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MORIN SINGULARITIES AND GLOBAL GEOMETRY IN A CLASS OF ORDINARY DIFFERENTIAL OPERATORS

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Introduction

In this paper we consider the differential equation

$$u'(t) + f(t, u(t)) = g(t), \tag{$*$}$$

where the unknown u is a real function on $\mathbb{S}^1$ and the nonlinearity $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ can assume a number of forms.

Our approach is to study the global geometry of the operator

$$F : B^1 \to B^0, \qquad u \mapsto u' + f(t, u)$$

where the domain is either $C^1(\mathbb{S}^1)$ (the Banach space of periodic functions with continuous derivatives) or the Hilbert space $H^1(\mathbb{S}^1)$ of periodic functions with square integrable derivative. Ideally, we search for global changes of variables in both domain and image taking the operator F to a simple normal form. This goal has been achieved in previous occasions, starting with the seminal work of A. A. Ambrosetti and G. Prodi ([AP]) and its geometric interpretation by M. S. Berger and P. T. Church ([BC]), who showed that the operator associated to a certain nonlinear Dirichlet problem gives rise to a global fold between infinite

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dimensional spaces. Topological global cusps have appeared already in operators related to partial differential equations with a parameter ([BCT], [CDT]). Closer to the subject of this paper, H. P. McKean and J. C. Scovel ([McKS]) showed that the operator F for $f(t,x) = x^2$ (or more generally, for convex nonlinearities) is also a global fold, and raised the question of the global nature of F for $f(t,x) = x^3 - x$. The same question was asked by V. Cafagna and F. Donati ([CD]) and P. T. Church and J. G. Timourian ([CT]), who state and prove partial results for the more general Cafagna–Donati equation ([CD]), for which $f(t,x) = ax+bx^2+cx^{2k+1}$ for appropriate choices of a, b and c. In Theorem 5.1 and Corollary 5.5, we show that these nonlinearities indeed obtain global cusps. With some additional effort, we show that in Hilbert spaces the requested global changes of variables can be taken to be smooth.

Actually, the operator F is simple enough that substantial insight into its global geometry can be obtained without having to specialize f. First, we construct a global Lyapunov–Schmidt type decomposition of F. Split a function u as a sum of a function of average zero ($\widetilde{u}$) and a constant ($\overline{u}$) and decompose domain and image accordingly: $B^i = \widetilde{B}^i \oplus \overline{B}^i$. Writing the action of F as

$$u = (\widetilde{u}, \overline{u}) \mapsto v = (\widetilde{v}, \overline{v}),$$

we show in Theorem 1.2 that, for each $\overline{u}$, the correspondence $\widetilde{u} \mapsto \widetilde{v}$ is a global diffeomorphism. This provides a change of coordinates in the domain of F bringing it to (global) *adapted coordinates*

$$(\widetilde{v}, \overline{u}) \mapsto (\widetilde{v}, \overline{v}).$$

We immediately infer that the inverse images of vertical lines under F are *fibres*, curves foliating the domain and intersecting every horizontal plane exactly once and transversally. The study of F in a sense boils down to the study of its behaviour on the fibres: for example, $f(t,x) = x^2$ produces a fold on every fibre and thus F is a global fold.

No hypothesis on the behaviour of f at infinity is necessary to obtain adapted coordinates: in particular, we obtain some results about the global geometry of F even when it is not proper. From Proposition 1.4, properness of f implies properness of F but, from Proposition 4.1, the converse is false. Adapted coordinates combined with properness make clear the possibility of definining a *topological degree* for F: the degree of F is just the degree of any of its restrictions to fibres.

It is easy to see that (generically) S_1, the critical set of F, is a manifold. Rather surprisingly, the global geometry of S_1 does not depend on the nonlinearity: generically, it is connected and contractible (Corollary 1.9). This follows from a more general theorem (Theorem 1.8 or [MST]) on the contractibility of regular level sets of a class of functionals defined by integration. From contractibility, by topological arguments often using the infinite dimension of the spaces involved ([Ka], [Ku], [S]), there is a change of variables in the domain of F taking S_1 to a closed hyperplane; in the Hilbert case, this change of variables can be taken to be a diffeomorphism but in the Banach case, it is merely a homeomorphism.

We then proceed to study the critical points of F in detail. From adapted coordinates, $\ker DF$ has dimension 1 at critical points and $\operatorname{im} DF$ is then a closed subspace of codimension 1. This restricts considerably the possible nature of a generic critical point of F: it has to be an infinite dimensional *Morin singularity* ([M]). More precisely, after changes of coordinates F near a generic singularity u can be written as

$$(Z, x_1, \ldots, x_{k-1}, y) \mapsto \left(Z, x_1, \ldots, x_{k-1}, y^{k+1} + \sum_{i=1,\ldots,k-1} x_i y^i\right)$$

near zero, where Z is an element of an infinite dimensional space and x_i and y are real numbers. The integer k is the *order* of the singularity: folds and cusps are Morin singularities of orders 1 and 2. Morin's classification and proof carry over to the infinite dimensional case by making use of a version of the Malgrange preparation theorem with an (infinite dimensional) parameter: this approach has been used in [CDT] to obtain a characterization of infinite dimensional cusps. The description of a Morin singularity is given more explicitly in Propositions 2.1 and 2.2 in terms of a collection of functionals $\Sigma_i, i = 1, 2, \ldots$: at a singular point of order k, the first k functionals have to be zero and some transversality relations have to hold. Given v, we may define a *return map* ρ_v taking x_0 to x_1 if a (possibly non-periodic) solution u of $(*)$ satisfies $u(0) = x_0$, $u(1) = x_1$. In Proposition 2.3, we relate the order of a singularity u to the order of contact between $\rho_{F(u)}$ and the identity at $u(0)$.

In the autonomous case, when the nonlinearity does not depend on t, also S_2, the set of critical points which are not folds, is (generically) a connected contractible manifold. To show this, we need again Theorem 1.8 and Lemma 3.1, stating that S_1 and S_2 are diffeomorphic to the simpler sets $\widehat{S}_1$ and $\widehat{S}_2$, critical and non-fold points of the *simplified operator*

$$\widehat{F} : B^1 \to B^0, \qquad u \mapsto u' + \int f(t, u(t))\, dt.$$

Again, contractibility yields a change of variables flattening S_1 and S_2. Here, and throughout the paper, the indefinite integral represents integration over $\mathbb{S}^1$.

The functionals Σ_i are rather complicated and we do not know of a simple procedure to decide if singularities of a given order exist for a fixed f. However, in the autonomous case, we describe in Lemma 3.5 a necessary (and essentially sufficient) criterion for the existence of singularities of order k for the simplified operator $\widehat{F}$. In the same lemma, we show that if $\widehat{F}$ has a singularity of order k then F also does.

In Section 4, we consider some special types of functions f: if f is either monotonic or convex for each value of the first coordinate t, we give a global description of the behaviour of F. Even though some of the results are simple or well known (from [McKS]), they provide a convenient introduction to our approach of studying F fibre by fibre. Using Lemma 3.5, we give a criterion (Theorem 4.4) for autonomous nonlinearities to decide whether the operator F is a global fold. In particular (Corollary 4.5), polynomial non-convex nonlinearities f give rise to operators F with $S_2 \neq \emptyset$ but there are fast-growing non-convex nonlinearities for which the operator is a global fold. Also, local behaviour characterizes global folds (Theorem 4.6): generically, if all singularities of F are folds then F is a global fold.

In Section 5, the autonomous nonlinearities f satisfy $f''' \geq 0$ with isolated zeros. The related operator F is then a global cusp: here we make full use of our techniques. In this case there does not seem to be an explicit description of the requested (global) changes of variables: their existence follows by topological arguments similar to those used in the study of the sets S_1 and S_2. Lemma 5.4 is a global parametrized version of Whitney's normal form for cusps ([W]); the proof appears to be cumbersome but its main difficulty lies in verifying that Whitney's construction can be performed smoothly in a parameter. We present only a sketch of argument and we thank John Mather for helpful discussions.

We finish the paper with an example of a different kind. The results in Sections 4 and 5 are enough to show that if $f(t,x)$ is a polynomial in x of degree $d \leq 3$ (with coefficients depending on t and non-zero coefficient of highest degree), then the related operator F is a diffeomorphism, a global fold or a global cusp. In this case, thus, equation $(*)$ has at most d periodic solutions. The number of solutions of $(*)$ when f is such a polynomial was considered by Pugh, Lins Neto and Smale ([L]) who proved the bounds above and that the number of solutions may be arbitrarily large for $d = 4$. We instead exhibit a numerical example of an autonomous polynomial f of degree four and a function u which is a Morin singularity of order four (a *butterfly*). This is accomplished by requesting that u be a root of the first four functionals Σ_i. By the normal form of F at a butterfly, there are points g near $F(u)$ with five pre-images; one is presented. By a degree-theoretic argument, such a (regular) point ought to have an even number of pre-images, and we verified by solving the differential equation with a Runge–Kutta method that there are exactly six initial conditions giving rise to periodic solutions.

1. Adapted coordinates and the critical set

We consider the smooth nonlinear operator $F : B^1 \to B^0$ given by

$$F(u)(t) = u'(t) + f(t, u(t)),$$

where $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Here B^1 and B^0 can be chosen in two different ways. In the H case, they are the Sobolev spaces $B^1 = H^1 = H^1(\mathbb{S}^1; \mathbb{R})$ (the periodic absolutely continuous real valued functions with derivative in L^2) and $B^0 = H^0 = L^2(\mathbb{S}^1; \mathbb{R})$. In the C case, $B^1 = C^1 = C^1(\mathbb{S}^1; \mathbb{R})$ and $B^0 = C^0 = C^0(\mathbb{S}^1; \mathbb{R})$. For notational convenience, inner products are to be interpreted in the L^2 sense even in other spaces. An interesting special situation is the *autonomous* case, in which f does not depend on the t coordinate. We denote the partial derivative of f with respect to the second variable by $D_2 f$.

Proposition 1.1 below obtains a formula for DF at arbitrary points, a description of the critical set $S_1 F$ and a Lyapunov-Schmidt decomposition for the operator F in a neighbourhood of a critical point in the domain. In Theorem 1.2 we show the existence of a convenient global decomposition of F.

Recall the familiar Green kernel $k(x) = x - \lfloor x \rfloor - 1/2$ (where $\lfloor x \rfloor$, following Knuth, is the largest integer not larger than x). If h is periodic (with period 1) then $h_1(t) = \int k(s - t)h(s)\, ds$ is also periodic and $h_1'(t) = h(t) - \int h(s)\, ds$, a function of average 0, so that k is a kernel for the inverse of derivative, restricted to functions of average zero.

Proposition 1.1. *The derivative*

$$(DF(u)v)(t) = v'(t) + D_2 f(t, u(t))v(t)$$

is a Fredholm operator of index 0 *from* B^1 *to* B^0*. Furthermore,* $\int D_2 f(s, u(s))\, ds$ *is the unique real eigenvalue of* $DF(u)$*, which is simple, with corresponding eigenvector is*

$$w(t) = e^{-\int k(s-t) D_2 f(s,u(s))\, ds}.$$

In particular, the critical set of F *is*

$$S_1 F = \left\{ u \in B^1 \;\middle|\; \int D_2 f(t, u(t))\, dt = 0 \right\}.$$

The subspace $\langle 1/w \rangle^\perp$ *has codimension* 1*, is transversal to* $\langle w \rangle$ *and is also invariant under* $DF(u)$*. Thus, the restriction*

$$DF(u) : \langle 1/w \rangle^\perp \subset B^1 \to \langle 1/w \rangle^\perp \subset B^0$$

is bijective.

By an eigenvector of $DF(u) : B^1 \to B^0$ we mean a solution of $DF(u)v = v' + D_2 f(t, u(t))v = \lambda v$; by standard regularity arguments, solutions of this equation are always in B^1.

Proof. The formula for the derivative is straightforward. The expression for w follows from the explicit solution of the first order periodic linear ODE and $1/w$ is the only real eigenvector of the adjoint operator

$$v \mapsto -v' + D_2 f(t, u(t))v. \qquad \square$$

Let $B^1 = \widetilde{B}^1 \oplus \langle 1 \rangle$ and $B^0 = \widetilde{B}^0 \oplus \langle 1 \rangle$ (the tilde denotes integral equal to 0) defining complementary projections $\Pi_{\widetilde{B}}$ and $\Pi_{\overline{B}}$. More concretely, $\widetilde{u} = \Pi_{\widetilde{B}} u = (u - \int u)$ and $\overline{u} = \Pi_{\overline{B}} u = \int u$. Notice that $\langle 1 \rangle$ is always transversal to $\langle 1/w \rangle^\perp$ and that $\langle w \rangle$ is likewise transversal to $\widetilde{B}^i$.

Theorem 1.2. *Let $F(\widetilde{u} + \overline{u}) = \widetilde{v} + \overline{v}$. The map $\Psi : B^1 \to B^0$, $\Psi(u) = \widetilde{v} + \overline{u}$, is a* (*global*) *diffeomorphism.*

We provide some equivalent, more geometric, readings for this rather dry statement. The following diagram may be helpful:

$$\begin{tikzcd} \widetilde{u}, \overline{u} \arrow[rr, "F"] \arrow[dr, "\Psi"'] & & \widetilde{v}, \overline{v} \\ & \widetilde{v}, \overline{u} \arrow[ur, "\mathbf{F}"'] & \end{tikzcd}$$

The map $\mathbf{F} = F \circ \Psi^{-1} : B^0 \to B^0$ takes $(\widetilde{v}, \overline{u})$ to $(\widetilde{v}, \overline{v}) = (\widetilde{v}, \phi(\widetilde{v}, \overline{u}))$. Horizontal hyperplanes $\widetilde{B}^1 + \{c\}$ are injectively taken by F onto *sheets*, i.e., hypersurfaces intersecting each vertical line $\{\widetilde{v}\} + \langle 1 \rangle$ transversally and exactly once. Equivalently, the inverse images under F of the vertical lines $\{\widetilde{v}\} + \langle 1 \rangle$ foliate B^1 by *fibres*, i.e., curves intersecting each horizontal hyperplane transversally and exactly once; we denote by τ_u the fibre containing u. Let T be the set of fibres: from transversality, we may identify T with any horizontal hyperplane in the domain, in particular with $\widetilde{B}^1$. The set of vertical lines in the image is naturally identified with $\widetilde{B}^0$ and F induces a diffeomorphism from T to $\widetilde{B}^0$.

The following proof applies to both cases but certain complications are relevant only in the H case.

Proof. In order to invert a vertical line and obtain a fibre, we consider the differential equation

$$(*) \qquad u'(t) + f(t, u(t)) = \widetilde{v}(t) + \nu,$$

where $\widetilde{v} \in \widetilde{B}^0$ is fixed and $\nu \in \mathbb{R}$ is a parameter. Local existence, uniqueness and continuous dependence on parameters hold even when $\widetilde{v}$ is only L^2. Also, solutions cease to exist only by going to infinity.

Given t_0 and $u(t_0)$ there are $\varepsilon > 0$, ν_+ and ν_- such that the two solutions u_+ and u_- of $(*)$ with initial condition $u(t_0)$ satisfy:

(+) *u_+ either goes to ∞ at some time t, $t_0 < t \le t_0 + \varepsilon$ or satisfies $u_+(t_0 + \varepsilon) > u_+(t_0)$.*

(−) *u_- either goes to $-\infty$ at some time t, $t_0 - \varepsilon \le t < t_0$ or satisfies $u_-(t_0 - \varepsilon) < u_-(t_0)$.*

We discuss only (+): item (−) is analogous. Notice that the claim is trivial in the C case: choose the parameter ν_+ so that the derivative at time t_0 of u_+ is positive. Clearly, if (+) is satisfied by some ν_+, it is satisfied by sufficiently positive ν_+.

Solve (∗) for $\nu = 0$ to obtain a solution u_0 defined on $[t_0, t_0 + \varepsilon]$. Without loss, $u_0(t_0) > u_0(t_0 + \varepsilon)$. Let $u_1(t) = u_0(t) + \varepsilon^{-1}(u_0(t_0) - u_0(t_0 + \varepsilon))(t - t_0)$. We choose ν_+ such that the vector field $(1, -f(t, u) + \widetilde{v}(t) + \nu_+)$ always crosses the graph of u_1 upwards, i.e., $-f(t, u_1(t)) + \widetilde{v}(t) + \nu_+ > u_1'(t) = -f(t, u_0(t)) + \widetilde{v}(t) + \varepsilon^{-1}(u_0(t_0) - u_0(t_0 + \varepsilon))$. This is clearly possible since f is continuous.

Given $u(0)$, there is some ν_+ for which the solution u of (∗) either goes to ∞ at some time t, $0 < t \le 1$, or satisfies $u(1) > u(0)$.

This follows from the previous claim by a compactness argument.

Given $u(0)$, there is a unique ν for which (∗) admits a periodic solution.

Consider the set A^+ (resp. A^-) of ν's such that the solution goes to ∞ (resp. $-\infty$) or satisfies $u(1) \ge u(0)$ (resp. $u(1) \le u(0)$). From the previous claim (and the obvious counterpart), both sets are non-empty. By continuous dependence on parameters, both are closed. As $A^+ \cup A^- = \mathbb{R}$, the sets intersect: any point in the intersection yields a periodic orbit. Uniqueness follows from the local behaviour of the solutions.

Given v, there is a unique ν for which (∗) admits a periodic solution u with $\int u = v$.

Consider all periodic solutions of (∗) as curves in $\mathbb{S}^1 \times \mathbb{R}$. Again, from the local behaviour of solutions, the curves are disjoint. By the previous claim, the union of all such curves contains the line $\{0\} \times \mathbb{R}$. By a similar argument applied to other lines, the union of the curves is $\mathbb{S}^1 \times \mathbb{R}$ and the curves form a continuous foliation of the cylinder by circles. Notice that circles in the foliation correspond to points in the fibre $F^{-1}(\{\widetilde{v}\} + \langle 1 \rangle)$. The integrals of the solutions are strictly increasing as a function of $u(0)$. Also, there are solutions with arbitrarily large (positive or negative) integrals, since the area between two curves goes to infinity as the initial condition of one of the curves does. □

At this point, we have that, given $\overline{u}$ and $\widetilde{v}$, there is a unique $\widetilde{u}$ with $F(\widetilde{u}+\overline{u}) = \widetilde{v}+\overline{v}$ (for some $\overline{v}$). Thus, the function $\Pi_{\widetilde{B}^0} \circ F$ is a bijection from any hyperplane $\widetilde{B}^1 + \{v\}$ to $\widetilde{B}^0$. This function is clearly smooth and, as discussed before the statement of this theorem, its derivative is always invertible. By the inverse function theorem, these bijections are diffeomorphisms.

Remarks. 1. Theorem 1.2 is the counterpart of the usual global domain decomposition found in the study of the equation $\Delta u = f(u)$, with Dirichlet boundary conditions and special resonance hypothesis on f (see [AP]). There, the task is simplified by the use of self-adjoint spectral theory. In our case, the derivative, unlike the Laplacian, is skew-symmetric, with purely imaginary spectrum containing 0, and the nonlinearity interacts at most with 0.

2. The hard part in an eventual functional analytic proof of Theorem 1.2 is the properness (and hence, from local behaviour, bijectivity) of the function $\Pi_{\widetilde{H}^0} \circ F : \widetilde{H}^1 + \{v\} \to \widetilde{H}^0$; the necessary estimates seem to be simple only in the C case.

For later use, we state as a lemma some consequences of the proof of Theorem 1.2.

Lemma 1.3. *Fibres are parametrized by average, i.e., the function*

$$\tau_{u_0} \to \mathbb{R}, \quad u \mapsto \int f(t, u(t))\, dt$$

is a diffeomorphism. Let u_a be the element of average a in τ_{u_0}. Then

$$\lim_{a\to\infty} \min_t u_a(t) = \infty, \qquad \lim_{a\to-\infty} \max_t u_a(t) = -\infty.$$

Given $t_0 \in \mathbb{S}^1$, the function

$$\tau_{u_0} \to \mathbb{R}, \quad u \mapsto u(t_0)$$

is also a diffeomorphism.

The study of the (global and local) geometry of F thus reduces to the study of $\mathbf{F} = F \circ \Psi^{-1}$ and therefore of $\phi : B^0 \to \mathbb{R}$. The diffeomorphism Ψ is said to provide F with *adapted coordinates*, i.e., $\mathbf{F}(\widetilde{v}, \overline{u}) = (\widetilde{v}, \phi(\widetilde{v}, \overline{u}))$. This change of variables is convenient to the classification of critical points of F, as we shall see in the next section. Notice that we do not have formulae for $\mathbf{F}$ or Ψ and have to make do with w and $\Phi(u) = (\Pi_{\overline{B}_0} \circ F)(u) = (\phi \circ \Psi)(u) = \int f(t, u(t))dt$ (a somewhat cumbersome formula for $\mathcal{W}$ is given in Lemma 1.5).

Proposition 1.4. *If f is proper then the operator $F : B^1 \to B^0$ is proper.*

Notice that $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ is proper if and only if $|f(t,x)|$ goes to infinity when (t,x) does. In particular, if f is proper, the restriction of F to a fibre takes infinity to infinity.

Proof. Any compact set $K \subseteq B^0$ is contained in the product of its (compact) projections, so without loss, K can be taken to be the product of a compact set $\widetilde{K} \subseteq \widetilde{B}^0$ and an interval $[-k, k]$. Since $F \circ \Psi^{-1}$ is of the form $(\widetilde{v}, \overline{u}) \mapsto (\widetilde{v}, \phi(\widetilde{v}, \overline{u}))$, the compactness of the preimage of K under $F \circ \Psi^{-1}$ (and hence under F) follows from the boundedness of $\overline{u}$ in $F^{-1}(K)$ or the uniform boundedness of $u \in F^{-1}(K)$.

In the C case, consider u at its global extrema: there, $v(t) = u'(t) + f(t,u(t)) = f(t,u(t))$ and properness of f gives us the required uniform bound. For the H case, assume by contradiction that there are $u_n \in F^{-1}(K)$ with $u_n(t_n) > 2^n$, where t_n is the global maximum of u_n, and, without loss of generality, that f is positive for large positive x. Consider the intervals $I_n = (t_n - 1/10, t_n)$. If $u_n(t) > 2^{n-1}$ for all t in I_n for all sufficiently large n then $\int_{I_n} v_n(t)\,dt = u_n(t_n) - u_n(t_n - 1/10) + \int_{I_n} f(t,u_n(t))\,dt > 1/10 \min_{x>2^{n-1}} f(t,x)$ goes to infinity with n; the L^2 norm of $v_n \in K$ is unbounded, and we are done with this case. Otherwise, let t_n' be the largest value in I_n for which $u_n(t_n') = 2^{n-1}$: $\int_{t_n'}^{t_n} v_n(t)\,dt = u_n(t_n) - u_n(t_n') + \int_{t_n'}^{t_n} f(t,u_n(t))\,dt > 2^{n-1}$ for sufficiently large n and again we have a contradiction. $\square$

Remark. There are simple a priori estimates yielding properness in the autonomous H case. Also, in the C case (even for non-autonomous f) easy estimates obtain Proposition 1.4 without invoking Theorem 1.2. The analogous proof in the general H case appears to be considerably more elaborate and we preferred making use of the more geometric Theorem 1.2.

The results above can be used to provide a simple definition of topological degree for the operator F in the case when f is proper:

$$\deg F = \deg \mathbf{F} = \sum_{w \in \mathbf{F}^{-1}(v)} \operatorname{sgn} \phi_2(w),$$

where v is an arbitrary regular value of F. As usual, the right hand side does not depend on the choice of v: it is the degree of $v \mapsto \phi(\widetilde{v}, v)$, a proper function from $\mathbb{R}$ to $\mathbb{R}$. From the behaviour of ϕ at infinity (proof of Proposition 1.4),

$$\deg F = \operatorname{sgn}\left(\lim_{x\to\infty} f(t,x)\right) - \operatorname{sgn}\left(\lim_{x\to-\infty} f(t,x)\right).$$

Adapted coordinates give another simple characterization of the critical set: $(\widetilde{v}, \overline{u})$ is a critical point of $\mathbf{F}$ if and only if $D_2\phi(\widetilde{v}, \overline{u}) = 0$. Equivalently, u is a critical point of F if and only if $D\Phi(u)\mathcal{W} = 0$ where $\mathcal{W}$ is the tangent vector to τ_u at u given by the pull-back $\mathcal{W}_u = (D\Psi(u))^{-1}(\mathbf{1}(\Psi(u)))$ ($\mathbf{1}$ is the vertical vector field consisting of the constant function 1 at each point).

Lemma 1.5. *Given $u \in B^1$ and $m \in \mathbb{R}$, there is a unique $\alpha \in \mathbb{R}$ such that the equation*

$$(*) \qquad \omega' + D_2 f(t,u(t))\omega = \alpha,$$

has a (unique) periodic solution ω of average m. The function $\mathcal{W}$ is the only such ω of average 1; furthermore, $\mathcal{W}$ is strictly positive.

Proof. The first claim follows from either solving $(*)$ or from arguments in the proof of Theorem 1.2. For the element u_a of average a in τ_u,

$$u_a'(t) + f(t, u_a(t)) - \int f(t, u_a(t))\, dt = \widetilde{v}.$$

Differentiating in a and setting $\mathcal{W} = \frac{\partial}{\partial a} u_a$, the equation $(*)$ for $\mathcal{W}$ follows. Since $\int u_a = a$, $\int \mathcal{W} = 1$. The graphs of $\mathcal{W}$ and the constant function 0 do not cross, implying positivity of $\mathcal{W}$. □

The following lemma introduces yet another characterization of the critical set S_1F and, under generic hypothesis, establishes convenient transversality properties of these characterizations. This lemma will be essential for the more detailed study of singularities of F in the next section.

Lemma 1.6. *Let $\Sigma_a, \Sigma_b, \Sigma_c : B_1 \to \mathbb{R}$ be given by*

$$\begin{aligned}
\Sigma_a(u) &= \int D_2 f(t, u(t))\, dt,\\
\Sigma_b(u) &= \int D_2 f(t, u(t))\, w_u(t) dt,\\
\Sigma_c(u) &= \int D_2 f(t, u(t)) \mathcal{W}_u(t)\, dt.
\end{aligned}$$

Then the three Σ's differ by strictly positive smooth multiplicative factors. In particular, S_1F is the zero level of each of these functionals and if 0 is a regular value of any of the functionals, it is a regular value of all of them.

Remarks. 1. For an open dense set of functions f, 0 is a regular value of Σ_a. Indeed, taking derivatives as usual, 0 is a singular value if and only if there is $u \in B^1$ with $\int D_2 f(t, u(t)) dt = 0$ and $D_2 D_2 f(t, u(t)) = 0$ for all $t \in \mathbb{S}^1$.

2. In the autonomous case, 0 is a singular value of Σ_a if and only if $D_2 f$ has a double root.

Proof. Here, P_i stands for a smooth strictly positive function. From the expression for w in Lemma 1.1,

$$w(t) = P_1(u) e^{-\int_0^t (D_2 f(s, u(s)) - \Sigma_a(u))\, ds},$$

Thus,

$$\Sigma_b(u) = P_1(u) \int_0^1 D_2 f(t, u(t)) e^{-\int_0^t (D_2 f(s, u(s)) - \Sigma_a(u))\, ds}\, dt.$$

On the other hand,

$$\begin{aligned}
0 &= \int_0^1 \frac{d}{dt} e^{-\int_0^t (D_2 f(s, u(s)) - \Sigma_a(u))\, ds}\, dt\\
&= \int_0^1 (\Sigma_a(u) - D_2 f(t, u(t))) e^{-\int_0^t (D_2 f(s, u(s)) - \Sigma_a(u))\, ds}\, dt
\end{aligned}$$

whence

$$\Sigma_b(u) = P_1(u)\Sigma_a(u) \int_0^1 e^{-\int_0^t (D_2 f(s,u(s)) - \Sigma_a(u))\, ds}\, dt = P_2(u)\Sigma_a(u).$$

Integrate from 0 to 1 the differential equation describing $\mathcal{W}$ to obtain $\alpha = \Sigma_c(u)$. Solving the equation, we have

$$\mathcal{W}(t) = \mathcal{W}(0)e^{-\int_0^t D_2 f(s,u(s))\, ds} + \Sigma_c(u)e^{-\int_0^t D_2 f(s,u(s))\, ds} \int_0^t e^{-\int_0^s D_2 f(r,u(r))\, dr}\, ds.$$

From $\mathcal{W}(1) = \mathcal{W}(0)$, we obtain

$$\mathcal{W}(0)(1 - e^{-\Sigma_a(u)}) = \Sigma_c(u)e^{-\Sigma_a(u)} \int_0^1 e^{-\int_0^s D_2 f(r,u(r))\, dr}\, ds.$$

Since $(1 - e^{-x})/x > 0$,

$$\Sigma_c(u) = P_3(u)\Sigma_a(u),$$

and we are done. □

From now on, we shall always assume that f is generic in the sense that 0 is a regular value of Σ_a; further generic properties will be required of f in Section 2 where we study in detail the singularities of F.

We shall later want to use the simpler w instead of $\mathcal{W}$: the following preparatory lemma allows for this interchange.

Lemma 1.7. *The vector fields w and $\mathcal{W}$ are positive multiples of each other on S_1F. Furthermore, given $u \in S_1F$ there is a neighborhood $U_u \subseteq B^1$ of u where we can write*

$$w = a_1\mathcal{W} + \Sigma_c z_1, \quad \mathcal{W} = a_2 w + \Sigma_b z_2$$

for smooth real functions $a_i : U_u \to \mathbb{R}$ and smooth vector fields z_i.

Proof. The first claim follows directly from the formulae for w and $\mathcal{W}$ when restricted to S_1F (where $\Sigma_a = \Sigma_c = 0$). The displayed equations are consequences of the regularity of Σ_b and Σ_c at S_1F. □

Remark. Actually, from results in [MST], U_u in the statement can be taken to be the whole space B^1.

It turns out that the global geometry of S_1 is very simple, as we shall see in Corollary 1.9. We need some preparation to state the key ingredient, Theorem 1.8.

Let M be a smooth compact manifold equipped with a unit measure μ. Given a continuous function $g_k : M \times \mathbb{R} \to \mathbb{R}^k$, define $G_k : B^1 \to \mathbb{R}^k$ to be the average of the related Nemytskiĭ operator: $G_k(v) = \int_M g_k(m, v(m))d\mu$. We request that g_k admits continuous partial derivatives of all orders with respect to the second variable, whence G_k is smooth.

Let $\Pi_i : \mathbb{R}^k \to \mathbb{R}^i$ be the projection to the first i coordinates. We say 0 is a *strong regular value* of G_k if it is a regular value of the composition $G_i = \Pi_i \circ G_k$ for all i, $1 \le i \le k$.

Theorem 1.8 ([MST]). *Assume* 0 *to be a strong regular value of* G_k. *Then the levels* Z_i $(1 \le i \le k)$ *are contractible manifolds. Furthermore, there is a global homeomorphism* Ξ *of* B^1 *taking each* Z_i *to a closed linear subspace of codimension* i*;* Ξ *can be chosen to be a diffeomorphism if* $B^1 = H^1$.

The contractibility of the levels Z_i essentially implies geometric triviality because of infinite dimension: recall that two infinite dimensional separable Hilbert manifolds are diffeomorphic if their homotopy groups coincide ([Ku]) and that all infinite dimensional separable Banach spaces are homeomorphic ([Ka]).

In the next corollary we have $k = 1$; Theorem 1.8 in its generality will be convenient in Section 3.

Corollary 1.9. *Assume that* 0 *is a regular value of* Σ_a. *Then* S_1 *is connected and contractible. Furthermore, there is a global homeomorphism* Ξ *of* B^1 *taking* S_1 *to a closed linear subspace of* B^1 *of codimension* 1*;* Ξ *can be chosen to be a diffeomorphism if* $B^1 = H^1$.

2. Morin theory

Morin classified generic singularities of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose derivative has kernel of dimension 1 ([M]). The first step in Morin's proof makes use of the implicit function theorem to write such a singularity at the origin in *adapted coordinates*, i.e., in the form

$$(\mathbf{x}, y) \mapsto (\mathbf{x}, \mu(\mathbf{x}, y)), \quad \mathbf{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ y \in \mathbb{R},$$

after composing with suitable diffeomorphisms in the neighborhoods of zero in both domain and image. Morin's central result is that such singularities are classified by their *order*: a *Morin singularity* of order k is a point $(\mathbf{x}, y)$ for which

(a) $D_2\mu(\mathbf{x}, y) = \ldots = D_2^k\mu(\mathbf{x}, y) = 0$,
(b) $D_2^{k+1}\mu(\mathbf{x}, y) \neq 0$,
(c) the Jacobian $D(D_2\mu, \ldots, D_2^{k-1}\mu)(\mathbf{x}, y)$ is surjective.

Set $S_k = \{(\mathbf{x}, y) \mid D_2\mu(\mathbf{x}, y) = \ldots = D_2^k\mu(\mathbf{x}, y) = 0\}$. Thus, S_0 is the domain, S_1 is the critical set $\{(\mathbf{x}, y) \mid D_2\mu(\mathbf{x}, y) = 0\}$ (consistently with previous notation). Also, a Morin singularity of order i belongs to S_k if and only if $i \ge k$. In a neighborhood of a Morin singularity, the sets S_k stratify the domain: the sets are nested and S_i is a submanifold of codimension i. Notice that a point $(\mathbf{x}, y) \in S_k - S_{k+1}$ is a Morin singularity (of order k) only if condition (c) above holds.

Composing by appropriate diffeomorphisms in the domain and image, a Morin singularity in dimension n and order k acquires the normal form

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^{k+1} + x_1 y^{k-1} + \ldots + x_{k-1} y).$$

Morin singularities of order 1, 2, 3 and 4 are called, respectively, folds, cusps, swallowtails and butterflies.

We shall need an equivalent classification for singularities of functions between infinite-dimensional spaces. Let $G : Z_1 \to Z_2$ be a smooth map between Banach spaces so that $DG(z_0)$ is Fredholm operator of index 0 and kernel of dimension 1. Again, after changes of variables in the domain and image, we may assume G near z_0 to be written in adapted coordinates as

$$G : X \times \mathbb{R} \to X \times \mathbb{R}, \quad (\mathbf{x}, y) \mapsto (\mathbf{x}, \mu(\mathbf{x}, y))$$

and if conditions (a), (b) and (c) above hold, the same normal form applies for an appropriate splitting $X = \mathbb{R}^{k-1} \oplus X'$ — we then call z_0 a Morin singularity of order k. The proof of this last fact follows Morin's ([M]), making use of a parameterized version of Malgrange's preparation theorem, the parameter taking values in a Banach space (see [CDT]).

We already saw in the previous section that the composition $\mathbf{F} = F \circ \Psi^{-1}$ is in adapted coordinates:

$$\mathbf{F} : \widetilde{B}^0 \oplus \langle 1 \rangle \to \widetilde{B}^0 \oplus \langle 1 \rangle, \quad (\widetilde{v}, \overline{u}) \mapsto (\widetilde{v}, \overline{v} = \phi(\widetilde{v}, \overline{u})).$$

From the previous paragraph, a point is a Morin singularity of $\mathbf{F}$ (or F) of order k if and only if conditions (a), (b) and (c) hold for μ replaced by ϕ. This criterion, however, can not be used directly since we have no formula for ϕ; we rephrase it in terms of $\Phi = \phi \circ \Psi$ and w. Following the usual notation, we write $w\xi$ for the Lie derivative $D\xi(u) \cdot w_u$. The following result shows that we may substitute w for $\mathcal{W}$ (alternative generators for the kernel of DF over S_1) — a fact which in finite dimension would be unsurprising.

PROPOSITION 2.1. *The point $u \in B^1$ is a Morin singularity of order k for F if and only if*

(a) $w\Phi(u) = \ldots = w^k\Phi(u) = 0$,
(b) $w^{k+1}\Phi(u) \neq 0$,
(c) $D(w\Phi, \ldots, w^{k-1}\Phi)(u)$ *is surjective.*

PROOF. Consider in $\widetilde{B}^0 \oplus \langle 1 \rangle$ the constant vertical vector field $\mathbf{1}$, consisting of the constant function 1 at each point. In the notation we just introduced, $D_2\xi = \mathbf{1}\xi$. In terms of the pull-back $\mathcal{W}(u) = (D\Psi(u))^{-1}(\mathbf{1}(\Psi(u)))$, the conditions for u to be a Morin singularity of order k are:

(a') $\mathcal{W}\Phi(u) = \ldots = \mathcal{W}^k\Phi(u) = 0$, (b') $\mathcal{W}^{k+1}\Phi(u) \neq 0$

(c') $D(\mathcal{W}\Phi, \dots, \mathcal{W}^{k-1}\Phi)(u)$ is surjective.

We are left with showing that we can substitute $\mathcal{W}$ by w in these conditions. Notice first that $w\Phi = \Sigma_b$ and $\mathcal{W}\Phi = \Sigma_c$, proving the case $k = 0$ (regular points). From now on, we assume $u \in S_1 F$ and write, making use of Lemma 1.7, $w = a_1\mathcal{W} + (\mathcal{W}\Phi)z_1$ and $\mathcal{W} = a_2 w + (w\Phi)z_2$ in a small neighborhood U_u of u.

For each k, the ideals in $C^\infty(U_u, \mathbb{R})$ generated by $w\Phi, \dots, w^k\Phi$ and $\mathcal{W}\Phi, \dots, \mathcal{W}^k\Phi$ are equal. Assuming by induction that the result holds for $k-1$,

$$\begin{aligned} w^k\Phi &= w(w^{k-1}\Phi) \\ &= (a_1\mathcal{W} + (\mathcal{W}\Phi)z_1)(b_1\mathcal{W}\Phi + \dots + b_{k-1}\mathcal{W}^{k-1}\Phi) \\ &= a_1(\mathcal{W}b_1)(\mathcal{W}\Phi) + a_1 b_1 \mathcal{W}^2\Phi + \dots a_1(\mathcal{W}b_{k-1})(\mathcal{W}^{k-1}\Phi) \\ &\quad + a_1 b_{k-1}\mathcal{W}^k\Phi + (\mathcal{W}\Phi)z_1(b_1\mathcal{W}\Phi + b_{k-1}\mathcal{W}^{k-1}\Phi) \end{aligned}$$

which is clearly in the ideal with generators $\mathcal{W}\Phi, \dots, \mathcal{W}^k\Phi$, proving one inclusion; the opposite inclusion is analogous.

The equality of the two ideals with k generators implies

$$w^k\Phi = b_k\mathcal{W}^k\Phi + \dots + b_1\mathcal{W}\Phi, \qquad \mathcal{W}^k\Phi = c_k w^k\Phi + \dots + c_1 w\Phi,$$

for smooth functions b_i and c_i where b_k and c_k are non-zero. The equivalence between the conditions (a) and (a') or (b) and (b') is clear. The third equivalence follows from repeated use of the simple fact that the spans of $D(g_1(u)g_2(u))$ and $D(g_1(u)g_2(u) + \alpha(u)g_1(u))$ coincide for points u such that $g_1(u) = g_2(u) = 0$ (α being a smooth real function). $\square$

Proposition 2.2. *For*

$$\begin{aligned} \Sigma_1(u) &= \int D_2 f(t, u(t))\, dt, \\ \Sigma_2(u) &= \int D_2^2 f(t, u(t)) w(t)\, dt, \\ \Sigma_3(u) &= \int D_2^3 f(t, u(t)) w^2(t)\, dt, \\ \Sigma_4(u) &= \int D_2^4 f(t, u(t)) w^3(t) \\ &\qquad - 2D_2^3 f(t, u(t)) w^2(t) \bigg(\int_0^t D_2^2 f(s, u(s)) w(s)\, ds \bigg)\, dt, \\ \Sigma_5(u) &= \int D_2^5 f(t, u(t)) w^4(t) \\ &\qquad - 5D_2^4 f(t, u(t)) w^3(t) \bigg(\int_0^t D_2^2 f(s, u(s)) w(s)\, ds \bigg) \\ &\qquad + 5D_2^3 f(t, u(t)) w^2(t) \bigg(\int_0^t D_2^2 f(s, u(s)) w(s) \bigg)^2 dt, \end{aligned}$$

we have

$$S_k = \{u \in B^1 \mid \Sigma_i(u) = 0,\ i = 1, \dots, k\} \quad \text{for } k = 1, \dots, 5.$$

Furthermore, for $k = 1, \dots, 4$, u is a Morin singularity of order k if and only if

(a) $\Sigma_i(u) = 0$, $i = 1, \dots, k$,

(b) $\Sigma_{k+1}(u) \neq 0$,

(c) *the derivative $D\Sigma(u)$ of the function*

$$\Sigma : B^1 \to \mathbb{R}^{k-1}, \quad u \mapsto (\Sigma_1(u), \dots, \Sigma_{k-1}(u))$$

is surjective.

Proof. From Proposition 2.1, we want to compute $w^k\Phi(u)$. The expressions for $\Sigma_k(u)$ follow from repeated integration by parts, discarding elements in the ideal generated by $w\Phi(u), \dots, w^{k-1}\Phi(u)$ and non-zero multiplicative factors. $\square$

In particular, u is a fold point if and only if $\Sigma_1(u) = 0$ and $\Sigma_2(u) \neq 0$. Also, u is a cusp point if and only if $\Sigma_1(u) = \Sigma_2(u) = 0$, $\Sigma_3(u) \neq 0$ and $D\Sigma_1(u) \neq 0$. Clearly, $D\Sigma_1(u) \cdot v = \int D_2^2 f(t, u(t))v(t)\, dt$, and hence $D\Sigma_1(u) = 0$ if and only if the function $D_2^2 f(t, u(t))$ is identically 0.

There is a simple relationship between Morin singularities and the *return map.* Given $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ and $v \in B^0$, the return map $\rho_v : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, sends $u(0)$ to $u(1)$ if $u : [0,1] \to \mathbb{R}$ satisfies

$$u'(t) + f(t, u(t)) = v(t).$$

Here I is the maximal domain, i.e., the set of initial conditions such that the solution u extends to $t = 1$.

Proposition 2.3. *For $u \in B^1$ and $v = F(u)$, $\rho_v(u(0)) = u(0)$. Also, $u \in S_1$ if and only if $\rho_v'(u(0)) = 1$ and $u \in S_k$ if and only if $\rho_v'(u(0)) = 1$ and $\rho_v^{(i)}(u(0)) = 0$ for $i = 2, \dots, k$.*

Proof. Consider the fibre τ_u through u. Let u_a be the element of average a of τ_u and $u = u_{a_0}$. By Lemma 1.3, τ_u is smoothly parametrized by a. Let $g(a) = \Phi(u_a)$, the average of $F(u_a)$. Clearly, $g^{(k)}(a) = \mathcal{W}^k\Phi(u_a)$ and therefore $u \in S^k$ if and only if $g^{(i)}(a_0) = 0$ for $i = 1, \dots, k$. Let u^c be the element of τ_u with $u^c(0) = c$. By Lemma 1.3, τ_u is also parametrized by c. Let $h(c) = \Phi(u^c) - \overline{v} = F(u_c) - F(u)$: by the chain rule, $u \in S^k$ if and only if $h^{(i)}(u(0)) = 0$ for $i = 1, \dots, k$.

For $c, b \in \mathbb{R}$, let $\beta(c, b) = \rho_{v+b}(c)$, that is, if $U(c, b, t)$ satisfies

$$(*) \qquad D_3U(c, b, t) + f(t, U(c, b, t)) = v(t) + b, \quad U(c, b, 0) = c$$

we have $\beta(c, b) = U(c, b, 1)$. The periodicity of u^c yields $\beta(c, h(c)) = c$. Points in the curve $(c, h(c))$ thus correspond to points in τ_u and the largest k for which

$u \in S_k$ is the order of contact between this curve and the horizontal axis $(c, 0)$ at the common point $(u(0), 0)$. Differentiating $(*)$,

$$D_3 D_2 U(u(0), 0, t) + D_2 f(t, u(t)) D_2 U(u(0), 0, t) = 1, \quad D_2 U(u(0), 0, 0) = 0$$

and, by explicitly solving for $D_2 U$, we obtain $D_2 U(u(0), 0, 1) = D_2 \beta(u(0), 0) > 0$. Thus, $G(c, b) = (c, \beta(c, b))$ is a local diffeomorphism near $(u(0), 0)$ taking the curve $(c, h(c))$ to the diagonal (c, c) and the horizontal axis to $(c, \rho_v(c))$. The order of contact between the curves is preserved by G and we are done. $\square$

3. The autonomous case

This section is dedicated to a number of special properties of the *autonomous case*, when f depends on x only.

The sets S_k are described by the rather complicated formulae Σ_k. In the autonomous case it is convenient to consider the *simplified operator*

$$\widehat{F} : B^1 \to B^0 \quad u \mapsto u' + \int f(u(t))\, dt$$

whose critical strata $\widehat{S}_k$ are far easier to handle but still convey significant information about F and S_k.

Since $\widehat{F}$ is already given in adapted coordinates, straightforward application of Morin's characterization obtains

$$\widehat{S}_k = \left\{ v \in B^1 \;\middle|\; \int \widehat{\gamma}_k(v(t)) = 0 \right\},$$

where $\widehat{\gamma}_k(x)$ is the k-dimensional vector $(f'(x), \ldots, f^{(k)}(x))$. Notice that, from Lemma 1.6, $\widehat{S}_1 = S_1$. However, $\widehat{S}_2$ is usually different from S_2: the following lemma relates both sets and is a key ingredient in Section 5.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, $F : B^1 \to B^0$ be the operator $(F(u))(t) = u'(t) + f(u(t))$. Then there exists a global diffeomorphism of B^1 taking S_1 to itself and S_2 to $\widehat{S}_2$.*

Proof. The diffeomorphism has the form $u \mapsto v = u \circ \alpha$, with inverse $v \mapsto u = v \circ \beta$, where α and β are orientation preserving C^1-diffeomorphisms of $\mathbb{S}^1$ fixing 0: such compositions take functions in B^1 to functions in B^1. Clearly, $\beta = \alpha^{-1}$. We need the following characterizations, which follow easily from Lemma 1.6 and Proposition 2.2

$$\begin{aligned} S_1 &= \left\{ u \in B^1 \;\middle|\; \int f'(u(t)) w(t)\, dt = 0 \right\}, \\ S_2 &= \left\{ u \in B^1 \;\middle|\; \int f'(u(t)) w(t)\, dt = \int f''(u(t)) w(t)\, dt = 0 \right\}. \end{aligned}$$

We first obtain β from u that

$$\beta(s) = \frac{\int_0^s w(\sigma)\,d\sigma}{\int_0^1 w(\sigma)\,d\sigma}$$

is clearly a C^1-diffeomorphism ($w > 0$). For any continuous function g, a change of variables gives

$$(*) \qquad \int_0^1 g(u(s))w(s)\,ds = 0 \Leftrightarrow \int_0^1 g(v(t))\,dt = 0,$$

where $u = v \circ \beta$. Thus, if $u \in S$ then $v \in S$ and if $u \in C$ then $v \in \widehat{S}_2$. The smooth dependence of β on u is obvious.

To show invertibility of the map $u \mapsto v$, we obtain α from v. If α satisfies

$$\alpha'(t) = \frac{1}{A - 1\int_0^t f'(v(\tau))\,d\tau}, \quad \alpha(0) = 0$$

for an arbitrary positive constant A, standard algebra shows that

$$(\dagger) \qquad \alpha'(t) = \frac{1}{A}\exp\left(\int_0^t f'(v(\tau))\alpha'(\tau)\,d\tau\right), \quad \alpha(0) = 0.$$

This again implies the equivalence $(*)$ (where, of course, $v = u \circ \alpha$) and hence $v \in S$ (resp., $\widehat{S}_2$) implies $u \in S$ (resp., C) provided $\alpha(1) = 1$. We have to show that for each v there is a unique A with $\alpha(1) = 1$ and that the dependence of A on v is smooth.

Let $h(t) = \int_0^t f'(v(\tau))\,d\tau$. h is C^1 with $h(0) = 0$. The function α is defined in $[0, 1]$ if $A > \max_t h(t)$. From

$$\alpha(1) = \int_0^1 \frac{dt}{A - h(t)},$$

the derivative of $\alpha(1)$ with respect to A is strictly negative. When A tends to infinity, $\alpha(1)$ becomes small and when A approaches $\max_t h(t)$, $\alpha(1)$ tends to infinity. This settles existence and uniqueness of the required A. Smoothness follows from the implicit function theorem applied to the smooth function $(v, A) \mapsto \alpha(1)$ and the fact that the derivative with respect to A is not zero.

It remains only to show that the two smooth maps constructed above are the inverse of each other. Consider the sequence of maps

$$v \xrightarrow{\alpha} u = v \circ \alpha^{-1} \xrightarrow{\beta} \widetilde{v} = u \circ \beta^{-1}.$$

Since $\tau = \alpha^{-1}(\sigma))$ and $(\dagger)$, for positive constants C_1 and C_2 we have

$$\begin{aligned} \beta'(s) &= C_1 w(s) = C_1 \exp\left(-\int_0^s f'(v(\alpha^{-1}(\sigma)))\,d\sigma\right) \\ &= C_1 \exp\left(-\int_0^{\alpha^{-1}(s)} f'(v(\tau))\alpha'(\tau)\,d\tau\right) = \frac{C_2}{\alpha'(\alpha^{-1}(s))} = C_2(\alpha^{-1})'(s) \end{aligned}$$

and, since $\alpha(0) = \beta(0) = 0$ and $\alpha(1) = \beta(1) = 1$, it follows that $\alpha^{-1} = \beta$ and $\widetilde{v} = v$. Similarly, for

$$u \xrightarrow{\beta} v = u \circ \beta^{-1} \xrightarrow{\alpha} \widetilde{u} = v \circ \alpha^{-1},$$

we have

$$\begin{aligned} \alpha'(t) &= C_3 \exp\left(\int_0^t f'(v(\tau))\alpha'(\tau)\,d\tau\right) \\ &= C_3 \exp\left(\int_0^{\alpha(t)} f'(v(\alpha^{-1}(\sigma)))\,d\sigma\right) = \frac{C_3}{w(\alpha(t))} = \frac{C_4}{\beta'(\alpha(t))}, \end{aligned}$$

hence $(\beta \circ \alpha)'$ is a constant and the result follows. □

As with $S_1 = \widehat{S}_1$, the global geometry of the sets $\widehat{S}_k$ is very simple, as we learn from the following application of Theorem 1.8. We say that the simplified operator $\widehat{F}$ is *k-regular* if 0 is a strong regular value of

$$\widehat{\Sigma} : B^1 \to \mathbb{R}^k, \quad u \mapsto (\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_k)$$

COROLLARY 3.2. *Assume $\widehat{F}$ is k-regular. Then there is a global homeomorphism Ξ of B^1 taking each $\widehat{S}_i$ $(1 \le i \le k)$ to a closed linear subspace of B^1 of codimension i; Ξ can be chosen to be a diffeomorphism if $B^1 = H^1$.*

Combining Lemma 3.1 and Corollary 3.2 we have

COROLLARY 3.3. *Assume $\widehat{F}$ is 2-regular. Then there is a global homeomorphism Ξ of B^1 taking each S_i, $i = 1, 2$, to a closed linear subspace of B^1 of codimension i; Ξ can be chosen to be a diffeomorphism if $B^1 = H^1$.*

It seems hard to give an operational criterion to decide for larger k even whether S_k is non-empty. We now present a partial criterion.

DEFINITION 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. f is said to be *k-good* if $\widehat{\gamma}_k$ never vanishes and the image of any open interval by $\widehat{\gamma}_k$ is not contained in a hyperplane through the origin in $\mathbb{R}^k$.

Generic smooth functions are k-good, as well as generic polynomials of fixed degree at least k. It is easy to see that if f is $k+1$-good, then the simplified operator $\widehat{F}$ is k-regular.

LEMMA 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a k-good function and F be the related operator. Then $\widehat{S}_k \ne \emptyset$ if and only if $0 \in \mathbb{R}^k$ is in the interior of the convex hull of the image of $\widehat{\gamma}_k$. Also, $\widehat{S}_k \ne \emptyset$ implies $S_k \ne \emptyset$.*

In Section 6, we give an example of a polynomial of degree 4 (and hence for which $\widehat{S}_4 = \emptyset$) having $S_4 \ne \emptyset$.

PROOF. Clearly, if $\widehat{S}_k \ne \emptyset$, 0 is in the convex hull of the curve $\widehat{\gamma}_k$ (but not on the curve itself). If 0 is on the boundary of the convex hull, by a standard support theorem ([G, p. 12]), the image of $\widehat{\gamma}_k$ is contained in a closed half-space defined by $\nu(p) \geq 0$ for some linear functional ν. Since f is k-good, for any non-constant $v \in H^1$, $\nu(\int \widehat{\gamma}_k(v)) > 0$ and $\widehat{S}_k = \emptyset$.

Conversely, assume 0 in the interior of the convex hull of the image of $\widehat{\gamma}_k$. By Steinitz's theorem ([G]), there are points $\widehat{\gamma}_k(x_j)$, $j = 0, \ldots, 2k-1$ such that 0 is in the interior of their convex hull. For $\varepsilon \in (0,1)$ and $a_j \geq 0$, $j = 0, \ldots, 2k-1$, $\sum a_j = 1-\varepsilon$, consider a smooth function $v_{\varepsilon,a}$ of period 1, defined as follows. We split the domain $[0,1]$ into intervals $I_0, J_0, \ldots, I_{2k-1}, J_{2k-1}$ of lengths $a_0, \varepsilon/2k, \ldots, a_{2k-1}, \varepsilon/2k$; inside I_j, $v_{\varepsilon,a}$ is constant equal to x_j and inside each J_j, $v_{\varepsilon,a}$ is the appropriate affine transformation of a fixed smooth arc joining two steps. As ε tends to 0, $v_{\varepsilon,a}$ approaches a step function. Let $\phi(\varepsilon, a) = \int \widehat{\gamma}_k(v_{\varepsilon,a})$: this function is affine in ε and a (i.e., linear plus constant) and to show that $\widehat{S}_k$ is non-empty, we need to find a zero of ϕ. The function ϕ extends continuously to $\varepsilon = 0$ and 0 is then in the interior of the image of the simplex spanned by the a_j's: there exists therefore a straight segment parametrized by small positive values of ε on which ϕ is zero and $\widehat{S}_k$ is thus non-empty. More, for a fixed small ε_0, take a k-subspace V of $\mathbb{R}^{2k}$ such that, for $a \in V$, $\phi(\varepsilon_0, a)$ is surjective. The image under ϕ of a small sphere around the origin in V is some ellipsoid containing the origin in its interior. □

Remark. For $k = 2$, k-goodness may be weakened to f' and f'' having no common zeros.

4. Some examples

In this section, we describe the global geometry of $F : B^1 \to B^0$ for several special classes of smooth functions $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$. The first rather simple proposition illustrates the use of fibres and adapted coordinates in three technically different scenarios.

Proposition 4.1.

(a) *If f is proper and $D_2 f(t,x) > 0$ then F is a diffeomorphism.*

(b) *Assume $D_2 f(t,x) > 0$. Let $f_\pm(t) = \lim_{x\to\pm\infty} f(t,x)$. Then F is a diffeomorphism from B^1 to the horizontal strip*

$$\mathcal{S} = \left\{ v \in B^0 \;\middle|\; \int f_-(t)\,dt < \int v(t)\,dt < \int f_+(t)\,dt \right\}.$$

(c) *If f is proper and strictly increasing in the second variable then F is a homeomorphism.*

Proof. (a) Recall that, by Theorem 1.2, B^1 is foliated by fibres, B^0 is foliated by vertical lines and there is a diffeomorphism between the space of fibres and the space of vertical lines. From the characterization of S_1 in Proposition 1.1, we see that F has no critical points. Thus, F takes each fibre strictly monotonically to its related vertical line in B^0 — by Proposition 1.4, F is actually a diffeomorphism from fibre to vertical line and the result follows.

(b) As in the previous item, fibres are bijectively taken to open subintervals of vertical lines. More explicitly, a function u in the fibre $(\Pi_{\widetilde{B}^0} \circ F)^{-1}(\widetilde{v})$ is taken to $u' + f(t, u(t)) = \widetilde{v} + \int f(t, u(t))\, dt$ and the extremes of the image of this fibre are $\lim_{a\to\infty} \int f(t, u_a(t))\, dt$ and $\lim_{a\to-\infty} \int f(t, u_a(t))\, dt$, where u_a is the element of average a in the fibre. From Proposition 1.3

$$\lim_{a\to\infty} \min_t u_a(t) = \infty, \qquad \lim_{a\to-\infty} \max_t u_a(t) = -\infty.$$

Thus

$$\lim_{a\to\infty} \int f(t, u_a(t))\, dt = \int f_+(t)\, dt, \qquad \lim_{a\to-\infty} \int f(t, u_a(t))\, dt = \int f_-(t)\, dt,$$

and the result follows.

(c) By properness (Proposition 1.4), it suffices to prove that F is strictly increasing on each fibre (notice that F restricted to fibres may have critical points where $\mathcal{W}F = 0$). Let $u_0 < u_1$ be elements of the fibre $(\Pi_{\widetilde{B}^0} \circ F)^{-1}(\widetilde{v})$ so that

$$F(u_i)(t) = u_i'(t) + f(t, u_i(t)) = \widetilde{v} + \overline{v}_i.$$

Integrating in $t \in \mathbb{S}^1$ and using the monotonicity of f, we obtain $\overline{v}_0 < \overline{v}_1$, concluding the proof. □

Remarks. 1. A more standard proof of (a), without making use of Theorem 1.2 (and the consequent fibre-sheet-adapted coordinates vocabulary), could be as follows. As before, $S_1 = \emptyset$ and from Proposition 1.4, F is proper. Since B^0 is simply connected, by covering space theory, F is a diffeomorphism. Notice that this argument does not extend easily to the other items.

2. Similar results and proofs hold if instead $D_2 f < 0$.

3. In item (b), if both $\int f_+(t)\, dt$ and $\int f_-(t)\, dt$ diverge, then F is a global diffeomorphism. In particular, F may be proper even if f is not.

Theorem 4.2. *If f is proper and $D_2^2 f(t, x) > 0$ then F is a global fold.*

We call an operator $G : B^1 \to B^0$ a *global fold* if there exist diffeomorphisms $\Xi_1 : B^1 \to \mathbb{R} \times \widetilde{B}^0$ and $\Xi_0 : B^0 \to \mathbb{R} \times \widetilde{B}^0$ such that $(\Xi_0 \circ G \circ \Xi_1^{-1})(x, \widetilde{v}) = (x^2, \widetilde{v})$, for all $(x, \widetilde{v}) \in \mathbb{R} \times \widetilde{B}^0$. Similarly, we call G a *topological* global fold if there exist homeomorphisms Ξ_i as above.

Proof. From $D_2^2 f(t, x) > 0$, we conclude that $D_2(t, x)$ is strictly increasing in x for any fixed t and hence that $\Sigma_a(u) = \int D_2(t, u(t))\, dt$ is strictly increasing on fibres. Thus, each fibre contains a unique critical point u_0 and, for arbitrary u_- and u_+ in the same fibre as u_0 satisfying $u_- < u_0 < u_+$, we have $\Sigma_a(u_-) < 0$ and $\Sigma_a(u_+) > 0$ and, from Lemma 1.6, $\mathcal{W}\Phi(u_-) = \Sigma_c(u_-) < 0$ and $\mathcal{W}\Phi(u_+) =$

$\Sigma_c(u_+) > 0$ and the restriction of Φ to a fibre is a global fold from $\mathbb{R}$ to $\mathbb{R}$. Thus, on each fibre, we have diffeomorphisms Ξ_i as above and the problem is whether such diffeomorphisms can be chosen so as to depend smoothly on the fibre.

In adapted coordinates, we must define $\xi_1 : B^0 \times \mathbb{R} \to \mathbb{R}$ and $\xi_0 : B^0 \times \mathbb{R} \to \mathbb{R}$ so that the vertical columns of the diagram below are diffeomorphisms and the diagram commutes, i.e.,

$$\xi_0(\widetilde{v}, \phi(\widetilde{v}, \overline{u})) = (\xi_1(\widetilde{v}, \overline{u}))^2.$$

We construct the ξ_i explicitly. For each $\widetilde{v}$, let $a_{\widetilde{v}}$ be the unique critical point of $\overline{u} \mapsto \phi(\widetilde{v}, \overline{u})$. Clearly, $a_{\widetilde{v}}$ and its image $b_{\widetilde{v}} = \phi(\widetilde{v}, a_{\widetilde{v}})$ depend smoothly on $\widetilde{v}$. Also, write $\phi(\widetilde{v}, \overline{u}) - b_{\widetilde{v}} = (\overline{u} - a_{\widetilde{v}})^2 g(\widetilde{v}, \overline{u})$. From the previous paragraph, g is a smooth positive function. Set $\xi_0(\widetilde{v}, \overline{v}) = \overline{v} - b_{\widetilde{v}}$ and $\xi_1(\widetilde{v}, \overline{u}) = (\overline{u} - a_{\widetilde{v}})\sqrt{g(\widetilde{v}, \overline{u})}$.□

REMARKS. 1. McKean and Scovel ([McKS]) studied this scenario with a different set of fibres for B^1 and its image under F. Our choice of fibering B^0 by vertical lines and inverting them under F to get a fibration of B^1 is more helpful in our examples.

2. Slight variations (as in Proposition 4.1) are possible and can be handled similarly; we omit the tedious details.

In the autonomous case, Theorem 4.2 admits a partial converse.

THEOREM 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function which is both* 2- *and* 3-*good, with* $\lim_{x\to\pm\infty} f(x) = \infty$. *If* 0 *is in the interior of the convex hull of the image of $\widehat{\gamma}_2$ then F has cusps. Otherwise, F is a* (*differentiable*) *global fold.*

Recall that $\widehat{\gamma}_2(t) = (f'(t), f''(t))$. Notice that once F is known to have a cusp, from the normal form we have image points with three regular pre-images near the cusp. Since $\deg(F) = 0$, such points have at least one additional pre-image.

PROOF. If 0 is in the interior of the convex hull of the image of $\widehat{\gamma}_2$, $S_2 \neq \emptyset$ by Lemma 3.5. We must prove that some points u in S_2 are differentiable cusps, i.e., satisfy $\Sigma_3(u) \neq 0$ and $D\Sigma_1(u) \neq 0$. As remarked after Proposition 2.2, $D\Sigma_1(u) = 0$ only when $f''(u(t)) = 0$ for all t, which implies, given 2-goodness, that u is constant equal to a root of f''. Again from 2-goodness, f' and f'' have no common roots and $\Sigma_1(u) = \int f'(u(t))\,dt \neq 0$, thus u is not in the critical set. It remains to verify that we may choose u so that $\Sigma_3(u) \neq 0$. From 3-goodness, the curve $\widehat{\gamma}_3(t) = (f'(t), f''(t), f'''(t))$ does not intersect the origin. The convex hull of the image of $\widehat{\gamma}_3$ meets the vertical axis and must contain points $(0, 0, A)$ distinct from the origin, otherwise the image of $\widehat{\gamma}_3$ would have to be contained in a hyperplane, contradicting 3-goodness. Imitating the proof of Lemma 3.5, we may construct u with $\Sigma_1(u) = \Sigma_2(u) = 0$, $\Sigma_3(u) \approx A$, which is the required cusp.

Conversely, assume 0 not to be in the interior of the convex hull of the image of $\widehat{\gamma}_2$. Again from Lemma 3.5, all critical points are folds. Notice that if m is the minimum of f, $f'(m) = 0$ and, by 2-goodness, $f''(m) > 0$. Thus, the intersection of the convex hull with the second axis consists of points with non-negative second coordinate. From the proof of Lemma 3.5, $\Sigma_1(u) = 0$ now implies $\Sigma_2(u) > 0$. Thus the fold points have concavity upwards in the restriction of F to each fibre. From the behaviour of f at infinity, F has exactly one fold point per fibre. The rest of the argument is similar to the proof of Theorem 4.2. $\square$

Corollary 4.4.

(a) *Let $f : \mathbb{R} \to \mathbb{R}$ be a* 2- *and* 3-*good polynomial of even degree and positive leading coefficient. If f'' assumes both signs then the operator F has a cusp and there are points in the image of F with four pre-images.*

(b) *There are non-convex smooth proper functions $f : \mathbb{R} \to \mathbb{R}$ for which F is a global fold.*

Proof. To prove (a), notice that, by hypothesis, there is a point in the lower half-plane ($f'' < 0$). Also, since f is a polynomial, the argument of $\widehat{\gamma}_2(t)$ approaches 0 from above (resp., π from below) when t goes to ∞ (resp., $-\infty$). This suffices to show that 0 is in the interior of the convex hull of the image of $\widehat{\gamma}_2(t)$.

As for (b), we deviate from the previous argument by considering functions for which f' and f'' are comparable for large x, such as $\cosh x$. More precisely, let f be such that f'' coincides with $\cosh t$ outside a small interval of large positive numbers in which we subtract from $\cosh t$ a narrow positive bump — if the bump is sufficiently high and narrow, f'' changes sign, $f'(x) > 0$ for all positive x and 0 is not in the convex hull of the image of $\widehat{\gamma}_2$. $\square$

We can also characterize global folds in a purely local way.

Theorem 4.5. *If $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ is proper, 0 is a regular value of Σ_1 and all singularities of F are folds, then F is a global fold.*

Proof. For the operators F being considered, there is a sign associated to each fold: it is the sign of Σ_2, which can also be interpreted as saying whether the concavity of the restriction of F to fibres points up or down. This splits the set of folds into two open subsets. Since by hypothesis $S_2 = \emptyset$ and by Corollary 1.9 S_1 is connected it follows that one of these sets is empty. In other words, all folds are concave upwards, say, and the restriction of F to any fibre thus has at most one critical point. The result follows by properness and juxtaposition of fibres as in Theorems 4.2 and 4.3. $\square$

5. Global cusps

The results in the previous section describe the global geometry of F when f satisfies $D_2 f > 0$ or $D_2^2 f > 0$. In the autonomous case, we are able to handle another kind of nonlinearity.

THEOREM 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a proper function such that*

(a) $f'''(x) \geq 0$,
(b) $f'''(x)$ *has isolated roots,*
(c) $f'(x)$ *assumes both signs.*

Then F is a topological global cusp; in the H case, F is a smooth global cusp.

A similar result holds if (a) is replaced by $f'''(x) \leq 0$.

The scenarios of the previous section are simple enough to allow for rather explicit global changes of variable to normal form. This is partly due to the fact that restrictions of F to arbitrary fibres have similar behaviours: in a sense, we may consider fibres individually. For the operators in the statement of Theorem 5.1, instead, such restrictions vary according to the fibre, as illustrated in Figure 5.1: we must therefore treat them collectively. In the process, explicitness is lost: from the theorem (and its proof), in the H case the domain and image are foliated by smooth surfaces, diffeomorphic to $\mathbb{R}^2$, which are in turn foliated by fibres. More, F takes surfaces to surfaces and, on each surface, F is a global cusp. Still, we have no idea how to exhibit such foliations: they are shown to exist by topological methods in Hilbert manifolds ([Ku], [MST]). For Banach spaces, we use an additional existential argument depending on the homeomorphism of all infinite dimensional separable Banach spaces ([Ka]).

An operator $G : H^1 \to H^0$ is a *smooth global cusp* if there exist diffeomorphisms $\Xi_1 : H^1 \to H \times \mathbb{R}^2$ and $\Xi_0 : H^0 \to H \times \mathbb{R}^2$ such that $(\Xi_0 \circ G \circ \Xi_1^{-1})(Z, x, y) = (Z, x, y^3 + xy)$, for all $(Z, x, y) \in H \times \mathbb{R}^2$, where H is a separable infinite dimensional Hilbert space. Similarly, we call $G : C^1 \to C^0$ a *topological* global cusp if there exist homeomorphisms $\Xi_i : C^i \to H \times \mathbb{R}^2$ as above.

The following lemma is an exercise in integration by parts.

LEMMA 5.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function. Given $a < b$, we have*

$$(b-a)(g'(a) + g'(b)) - 2(g(b) - g(a)) = -\int_a^b (t-a)(t-b)g'''(t)\, dt.$$

An additional topological ingredient is an infinite-dimensional version of the corollary to Theorem 1 in [S]. The proof is similar for Hilbert spaces and makes use of results in [BH] for Banach spaces (see [MST] for additional information).

Lemma 5.3.

(a) *Given a contractible connected smooth submanifold H' of codimension* 1 *of a separable Hilbert space H of infinite dimension, there is a diffeomorphism of H to itself taking H' to a closed subspace of codimension* 1*.*

(b) *Let B' be a closed subset of a separable infinite-dimensional Banach space B. Assume that B' is connected, contractible, and a bicollared topological submanifold of codimension* 1 *in B. Then there is a homeomorphism from B to itself taking B' to a closed subspace of codimension* 1*.*

Actually, a contractible connected closed bicollared topological submanifold of codimension 1 always splits the ambient space in (exactly) two components.

Finally, we make use of a canonical construction to bring planar global cusps to normal form. A sketch of proof is given at the end of this section.

Lemma 5.4. *Let $\mathcal{Z}$ be a topological space of parameters. Let $G : \mathcal{Z} \times \mathbb{R}^2 \to \mathcal{Z} \times \mathbb{R}^2$ be a continuous function of the form*

$$G(Z, x, y) = (Z, x, g_Z(x, y))$$

with the following properties:

- *For all $Z \in \mathcal{Z}$, $g_Z(0, 0) = 0$.*
- *For all $Z \in \mathcal{Z}$ and $x \in \mathbb{R}$, $\lim_{y\to\pm\infty} g_Z(x, y) = \pm\infty$.*
- *For all $Z \in \mathcal{Z}$ and $x \le 0$ the function $y \mapsto g_Z(x, y)$ is strictly increasing.*
- *There exist continuous functions $m, M : \mathcal{Z} \times [0, \infty) \to \mathbb{R}$ with $m(Z, 0) = M(Z, 0) = 0$ such that, for all $Z \in \mathcal{Z}$ and $x > 0$ the function $y \mapsto g_Z(x, y)$ is strictly increasing in $(-\infty, m(Z, x)]$ and $[M(Z, x), \infty)$ but strictly decreasing in $[m(Z, x), M(Z, x)]$.*

(a) *There exist homeomorphisms W_d and W_i of $\mathcal{Z} \times \mathbb{R}^2$ keeping the Z and x coordinates fixed such that*

$$(W_i \circ G \circ W_d^{-1})(Z, x, y) = (Z, x, y^3 + xy).$$

(b) *If $\mathcal{Z}$ is a smooth Hilbert manifold and G is smooth with*

$$D_2 g_Z(0, 0) = D_2^2 g_Z(0, 0) = 0, \quad D_2^3 g_Z(0, 0) > 0, \quad D_1 D_2 g_Z(0, 0) < 0,$$

then W_d and W_i can be taken to be diffeomorphisms.

Proof of Theorem 5.1. We first classify the singularities, next we study the behaviour of the restriction of F to fibres and finally obtain global results by juxtaposing fibres.

All critical points of F are folds or cusps. Clearly, Σ_3 is always positive and we only have to check transversality when $\Sigma_1 = \Sigma_2 = 0$, i.e., we have to verify that $D\Sigma_1 \neq 0$ at such points. Since $\Sigma_1 = \int f'(u(t))\,dt$,

$$D\Sigma_1(u) \cdot v = \int f''(u(t))v(t)\,dt.$$

We show that the function $f''(u(t))$ is not identically zero. First, there is a unique x_0 for which $f''(x_0) = 0$; from (c), we have $f'(x_0) < 0$. Thus, the function $f''(u(t))$ is identically zero only if $u(t) = x_0$ for all t but then $\int f'(x_t)dt < 0$ and $u \notin S_1$.

We now consider F restricted to fibres. From Theorem 1.4, such restrictions take $\pm\infty$ to $\pm\infty$. Also, regular and fold points of F are regular or fold points of the restriction. Furthermore, the restrictions are locally increasing at cusp points. Indeed, $\Sigma_3 > 0$ and, at cusp points, Σ_3 is a positive multiple of $\mathcal{W}^3\Phi$, the third derivative of the restriction.

The operator F has at most two critical points per fibre. Let $u_1 < u_2$ be two critical points of F in the same fibre:

$$(*) \qquad \int f'(u_1(t))\,dt = \int f'(u_2(t))\,dt = 0$$

and

$$(\dagger) \qquad u_1'(t) + f(u_1(t)) = u_2'(t) + f(u_2(t)) + C$$

for some constant C. We first show that $C > 0$. From Lemma 5.2

$$f'(u_2(t)) + f'(u_1(t)) - \frac{f(u_2(t)) - f(u_1(t))}{u_2(t) - u_1(t)} > 0$$

and, integrating and making use of $(*)$,

$$\int \frac{f(u_2(t)) - f(u_1(t))}{u_2(t) - u_1(t)}\,dt < 0.$$

From $(\dagger)$

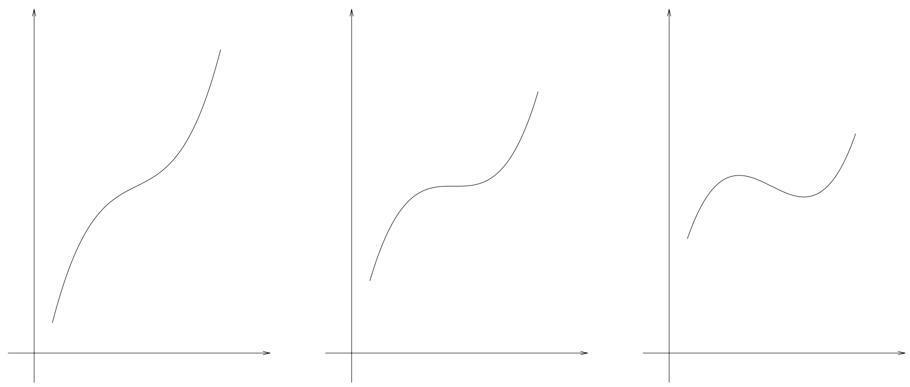
$$\frac{(u_2 - u_1)'(t)}{(u_2 - u_1)(t)} + \frac{f(u_2(t)) - f(u_1(t))}{u_2(t) - u_1(t)} + \frac{C}{u_2(t) - u_1(t)} = 0$$

and, integrating, we obtain

$$C \int \frac{dt}{u_2(t) - u_1(t)} > 0$$

implying $C > 0$. This means that the restriction of F to fibres, if further restricted to critical points, is decreasing: $u_1 < u_2$ implies $F(u_1) > F(u_2)$. It follows that, if there are at least three critical points on a fibre, F is decreasing near the second one: from the previous paragraph, this second critical point can thus be neither a fold nor a cusp.

At this point we already know that regular values have one or three pre-images, images of folds have two pre-images and images of cusps have a single pre-image (as in [CD] and [CT]). But we have more: F restricted to a fibre is topologically equivalent to one of the three graphs on Figure 5.1. Fibres containing a cusp, on which F behaves as depicted in (b), split the space of fibres into two subsets, on which restrictions of F behave as in either (a) (no critical points) or (c) (two folds). Let $\mathcal{F} = \widetilde{B}^1$ be, as before, the space of fibres: we have the natural partition $\mathcal{F} = \mathcal{F}_a \cup \mathcal{F}_b \cup \mathcal{F}_c$ into sets of fibres of types (a), (b) and (c), respectively.



(a) (b) (c)

Figure 5.1.

The sets $\mathcal{F}_a$ and $\mathcal{F}_c$ are open in $\mathcal{F}$; $\mathcal{F}_b$ is closed. Let $u_0 \in \mathcal{F}_c$; there exist elements $u_1 < u_2$ in the u_0-fibre with $F(u_1) > F(u_2)$. The hyperplanes parallel to $\widetilde{B}^1$ passing through u_1 and u_2 transversally intersect fibres. In particular, fibres sufficiently close to the u_0-fibre contain point $u_1' < u_2'$ in these hyperplanes for which $F(u_1') > F(u_2')$, proving the openness of $\mathcal{F}_c$.

Assuming by contradiction that $\mathcal{F}_a$ is not open, let u_n be a sequence of critical points whose corresponding fibres converge to the fibre $u_\infty \in \mathcal{F}_a$. If the averages of u_n are bounded, we may assume by compactness that these averages converge; this, however, implies that the sequence u_n itself converges to the element of the u_∞-fibre with limiting average. Since this limit is clearly a critical point of F we have the contradiction in this case and may assume from now on that the averages of u_n tend monotonically to ∞.

Let M be such that $x > M$ implies $f'(x) > 0$. Notice in particular that for all $u \in S_1$ there is a $t \in \mathbb{S}^1$ such that $f'(u(t)) = 0$ and therefore $u(t) < M$. For each n, let $t_n \in \mathbb{S}^1$ be such that $u_n(t_n) < M$. From the compactness of $\mathbb{S}^1$, we may assume that the sequence t_n converges to, say, t_∞. Let u_{nm} $(m < n)$ be the element in the u_n-fibre with average $\overline{u}_m$; clearly, $u_{nm} < u_n$. Define similarly

$u_{\infty m}$: the sequence u_{nm} (for fixed m) tends to $u_{\infty m}$. Since $u_{nm}(t_n) < M$ for all n, $u_{\infty m}(t_\infty) \le M$ for all m, in contradiction with Lemma 1.3.

The set S_2 of cusps of F is a smooth contractible submanifold of codimension 2 of B^1. From Lemma 3.1, there is a diffeomorphism of B^1 to itself taking the sets S_1 and S_2 to $S_1 = \widehat{S}_1$ and $\widehat{S}_2$, respectively and it suffices to show that $\widehat{S}_2$ is contractible. We may thus use Corollary 3.3: we check that $\widehat{F}$ is 2-regular, i.e., that 0 is a regular value for both $\widehat{\Sigma}_1$ and $(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$. We have already seen that $D\Sigma_1 = D(\widehat{\Sigma}_1)$ is never zero in the critical set. Also,

$$D(\widehat{\Sigma}_1, \widehat{\Sigma}_2)(u) \cdot v = \left(\int f''(u(t))v(t)\, dt, \int f'''(u(t))v(t)\, dt \right)$$

and we are left with showing the linear independence of the functions $f''(u(t))$ and $f'''(u(t))$ for u satisfying $\int f'(u(t))\, dt = \int f''(u(t))\, dt = 0$. The first function is non-zero but of average zero and the second is strictly positive.

In particular, from Corollary 3.3, there is a homeomorphism of B^1 to itself taking S_1 and S_2 to nested subspaces of codimensions 1 and 2. This homeomorphism, however, does not respect fibres: we now show how to do better. It is convenient from this point on to work in adapted coordinates, i.e., to consider $\mathbf{F} : B^0 \to B^0$, its critical set $\mathbf{S}_1 = \Psi(S_1)$ and set of cusps $\mathbf{S}_2 = \Psi(S_2)$. Recall that $\Psi : B^1 \to B^0$ takes fibres to vertical lines and that $\mathbf{F}$ is in adapted coordinates: in other words, vertical lines are fibres for $\mathbf{F}$. Similarly, $\widetilde{B}^0 = \mathcal{F}_a \cup \mathcal{F}_b \cup \mathcal{F}_c$, the disjoint images under Ψ of $\mathcal{F}_a$, $\mathcal{F}_b$ and $\mathcal{F}_c$.

In the next claims, we construct a number of auxiliary changes of variable, leading eventually to the global normal form for the cusp. In the H case, all constructions are smooth. In the C case, however, we make use of homeomorphisms and folds and cusps have to be interpreted topologically. Figure 5.2 may be helpful.

There is a homeomorphism Υ_1 of B^0 to itself taking vertical lines to vertical lines and $\mathcal{F}_b$ to a subspace $\mathcal{C}$ of codimension 1 of $\widetilde{B}^0$. In the H case, this homeomorphism can be taken to be a diffeomorphism. From the local normal form for cusps, vertical lines intersect $\mathbf{S}_2$ transversally: the vertical projection is then a natural diffeomorphism between $\mathbf{S}_2$ and $\mathcal{F}_b$. In particular, $\mathcal{F}_b \subset \widetilde{B}^0$ is a contractible submanifold of codimension 1 for which Lemma 5.3 applies: smoothness guarantees the existence of local tubular neighbourhoods which can be consistently glued because the complement of $\mathcal{F}_b$ has two connected components. Thus, there is a homeomorphism $\Upsilon_0 : \widetilde{B}^0 \to \widetilde{B}^0$ taking $\mathcal{F}_b$ to a closed subspace $\mathcal{C}$ of codimension 1. Define $\Upsilon_1 : B^0 \to B^0$ as the only extension of Υ_0 respecting vertical lines and horizontal hyperplanes — Υ_1 is clearly a homeomorphism.

Notice that the conjugation $\mathbf{F}_1 = \Upsilon_1 \circ \mathbf{F} \circ \Upsilon_1^{-1}$ is still in adapted coordinates, i.e., vertical lines are invariant. Also, the vertical projection of $\Upsilon_1(\mathbf{S}_2)$ and

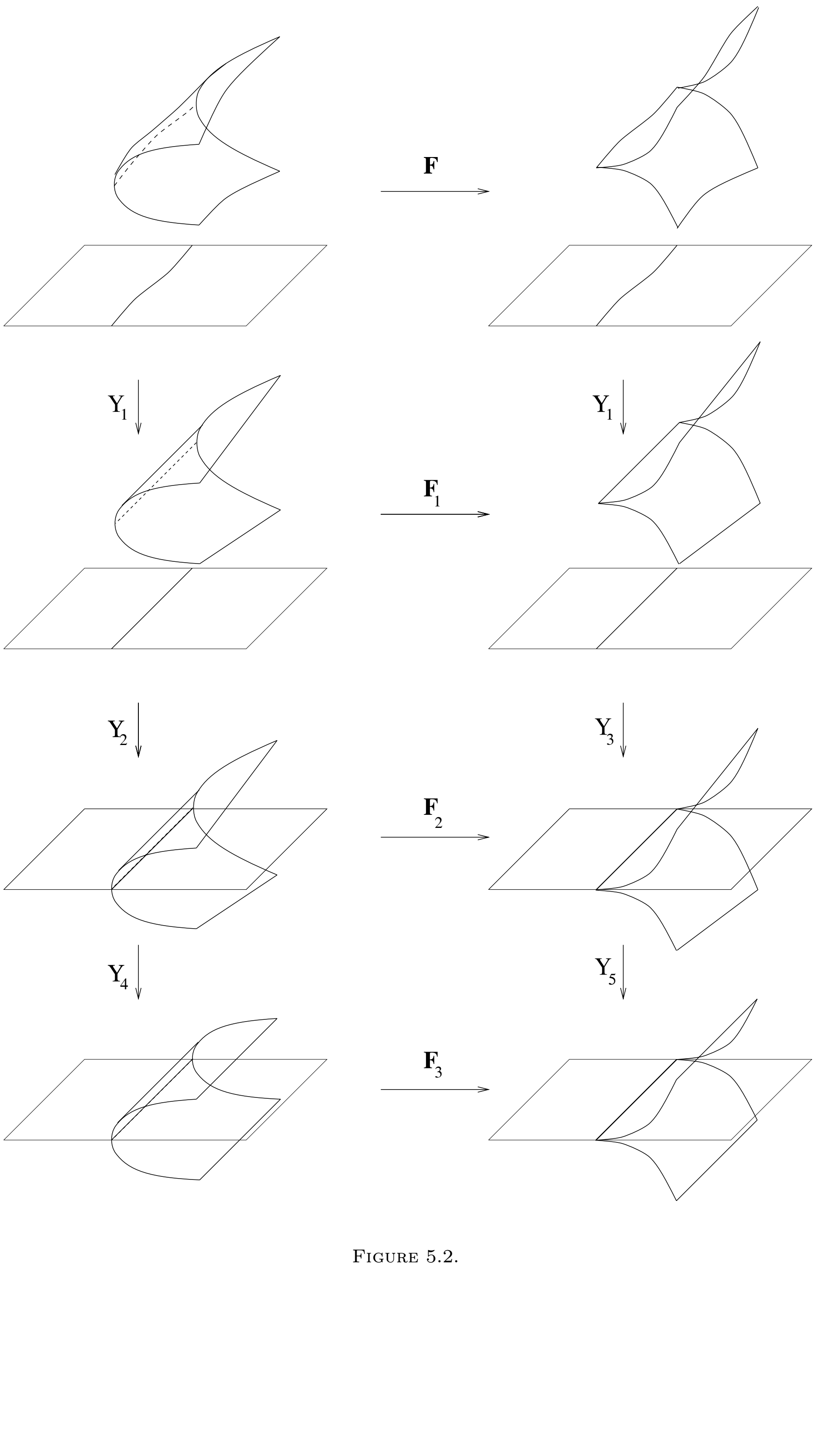


Figure 5.2.

$\Upsilon_1(\mathbf{F}(\mathbf{S}_2))$ are both equal to $\mathcal{C}$; $\Upsilon_1(\mathbf{S}_2)$ is the set of cusps (or, in the C case, topological cusps) of $\mathbf{F}_1$ and $\Upsilon_1(\mathbf{F}(\mathbf{S}_2))$ is its image.

Identify

$$B^0 = \mathcal{C} \qquad r\rangle \oplus \langle 1\rangle,$$

for $r \in \widetilde{B}^0$ not in $\mathcal{C}$ and now write a typical element of B^0 as $(Z, x, y) \in \mathcal{C} \quad \mathbb{R}^2$ making use of the natural projections. Notice that the sign of x determines the type of the $(Z, x, 0)$-fibre: without loss, the cases $x < 0$, $x = 0$ and $x > 0$ are set to correspond to fibres of types (a), (b) and (c), respectively.

There are homeomorphisms Υ_2 and Υ_3 of B^0 to itself keeping each vertical line invariant and such that $\Upsilon_2(\Upsilon_1(\mathbf{S}_2)) = \Upsilon_3(\Upsilon_1(\mathbf{F}(\mathbf{S}_2))) = \mathcal{C}$. In the H case, these maps are diffeomorphisms. Set $\Upsilon_2(Z, x, y) = (Z, x, y - y')$ where y' is the only real number such that $\Upsilon_1^{-1}(Z, 0, y') \in \mathbf{S}_2$. Similarly, set $\Upsilon_3(Z, x, y) = (Z, x, y - y'')$ where y'' satisfies $\Upsilon_1^{-1}(Z, 0, y'') \in \mathbf{F}(\mathbf{S}_2)$.

The composition $\mathbf{F}_2 = \Upsilon_3 \circ \mathbf{F}_1 \circ \Upsilon_2^{-1}$ is almost in normal form: for each Z, $\mathbf{F}_2$ restricted to the Z-plane is a global 2-dimensional cusp. We are now ready to apply Lemma 5.4 to get two further changes of variable Υ_4 and Υ_5 such that $\mathbf{F}_3 = \Upsilon_5 \circ \mathbf{F}_2 \circ \Upsilon_4^{-1}$ is the desired normal form

$$\mathbf{F}_3(Z, x, y) = (Z, x, y^3 + xy). \qquad \square$$

Sketch of proof of Lemma 5.4. Item (a) is straightforward. As to item (b), we begin by invoking Whitney's construction ([W]) which obtains, for any given Z, diffeomorphisms $\upsilon_{Z,6}$ and $\upsilon_{Z,7}$ of neighbourhoods of the origin taking the restriction $G_Z(x, y) = (x, g_Z(x, y))$ to the normal form $k(x, y) = (x, y^3 + xy)$, i.e., $k = \upsilon_{Z,7} \circ G_Z \circ \upsilon_{Z,6}^{-1}$. The diffeomorphisms $\upsilon_{Z,6}$ and $\upsilon_{Z,7}$ so constructed are of the form $(x, y) \mapsto (x, y')$ and preserve orientation.

Actually (and here we omit the verification), the construction allows $\upsilon_{Z,6}$, $\upsilon_{Z,7}$ and the size of the neighbourhoods to be chosen smoothly as functions of the parameter Z. More exactly, we have diffeomorphisms $\Upsilon_{T,6}$ and $\Upsilon_{T,7}$ defined on tubular neighbourhoods of $\mathcal{Z} \times (0, 0)$ taking G to the normal form $K(Z, x, y) = (Z, x, y^3 + xy)$ near $\mathcal{Z} \times (0, 0)$, i.e., $K = \Upsilon_{T,7} \circ G \circ \Upsilon_{T,6}^{-1}$ whenever the right hand side is defined. Now extend $\Upsilon_{T,6}$ and $\Upsilon_{T,7}$ to diffeomorphisms Υ_6 and Υ_7 from $\mathcal{Z} \times \mathbb{R}^2$ to itself of the form $(Z, x, y) \mapsto (Z, x, y')$ which coincide with the identity outside a tubular neighbourhood of $\mathcal{Z} \times (0, 0)$ — notice that this extension is just a one-dimensional problem, parametrized by (Z, x). The composition $G_1 = \Upsilon_7 \circ G \circ \Upsilon_6^{-1}$ satisfies all the original conditions in Lemma 5.4 and coincides with the normal form K in a tubular neighbourhood of $\mathcal{Z} \times (0, 0)$. The hard part of bringing the cusps into normal form being done, we give explicit instructions to take G_1 to normal form.

Let S_{G_1} and S_K be the critical sets of G_1 and K. Their images $G_1(S_{G_1})$ and $K(S_K)$ have two points in each vertical line $(Z, x, \cdot)$, $x > 0$. Construct Υ_9 by juxtaposing 1-dimensional maps to be a diffeomorphism of $\mathcal{Z} \times (0,0)$ satisfying

- $\Upsilon_9(Z, x, y) = (Z, x, y')$,
- for each Z there is a positive x_Z such that $\Upsilon_9(Z, x, y) = (Z, x, y)$ for $x < x_Z$,
- $\Upsilon_9(G_1(S_{G_1})) = K(S_K)$.

Let $G_2 = \Upsilon_9 \circ G_1$. Notice that $G_2(S_{G_2}) = K(S_K)$, where S_{G_2} is the critical set of G_2. There is now a unique diffeomorphism Υ_8 satisfying $G_2 \circ \Upsilon_8^{-1} = K$. Indeed, if $x \leq 0$, let $\Upsilon_8(Z, x, y)$ be the only point (Z, x, y') for which $G_2(Z, x, y) = K(Z, x, y')$. For positive x, given (Z, x, y) there are at most three points satisfying $G_2(Z, x, y) = K(Z, x, y')$: we define $\Upsilon_8(Z, x, y)$ to be the only point (Z, x, y') which is in the same position with respect to the two critical points of K in the vertical line $(Z, x, \cdot\,)$ as (Z, x, y) is with respect to the two critical points of G_2 in the same vertical line. Clearly, Υ_8 is a homeomorphism: we check smoothness of Υ_8 and its inverse. At regular points, this is the inverse function theorem. At fold points, one may use the square root trick in the proof of Theorem 4.2, but we omit the details. Finally, smoothness at cusps is guaranteed from the simple fact that Υ_8 turns out to be the identity near $\mathcal{Z} \times (0,0)$. □

As a corollary, we obtain a global cusp form for the Cafagna–Donati operator ([CD], [CT]):

Corollary 5.5. *Let $f(x) = ax + bx^2 + cx^{2k+1}$ where k is a positive integer, $a \geq 0$, $a^2 + b^2 > 0$ and $c < 0$. Then the operator $F : H^1 \to H^0$ is a smooth global cusp and $F : C^1 \to C^0$ is a topological global cusp.*

6. A numerical counter-example

It is of course tempting to speculate about the possible consequences of $D_2^4 f > 0$: does this condition at least guarantee that points have at most four pre-images? In [L], Lins Neto shows that, if $f(x,t)$ is a polynomial of degree four in x with coefficients depending on t and positive highest degree coefficient, then the number of solutions may be arbitrarily large.

In this section, we obtain a polynomial f of degree 4 and a smooth periodic u_b such that u_b is a Morin singularity of order 4 (a *butterfly*). From Morin's normal form, some points v near $F(u_b)$ have five regular pre-images close to u_b. Since the degree of F is zero, there is yet a sixth pre-image and we thus obtain a smooth periodic function v for which the equation

$$u' + f(u) = v(t), \quad u(0) = u(1)$$

has six solutions.

We briefly describe the numerical procedure used in the identification of u_b. Without loss of generality, $f(x) = x^4 - bx^2 + cx$ and we try to find a butterfly u_b of the form

$$u_b(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + b_2 \sin(2t) + \ldots + b_4 \sin(2t).$$

We now write the four scalar equations $\Sigma_i(u_b) = 0$, $i = 1, \ldots, 4$, in terms of the ten parameters $b, c, a_0, \ldots, b_4$ and search for a zero with a Newton-like method with pseudo-inverses [AG]. Actually, for appropriate b and c, four extra parameters should be enough to locate a butterfly, but the numerical analysis becomes more robust with additional parameters. One example is $b = 4$, $c = -0.3$, $a_0 = -0.01173378$, $a_1 = -0.8836063$, $a_2 = 0.2428734$, $b_2 = -0.6855379$, $a_3 = 0.4465347$, $b_3 = 0.1853376$, $a_4 = -0.01881213$ and $b_4 = 0.2105862$. The Newton method itself checks for the surjectivity of (the restriction of) the derivative of $(\Sigma_1, \ldots, \Sigma_4)$ and the program also verifies that $\Sigma_5(u_b) \neq 0$.

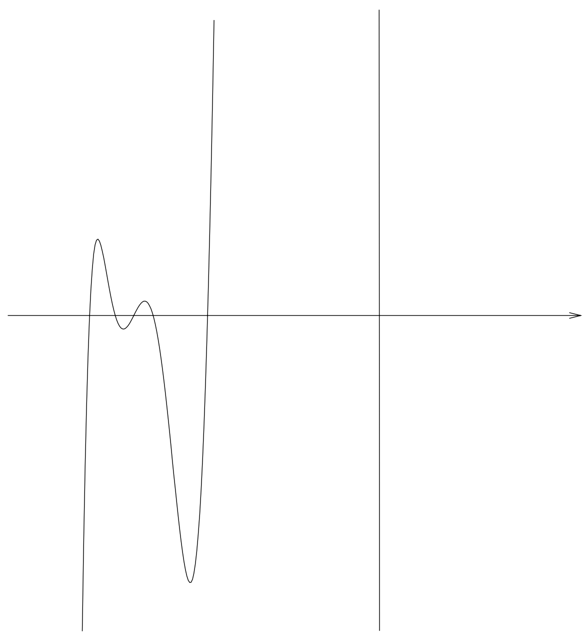


Figure 6.1.

Again by Newton's method, we try to solve

$$(\Sigma_1, \ldots, \Sigma_4)(u_1) = (-0.0000005, 0, 0.00008, 0),$$

where the non-zero constants on the right hand side were adjusted somewhat empirically — in a nutshell, we are trying to perturb the polynomial x^5 to get five distinct real roots, which can be accomplished by adding small multiples of x^3 and x. The parameters for u_1 are

$$\begin{aligned} a_0 &= -0.011367708203969, & a_1 &= -0.883600656945802,\\ a_2 &= 0.243308077825844, & a_3 &= 0.446085678376277,\\ a_4 &= -0.018458472190807, & &\\ b_2 &= -0.685621717642052, & b_3 &= 0.185481811055651,\\ b_4 &= 0.210509692732880. & & \end{aligned}$$

In Figure 6.1 we plot $\rho_v(x) - x$, where $v = u_1' + f(u_1) = F(u_1)$, so that roots of this auxiliary function correspond to periodic solutions of $u' + f(u) = v$. This graph was obtained by solving the differential equation with a Runge–Kutta method of order 4 for initial conditions ranging from -0.4 to 0.4. The vertical scale is stretched by a factor of $2.5 \cdot 10^6$ and the time step for the method had to be taken as $2 \cdot 10^{-4}$. Notice the clustering of the first five roots, stemming from the butterfly: actually it is this clustering and the quintic behaviour of the butterfly which account for the need of a huge vertical stretching factor. Another consequence of the quintic behaviour is the great sensitivity of the coefficients: for instance, a change of 10^{-6} in a_0 destroys four of the six solutions. Still, this final direct check is far easier (and more reliable) than the process of obtaining the coefficients for the example.

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