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MULTIPLE SEMICLASSICAL STANDING WAVES FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS

Silvia Cingolani — Monica Lazzo

1. Introduction and statement of the results

In recent years, much interest has been paid to the nonlinear Schrödinger equation in $\mathbb{R}^N$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + U(x)\psi - |\psi|^{p-2}\psi, \quad x\in\mathbb{R}^N; \tag{1.1}$$

i is the imaginary unit, $\hbar$ is the Planck constant, Δ denotes the Laplace operator, $p>2$ if $N=1,2$ and $2<p<2N/(N-2)$ if $N\geq 3$.

When looking for *standing waves* of (1.1), namely solutions of the form $\psi(t,x)=\exp\left(-i\lambda\hbar^{-1}t\right)u(x)$, with $\lambda\in\mathbb{R}$ and u real valued function, one has to deal with an elliptic equation in $\mathbb{R}^N$. Precisely, replacing $\hbar$ by ε leads to look for solutions of the problem

$$(\mathrm{P}_\varepsilon)\qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u, \quad x\in\mathbb{R}^N,\\ u>0,\\ \lim\limits_{|x|\to\infty} u(x)=0, \end{cases}$$

where $V(x)=U(x)+\lambda$.

Solutions of (P_ε) corresponding to small values of the parameter ε are usually referred to as *semiclassical* solutions of the Schrödinger equation. The existence of semiclassical solutions for (P_ε) has been proved for the first time by Floer and

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Weinstein in [8] when $N = 1$ and $p = 4$. They consider a bounded potential V with a nondegenerate critical point x_0, and their method is based on a Lyapunov-Schmidt finite dimensional reduction. We also refer to [16] for some extensions to higher dimensions and to a wider class of potentials.

Some years later, by means of a mountain-pass type argument, Rabinowitz proved in [17] the existence of "least-energy" solutions to (P_ε) for ε sufficiently small, under the assumption

$$\liminf_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x). \tag{1.2}$$

Afterwards, several authors studied the concentration behaviour of solutions to (P_ε). For example, in [18] it is shown that the mountain-pass solution found in [17] concentrates near the global minima of V as ε tends to 0. In [7] a local version of the results in [17] and [18] is obtained, via variational methods. In [1] problem (P_ε) is studied by perturbation arguments, for a bounded potential V having at x_0 a possibly degenerate local minimum (or maximum).

Finally, in [11] some previous results are extended and existence results of multi-bump solutions to (P_ε) are presented. Incidentally, we note that multi-bump solutions have been widely studied; for an extensive bibliography on this subject we refer again to [11].

Let us point out that in many results mentioned above the existence of solutions for (P_ε) is related to the existence of a minimum point of the potential V. As a consequence, it seems rather natural to ask whether it is possible to relate the multiplicity of solutions for (P_ε) to the "richness" (intended in a suitable sense) of the set of minimum points of V. The aim of the present paper is to give an affirmative answer to such a question.

Before stating our main result, we need some notations. Let

$$V_0 = \inf_{x\in\mathbb{R}^N} V(x), \quad M = \{x \in \mathbb{R}^N : V(x) = V_0\}.$$

For any $\delta > 0$, let $M_\delta = \{x \in \mathbb{R}^N : d(x, M) \le \delta\}$.

Theorem 1.1. *Assume that V is a continuous map in $\mathbb{R}^N$ and that*

$$\liminf_{|x|\to\infty} V(x) > V_0 > 0. \tag{V}$$

Then, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that (P_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ solutions, for any $\varepsilon < \varepsilon_\delta$.

Remark 1.2. We recall that, if Y is a closed subset of a topological space X, the Ljusternik–Schnirelman category $\operatorname{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. In some situations this results in $\operatorname{cat}_{M_\delta}(M) = \operatorname{cat}_M(M)$, for δ small. That is the case, for instance, if M is the closure of a bounded open set with smooth boundary, or a smooth and compact submanifold of $\mathbb{R}^N$. If M is a finite set, then $\mathrm{cat}_{M_\delta}(M) = \mathrm{cat}_M(M) =$ cardinality of M, for δ small.

Remark 1.3. As an example, let us show a case in which Theorem 1.1 permits to find an arbitrarily large number of solutions to (P_ε). Suppose that V fulfills (V) and, in addition, $M = \{x_n : n \geq 1\} \cup \{x\}$, where x_n converges to x and $x_n \neq x$ for infinitely many indices. Fix any integer m. It is easy to check that there exists $\delta = \delta(m) > 0$ such that $\mathrm{cat}_{M_\delta}(M) \geq m$. By Theorem 1.1, (P_ε) has at least m solutions for any $\varepsilon < \varepsilon_\delta$. Such a result holds, for example, for any continuous extension in $\mathbb{R}^N$ of the map

$$V(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1 + |x| \sin(1/|x|) & \text{if } 0 < |x| < 1, \end{cases}$$

which satisfies (V).

Remark 1.4. Let us point out that in Theorem 1.1 we do not require V to be smooth; the assumptions on V in our result are the same as in [17], where one solution to (P_ε) is found. Let us remark that (V) is fulfilled by a large class of potentials, including unbounded and oscillating ones.

Remark 1.5. As we have already mentioned, in [18] the concentration behaviour of mountain-pass type solutions to (P_ε) is investigated. By similar arguments, it is possible to prove that also the solutions found in Theorem 1.1 concentrate as ε tends to zero. Roughly speaking, if ε is small, such solutions look like ground state solutions of the equation $-\Delta u + V_0 u = |u|^{p-2}u$ in $\mathbb{R}^N$, highly concentrated around some point of M. We refer to Remark 5.1 below for further details.

In proving Theorem 1.1 we will apply some variational arguments due to Benci and Cerami (see [2], [3], [4]) and used by many authors to deal with boundary value problems for semilinear elliptic equations. For example, see [14], [19] and, in particular, [15] where the influence of a coefficient in the nonlinear part of the equation is studied.

2. Preliminaries

Let $H^1(\mathbb{R}^N)$ be the standard Sobolev space endowed with the usual norm. The set

$$\mathcal{H} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 < \infty \right\},$$

endowed with the inner product

$$(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv,$$

is a Hilbert space, continuously embedded in $H^1(\mathbb{R}^N)$. We will denote by $\|\cdot\|$ the norm associated with the scalar product defined above.

Let us consider the manifold

$$\Sigma = \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^N} |u|^p = 1 \right\}$$

and the functional

$$J_\varepsilon(u) = \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2), \quad u \in \Sigma.$$

It is easy to see that J_ε is well defined and smooth on Σ. Furthermore, if u is a critical point of J_ε on Σ and $u > 0$, then $(J_\varepsilon(u))^{1/(p-2)} u$ is a weak solution for (P_ε).

Let us recall some facts about ground states of the equation

$$-\varepsilon^2 \Delta u + \mu u = |u|^{p-2} u, \quad x \in \mathbb{R}^N, \tag{2.1}$$

with $\varepsilon, \mu > 0$. It is well known that (2.1) has (up to translations) a unique positive solution $\widetilde{\omega}(\varepsilon;\mu) \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, which is radially symmetric around the origin and which decays exponentially at infinity (see [5], [6], [9]). The infimum

$$m(\varepsilon;\mu) \equiv \inf \left\{ \frac{\varepsilon^2 \int_{\mathbb{R}^N} |\nabla u|^2 + \mu \int_{\mathbb{R}^N} |u|^2}{\left(\int_{\mathbb{R}^N} |u|^p \right)^{2/p}} : \ u \in H^1(\mathbb{R}^N), \ u \neq 0 \right\}$$

is achieved in $\omega(\varepsilon;\mu) = \widetilde{\omega}(\varepsilon;\mu)/\|\widetilde{\omega}(\varepsilon;\mu)\|_{L^p(\mathbb{R}^N)}$. It is easy to see that the map $m(\varepsilon;\,\cdot\,)$ is strictly increasing. For convenience, we will denote $\omega = \omega(1;V_0)$. We explicitly note that $\omega(x) \le C_1 e^{-|x|}$ for any $x \in \mathbb{R}^N$, for some $C_1 > 0$.

In the next two sections we will introduce two maps Φ_ε and β which permit to compare the topology of M and the topology of a suitable sublevel of the functional J_ε.

3. The map Φ_ε

Let $\delta > 0$ be fixed. Let η be a smooth non increasing cut-off function, defined in $[0,\infty)$, such that $\eta(t) = 1$ if $0 \le t \le \delta/2$, $\eta(t) = 0$ if $t \ge \delta$, $0 \le \eta \le 1$ and $|\eta'(t)| \le c$ for some $c > 0$.

For any $y \in M$, let us define

$$\psi_{\varepsilon,y}(x) = \eta(|x-y|)\varepsilon^{-N/p} \omega\left(\frac{x-y}{\varepsilon} \right)$$

and

$$\varphi_{\varepsilon,y}(x) = \frac{\psi_{\varepsilon,y}}{|\psi_{\varepsilon,y}|_p}. \tag{3.1}$$

Finally, let us define the map $\Phi_\varepsilon : M \to H^1(\mathbb{R}^N)$ by $\Phi_\varepsilon(y) = \varphi_{\varepsilon,y}$.

REMARK 3.1. By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$. As a consequence, $\Phi_\varepsilon(y)$ is in $\mathcal{H}$ and, by (3.1), in Σ.

LEMMA 3.2. *We have*

$$\lim_{\varepsilon\to 0} \varepsilon^{N\left(2/p-1\right)} J_\varepsilon\big(\Phi_\varepsilon(y)\big) = m(1;V_0), \tag{3.2}$$

uniformly in $y \in M$.

PROOF. Let $y \in M$. By taking into account the exponential decay of ω, it is easy to check that

$$J_\varepsilon(\Phi_\varepsilon(y)) = \varepsilon^{N(1-2/p)} \frac{\int_{\mathbb{R}^N}(|\nabla\omega|^2 + V_0|\omega|^2) + o(1)}{\int_{\mathbb{R}^N}|\omega|^p + o(1)} = \varepsilon^{N(1-2/p)} \frac{m(1;V_0)+o(1)}{1+o(1)}.$$

Letting $\varepsilon \to 0$ implies (3.2). Moreover, the limit is uniform in y since M is a compact set. □

4. The map β

Let $\rho > 0$ be such that $M_\delta \subset B_\rho = \{x \in \mathbb{R}^N : |x| \le \rho\}$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be such that $\chi(x) = x$ for $|x| \le \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \ge \rho$. Finally, let us define $\beta : \Sigma \to \mathbb{R}^N$ by

$$\beta(u) = \int_{\mathbb{R}^N} \chi(x)|u(x)|^p.$$

Let us remark that

$$\beta(\Phi_\varepsilon(y)) = y + \int_{\mathbb{R}^N} (\chi(\varepsilon x + y) - y)|\omega(x)|^p = y + o(1), \tag{4.1}$$

as $\varepsilon \to 0$, uniformly for $y \in M$.

Let $h(\varepsilon)$ be any positive function tending to 0 as $\varepsilon \to 0$ and let

$$\Sigma_\varepsilon = \{u \in \Sigma : J_\varepsilon(u) \le m(\varepsilon;V_0) + \varepsilon^{N(1-2/p)}h(\varepsilon)\}. \tag{4.2}$$

Next result is based on the Concentration–Compacteness Lemma by Lions (see [12], [13]).

LEMMA 4.1. *We have*

$$\lim_{\varepsilon\to 0} \sup_{u\in\Sigma_\varepsilon} \inf_{y\in M_\delta} [\beta(u) - \beta(\varphi_{\varepsilon,y})] = 0. \tag{4.3}$$

PROOF. Let $\{\varepsilon_n\}$ be such that $\varepsilon_n \to 0$ as $n \to \infty$. For any n there exists $u_n \in \Sigma_{\varepsilon_n}$ such that

$$\inf_{y\in M_\delta} [\beta(u_n) - \beta(\varphi_{\varepsilon_n,y})] = \sup_{u\in\Sigma_{\varepsilon_n}} \inf_{y\in M_\delta} [\beta(u) - \beta(\varphi_{\varepsilon_n,y})] + o(1).$$

In order to prove (4.3) it suffices to find points $y_n \in M_\delta$ such that

$$\lim_{n\to\infty} [\beta(u_n) - \beta(\varphi_{\varepsilon_n,y_n})] = 0, \tag{4.4}$$

possibly up to a subsequence. For any n, let us consider $v_n(x) = \varepsilon_n^{N/p} u_n(\varepsilon_n x)$.

Claim 4.2. *There exists* $\{z_n\} \subset \mathbb{R}^N$ *such that* $\varepsilon_n z_n \to \widehat{y} \in M$ *and* $v_n(\,\cdot\, + z_n)$ *converges to* ω *strongly in* $H^1(\mathbb{R}^N)$*, as* $n \to \infty$.

For the proof of the Claim, we refer to the Appendix. As $\varepsilon_n z_n \to \widehat{y} \in M$, we can assume $y_n = \varepsilon_n z_n \in M_\delta$. This results in

$$\begin{aligned} |\beta(u_n) - \beta(\varphi_{\varepsilon_n, y_n}| &= \bigg| \int_{\mathbb{R}^N} \chi(x)|u_n(x)|^p - \int_{\mathbb{R}^N} \chi(x)|\varphi_{\varepsilon_n,y_n}(x)|^p \bigg| \\ &\leq \rho \int_{\mathbb{R}^N} |\, |u_n(x)|^p - |\varphi_{\varepsilon_n,y_n}(x)|^p| \\ &= \rho \int_{\mathbb{R}^N} ||v_n(x+z_n)|^p - |\omega(x)|^p|. \end{aligned}$$

Since $v_n(\,\cdot\, + z_n) \to \omega$ strongly in $L^p(\mathbb{R}^N)$, Lebesgue Theorem now implies (4.4).

5. Palais–Smale condition

For convenience, we discuss Palais–Smale condition for the unconstrained functional associated with (P_ε), namely

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p, \quad u \in \mathcal{H}.$$

As the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous but not compact, it is well known that, in general, I_ε does not satisfy Palais–Smale condition in $\mathcal{H}$. For example, if $V(x) \to \overline{V}$ as $|x| \to \infty$, then I_ε does not satisfy Palais–Smale condition at the level $[(p-2)/2p] m(\varepsilon; \overline{V})^{p/(p-2)}$.

Let V_∞ be such that

$$V_0 < V_\infty \leq \liminf_{|x|\to\infty} V(x). \tag{5.1}$$

Lemma 5.1. *For any* $\varepsilon > 0$*, the functional* I_ε *satisfies Palais–Smale condition in the sublevel*

$$\bigg\{ u \in \mathcal{H} : I_\varepsilon(u) < \frac{p-2}{2p} m(\varepsilon; V_\infty)^{p/(p-2)} \bigg\}.$$

Proof. Let $\{u_n\} \subset \mathcal{H}$ be a Palais–Smale sequence for I_ε at the level C, namely

$$I_\varepsilon(u_n) = C + o(1), \quad I'_\varepsilon(u_n) = o(1) \quad \text{in } \mathcal{H}^{-1} \tag{5.2}$$

as $n \to \infty$, and assume $C < [(p-2)/2p] m(\varepsilon; V_\infty)^{p/(p-2)}$. It is easy to see that $\{u_n\}$ is bounded in $\mathcal{H}$. Up to a subsequence, $\{u_n\}$ has a weak limit $u \in \mathcal{H}$. We have to prove that $\{u_n\}$ converges to u strongly in $\mathcal{H}$. As the Sobolev embedding is compact on bounded sets, it suffices to show that for any $\delta > 0$ there exists $R > 0$ such that

$$\int_{|x|\geq R} (|\nabla u_n|^2 + V(x)|u_n|^2) < \delta \quad \text{for any } n \geq R. \tag{5.3}$$

By contradiction, assume that (5.3) does not hold, namely there exists δ_0 such that for any $R > 0$ we have

$$\int_{|x|\geq R} (|\nabla u_n|^2 + V(x)|u_n|^2) \geq \delta_0 \tag{5.4}$$

for some $n = n(R) \geq R$. As a consequence, there exists a subsequence $\{u_{n_k}\}$ such that

$$\int_{|x|\geq k} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) \geq \delta_0 \tag{5.5}$$

for any $k \in \mathbb{N}$. For any $r > 0$, let us introduce the annulus

$$A_r = \big\{x \in \mathbb{R}^N : r \leq |x| \leq r + 1\big\}.$$

Claim 5.2. *For any $\xi > 0$ and for any $R > 0$ there exists $r > R$ such that*

$$\int_{A_r} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) < \xi \tag{5.6}$$

for infinitely many $k \in \mathbb{N}$.

By contradiction, assume that for some $\xi_0, R_0 > 0$ and for any integer $m \geq [R_0]$ there exists $\nu(m) \in \mathbb{N}$ such that

$$\int_{A_m} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) \geq \xi_0$$

for any $k \geq \nu(m)$. Plainly, we can assume that the sequence $\nu(m)$ is non decreasing. Therefore, for any integer $\overline{m} \geq [R_0]$ there exists an integer $\nu(\overline{m})$ such that

$$\int_{\mathbb{R}^N} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) \geq \int_{[R_0]\leq|x|\leq\overline{m}} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) \geq (\overline{m} - [R_0])\xi_0$$

for any $k \geq \nu(\overline{m})$, which contradicts the boundedness of $\|u_{n_k}\|$ and proves Claim 5.2.

Now, let $\xi > 0$ be fixed. By (5.1) there exists $R(\xi) > 0$ such that

$$V(x) \geq V_\infty - \xi \quad \text{for any } |x| \geq R(\xi). \tag{5.7}$$

Let $r = r(\xi) > R(\xi)$ be as in (5.6) and let $A = A_r$; up to a subsequence, we have

$$\int_{A} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) < \xi \tag{5.8}$$

for any $k \in \mathbb{N}$. Now let us choose any function $\rho \in C^\infty(\mathbb{R}^N, [0,1])$ such that $\rho(x) = 1$ for $|x| \le r$, $\rho(x) = 0$ for $|x| \ge r+1$ and $|\nabla\rho(x)| \le 2$ for any $x \in \mathbb{R}^N$. For any $k \in \mathbb{N}$, let $v_k = \rho u_{n_k}$ and $w_k = (1-\rho)u_{n_k}$. It is not difficult to see that

$$|\langle I'_\varepsilon(u_{n_k}), v_k\rangle - \langle I'_\varepsilon(v_k), v_k\rangle| \le C_1 \int_A (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2),$$
$$|\langle I'_\varepsilon(u_{n_k}), w_k\rangle - \langle I'_\varepsilon(w_k), w_k\rangle| \le C_2 \int_A (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2),$$

where C_1 and C_2 are positive constants which do not depend on r. By (5.2) and (5.8), we deduce

$$o(1) = \langle I'_\varepsilon(v_k), v_k\rangle + O(\xi), \quad o(1) = \langle I'_\varepsilon(w_k), w_k\rangle + O(\xi),$$

whence

$$\|v_k\|^2 = |v_k|_p^p + O(\xi), \quad \|w_k\|^2 = |w_k|_p^p + O(\xi). \tag{5.9}$$

By (5.2), (5.7), (5.9) we have

$$\begin{aligned} C + o(1) &= I_\varepsilon(u_{n_k}) = I_\varepsilon(v_k) + I_\varepsilon(w_k) + O(\xi) \\ &\ge \frac{p-2}{2p}\|w_k\|^2 + O(\xi) \\ &\ge \frac{p-2}{2p}\int_{\mathbb{R}^N} (|\nabla w_k|^2 + V_\infty |w_k|^2) + O(\xi). \end{aligned} \tag{5.10}$$

By (5.5) we have

$$\int_{\mathbb{R}^N} |w_k|^p \ge \int_{|x|\ge r+1} (|\nabla u_{n_k}|^2 + V(x)|u_{n_k}|^2) + O(\xi) \ge \delta_0/2$$

for ξ small, whence, by (5.10)

$$C + o(1) \ge \frac{p-2}{2p} m(\varepsilon; V_\infty)^{p/(p-2)} + O(\xi).$$

Letting $k \to \infty$ and $\xi \to 0$ yields a contradiction and concludes the proof. □

We remark that similar arguments are developed in [10] to discuss Palais–Smale condition in a different setting. At this point it is easy to prove the following lemma.

Lemma 5.3. *For any $\varepsilon > 0$ sufficiently small, the functional J_ε satisfies Palais–Smale condition on $\{u \in \Sigma : J_\varepsilon(u) < m(\varepsilon; V_\infty)\}$.*

Proof. It follows from Lemma 5.1 and standard computations. Here we only remark that, for $\varepsilon > 0$ sufficiently small, the sublevel $\{u \in \Sigma : J_\varepsilon(u) < m(\varepsilon; V_\infty)\}$ is not empty, since

$$\inf_{u\in\Sigma} J_\varepsilon(u) < m(\varepsilon; V_\infty). \tag{5.11}$$

Indeed, if there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$m(\varepsilon_n; V_\infty) \leq \inf_{u\in\Sigma} J_{\varepsilon_n}(u)$$

for any $n \in \mathbb{N}$, then Lemma 3.2 implies

$$m(\varepsilon_n; V_\infty) \leq m(\varepsilon_n; V_0) + o(\varepsilon_n^{N(p-2)/p})$$

for any n. If we divide by ε_n and let $n \to \infty$ we get

$$m(1; V_\infty) \leq m(1; V_0). \tag{5.12}$$

On the other hand, $m(1; V_0) < m(1; V_\infty)$, which contradicts (5.12). □

Remark 5.4. By Lemma 5.3 and the choice of V_∞ it follows that if V is coercive, namely $V(x) \to \infty$ as $|x| \to \infty$, then the functional J_ε satisfies Palais–Smale condition on Σ, at any level.

6. Proof of Theorem 1.1

In order to compare the topology of M and the topology of a suitable energy sublevel we will use the maps Φ_ε and β introduced in Sections 3 and 4. Let us choose a function $h(\varepsilon) > 0$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $m(\varepsilon; V_0) + h(\varepsilon)\varepsilon^{N(p-2)/p}$ is not a critical level for J_ε. For such $h(\varepsilon)$, let us consider the set Σ_ε, introduced in (4.2).

By Lemma 4.1 and 5.3, we can find $\overline{\varepsilon} > 0$ such that J_ε satisfies Palais–Smale condition on Σ_ε and

$$\sup_{u\in\Sigma_\varepsilon} \inf_{y\in M_\delta} [\beta(u) - \beta(\varphi_{\varepsilon,y})] \leq \delta/2 \tag{6.1}$$

for any $\varepsilon < \overline{\varepsilon}$. By Lemma 3.2, we can assume that for such ε we have

$$J_\varepsilon(\Phi_\varepsilon(y)) \leq m(\varepsilon; V_0) + h(\varepsilon)\varepsilon^{N(p-2)/p},$$

thus $\Phi_\varepsilon(M) \subset \Sigma_\varepsilon$. By (6.1) and (4.1) we can assume that $\operatorname{dist}(\beta(u), M_\delta) < \delta/2$ for every $\varepsilon < \overline{\varepsilon}$ and for every $u \in \Sigma_\varepsilon$. Thus $\beta(\Sigma_\varepsilon) \subset M_\delta$.

In conclusion, the map $\beta \circ \Phi_\varepsilon$ is homotopic to the inclusion $j : M \to M_\delta$ in M_δ, for any $\varepsilon \in (0, \overline{\varepsilon})$. Now set $\Sigma_\varepsilon^+ = \Sigma_\varepsilon \cap \{u \in \Sigma : u \geq 0 \text{ in } \mathbb{R}^N\}$. Standard arguments (for example, see [4]) show that $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon^+) \geq \operatorname{cat}_{M_\delta}(M)$. By the opposite map $-\Phi_\varepsilon$ and the same arguments we get $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon^-) \geq \operatorname{cat}_{M_\delta}(M)$, where $\Sigma_\varepsilon^- = \Sigma_\varepsilon \cap \{u \in \Sigma : u \leq 0 \text{ in } \mathbb{R}^N\}$. Since Σ_ε^+ and Σ_ε^- are disjoint in Σ_ε, it follows that

$$\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) = \operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon^+) + \operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon^-) \geq 2\operatorname{cat}_{M_\delta}(M).$$

Ljusternik–Schnirelman theory implies that J_ε has at least $2\operatorname{cat}_{M_\delta}(M)$ critical points on Σ. By construction, for any such point, say u, we have

$$J_\varepsilon(u) \leq m(\varepsilon; V_0) + h(\varepsilon)\varepsilon^{N(p-2)/p}. \tag{6.2}$$

We aim at proving that (6.2) implies that u cannot change sign. Indeed, if $u = u^+ + u^-$ with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, then

$$(6.3)\quad \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u^\pm|^2 + V(x)|u^\pm|^2) \geq \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u^\pm|^2 + V_0|u^\pm|^2) \geq m(\varepsilon; V_0)\|u^\pm\|_p^2.$$

Since u is a critical point of J_ε on Σ, it satisfies

$$-\varepsilon^2\Delta u + V(x)u = J_\varepsilon(u)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

whence

$$(6.4)\qquad \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u^\pm|^2 + V(x)|u^\pm|^2) = J_\varepsilon(u)\int_{\mathbb{R}^N} |u^\pm|^p.$$

By (6.3) and (6.4) we get $\|u^\pm\|_p^{p-2} \geq m(\varepsilon; V_0)/J_\varepsilon(u)$ which implies

$$1 = \|u\|_p^p = \|u^+\|_p^p + \|u^-\|_p^p \geq 2\bigg(\frac{m(\varepsilon; V_0)}{J_\varepsilon(u)}\bigg)^{p/(p-2)}.$$

As a consequence,

$$m(\varepsilon; V_0) \leq 2^{(2-p)/p} J_\varepsilon(u),$$

which contradicts (6.2). Thus we can assume that there exist at least $\mathrm{cat}_{M_\delta}(M)$ critical points that are positive on $\mathbb{R}^N$; by standard maximum principle in $\mathbb{R}^N$ they are strictly positive. The proof of Theorem 1.1 is now complete.

Remark 6.1. For any $\varepsilon \in (0, \overline{\varepsilon})$ let u_ε be a solution to (P_ε) found in Theorem 1.1. By slight changes in the proof of Theorem 2.1 and 2.3 in [18], taking into account the energy estimate

$$\varepsilon^{-N}\int_{\mathbb{R}^N} (\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2) \to m(1; V_0)^{p/(p-2)} \quad \text{as } \varepsilon \to 0,$$

it is possible to prove that $\{u_\varepsilon\}$ has a concentration behaviour. Indeed, for ε small, u_ε has a unique maximum point x_ε. As $\varepsilon \to 0$, the points x_ε converge to a suitable $x_0 \in M$ and the functions $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ approach in $H^1(\mathbb{R})$ the ground state of the equation

$$-\Delta u + V_0 u = |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

Appendix

In this section we will prove Claim 4.2. For any $n \in \mathbb{N}$, $\rho_n = |v_n|^p$ satisfies the following properties:

$$(A.1)\qquad \rho_n \in L^1(\mathbb{R}^N), \quad \rho_n \geq 0, \quad \int_{\mathbb{R}^N} \rho_n = 1,$$

thus the Concentration–Compactness Lemma applies. Since

$$\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(\varepsilon_n x)|v_n|^2) \leq m(1; V_0) + h(\varepsilon_n),$$

v_n and ∇v_n are bounded in $L^2(\mathbb{R}^N)$; by Lemma I.1 in [13] we can exclude that vanishing occurs. If dichotomy occurs, there exists $\alpha \in (0,1)$ such that for any $\xi > 0$ the function ρ_n splits into $\rho_n^1 = \chi_{B_R(z_n)}\rho_n$ and $\rho_n^2 = \chi_{\mathbb{R}^N \setminus B_{R_n}(z_n)}\rho_n$ for some $R > 0$, $R_n \to \infty$ and $z_n \in \mathbb{R}^N$, with the following properties:

$$\int_{\mathbb{R}^N} \rho_n^1 \geq \alpha - \xi, \qquad \int_{\mathbb{R}^N} \rho_n^2 \geq 1 - \alpha - \xi. \tag{A.2}$$

If we denote $v_n^1 = \chi_{B_R(z_n)} v_n$ and $v_n^2 = \chi_{\mathbb{R}^N \setminus B_{R_n}(z_n)} v_n$, (A2) becomes

$$\int_{\mathbb{R}^N} |v_n^1|^p \geq \alpha - \xi, \qquad \int_{\mathbb{R}^N} |v_n^2|^p \geq 1 - \alpha - \xi.$$

After smoothing v_n^1 and v_n^2 we can assume that they belong to $H^1(\mathbb{R}^N)$ and the inequalities above still hold. This results in

$$\begin{aligned} m(1; V_0) + h(\varepsilon_n) &\geq \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V_0 |v_n|^2) \\ &\geq \int_{\mathbb{R}^N} (|\nabla v_n^1|^2 + V_0 |v_n^1|^2) + \int_{\mathbb{R}^N} (|\nabla v_n^2|^2 + V_0 |v_n^2|^2) - \xi \\ &\geq m(1; V_0) \left[\left(\int_{\mathbb{R}^N} |v_n^1|^p \right)^{2/p} + \left(\int_{\mathbb{R}^N} |v_n^2|^p \right)^{2/p} \right] - \xi \\ &\geq m(1; V_0)[(\alpha - \xi)^{2/p} + (1 - \alpha - \xi)^{2/p}] - \xi. \end{aligned}$$

For $\xi \to 0$ and $n \to \infty$ we get $1 \geq \alpha^{2/p} + (1-\alpha)^{2/p} > 1$, a contradiction. As a consequence, the sequence $\{\rho_n\}$ is tight, namely there exists $\{z_n\} \subset \mathbb{R}^N$ such that for any $\xi > 0$ we have

$$\int_{B_R(z_n)} |v_n(x)|^p \geq 1 - \xi$$

for a suitable $R > 0$. Let us define $\widehat{v}_n = v_n(\cdot + z_n)$. As $\widehat{v}_n$ is bounded in $H^1(\mathbb{R}^N)$, it weakly converges to some $\widehat{v}$ in $H^1(\mathbb{R}^N)$. Since

$$\int_{B_R(0)} |\widehat{v}_n|^p \geq 1 - \eta \quad \text{and} \quad \int_{\mathbb{R}^N} |\widehat{v}_n|^p = 1,$$

Rellich Theorem implies

$$\int_{\mathbb{R}^N} |\widehat{v}_n - \widehat{v}|^p = o(1) \quad \text{and} \quad \int_{\mathbb{R}^N} |\widehat{v}|^p = 1. \tag{A.3}$$

for n large. Let us prove that the sequence $\varepsilon_n z_n$ is bounded. Arguing by contradiction, assume that $|\varepsilon_n z_n| \to \infty$ as $n \to \infty$. This results in

$$\begin{aligned} m(1; V_0) + h(\varepsilon_n) &\geq \int_{\mathbb{R}^N} (|\nabla \widehat{v}_n|^2 + V(\varepsilon_n(x + z_n))|\widehat{v}_n|^2) \\ &\geq \int_{\mathbb{R}^N} |\nabla \widehat{v}|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n(x + z_n))|\widehat{v}_n|^2 + o(1). \end{aligned}$$

As $\widehat{v}_n(x) \to \widehat{v}(x)$ a.e. in $\mathbb{R}^N$, letting $n \to \infty$ and (A.3) give

$$\begin{aligned} m(1;V_0) &\geq \int_{\mathbb{R}^N} |\nabla \widehat{v}|^2 + \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon_n(x+z_n))|\widehat{v}_n|^2 \\ &\geq \int_{\mathbb{R}^N} |\nabla \widehat{v}|^2 + \int_{\mathbb{R}^N} \liminf_{n\to\infty} V(\varepsilon_n(x+z_n))|\widehat{v}|^2. \end{aligned} \tag{A.4}$$

By assumption (V), we can choose some V_∞ such that

$$\liminf_{|x|\to\infty} V(x) \geq V_\infty > V_0;$$

plainly, $m(1;V_\infty) > m(1;V_0)$ (cf. Section 2). By (A.3) and (A.4) it follows

$$m(1;V_0) \geq \int_{\mathbb{R}^N} |\nabla \widehat{v}|^2 + \int_{\mathbb{R}^N} V_\infty |\widehat{v}|^2 \geq m(1;V_\infty),$$

a contradiction. Thus we can assume that $\varepsilon_n z_n \to \widehat{z}$; we aim to prove that $\widehat{z} \in M$ and $\widehat{v} = \omega$ (cf. Section 2). Arguing as before yields

$$m(1;V_0) \geq \int_{\mathbb{R}^N} (|\nabla \widehat{v}|^2 + V(\widehat{z})|\widehat{v}|^2) \geq m(1;V(\widehat{z})) \geq m(1;V_0), \tag{$A.5$}$$

whence $V_0 = V(\widehat{z})$, that is $\widehat{z} \in M$. Moreover, (A.5) also gives

$$\int_{\mathbb{R}^N} (|\nabla \widehat{v}|^2 + V_0|\widehat{v}|^2) = m(1;V_0),$$

and the uniqueness of ground state solutions of equation (2.1) implies $\widehat{v} = \omega$. Finally, let us note that

$$m(1;V_0) \leq \int_{\mathbb{R}^N} (|\nabla \widehat{v}_n|^2 + V_0|\widehat{v}_n|^2) \leq \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(\varepsilon_n x)|v_n|^2) \leq m(1;V_0) + h(\varepsilon_n)$$

yields

$$\int_{\mathbb{R}^N} (|\nabla \widehat{v}_n|^2 + V_0|\widehat{v}_n|^2) \to \int_{\mathbb{R}^N} (|\nabla \omega|^2 + V_0|\omega|^2)$$

as $n \to \infty$, whence $\widehat{v}_n$ converges to ω strongly in $H^1(\mathbb{R}^N)$.

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Silvia Cingolani
Dipartimento di Matematica
Politecnico di Bari
via E. Orabona 4
70125 Bari, ITALY

E-mail address: cingolan@pascal.dm.uniba.it

Monica Lazzo
Dipartimento di Matematica
Università di Bari
via E. Orabona 4
70125 Bari, ITALY

E-mail address: lazzo@pascal.dm.uniba.it