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ON THE ATTRACTION AND STABILITY OF SETS WITH RESPECT TO SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

An analysis of the concepts of attraction and stability of solutions of ordinary differential equations is normally concerned with a study of *one* equation near its stationary point or invariant set. This paper is concerned with a more general situation: we consider attraction and stability of sets with respect to solutions of a *family* of differential equations.

Our analysis will be carried out within the framework of an axiomatic theory of spaces of solutions of ordinary differential equations and inclusions suggested by V. V. Filippov (see survey [10] and the references therein). This theory is based upon singling out as axioms some basic properties of the solutions of ordinary differential equations and studying sets of continuous functions having these properties. Topological structures introduced in the framework of the theory, which allow one to deal with sets of solutions of a differential equation (or inclusion) as with points of a topological space, play the most important role in the apparatus of the theory. These structures will provide the environment we shall be working in throughout the paper.

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Our results on the properties of attraction and stability of sets with respect to solutions of ordinary differential equations can be divided into two groups.

The first contains theorems in which these properties are studied with regard to a certain family of solution spaces of differential equations. This family is assumed to be compact (with respect to the convergence introduced in the framework of the axiomatic theory).

The other group is concerned with the study of these properties for the space of solutions of a differential equation via the asymptotic behaviour of the family of its limiting spaces of solutions. Since the analysis of the limiting equations is often simpler than that of the original equation, these results are useful in the study of perturbed equations, with the perturbing term vanishing as time increases.

The topological structures introduced in the theory of solution spaces have allowed one to approach the study of the concept of a limiting equation in a new way. The systematic study of limiting equations was initially undertaken by Sell [18], and the theory of limiting equations has since been developed in many papers (for the references, see, e.g., [1]). In the framework of this theory, the limiting equations for the differential equation $dy/dt = f(t, y)$ are defined as the equations of the type $dy/dt = f^*(t, y)$, where $f^*(t, y)$ is the limit of a sequence of translates $\{f(t + t_n, y) : n \in \mathbb{N}\}$ for some sequence $t_n \to \infty$. To make the above definition complete, the meaning of the convergence of translates has to be specified: namely, the set of translates is embedded as a subspace in a function space endowed with some convergence structure, and the limit is taken with respect to this convergence. Thus, the notion of a limiting equation depends on the choice of specific convergence in a function space associated with the right-hand side of a differential equation.

In the theory of solution spaces under consideration, the description of convergence structures is transferred from the level of the right-hand sides of equations directly to the level of sets of solutions. This enables one to address the study of the limiting equations also at the level of sets of solutions. This transition is of the utmost importance, since it allows one to employ methods of analysis developed in the framework of the axiomatic theory that are applicable to broad classes of differential equations and inclusions, including those with singularities (see [10]).

We conclude the introduction with some bibliographical notes on papers in which problems of attraction and stability were studied via the limiting equations.

In [15], [16], [19] these problems were considered with regard to the asymptotically autonomous equations (although in these papers the concept of a limiting equation was not presented explicitly, ideas of this approach actually appear there). In the paper [18] which laid the foundations for a systematic theory of limiting equations, Sell studied properties of stability and asymptotic stability from the point of view of this theory. The studies by Sell were continued in [4]. A series of remarkable results on attraction and uniform asymptotic stability was obtained by Artstein [1], [2]. A study of attraction employing Lyapunov functions was carried out by Ball [3]. Thieme [20] studied attraction in the theory of dynamical systems via the limiting flows. A study of attraction and stability problems in the framework of the axiomatic theory of spaces of solutions via the limiting spaces was carried out by Filippov [9]–[11]. The results of this paper are closely related to his studies.

2. Notation

For a function z, we denote the domain of definition of z by $\pi(z)$ and the set of its values by $\mathrm{Im}(z)$. If $\pi(z)$ is a closed subset of the real line $\mathbb{R}$ and bounded below (respectively, above), we denote by $\alpha(z)$ (respectively, by $\omega(z)$) the initial (respectively, final) point of $\pi(z)$.

The closure of a set $A \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$ is denoted by $\overline{A}$, the boundary of A is denoted by ∂A.

The letter $\mathbb{N}$ stands for the positive integers.

3. Prerequisites from the axiomatic theory of solution spaces

3.1. Let U be the product of $\mathbb{R}$ and an open set $L \subseteq \mathbb{R}^n$. Consider the set of all continuous mappings of all possible finite closed intervals $[a,\ b]$, $a \leq b$, of $\mathbb{R}$ into L. Define a metric on this set by setting the distance between any two mappings equal to the Hausdorff distance between their graphs. The metric space described above is denoted by $C_s(U)$.

Denote by $R(U)$ the set of all subspaces Z of $C_s(U)$ satisfying the following two conditions (axioms).

- (R1) If $z \in Z$ and the closed interval I (possibly degenerate) lies in $\pi(z)$, then the restriction $z|_I$ belongs to Z.
- (R2) If the domains of definition of functions $z_1, z_2 \in Z$ intersect and these functions coincide on $\pi(z_1) \cap \pi(z_2)$, then the function z defined on $\pi(z_1) \cup \pi(z_2)$ by the formula $z(t) = z_i(t)$ if $t \in \pi(z_i)$ $(i = 1, 2)$, also belongs to Z.

We denote by $R_c(U)$ the set of all $Z \in R(U)$ satisfying the following condition:

- (c) for any compact set $K \subseteq U$ the set of all elements of Z with graphs in K is compact.

The set of all $Z \in R(U)$ satisfying the condition

(e) for any point $(t, y) \in U$ there exists a function $z \in Z$ defined on an interval containing t in its interior such that $z(t) = y$,

is denoted by $R_e(U)$.

The intersection $R_c(U) \cap R_e(U)$ is denoted by $R_{ce}(U)$.

3.2. Given a differential equation

$$\frac{dy}{dt} = f(t, y), \tag{E}$$

or a differential inclusion

$$\frac{dy}{dt} \in F(t, y), \tag{I}$$

in the set U, we define its solutions to be generalized absolutely continuous functions [17] defined on arbitrary finite closed intervals $[a, b]$, $a \leq b$, that satisfy it almost everywhere. These functions form a subspace of $C_s(U)$ called *the space of solutions* of this equation (respectively, inclusion) and denoted by $D(f)$ (respectively, by $D(F)$). Under the hypotheses of the classical theorems of Peano and Carathéodory, the above definition of solutions agrees well with the standard definitions [12].

The spaces of solutions defined above belong to the set $R(U)$. The space of solutions of (E) belongs to $R_{ce}(U)$ if f satisfies the hypotheses of the theorems of Peano and Carathéodory. Moreover, $R_{ce}(U)$ contains the space of solutions of (I) if the multivalued function F satisfies the hypotheses of Davy's theorem [5].

If f is continuous in U everywhere except for a closed, at most countable set of points, then the space of solutions of (E) belongs to $R_{ce}(U)$ [7], [10]. This is an example of a space of solutions which belongs to $R_{ce}(U)$ under non-classical assumptions on the right-hand side of the differential equation.

3.3. For $Z \in R(U)$ denote by Z^+ (respectively, by Z^-) the set of all continuous functions $z : [a, b) \to L$, $-\infty < a < b \leq \infty$ (respectively, $z : (a, b] \to L$, $-\infty \leq a < b < \infty$) such that $z|_I \in Z$ for any finite closed interval $I \subseteq \pi(z)$ and there is no function in Z extending the function z. Denote by Z^{-+} the set of all continuous functions $z : (a, b) \to L$, $-\infty \leq a < b \leq \infty$ such that for some $t \in (a, b)$ we have $z|_{(a,t]} \in Z^-$ and $z|_{[t,b)} \in Z^+$.

It is clear that the sets Z^+, Z^-, and Z^{-+} consist of analogues of the solutions of a differential equation which are maximally extended forward, backward, and both forward and backward, respectively.

We note that if $Z \in R_e(U)$, for any $z \in Z$ there exist functions in Z^+ and Z^- extending z.

3.4. For any function $z \in Z^+ \cup Z^{-+}$, its *positive limit set* $\Lambda^+(z)$ is defined by the formula

$$\Lambda^+(z) = \bigcap \{\overline{\mathrm{Im}(z|_{[t,\infty)\cap\pi(z)})} : t \in \pi(z)\}.$$

The *negative limit set* $\Lambda^-(z)$ of a function $z \in Z^- \cup Z^{-+}$ is defined by the formula

$$\Lambda^-(z) = \bigcap \{\overline{\mathrm{Im}(z|_{(-\infty,t]\cap\pi(z)})} : t \in \pi(z)\}.$$

3.5. The following concept of convergence of subspaces of $C_s(U)$ plays a key role in the axiomatic theory of solution spaces. It is adequate for the continuous dependence of a solution to the Cauchy problem on parameters in the right-hand side of an ordinary differential equation.

Definition. A sequence $\{Z_n : n \in \mathbb{N}\}$ of subspaces of $C_s(U)$ *converges* in U to a space $Z \subseteq C_s(U)$ if every sequence of functions $z_k \in Z_{n_k}$ $(n_1 < n_2 < \dots)$ with graphs lying in an arbitrary compact set $K \subseteq U$ has a subsequence convergent to a function belonging to Z.

As noted in [10, Section 4], one can define a topology $\mathcal{T}$ on the set $R_c(U)$ so that the convergence in the topological space $(R_c(U), \mathcal{T})$ is precisely adequate to the convergence in the sense specified in the above definition.

4. Limiting spaces

Following the ideas of [11], we now proceed to define the notion of a limiting space of a subspace of $C_s(U)$.

Let a be an arbitrary real number. For any function z with $\pi(z) \subseteq \mathbb{R}$, we denote by $\Psi_a(z)$ its translate defined on the set $\{s - a : s \in \pi(z)\}$ by the formula $\Psi_a(z)(t) = z(t + a)$. For a space $X \subseteq C_s(U)$ we set $\Psi_a(X) = \{\Psi_a(z) : z \in X\}$. Clearly, $\Psi_a(X) \in R_{ce}(U)$ if $X \in R_{ce}(U)$.

Let X be a subspace of $C_s(U)$.

Definition. A space $Z \subseteq C_s(U)$ is a *limiting space* of the space X as $t \to \infty$ if there exists a sequence $t_n \to \infty$ such that the sequence of spaces $\{\Psi_{t_n}(X) : n \in \mathbb{N}\}$ converges in U to Z.

The above concept of a limiting space naturally corresponds to the notion of a limiting equation introduced by Sell [18] and studied in many papers. Namely, if X is the space of solutions of an ordinary differential equation and Z is the space of solutions of its limiting equation, then Z is the limiting space of X as defined above.

The following important notion of convergence was introduced by Filippov [11].

DEFINITION. A space X converges to a family γ of its limiting spaces as $t \to \infty$ if, for each sequence $t_n \to \infty$, the sequence of spaces $\{\Psi_{t_n}(X) : n \in \mathbb{N}\}$ has a subsequence convergent in U to a space $Z \in \gamma$.

The concept of convergence of the space $X \subseteq C_s(U)$ to a family γ of its limiting spaces as $t \to \infty$ plays a significant role in the study of asymptotic properties of differential equations and inclusions in the framework of the axiomatic theory of solution spaces (results employing this concept can be found in [9], [11], [14]).

In our results presented below the family γ will be assumed to be a compact subset of the space $R_{ce}(U)$ (with respect to the introduced notion of convergence of a sequence of spaces). We now present, following [11], an example of a family γ which is a compact set in $R_{ce}(U)$ and a space $X \in R_{ce}(U)$ converging to γ as $t \to \infty$.

EXAMPLE. For any $i \in \mathbb{N}$ let the function $g_i : U \to \mathbb{R}^n$ be periodic in $t \in \mathbb{R}$ with period q_i and continuous everywhere except in an at most countable closed set. Suppose that there exist real numbers m_i, $i \in \mathbb{N}$, with $\sum_{i=1}^{\infty} m_i < \infty$ such that $\|g_i(t, y)\| \leq m_i$ for all $(t, y) \in U$.

For any sequence $\{\nu_i : i \in \mathbb{N}\}$ with $\nu_i \in [0, q_i)$ denote by $Z(\{\nu_i\})$ the space of solutions of the equation

$$\frac{dy}{dt} = \sum_{i=1}^{\infty} g_i(t + \nu_i, y).$$

All members of the family $\gamma = \{Z(\{\nu_i\}) : \nu_i \in [0, q_i), i \in \mathbb{N}\}$ belong to $R_{ce}(U)$ (see [7, Sections 3 and 4 of Chapter 9]). The family γ is a compact subset of $R_{ce}(U)$ [11]. Note also that γ is invariant under translations along the t-axis, that is, $\Psi_t(Z) \in \gamma$ for all $Z \in \gamma$ and $t \in \mathbb{R}$.

Let $h : U \to \mathbb{R}^n$ be a continuous function such that for some real-valued function $\varphi(t)$ defined on $[t_0, \infty)$, with $\int_{t_0}^{\infty} \varphi(t)dt < \infty$, the inequality $\|h(t, y)\| \leq \varphi(t)$ holds for all $t \geq t_0$ and $y \in L$. The space X of solutions of the equation

$$\frac{dy}{dt} = \sum_{i=1}^{\infty} g_i(t, y) + h(t, y),$$

converges to the family γ as $t \to \infty$ (see [11]).

5. Invariance, stability and attraction concepts

In this section we define analogues of the above named concepts from the classical theory of ordinary differential equations, with regard to the axiomatic theory of solution spaces.

Let $\emptyset \neq H \subseteq L$, $\gamma \subseteq R(U)$, and $X \in R(U)$.

5.1. Invariance. The set H is called *positively γ-invariant* if for any point $y \in H$ and any function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ such that $z(\alpha(z)) = y$ the set $\mathrm{Im}(z)$ lies in H.

The set H is called *negatively γ-invariant* if for any point $y \in H$ and any function $z \in \bigcup\{Z^- : Z \in \gamma\}$ such that $z(\omega(z)) = y$ the set $\mathrm{Im}(z)$ lies in H.

If the set is both positively and negatively γ-invariant, it is said to be *γ-invariant.*

The set H is called *weakly γ-invariant* if for any point $y \in H$ there exists a function $z \in \bigcup\{Z^{-+} : Z \in \gamma\}$ such that the set $\mathrm{Im}(z)$ contains y and lies in H.

5.2. Uniform stability. The set H is called *uniformly γ-stable* if for every neighbourhood V of H there exists a neighbourhood O of H such that for any function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ with $z(\alpha(z)) \in O$ we have $z(t) \in V$ for all $t \in \pi(z)$.

Remark. Throughout the paper the neighbourhoods of H are always assumed to be open subsets of L.

We say that H is *eventually uniformly X-stable* if for every neighbourhood V of H there exists a $t_0 \in \mathbb{R}$ and a neighbourhood O of H such that for any function $z \in X^+$ with $\alpha(z) \geq t_0$ and $z(\alpha(z)) \in O$ we have $z(t) \in V$ for all $t \in \pi(z)$.

5.3. Attraction. The set $W^+_\gamma(H)$, *the basin of attraction of H with respect to γ*, consists of all points $y \in L$ such that for every function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ with $z(\alpha(z)) = y$ the distance $\mathrm{dist}(z(t), H)$ tends to 0 as $t \to \omega(z)$.

Remark. If H is compact, the condition "$\mathrm{dist}(z(t), H)$ tends to 0 as $t \to \omega(z)$" in the above definition is equivalent to "$\emptyset \neq \Lambda^+(z) \subseteq H$".

We say that H is a *γ-attractor* if there exists a neighbourhood O of H lying in $W^+_\gamma(H)$, i.e., there exists a neighbourhood O of H such that for any neighbourhood G of H and every function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ with $z(\alpha(z)) \in O$ there is a real number $\sigma > 0$ such that $z(t) \in G$ for all $t \in \pi(z) \cap [\sigma, \infty)$.

5.4. Uniform attraction. The set H is called a *uniform γ-attractor* if there exists a neighbourhood O of H such that for any neighbourhood G of H there is a $\sigma > 0$ such that for any function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ with $z(\alpha(z)) \in O$ we have $z(\alpha(z) + t) \in G$ when $t \in [\sigma, \infty)$ and $\alpha(z) + t \in \pi(z)$.

The set H is said to be an *eventually uniform X-attractor* if there exists a neighbourhood O of H such that for any neighbourhood G of H there is a $t_0 \in \mathbb{R}$ and a $\sigma > 0$ such that for any function $z \in X^+$ with $\alpha(z) \geq t_0$ and $z(\alpha(z)) \in O$ we have $z(\alpha(z) + t) \in G$ when $t \in [\sigma, \infty)$ and $\alpha(z) + t \in \pi(z)$.

We say that H is a *weak eventually uniform X-attractor* if there exists a neighbourhood O of H such that for any neighbourhood G of H there is a $t_0 \in \mathbb{R}$ and a $\sigma > 0$ such that for any function $z \in X^+$ with $\alpha(z) \geq t_0$, $z(\alpha(z)) \in O$ and $[\alpha(z), \alpha(z) + \sigma] \subseteq \pi(z)$ we have $z([\alpha(z), \alpha(z) + \sigma]) \cap G \neq \emptyset$.

REMARK. One can easily prove that if H is eventually uniformly X-stable and a weak eventually uniform X-attractor, then H is an eventually uniform X-attractor.

5.5. Uniform asymptotic stability. The set H is said to be *uniformly asymptotically γ-stable* if it is uniformly γ-stable and a uniform γ-attractor.

The notion of *negative uniform asymptotic γ-stability* is defined by reversal of time.

The set H is said to be *eventually uniformly asymptotically X-stable* if it is eventually uniformly X-stable and an eventually uniform γ-attractor.

6. Results

6.1. Consider the following assumptions.

(1) U is the product of the real line $\mathbb{R}$ and an open set $L \subseteq \mathbb{R}^n$.
(2) H is a nonempty positively γ-invariant compact subset of L.
(3) The family γ is a compact subset of $R_{ce}(U)$ such that $\Psi_t(Z) \in \gamma$ for all $Z \in \gamma$ and $t \in \mathbb{R}$.
(4) The space $X \in R_{ce}(U)$ converges to γ as $t \to \infty$.

The following theorem is a corollary to a result due to Filippov (see [7, Theorem 9.3.12] or [10, Theorem 3.2]) which plays an important role in the analysis carried out within the framework of the axiomatic theory of solution spaces. The theorem will be repeatedly used below.

CONVERGENCE THEOREM (Filippov). *Suppose that a sequence of spaces $\{Z_n : n \in \mathbb{N}\} \subseteq R_{ce}(U)$ converges in U to the space $Z \in R_{ce}(U)$, $z_n \in Z_n^+$ (respectively, $z_n \in Z_n^-$), $t \in \pi(z_n)$ for all $n \in \mathbb{N}$, and the sequence of points $\{z_n(t) : n \in \mathbb{N}\}$ converges to a point $y \in L$. Then there exists a function $z^* \in Z^+$ (respectively, $z^* \in Z^-$) with $t \in \pi(z)$ and $z(t) = y$, and a subsequence $\{z_{n_k} : k \in \mathbb{N}\}$ such that for any finite interval $I \subseteq \pi(z^*)$, beginning with some $k = k_0$, we have $I \subseteq \pi(z_{n_k})$ and the sequence $\{z_{n_k}|_I : k \geq k_0\}$ converges uniformly to the function $z^*|_I$.*

6.2. We first prove a topological result on attractors of compact families of spaces.

THEOREM 1. *Suppose that assumptions* (1)–(3) *hold. Assume that H is a γ-attractor. Then the set $W_\gamma^+(H)$ is a neighbourhood of H.*

PROOF. I. Let O be an arbitrary neighbourhood of H and x be an arbitrary point in $W_\gamma^+(H)$. We claim that there exists a neighbourhood $E = E(O)$ of x such that for any function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ with $z(\alpha(z)) \in E$ we have $\mathrm{Im}(z) \cap O \neq \emptyset$.

Assume the contrary. Then there exists a sequence $\{x_n : n \in \mathbb{N}\} \subseteq L$ converging to x, a sequence $\{t_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, and a sequence of functions $z_n \in Z_n^+$, $Z_n \in \gamma$, such that $t_n = \alpha(z_n), z_n(t_n) = x_n$, and $\mathrm{Im}(z_n) \cap O = \emptyset$. We set $\Xi_n = \Psi_{t_n}(Z_n)$. By virtue of (3), $\Xi_n \in \gamma$ for any $n \in \mathbb{N}$. Define for all $n \in \mathbb{N}$ functions ξ_n by the formula $\xi_n = \Psi_{t_n}(z_n)$. The assumption $\gamma \in R(U)$ implies that $\xi_n \in \Xi_n^+$. For the functions ξ_n we have $\alpha(\xi_n) = 0, \xi_n(0) = z_n(t_n) = x_n$, and $\mathrm{Im}(\xi_n) \cap O = \emptyset$.

Since the family γ satisfies (3), there exists a sequence of spaces $\{\Xi_{n_k} : k \in \mathbb{N}\}$ which converges in U to a space from γ. We may apply the Convergence Theorem to the sequence of functions $\xi_{n_k} \in \Xi_{n_k}^+$ to find a function $z \in \bigcup\{Z^+ : Z \in \gamma\}$ such that $\alpha(z) = 0, z(0) = x$, and $\mathrm{Im}(z) \cap O = \emptyset$. Therefore, the set $\Lambda^+(z)$ is disjoint from O. However, the assumption $x \in W_\gamma^+(H)$ implies (see the remark in 5.3) that $\emptyset \neq \Lambda^+(z) \subseteq H \subseteq O$. The contradiction obtained proves the claim.

II. Since H is a γ-attractor, there exists a neighbourhood O^* of H which lies in $W_\gamma^+(H)$. One can easily see that for any point $x \in W_\gamma^+(H)$ the neighbourhood $E(O^*)$ of x chosen according to Part I is contained in $W_\gamma^+(H)$.

Thus we have proved that $W_\gamma^+(H)$ is open. Since H satisfies (2), $H \subseteq W_\gamma^+(H)$. Hence $W_\gamma^+(H)$ is a neighbourhood of H. The proof is complete. □

6.3. In [15, Theorem 2] Markus proved a result on the attraction, with respect to the solutions of an asymptotically autonomous differential equation, of a critical point of its limiting autonomous equation (further results were obtained in [16], [19], [20]). In Theorem 2 below we extend these results to a case of nonautonomous limiting equations.

Theorem 2. *Suppose the assumptions of Theorem* 1 *hold and H is uniformly γ-stable. Assume that the space X satisfies* (4)*, and let $z \in X^+$. If $\Lambda^+(z) \cap W_\gamma^+(H) \neq \emptyset$, then $\Lambda^+(z) \subseteq H$ (or equivalently,* $\mathrm{dist}(z(t), H) \to 0$ *as $t \to \infty$).*

Proof. I. Let us prove first that $\Lambda^+(z) \cap W_\gamma^+(H) \subseteq H$. Assume the contrary, that is, there exists a point $x \in \Lambda^+(z) \cap W_\gamma^+(H) \setminus H$. Let V be a neighbourhood of H whose closure is a compact subset of L, disjoint from x. Since the set H is uniformly γ-stable, there is a neighbourhood G of H such that if ξ is any function from $\bigcup\{Z^+ : Z \in \gamma\}$ with $\xi(\alpha(\xi)) \in \overline{G}$, then $\mathrm{Im}(\xi) \subseteq V$ (clearly, $\overline{G}$ lies in V and is compact). Since H is a γ-attractor, without any loss of generality we may assume that $\overline{G} \subseteq W_\gamma^+(H)$.

II. We note that since $\Lambda^+(z)$ is a nonempty subset of L, $\sup\{t : t \in \pi(z)\} = \infty$ (this follows from [10, Theorem 2.8]). Let us prove that there exists a sequence $\{t_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, $t_n \to \infty$ as $n \to \infty$, such that $z(t_n) \in G$ for all $n \in \mathbb{N}$.

Suppose such a sequence does not exist. Then there is a $T \in \pi(z)$ such that $z(t) \in L \setminus G$ for all $t > T$. Since $x \in \Lambda^+(z)$, there exists a sequence $\{s_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, $s_n \to \infty$ as $n \to \infty$, such that $z(s_n) \to x$ as $n \to \infty$. Denote by z_n the restriction of z to the set $[s_n, \infty)$. Without any loss of generality we may assume that $s_n > T$ for all $n \in \mathbb{N}$. Then $\mathrm{Im}(z_n) \subseteq L \setminus G$ for every $n \in \mathbb{N}$.

We now define $\Xi_n = \Psi_{s_n}(X)$ and $\xi_n = \Psi_{s_n}(z_n)$. Note that $\xi_n \in \Psi_{s_n}(X^+) = \Xi_n^+$ and $\Xi_n \in R_{ce}(U)$. For the functions ξ_n we have: $\pi(\xi_n) = [0, \infty), \mathrm{Im}(\xi_n) \subseteq L \setminus G$, and $\xi_n(0) \to x$ as $n \to \infty$. Since X converges to γ as $t \to \infty$, there exists a sequence $\{\Xi_{n_k} : k \in \mathbb{N}\}$ which converges in U to a space $Z \in \gamma$. We now apply the Convergence Theorem to the sequence $\{\xi_{n_k} : k \in \mathbb{N}\}$ to find a function $\xi \in Z^+$ such that $\pi(\xi) = [0, \infty), \mathrm{Im}(\xi) \subseteq L \setminus G$, and $\xi(0) = x$. The last equality together with the condition $x \in W_\gamma^+(H)$ imply that $\emptyset \neq \Lambda^+(\xi) \subseteq H$. But the inclusion $\mathrm{Im}(\xi) \subseteq L \setminus G$ yields that $\Lambda^+(\xi) \cap G = \emptyset$, so $\Lambda^+(\xi)$ should be disjoint from $H \subseteq G$. The contradiction obtained proves what is required.

III. The statement proved in II, together with the conditions $x \in \Lambda^+(z)$ and $x \notin V$, imply that there exist sequences $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ lying in $\pi(z)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$, and, for all n, we have $a_n < b_n, z(a_n) \in \partial G$, $z(b_n) \in \partial V$, and $z(t) \in V \setminus \overline{G}$ for any $t \in (a_n, b_n)$. The set $\overline{G}$ is compact, so one may assume, by passing to a subsequence if necessary, that the sequence $\{z(a_n) : n \in \mathbb{N}\}$ converges to some point $y \in \partial G$.

Let $\Xi_n = \Psi_{a_n}(X)$ and $\xi_n = \Psi_{a_n}(z_n)$, where z_n is the restriction of z to $[a_n, b_n]$. The functions $\xi_n \in \Xi_n$ have the following properties: $\pi(\xi_n) = [0, b_n - a_n]$, $\xi_n(0) \to y$ as $n \to \infty$, $\xi_n(b_n - a_n) \in \partial V$, and $z(t) \in V \setminus \overline{G}$ for any $t \in (0, b_n - a_n)$. We distinguish two cases: the sequence $\{b_n - a_n : n \in \mathbb{N}\}$ either contains a bounded subsequence (Case 1) or does not contain it (Case 2).

IV. In Case 1, there exists a sequence $S = \{n_k : k \in \mathbb{N}\}$ such that the sequence $\{b_{n_k} - a_{n_k} : k \in \mathbb{N}\}$ converges to some number c. Since for all $n \in \mathbb{N}$ the sets $\mathrm{Im}(\xi_n)$ lie in the compact set $\overline{V}$, we may use the convergence of X to γ as $t \to \infty$ to find a subsequence S_1 of S such that $\{\xi_i : i \in S_1\}$ converges to a function $\xi \in \bigcup\{Z : Z \in \gamma\}$ having the following properties: $\pi(\xi) = [0, c]$, $\xi(0) = y \in \partial G$, and $\xi(c) \in \partial V$. The existence of such a function ξ contradicts the choice of G in Part I of the proof.

V. In Case 2, for each $n \in \mathbb{N}$, let $\xi_n^* \in \Xi_n^+$ be any function which extends ξ_n (since $\xi_n \in \Xi_n$ and $\Xi_n \in R_e(U)$, such a function ξ_n^* exists — see 3.3). The convergence of X to γ as $t \to \infty$ implies that there exists a sequence $\{\Xi_{n_k} : k \in \mathbb{N}\}$ converging in U to a space $Z \in \gamma$. We may apply the Convergence Theorem to the sequence of functions $\{\xi_{n_k}^* : k \in \mathbb{N}\}$ to find a function $\xi \in Z^+$ such that $\alpha(\xi) = 0$, $\xi(0) = y \in \partial G$, and $\mathrm{Im}(\xi) \in \overline{V} \setminus G$. Since $y \in \overline{G}$, and $\overline{G} \subseteq W_\gamma^+(H)$ (see Part I), $\emptyset \neq \Lambda^+(\xi) \subseteq H$. However, the inclusion $\mathrm{Im}(\xi) \in \overline{V} \setminus G$ implies that $\Lambda^+(\xi)$ is disjoint from the neighbourhood G of H. Thus Case 2 is impossible, too.

The contradiction obtained proves the statement formulated in Part I of the proof.

VI. Taking into account the statement just proved, to conclude the proof of the theorem, we have to show that $\Lambda^+(z) \subseteq W^+_\gamma(H)$.

Note that by the statement formulated in Part I, $\Lambda^+(z) \cap W^+_\gamma(H) = \Lambda^+(z) \cap H$, so this intersection is a compact set. By a hypothesis of the theorem, this set is not empty. Note also that by Theorem 1 the set $F = \Lambda^+(z) \setminus W^+_\gamma(H)$ is closed. If F were not empty, $\Lambda^+(z)$ could be represented as the union of two closed nonempty sets, F and $\Lambda^+(z) \cap H$, the latter being compact. However, the positive limit set of a continuous curve in $\mathbb{R}^n$ cannot have such a representation (see Lemma 2 in Section 12 of [8]). Thus F is empty, so that $\Lambda^+(z) \subseteq W^+_\gamma(H)$.

The theorem is proved. □

6.4. The next proposition, to be used in the sequel, is concerned with the weak invariance of positive and negative limit sets.

Proposition. *Suppose that γ satisfies* (3), *and let $z \in \bigcup\{Z^- : Z \in \gamma\}$* (*respectively, $z \in \bigcup\{Z^+ : Z \in \gamma\}$*). *If $\Lambda^-(z)$* (*respectively, $\Lambda^+(z)$*) *is a compact subset of L, then this set is weakly γ-invariant.*

The Proposition is a special case of Theorem 2.1 in [14] (formulated there for positive limit sets). To derive the Proposition from this theorem it suffices to note that the assumption (3) implies that an arbitrary space from γ converges to γ both as $t \to \infty$ and $t \to -\infty$ (the convergence as $t \to -\infty$ is defined by an obvious modification of the definition for the case $t \to \infty$).

Theorem 3. *Suppose that assumptions* (1)–(4) *hold. Assume that there exists a neighbourhood $I(H)$ of H such that any weakly γ-invariant closed set in $I(H)$ is contained in H. Suppose that the set*

$$S^-_\gamma(H) = \Big\{ y \in L : \exists z \in \bigcup\{Z^- : Z \in \gamma\} \quad \textit{such that } z(\omega(z)) = y \textit{ and } \emptyset \neq \Lambda^-(z) \subseteq H \Big\}$$

lies in H. Then the set H is eventually uniformly asymptotically X-stable.

Proof. By the remark in 5.4, to prove the theorem it suffices to show that

(i) H is eventually uniformly X-stable and
(ii) H is a weak eventually uniform X-attractor.

I. Let us prove that (i) holds. Assume the contrary. Then there exists a neighbourhood V of H and a sequence of functions $\{z_n : n \in \mathbb{N}\} \subseteq X$, $\pi(z_n) = [t_n, T_n]$, such that $t_n \to \infty$, $\operatorname{dist}(z_n(t_n), H) \to 0$ as $n \to \infty$, $z_n(T_n) \in \partial V$, and $\operatorname{Im}(z_n) \subseteq \overline{V}$. Without any loss of generality one may assume that $\overline{V}$ is compact and lies in $I(H)$. The set $\overline{V}$ being compact, one may also assume that the sequence $\{z_n(T_n) : n \in \mathbb{N}\}$ converges to some point $y \in \partial V$. We distinguish two cases: the sequence $\{T_n - t_n : n \in \mathbb{N}\}$ either contains a bounded subsequence (Case 1) or does not contain such a subsequence (Case 2).

Consider first Case 1. In this case the convergence of X to γ as $t \to \infty$ implies that the sequence $\{\Psi_{t_n}(z_n) : n \in \mathbb{N}\}$ has a subsequence converging to some function $z \in \bigcup\{Z : Z \in \gamma\}$. One can easily see that $\alpha(z) = 0$, $z(0) \in H$, and $z(\omega(z)) = y \in \partial V \subseteq L \setminus H$. This contradicts (2). Hence Case 1 is impossible.

Consider now Case 2. Let $\Xi_n = \Psi_{T_n}(X)$ and $\xi_n = \Psi_{T_n}(z_n)$. Clearly $\xi_n \in \Xi_n$, $\pi(\xi_n) = [t_n - T_n, 0]$, $\xi_n(0) \in \partial V$, and $\mathrm{Im}(\xi_n) \subseteq \overline{V}$. Let $\xi_n^* \in \Xi_n^-$ be any function which extends ξ_n. Since X converges to γ as $t \to \infty$, there exists a sequence of spaces $\{\Xi_{n_k} : k \in \mathbb{N}\}$ which converges in U to a space $Z \in \gamma$. The Convergence Theorem, applied to the sequence of functions $\{\xi_{n_k}^* : k \in \mathbb{N}\}$, implies that there exists a function $\xi \in Z^-$ such that $\omega(\xi) = 0$, $\xi(0) = y \in \partial V$, and $\mathrm{Im}(\xi) \subseteq \overline{V}$. The last inclusion and the compactness of $\overline{V}$ imply that $\Lambda^-(\xi)$ as a nonempty compact subset of $\overline{V}$. By the Proposition, the set $\Lambda^-(\xi)$ is weakly γ-invariant. Since $\Lambda^-(\xi) \subseteq \overline{V} \subseteq I(H)$, the hypothesis of the theorem on $I(H)$ implies that $\Lambda^-(\xi) \subseteq H$. Then, according to the definition of the set $S_\gamma^-(H)$, the point $y = \xi(0)$ belongs to this set. Now the hypothesis of the theorem on $S_\gamma^-(H)$ implies that $y \in H$, which is impossible, since $y \in \partial V$. The contradiction obtained proves that Case 2 also cannot take place. Thus (i) is proved.

II. Let us now prove that (ii) is valid. Let V_0 be a neighbourhood of H whose closure is compact and lies in $I(H)$. We proved in Part I above that there exists a neighbourhood O of H and a real number t_0 such that for every function $z \in X^+$ with $\alpha(z) \geq t_0$ and $z(\alpha(z)) \in O$ we have $\mathrm{Im}(z) \subseteq V_0$. We note that since $\overline{V_0}$ is compact, the last inclusion implies that $\omega(z) = \infty$ for any such function z (see [10, Theorem 2.8]).

The validity of (ii) will be established if we prove the following statement: for any neighbourhood G of H there exists an $s_0 \geq t_0$ and a $\sigma > 0$ such that if $z \in X^+$ is any function such that $\alpha(z) \geq s_0$ and $z(\alpha(z)) \in O$, then $z([\alpha(z), \alpha(z) + \sigma]) \cap G \neq \emptyset$.

Let us prove it. Suppose the statement is wrong. Then there exists a neighbourhood G of H such that for any $n \in \mathbb{N}$ there is a function $z_n \in X^+$ with $\pi(z_n) = [t_n, \infty)$ having the following properties: $t_n \geq \max\{t_0, n\}, z_n(t_n) \in O$, and $z_n([t_n, t_n + n]) \subseteq L \setminus G$. The set O lies in the precompact set V_0, so we may assume without any loss of generality that the sequence $\{z_n(t_n) : n \in \mathbb{N}\}$ is convergent. By the choice of set O, $\mathrm{Im}(z_n) \subseteq V_0$, so that $z_n([t_n, t_n + n]) \subseteq V_0 \setminus G$ for all $n \in \mathbb{N}$.

We set $\Xi_n = \Psi_{t_n}(X)$ and $\xi_n = \Psi_{t_n}(z_n)$. For the functions $\xi_n \in \Xi_n^+$ we have $\pi(\xi_n) = [0, \infty), \xi_n(0) \in O$, and $\xi_n([0, n]) \subseteq V_0 \setminus G$. Since $t_n \to \infty$ as $n \to \infty$, we may use the convergence of X to γ as $t \to \infty$ to find a sequence of spaces $\{\Xi_{n_k} : k \in \mathbb{N}\}$ which converges in U to a space $Z \in \gamma$. We now apply the Convergence

Theorem to the sequence of functions $\{\xi_{n_k} : k \in \mathbb{N}\}$. By this theorem there exists a function $\xi \in Z^+$ with $\pi(\xi) = [0, \infty)$ such that $\mathrm{Im}(\xi)$ lies in the compact set $\overline{V_0} \setminus G$. Then $\Lambda^+(\xi)$ is a nonempty compact subset of $\overline{V_0}$, disjoint from G. By the Proposition, $\Lambda^+(\xi)$ is a weakly γ-invariant subset of H. Therefore, by the assumption of the theorem on $I(H)$, the set $\Lambda^+(\xi) \subseteq \overline{V}_0 \subseteq I(H)$ must lie in H. However, this is impossible, since we have established earlier that $\Lambda^+(\xi)$ is disjoint from the neighbourhood G of H. The contradiction obtained proves that (ii) holds. This completes the proof. □

6.5. The following result is an analogue of Proposition 2 from [13] formulated there for dynamical systems.

Theorem 4. *Suppose that assumptions* (1) *and* (3) *hold. Let $H \neq \emptyset$ be a compact subset of L. Assume that there exists a neighbourhood $I(H)$ of H such that any weakly γ-invariant closed set in $I(H)$ is contained in H.*

(a) *If H is positively γ-invariant and contains the set*

$$S_\gamma^-(H) = \Big\{ y \in L : \exists z \in \bigcup \{Z^- : Z \in \gamma\} \\ \text{such that } z(\omega(z)) = y \text{ and } \emptyset \neq \Lambda^-(z) \subseteq H \Big\},$$

then H is uniformly asymptotically γ-stable.

(b) *If H is negatively γ-invariant and contains the set*

$$S_\gamma^+(H) = \Big\{ y \in L : \exists z \in \bigcup \{Z^+ : Z \in \gamma\} \\ \text{such that } z(\alpha(z)) = y \text{ and } \emptyset \neq \Lambda^+(z) \subseteq H \Big\},$$

then H is negatively uniformly asymptotically γ-stable.

The proof of (a) in Theorem 4 can be obtained by a simple modification of the proof of Theorem 3 if one takes into account the following observation. The convergence of X to γ as $t \to \infty$ means that for any sequence $\{t_n : n \in \mathbb{N}\} \to \infty$ of real numbers the following property holds: the sequence $\{\Psi_{t_n}(X) : n \in \mathbb{N}\}$ has a subsequence converging to some space from γ, whereas the validity of (3) implies that this property holds for any sequence $\{t_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ if one takes for X an arbitrary space from γ. The proof of (b) can be derived from that of (a) by reversal of time.

The following theorem is a corollary to Theorem 4. It is an analogue of a theorem of Ura and Kimura proved for dynamical systems (see [21], cited in [6], [13]).

THEOREM 5. *Suppose that* (1) *holds. Let $H \neq \emptyset$ be a compact γ-invariant subset of L. Assume that there exists a neighbourhood $I(H)$ of H such that any weakly γ-invariant closed set in $I(H)$ is contained in H. Then one and only one of the following alternatives holds:*

(i) *H is uniformly asymptotically γ-stable,*

(ii) *H is negatively uniformly asymptotically γ-stable,*

(iii) *there exist functions $u \in \bigcup\{Z^- : Z \in \gamma\}$ and $v \in \bigcup\{Z^+ : Z \in \gamma\}$ with $z(\omega(u)) \in L \setminus H$ and $z(\alpha(v)) \in L \setminus H$ such that $\emptyset \neq \Lambda^-(u) \subseteq H$ and $\emptyset \neq \Lambda^+(v) \subseteq H$.*

REMARK. The results of this paper are extendable to the case where L in (1) is assumed to be a locally compact metric space.

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