sum_{j=0}^{k} c(j,k,n)A^j F_{k-j}[u_m] \geq F_k[u_m] + \binom{n}{k} A^k,$$

since $F_j[u_m] \geq 0$, $j = 1, \ldots, k-1$. Equivalently, if $u \in C^0(\Omega)$ satisfies the inequality

$$F_k[u] \geq -\binom{n}{k} A^k \tag{4.7}$$

in the viscosity sense ([11], [16]), then u is k-semi-convex with constant A. Consequently, we can define signed Borel measures μ_k for such functions, which extend the smooth case and are weakly continuous with respect to convergence in $C^0(\Omega)$.

Alternatively, the existence of the signed measure μ_k can be approached directly since Lemma 2.1 holds, more generally, for k-admissible functions $u, v \in C^2(\overline{\Omega})$. In Lemma 2.2, we obtain, in place of (2.11), for k-admissible $u \in C^2(\Omega)$,

$$\int_{\Omega'} F_k[u] \leq C\left\{ \int_{\Omega} (F_k[u])^- + (\operatorname{osc}_{\Omega} u)^k \right\}. \tag{4.8}$$

Consequently, by following the proof of Theorem 1.1, we see that Theorem 1.1 can be extended to the class $\Phi^k(\Omega; g)$ of k-admissible functions u which are limits in $C^0(\Omega)$ of sequences $\{u_m\} \subset C^2(\Omega)$ of k-admissible functions u_m satisfying

$$F_k[u_m] \geq -g, \tag{4.9}$$

where g is a fixed, non-negative, locally integrable function in Ω. Corollaries 2.4 and 2.5 then extend also to $\Phi^k(\Omega; g)$ with (2.19) replaced by

$$\mu_k[u](\Omega') \leq C\left\{ \int_{\Omega} g + (\operatorname{osc}_{\Omega} u)^k \right\}. \tag{4.10}$$

5. Hessian integrals

For $u \in C^2(\overline{\Omega})$, we define the Hessian integral $I_k[u]$ by

$$I_k[u] = I_k[u;\Omega] = -\int_{\Omega} u F_k[u]. \tag{5.1}$$

If $u = 0$ on $\partial\Omega$, we have by integration by parts,

$$I_k[u] = k\int_{\Omega} S_k^{ij} D_i u D_j u, \tag{5.2}$$

so that $I_k[u] \geq 0$ if, also, u is k-admissible. Imbedding properties of Hessian integrals are treated in the papers [7], [19], [20], [24]. Using Theorem 1, we define an extension of I_k to $\Phi^k(\Omega) \cap C^0(\overline{\Omega})$ by

$$I_k[u] = -\int_\Omega u\, d\mu_k[u]. \tag{5.3}$$

Clearly, $I_k[u]$ is finite if $\mu_k[u](\Omega) < \infty$. Letting $\Phi_0^k(\Omega)$ denote the subset of $\Phi^k(\Omega) \cap C^0(\overline{\Omega})$ of functions vanishing on $\partial\Omega$, we then obtain from the weak continuity of μ_k, the approximation result.

Theorem 5.1. *Let $\{u_m\} \subset \Phi_0^k(\Omega)$ converge uniformly to u and suppose $\{\mu_k[u_m](\Omega)\}$ is bounded. Then $I_k[u_m] \to I_k[u]$.*

Proof. For $\Omega' \Subset \Omega$, we have

$$\mu_k[u](\Omega') \leq \liminf_{m\to\infty} \mu_k[u_m](\Omega')$$

so that $\mu_k[u](\Omega) < \infty$. From (5.3) we have, for any $\eta \in C_0^0(\Omega)$, $0 \leq \eta \leq 1$,

$$\begin{aligned} I_k[u_m] - I_k[u] &= \int_\Omega (u-u_m)\, d\mu_k[u_m] + \int_\Omega u\,(d\mu_k[u] - d\mu_k[u_m]) \\ &\leq \sup_\Omega |u-u_m|\, \mu_k[u_m](\Omega) + \sup(1-\eta)|u|(\mu_k[u_m](\Omega) + \mu_k[u](\Omega)) \\ &\quad + \int_\Omega \eta u\,(d\mu_k[u] - d\mu_k[u_m]) \to 0, \end{aligned}$$

as $\eta \to 1$, $m \to \infty$. Interchanging u and u_m we obtain $I_k[u_m] \to I_k[u]$ as required. □

Remark. If we only assume $\{u_m\} \subset \Phi^k(\Omega)$ converges to u in $C^0(\Omega)$, we obtain $I_k[u_m;\Omega'] \to I_k[u;\Omega']$ for any subdomain $\Omega' \Subset \Omega$ satisfying $\mu_k[u_m](\partial\Omega') = 0$, $m \in \mathbb{N}$. If additionally, $\mu_k[u_m] \to \mu_k[u]$ *strongly* in $\Omega - \Omega'$ for some $\Omega' \Subset \Omega$, then we obtain $I_k[u_m;\Omega] \to I_k[u;\Omega]$ as above.

Monotonicity. Hessian integrals enjoy corresponding monotonicity properties to the Hessian measures. Assuming $u, v \in C^2(\overline{\Omega})$, $u = v$ on $\partial\Omega$, $\partial\Omega \in C^2$, and writing

$$w_t = (1-t)u + tv, \quad 0 \leq t \leq 1, \quad f(t) = I_k[w_t], \tag{5.4}$$

we calculate

$$\begin{aligned} I_k[u] - I_k[v] &= f(0) - f(1) \\ &= \int_0^1 \int_\Omega (v-u) F_k[w_t]\, dt + \int_0^1 \int_\Omega w_t S_k^{ij} D_{ij}(v-u)\, dt \\ &= (k+1) \int_0^1 \int_\Omega (v-u) F_k[w_t]\, dt \\ &\quad + \int_0^1 \int_{\partial\Omega} u S_k^{ij} \gamma_i \gamma_j D_\gamma (v-u)\, dt, \end{aligned} \tag{5.5}$$

where, as in Section 2, γ denotes the unit outer normal to $\partial\Omega$. Accordingly, if $u \geq v$ in Ω, $u = v \leq 0$ on $\partial\Omega$, with u and v both k-convex in Ω, we infer $I_k[u] \leq I_k[v]$. More generally, if $M = \max_{\partial\Omega} u > 0$, we replace u, v by $u - M$, $v - M$ respectively, to obtain

$$I_k[u] = I_k[u - M] + M\mu_k[u](\Omega) \leq I_k[v - M] + M\mu_k[v](\Omega) = I_k[v].$$

We therefore have the following analogue of Lemma 2.1.

LEMMA 5.2. *Let $u, v \in \Phi^k(\Omega) \cap C^2(\overline{\Omega})$ satisfy $u = v$ on $\partial\Omega$, $u \geq v$ in Ω. Then*

$$I_k[u] \leq I_k[v]. \tag{5.6}$$

By approximation, using Theorem 5.1, we then infer the analogue of Corollary 2.4.

COROLLARY 5.3. *Let $u, v \in \Phi_0^k(\Omega)$ satisfy $u \geq v$ in Ω. Then*

$$I_k[u] \leq I_k[v]. \tag{5.7}$$

REMARK. More generally, if $u, v \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$, $u \geq v$ in Ω, we obtain, using our previous remark after Theorem 5.1,

$$\liminf_{\delta\to 0} I_k[u; \Omega_\delta] \leq \limsup_{\delta\to 0} I_k[v; \Omega_\delta], \tag{5.8}$$

where $\Omega_\delta = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \delta\}$. Also if only $u, v \in \Phi^k(\Omega)$, then $I_k[u; \Omega'] \leq I_k[v; \Omega']$ for any subdomain $\Omega' \Subset \Omega$, where $u \geq v$ and $u = v$ on $\partial\Omega'$.

Variational derivatives. From (5.5) we have

$$f'(0) = (k+1)\int_\Omega (u - v)F_k[u] - \int_{\partial\Omega} uS_k^{ij}(D^2u)\gamma_i\gamma_j D_\gamma(v - u). \tag{5.9}$$

Furthermore, if $D_\gamma u = D_\gamma v$ (or $u = 0$) on $\partial\Omega$, we have

$$\begin{aligned} f'(0) &= (k+1)\int_\Omega (u - v)F_k[u], \\ f''(t) &= (k+1)\int_\Omega S_k^{ij}(D^2 w_t)D_i(u - v)D_j(u - v) \geq 0, \end{aligned} \tag{5.10}$$

if u, v are k-admissible in Ω. Moreover, if $\varphi = u - v$ has compact support in Ω_δ for some $\delta > 0$, we have an upper bound,

$$\begin{aligned} f''(t) &\leq (k+1)(n-k+1)\int_{\Omega_\delta} F_{k-1}[w_t] \max|D\varphi|^2 \\ &\leq C\big(\mathrm{osc}_\Omega u + \mathrm{osc}_\Omega \varphi\big)^{k-1} \max|D\varphi|^2 \end{aligned} \tag{5.11}$$

by Lemma 2.2. By approximation we then obtain the following variational formula.

Theorem 5.4. *Let $u, v \in \Phi^k(\Omega) \cap C^0(\overline{\Omega})$ with $\varphi = u - v \in C_0^2(\Omega)$ and let $f(t) = I_k[w_t]$, $0 \le t \le 1$. Then*

$$f'(0) = (k+1) \int_\Omega (u - v)\, d\mu_k[u]. \tag{5.12}$$

Further Remarks. Taking account of the preceding section, certain of the above results extend to semi-convex or admissible functions. In particular Theorem 5.1 extends to sequences $\{u_m\}$ of k-semi-convex functions (with same constant A) or sequences $\{u_m\} \subset \Phi^k(\Omega, g)$ for some $g \in L^1(\Omega)$, vanishing continuously on $\partial\Omega$. The variational formula (5.12) remains valid for u, v being k-semi-convex or k-admissible with $u \in \Phi^k(\Omega, g)$. Furthermore, if $u, v \in C^0(\overline{\Omega}) \cap \Phi^k(\Omega, g)$ with $u = v = 0$ on $\partial\Omega$, $u \ge v$ in Ω, we obtain from (5.10), the inequality

$$I_k[u] - I_k[v] \le (k+1) \int_\Omega (v - u)\, d\mu_k[u], \tag{5.13}$$

which complements Lemma 5.2. In the case $k = n$, inequalities (5.7) and (5.13) were proved by Krylov [14]. Accordingly, if we define the functional $J_{k;\mu}$ on $\Phi_0^k(\Omega)$ by

$$J_{k;\mu}[u] = \frac{1}{k+1} I_k[u] + \int_\Omega u\, d\mu \tag{5.14}$$

for $u \in \Phi_0^k(\Omega)$, where μ is a finite Borel measure on Ω, we obtain

$$J_{k,\mu}[u] = \min_{u \ge v} J_{k;\mu}[v], \tag{5.15}$$

provided $\mu_k[u] = \mu$. Consequently, the solution of the Dirichlet problem (1.7) for $\varphi \equiv 0$ solves the variational problem (5.15). Related variational problems are treated in [3], [8], [9] and [13].

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