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ON THE MULTIVALUED POINCARÉ OPERATORS

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(*Submitted by L. Górniewicz*)

1. Introduction

By Poincaré operators we mean the translation operator along the trajectories of the associated differential system and the first return (or section) map defined on the cross section of the torus by means of the flow generated by the vector field. The translation operator is sometimes also called as Poincaré–Andronov or Levinson or, simply, T-operator.

In the classical theory (see [K], [W], [Z] and the references therein), both these operators are defined to be single-valued, when assuming, among other things, the uniqueness of the initial value problems. At the absence of uniqueness one usually approximates the right-hand sides of the given systems by the locally lipschitzian ones (implying uniqueness already), and then applies the standard limiting argument. This might be, however, rather complicated and is impossible for the discontinuous right-hand sides.

On the other hand, set-valued analysis allows us to handle effectively also with such classically troublesome situations. In particular, the class of admissible maps in the sense of [G] has been shown to be very useful with this respect, because generalized topological invariants like the Brouwer degree, the fixed point

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index or the Lefschetz index with properties similar to those of their classical analogues can be defined, and subsequently applied for them.

Hence, in our contribution we develop at first the Rothe-type generalization of the Brouwer fixed-point theorem for admissible maps. Then we introduce some conditions under which the Marchaud right-hand sides of differential inclusions determine admissible Poincaré operators. Finally, we present simple applications of the obtained results to the existence of forced nonlinear oscillations and to the multiplicity criterium for the target problem.

2. Preliminaries

It will be very convenient to employ here the notion of admissibility as developed in [G].

Following [G, Definition 2.5, p. 27], an upper semi-continuous (u.s.c.) map $\varphi : X \multimap Y$ with compact values, where X and Y are two metric spaces, is called *admissible* if there exist a metric space Z and two continuous (single-valued) maps $p : Z \to X$, $q : Z \to Y$ such that the following conditions are satisfied:

- (i) p is a Vietoris map, i.e.
  - (a) p is proper (for each compact $A \subset X$, the counter image $p^{-1}(A)$ is a compact subset of Z),
  - (b) p is onto,
  - (c) for every point $x \in X$, the set $p^{-1}(x)$ is acyclic,
- (ii) $q(p^{-1}(x)) \subset \varphi(x)$ for every $x \in X$; the pair (p, q) is called a selected pair for φ.

The following "Rothe-type" generalization of the Brouwer fixed point theorem will be useful for us.

Theorem 1. *Let K be a compact, convex subset of $\mathbb{R}^n$, with nonempty interior, and let $\varphi : K \multimap \mathbb{R}^n$ be an admissible map such that $\varphi(\partial K) \subset K$, where ∂K denotes the boundary of K. Then φ has a fixed point, i.e. there exists $x \in K$ such that $x \in \varphi(x)$.*

Proof. We can assume without any loss of generality that $0 \in \operatorname{int} K$ and φ has no fixed points on the boundary ∂K.

Consider the homotopy

$$\chi : K \times [0,1] \multimap \mathbb{R}^n$$

defined by the formula $\chi(x,t) = t\varphi(x)$.

Our aim is to show that also χ has no fixed points on the boundary ∂K, i.e. for $x \in \partial K$ and $t \in [0,1] : x \notin \chi(x,t)$. In other words, it is enough to prove that $x \neq ty$ for any $y \in \varphi(x)$. Since for $t = 1$ our claim is implied by the hypothesis, let us assume that $t \in [0,1)$. However, because of $x \in \partial K$, $y \in K$ and K being a convex set, we already have $x \neq ty$ for such values of t. Now, by applying the homotopy property of the fixed point index, we obtain that the one of φ is equal 1, namely $i(\varphi, K) = 1$. Therefore, our assertion follows from the existence property for the fixed point index (see [BK]). $\square$

As a direct consequence, we can give the following

Corollary 1. *Let K be a compact, convex subset of $\mathbb{R}^n$ with nonempty interior. If $\varphi : K \rightsquigarrow K$ is an admissible map, then φ has a fixed point.*

Proof. An alternative proof directly follows from Corollary 5.2 in [G, p. 47]. Since K is an absolute neighbourhood retract, any admissible map $\varphi : K \rightsquigarrow K$ has the generalized Lefschetz number 1 (see [G]). Thus , the above statement follows also directly from the Lefschetz theorem for admissible maps (see [G, Theorem 5.1, p. 46]). $\square$

3. Multivalued translation operator

Now, consider the differential inclusion

$$\theta' \in f(t, \theta), \tag{1}$$

where $\theta = (\theta_1, \dots , \theta_n)$, $\theta' = (\theta'_1, \dots , \theta'_n)^T$ and $f(t, \theta) = (f_1(t, \theta), \dots , f_n(t, \theta))^T$. Let us assume that $f(t, \theta) : \mathbb{R} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is bounded in t, and linearly bounded in θ, upper semi-continuous with nonempty, compact, convex values. Then all solutions of (1) entirely exist in the sense of Carathéodory (i.e. are locally absolutely continuous and satisfy (1) a.e.) — see e.g. [F, Theorem 6.1]. Moreover, on any compact interval, the solution set is R_δ (see [BGP] and the references therein, where the appropriate definitions can also be found).

If $\theta(t, X) := \theta(t; 0, X)$ is a solution of (1) with $\theta(0, X) = X$, then we can define the *Poincaré-Andronov map* (*translation operator at the time T*) $\Phi_T : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ along the trajectories of (1) as follows:

$$\Phi_T(X) := \{\theta(T, X) : \theta(\,\cdot\,, X) \text{ is a solution of (1) satisfying } \theta(0, X) = X\}. \tag{2}$$

We recall the following important property (cf. [DG], [BGP]).

Lemma 1. *Φ_T given by* (2) *is admissible in the sense of the above definition.*

Proof. $\Phi_T : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ can be considered as the composition of two maps, namely $\Phi_T = \psi \circ \varphi$, or more precisely

$$\mathbb{R}^n \overset{\varphi}{\rightsquigarrow} AC([0, T], \mathbb{R}^n) \overset{\psi}{\rightsquigarrow} \mathbb{R}^n,$$

where $\varphi(X) : X \rightsquigarrow \{\theta(t, X) : \theta(t, X)$ is a solution of (1) with $\theta(0, X) = X\}$ is known to be acyclic (see e.g. [BGP]) and $\psi(y) : y(t) \to y(T)$, which is obviously continuous. Since every composition of an acyclic and continuous map is admissible as required (see [G]), we are done. $\square$

Observe that no uniqueness restriction has been imposed on. Hence, assuming furthermore that

$$f(t+T,\theta) \equiv f(t,\theta), \tag{3}$$

where T is a positive constant, system (1) admits a T-periodic solution as far as Φ_T in (2) has a fixed point.

If, for example, $\Phi_T(S^{n-1}) \subset B^n$, where $S^{n-1} = \partial B^n$, and $B^n \subset \mathbb{R}^n$ is a closed ball centered at the origin, or any other set with the fixed-point property as indicated in Theorem 1 or Corollary 1, then (1) admits, under the above assumptions, including (3), a harmonic, i.e. a T-periodic solution. Similarly, if for some $k \in \mathbb{N}$, $\Phi_{kT}(S^{n-1}) \subset B^n$, then by the same reasoning (1) admits a subharmonic, i.e. a kT-periodic solution. This is certainly also true because of $\operatorname{Deg}(X - \Phi_T, B^n) \neq \{0\}$, where Deg denotes the generalized Brouwer degree of an admissible map (see e.g. [BGP], [DG], [G]).

This can be expressed in terms of bounding functions or guiding functions as follows.

THEOREM 2. *Let a continuous T-periodic in t and locally lipschitzian in θ bounding function $V(t,\theta)$ exist such that*

(i) $V(t,\theta) = 0$ *for* $\|\theta\| = r$, *uniformly w.r.t.* $t \in [0,T]$,
(ii) $V(t,\theta) < 0$ *for* $\|\theta\| < r$, *uniformly w.r.t.* $t \in [0,T]$,
(iii) $\limsup_{h\to 0^+}[V(t+h,\theta+hY) - V(t,\theta)]/h < 0$ *for each* $Y \in f(t,\theta)$ *and* $\|\theta\| = r$, *uniformly w.r.t.* $t \in [0,T]$,

where r is a suitable positive constant which may be large. Then system (1) *admits, under* (3), *a harmonic.*

PROOF. In the single-valued case this result is well-known (see e.g. [GM]), when using C^1-bounding functions. For the differential inclusions, a similar type of results has also been developed in [DG], [F, Theorem 14.3] (see also the references therein), but using again only autonomous C^1-bounding functions. Thus, our statement represents only a slight generalization and can be proved quite analogously, when following the same geometrical ideas. □

REMARK 1. Replacing conditions (i)–(iii) by

$$\lim_{\|\theta\|\to\infty} V(t,\theta) = \infty, \quad \text{uniformly w.r.t. } t \in [0,T],$$

and

$$\lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,\theta+hY) - V(t,\theta)] < 0 \quad \text{for each } Y \in f(t,\theta),$$

and $\|\theta\| \geq r$, uniformly w.r.t. $t \in [0,T]$, where r is a positive constant which may be large, we obtain, under (3), a subharmonic of (1) (for the single-valued case see e.g. [A4], [Y] and the references therein).

Remark 2. In the single-valued case, the dissipativity in the sense of N. Levinson, i.e. the uniform ultimate boundedness of all solutions of (1) (which can be expressed quite equivalently in terms of guiding functions with the same properties as in Remark 1), is sufficient for the existence of harmonics (see e.g. [A4], [Y]). So, we can conjecture that the same is true in the set-valued case.

The situation is, however, much more interesting when, for example, (1) is only partially dissipative, i.e. if only

$$\lim_{t\to\infty}\sup\|(\theta_1(t),\dots,\theta_j(t))\|\le D\quad 1\le j<n$$

holds w.r.t. some part of components of every solution $\theta(t)$ of (1), where D is a positive constant common for all solutions of (1). It is clear that Theorem 1 is this time insufficient for applications.

In [AGL], we have developed the appropriate abstract apparatus for considering such a situation, mainly using the generalized fixed point index technique, which can be expressed in terms of two bounding functions as follows (for more details, in the single-valued case, see e.g. [GM]).

Theorem 3. *Let continuous T-periodic in t and locally lipschitzian in θ bounding functions $V(t,\theta)$ and $W(t,\theta)$ exist such that*

(i) $V(t,\theta)=0$ *for* $\|\theta_j\|=r$, *uniformly w.r.t. all* $\widehat{\theta_j}\in\mathbb{R}^{n-j}$ *and* $t\in[0,T]$,
(ii) $V(t,\theta)<0$ *for* $\|\theta_j\|<r$, *uniformly w.r.t. all* $\widehat{\theta_j}\in\mathbb{R}^{n-j}$ *and* $t\in[0,T]$,
(iii) $\limsup_{h\to0^+}[V(t+h,\theta+hY)-V(t,\theta)]/h<0$ *for each* $Y\in f(t,\theta)$, $\|\theta_j\|=r$, $\widehat{\theta_j}\in\mathbb{R}^{n-j}$ *and* $t\in[0,T]$,
(iv) $W(t,\theta)=0$ *for* $\|\widehat{\theta}_j\|=s$, *uniformly w.r.t.* $\|\theta_j\|\le r$ *and* $t\in[0,T]$,
(v) $W(t,\theta)>0$ *for* $\|\widehat{\theta}_j\|>s$, *uniformly w.r.t.* $\|\theta_j\|\le r$ *and* $t\in[0,T]$,
(vi) $\liminf_{h\to0^+}[W(t+h,\theta+hY)-W(t,\theta)]/h>0$ *for each* $Y\in f(t,\theta)$, $\|\widehat{\theta}_j\|=s$, $\|\theta_j\|\le r$ *and* $t\in[0,T]$, *where* $\theta=(\theta_j\oplus\widehat{\theta}_j)$, $1\le j<n$, *i.e.* $\theta_j:=(\theta_1,\dots,\theta_j)$, $\widehat{\theta}_j:=(\theta_{j+1},\dots,\theta_n)$ *and* r, s *are suitable positive constants which may be large.*

Then system (1) *admits, under* (3), *a harmonic.*

Remark 3. The application of the Dini derivatives above is more appropriate on the boundary of nonconvex bound sets (in the sense of [GM]) G, H. The approach developed in [AGL] allows us, certainly under a modification in the spirit of e.g. [GM], to take for this goal the domains which are star-shaped.

Remark 4. For the existence of subharmonics, the bounding functions V, W satisfying (i)–(vi) can be replaced by guiding functions like in Remark 1 for $j=n$ (see e.g. [AGG], [A4]). In the single-valued case, the existence of suitable guiding functions can be shown to imply again the existence of harmonics, when using the abstract results in [AGZ], provided $f(t,\theta)$ represents a periodic perturbation of an autonomous function. So, we can again conjecture that the same is true in the set-valued case.

4. Multivalued first return map

In this section, system (1) will be considered on the cylinder $\mathcal{C}^{n+1} = \mathbb{R}_0^+ \times \mathbb{T}^n$ or, in the autonomous case, on the torus $\mathbb{T}^n = \mathbb{R}^n/\omega\mathbb{Z}^n$, where $\omega\mathbb{Z}$ denotes all integer multiples of a positive constant ω. Thus, the natural restriction imposed on the right-hand side of (1), besides the boundedness in $t \in \mathbb{R}_0^+$, reads

$$f(t,\dots,\theta_j+\omega,\dots) \equiv f(t,\dots,\theta_j,\dots) \quad \text{for } j=1,\dots,n. \tag{4}$$

Consider still the $(n-1)$-dimensional subtorus $\Sigma \subset \mathbb{T}^n$ given by

$$\sum_{j=1}^{n} \theta_j = 0 \pmod{\omega}$$

and assume, additionally, that

$$\inf_{(t,\theta)\in\mathcal{C}^{n+1}} \sum_{i=1}^{n} f_i(t,\theta) > 0 \quad \text{or} \quad \sup_{(t,\theta)\in\mathcal{C}^{n+1}} \sum_{i=1}^{n} f_i(t,\theta) < 0. \tag{5}$$

Then we can define *the Poincaré* (*first-return*) *map* Φ on the cross section Σ as follows:

$$\Phi(p)_{\{\tau(p)\}} : \Sigma \multimap \Sigma, \quad \Phi(p)_{\{\tau(p)\}} := \{\theta(\tau(p),p)\}, \tag{6}$$

where $\Phi_0(p) = p \in \Sigma$ and $\{\tau(p)\}$ denotes the least time for p to return back to Σ, when taking into account each branch of $\theta(t,p)$. Indeed, (5) implies that $\sum_{i=1}^{n} \theta_i'(t,p) \neq 0$ for every solution $\theta(t,p)$ of (1) and almost all $t \geq 0$ by which the map $\tau(p) : \Sigma \multimap [\omega/E, \omega/\varepsilon]$ is well defined, where ε, E are positive constants such that

$$0 < \varepsilon \leq \inf_{(t,\theta)\in\mathcal{C}^{n+1}} \left| \sum_{i=1}^{n} f_i(t,\theta) \right| \leq \sup_{(t,\theta)\in\mathcal{C}^{n+1}} \left| \sum_{i=1}^{n} f_i(t,\theta) \right| \leq E.$$

Moreover, (5) means geometrically that the trajectories of (1), associated to (6), intersect Σ in a transversal way, which will be essential into the future.

Let us note that $\{\tau(p)\}$ is, even without (5), lower semi-continuous (see e.g. [CQS] and the references therein).

Observe that $\Phi_{\{\tau(p)\}}$ can be, as in the foregoing section, the fixed T time map. This appears if, for example,

$$\sum_{i=1}^{n} f_i(t,\theta) \equiv \{F(t)\},$$

where $F(t)$ is a T-periodic function such that $|\int_0^T F(t)\,dt| = \omega > 0$, because then $\tau(p) = T$ for all $p \in \Sigma$.

Lemma 2. *$\Phi_{\{\tau(p)\}}$ given by (6) is, under (5), admissible.*

Proof. $\Phi_{\{\tau(p)\}}$ can be considered as the composition of two maps, namely $\Phi_{\{\tau(p)\}} = \psi \circ \varphi$, or more precisely

$$\Sigma \overset{\varphi}{\rightsquigarrow} AC^{\star}([0,\omega/\varepsilon],\mathbb{R}^n) \overset{\psi}{\rightsquigarrow} \Sigma,$$

where $AC^{\star}$ means the space of all absolutely continuous functions with the properties (cf. (5))

$$E \geq \left|\sum_{i=1}^{n} y_i'(t,p)\right| \geq \varepsilon > 0, \quad \text{for almost all } t \in [0,\omega/\varepsilon], \tag{7}$$

$$\varepsilon t \leq \left|\sum_{i=1}^{n} y_i(t,p)\right| \leq Et \qquad \text{for } t \in [0,\omega/\varepsilon]. \tag{8}$$

Here, $\varphi(p) : p \rightsquigarrow \{\theta(t,p) : \theta(t,p)$ is a solution of (1) with $\theta(0,p) = p\}$ is known to be acyclic (see e.g. [BGP, Theorem 5.7]) and $\psi(y) : y(t,p) \to y(\tau(y),p) \in \Sigma$, which will obviously be continuous as far as $\tau(y)$ is so.

Observe that, because of the "asterisque" properties (7), (8), $\tau(y)$ is again well defined and, moreover,

$$\sum_{i=1}^{n} y_i(\tau(y)) - y_i(0) = \pm\omega. \tag{9}$$

Hence, applying to (9) a suitable implicit function theorem for maps without continuous differentiability (see e.g. [AE, Theorem 7.5.8]), the map $y \to \tau(y)$ can easily be verified, under (7), to be Lipschitz-continuous, as required.

Since the composition of acyclic and continuous maps is admissible (see[G]), the proof is complete. □

In the single-valued case, Lemma 2 has the following direct consequence.

Lemma 3. *If $f(t,\theta)$ is, additionally, continuous and locally lipschitzian in θ, then $\Phi_{\tau(p)}(p)$, associated to the equation $\theta' = f(t,\theta)$, is continuous.*

Since no torus has a fixed-point property (neither in the classical nor in the generalized sense), the analogue of Theorem 1 and Corollary 1 for Σ or $\mathbb{T}^n$ is impossible. Nevertheless, we can define the generalized Lefschetz index Λ for admissible maps (see [G]) and, moreover, $\Phi_{\{\tau(p)\}}(p)$ in (6) can be shown to be (see [A3]), under (5), homotopic in the sense of admissible maps to the identity I. Thus, if for example $\Lambda(C^{-1}(\Phi_{\{\tau(p)\}}(p))) \neq \{0\}$, where $C : \Sigma \to \Sigma$ is a diffeomorphism, and subsequently $C^{-1}(\Phi_{\{\tau(p)\}}(p)) : \Sigma \rightsquigarrow \Sigma$ is admissible, then $C^{-1} \circ \Phi$ has a fixed point, i.e. the "target problem" $C(p) \in \Phi_{\{\tau(p)\}}(p)$ is solvable. Because of the invariance under admissible homotopy, the problem turns out to be equivalent to the computation of $\Lambda(C^{-1}) \neq 0$. This can be however performed, under natural restrictions, by means of a sum of the local indices, (see e.g. [B]).

Therefore, we can give the following statement concerning the target problem for (1) on the torus Σ, expressed by means of the condition

$$C(\theta(0,p)) = \theta(t^\star, p) \quad \text{for some } t^\star > 0. \tag{10}$$

For more details see [A3].

THEOREM 4. *Let all the above regularity assumptions be satisfied, jointly with* (5). *Assume that $C : \Sigma \to \Sigma$ is a diffeomorphism having finitely many, but at least one, simple fixed points, $\gamma_1, \dots, \gamma_r$, on Σ and*

$$\Lambda := \sum_{k=1}^{r} \operatorname{sgn} \det(I - dC_{\gamma_k}^{-1}) \neq 0, \tag{11}$$

where $dC_{\gamma_k}^{-1}$ denotes the derivative of C^{-1} at $\gamma_k \in \Sigma$. Then problem (1)–(10) *admits a solution.*

COROLLARY 2. *Problem* (1)–(10) *admits, under the assumptions of Theorem* 4*, at least $|\Lambda|$ geometrically distinct solutions.*

REMARK 5. In the single-valued case, it is namely well-known (see [BBPT]) that $N(C^{-1} \circ \Phi) = |\Lambda(C^{-1} \circ \Phi)|$ on Σ. Therefore, since the Nielsen index $N(C^{-1} \circ \Phi)$ determines the lower estimate of fixed points of $C^{-1} \circ \Phi$ on Σ (see e.g. [B]), the absolute value of the nonzero number in (11) designates at the same time the lower estimate of desired solutions. In [AGJ], we have quite recently proved that the same is also true for differential inclusions, i.e. for problem (1)–(10).

Taking, in particular, $C := \sigma_{+,-} : \mathbb{T}^n \to \mathbb{T}^n$, where the shifts σ_+ or σ_- are defined by the rules $\sigma_{+,-}(\theta_1, \dots, \theta_n) = \pm(\theta_2, \dots, \theta_n, \theta_1)$, respectively, we can prove the following statement concerning the corresponding sort of nonlinear rotations (periodic oscillations of the second kind); for more details see [A2].

THEOREM 5. *Let an autonomous system* (1) *determine a $\sigma_{+,-}$-equivariant flow on $\mathbb{T}^n$, i.e. for $i = 1, \dots, n$*

$$\begin{aligned} &f_i(\theta_1, \dots, \theta_n) \\ &\quad = (\pm 1)^{i+1} f_i((\pm 1)^{i+1}\theta_i, \dots, (\pm 1)^{i+1}\theta_n, (\pm 1)^{i+1}\theta_1, \dots, (\pm 1)^{i+1}\theta_{i-1}). \end{aligned} \tag{12}$$

Let, furthermore, f_i be ω-periodic in each variable θ_j for $i, j = 1, \dots, n$, (*see* (4)) *and*

$$\sum_{i=1}^{n} (\pm 1)^i f_i(\theta) > 0 \quad \text{or} \quad \sum_{i=1}^{n} (\pm 1)^i f_i(\theta) < 0, \tag{5±}$$

respectively. Then system (1) *admits nontrivial splay-phase or anti-splay-phase orbits* $\theta(t)$*, respectively, provided* n *is even in the latter case, i.e.* $\theta'(t+T) = \theta'(t)$ *for almost all* $t \in (-\infty, \infty)$*, where* T *is a suitable positive constant and*

$$\theta(t) = \left(\varphi(t), \pm\varphi\left(t + \frac{1}{n}T\right), \dots, \varphi\left(t + \frac{n-2}{n}T\right), \pm\varphi\left(t + \frac{n-1}{n}T\right)\right),$$

respectively.

Corollary 3. *The assertion of Theorem* 5 *remains valid for* (*not necessarily autonomous*) T/n*-periodic in* t *system* (1)*, provided the nonautonomous analogue to* (12) *holds and, instead of* (5±)*,*

$$\sum_{i=1}^{n} (\pm 1)^i f_i(t, \theta) \equiv \{F(t)\},$$

where $F(t)$ *is a* T/n*-periodic function such that* $\int_0^{T/n} F(t)\,dt = \pm\omega$*, respectively. For nonautonomous,* T/n*-periodic in* t*, systems* (1)*, we obtain in fact the whole one-parameter family* (*i.e., generically, infinitely many*) *of such subharmonics of the second kind.*

It is so, because the associated first return map becomes the translation (fixed time) operator, $\Phi_{\{\tau(p)\}}(p) = \Phi_{T/n}(p)$, as already pointed out. In the nonautonomous case, it has meaning to apply the admissible translation operator $\Phi_{t_0+T/n}(p) := \{\theta(t_0 + T/n, t_0, p)\}$, for each $t_0 \in [0, T/n]$, where $\theta(t, t_0, p)$ is this time a solution of (1) with $\theta(t_0, t_0, p) = p \in \Sigma$. Thus, $\Phi_{t_0+T/n}(\Sigma) \subset \Sigma$, and using the same approach (see [A2]), we obtain for each value of $t_0 \in [0, T/n]$ the desired solutions.

5. Simple applications in examples

Example 1. As the simplest application of Theorem 3, consider the planar system under nonlinear, periodic in t, perturbation:

$$\theta' + A\theta^T \in g(t, \theta),$$

where $\theta = (\theta_1, \theta_2)$, $A = \operatorname{diag}(a_1, a_2)$ is a constant matrix and $g = (g_1, g_2)^T$ satisfies all the above regularity assumptions. Suppose, furthermore, that $a_1 > 0$, $a_2 < 0$,

$$\lim_{|\theta_1| \to \infty} \frac{g_1(t, \theta)}{|\theta_1|} = 0,$$

uniformly w.r.t. $t \in [0, T]$ and $\theta_2 \in (-\infty, \infty)$;

$$\lim_{|\theta_2| \to \infty} \frac{g_2(t, \theta)}{|\theta_2|} = 0,$$

uniformly w.r.t. $t \in [0, T]$ and $|\theta_1| \le r$.

Defining the bounding functions V and W as $V(\theta_1) := |\theta_1| - r$ and $W(\theta_2) = |\theta_2| - s$, where r, s are sufficiently big positive constants, one can readily check that Theorem 3 applies. Consequently, we have a harmonic.

Example 2. As an application of Corollary 2, consider the system

$$\theta' \in f(t,\theta) \quad \text{for } n = 2, \text{ satisfying (5)},$$

where f is 2π-periodic in each variable θ_j, $j = 1,2$ $(n = 2)$, and define the diffeomorphism $C : \Sigma \to \Sigma$ by

$$C(\theta) := (\theta_1 + c\sin\,\theta_1, \theta_2 + c\sin\,\theta_2), \quad \text{where } c \in (0,1).$$

One can easily check that $C^{-1}(\theta)$ as well as $C(\theta)$ have two fixed points on Σ, namely $\gamma_0 = (0,0)$ and $\gamma_\pi = (\pi,\pi)$.

Since there are still (using the theorem about the derivative of the inverse function)

$$dC_0^{-1} = \operatorname{diag}\left(\frac{1}{1+c}, \frac{1}{1+c}\right), \qquad dC_\pi^{-1} = \operatorname{diag}\left(\frac{1}{1-c}, \frac{1}{1-c}\right),$$

i.e.

$$I - dC_0^{-1} = \operatorname{diag}\left(\frac{c}{1+c}, \frac{c}{1+c}\right), \qquad I - dC_\pi^{-1} = \operatorname{diag}\left(\frac{-c}{1-c}, \frac{-c}{1-c}\right),$$

we obtain

$$\det(I - \mathrm{d}\,C_0^{-1}) = \left(\frac{c}{1+c}\right)^2 \quad \text{and} \quad \det(I - \mathrm{d}\,C_\pi^{-1}) = \left(\frac{c}{1-c}\right)^2.$$

So, we can conclude that the Nielsen index $N(C^{-1}) = 2$, by which our system admits, according to Corollary 2, at least two geometrically distinct solutions $\theta(t)$ satisfying (10), i.e.

$$[\theta_1(0) + c\sin\theta_1(0), \theta_2(0) + c\sin\theta_2(0)] = [\theta_1(t^*), \theta_2(t^*)]$$

for some $t^* > 0$.

Example 3. As an application of Corollary 3, consider the system

$$\theta_i' = (\pm 1)^{i+1}p(t) \mp nC\sin\theta_i + C\sum_{j=1}^{n}(\pm 1)^{j+i+1}\sin\theta_j, \quad i = 1,\dots,n,$$

where C is an arbitrary constant and $p(t)$ is a continuous function such that $p(t + T/n) \equiv p(t)$, $\int_0^{T/n} p(t)\,dt = \pm 2\pi$, respectively. Since all the assumptions of Corollary 3 are evidently satisfied, we have a one-parameter family of subharmonics of the second kind with the components equally staggered in phase.

6. Concluding remarks and acknowledgements

The discretization of the single-valued translation operator has been considered, as a possible modification of the Poincaré–Andronov map, w.r.t. the four-point boundary value problem in [A1]. M. Lewicka has proved in [L] that the set-valued analogue is also obtainable, of course under an appropriate elaboration.

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