Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 9, 1997, 325–382

A NEW COHOMOLOGY FOR THE MORSE THEORY OF STRONGLY INDEFINITE FUNCTIONALS ON HILBERT SPACES

Alberto Abbondandolo

A generalized cohomology, similar to Szulkin's cohomology but with more general functorial properties, is constructed. This theory is used to define a relative Morse index and to prove relative Morse relations for strongly indefinite functionals on Hilbert spaces.

Introduction

Take a C^2 function $f : M \to \mathbb{R}$ on a complete Hilbert manifold which satisfies the Palais–Smale condition. Assume that it is a Morse function, meaning that the second order differential $d^2 f(x)$ is non-degenerate at every critical point x. Recall that the Morse index $m(x, f)$ of a critical point x is the dimension of the maximal subspace on which $d^2 f(x)$ is negative definite.

Then the basic result of Morse theory, as generalized by Palais [14], is the following: if $c \in]a, b[$ is the only critical level in $[a, b]$ and x_0 is the only critical point at level c, then the set $f^b = \{x \in M \mid f(x) \leq b\}$ can be continuously deformed onto $f^a \cup B_{m(x,f)}$, where $B_{m(x,f)}$ is an $m(x, f)$-dimensional closed ball, attached to f^a by its boundary. This local result is used to prove the well

1991 *Mathematics Subject Classification.* Primary 58E05; Secondary 55N20.

known Morse relations, which can be written in the form

$$\sum_{\substack{a<f(x)<b \\ x\,\text{critical point}}} t^{m(x,f)} = P(f^b, f^a) + (1+t)Q(t) \tag{0.1}$$

where Q is a polynomial with non-negative integer coefficients and P is the Poincaré polynomial of the pair (f^b, f^a):

$$P(f^b, f^a) = \sum_{q\geq 0} \dim H_q(f^b, f^a)\, t^q.$$

Here H_* denotes the singular homology theory, with coefficients in a given field.

If a critical point has infinite Morse index, then the corresponding ball which has to be attached is infinite-dimensional. Since every infinite-dimensional ball can be continuously deformed onto its boundary (see, for example, [2]), attaching such a ball does not change the homotopy type, and therefore the homology, of the sublevels. As a consequence, critical points with infinite Morse index are not detected by formula (0.1), which still makes sense with the convention that $t^\infty = 0$.

The problem is relevant, because there are many variational problems in which all the solutions are critical points with infinite Morse index of some functional. This happens, for example, in the study of Hamiltonian systems, of symplectic geometry, of wave equations, of minimal submanifolds in semi-Riemannian geometries, of Dirac-type equations. Such functionals are called strongly indefinite.

In this paper we consider the case in which the domain of the functional is a Hilbert space H. In most applications one notes that, although infinite-dimensional, the negative eigenspace of $d^2 f(x)$ has a finite relative dimension with respect to a fixed subspace of H. To be more precise, there exists a fixed orthogonal splitting $H = E\oplus E^\perp$ such that, for every critical point x, the positive eigenspace V_x of $d^2 f(x)$ is commensurable with E, while the negative eigenspace W_x is commensurable with $E^\perp$ (two closed subspaces E and E' are said to be commensurable if the projections $E \to H/E'$ and $E' \to H/E$ are compact).

In this case it seems reasonable to define the relative dimension of W_x, which may be called E-dimension, as

$$E\text{-}\dim W_x = \dim W_x \cap E - \dim W_x^\perp \cap E^\perp = \dim W_x \cap E - \operatorname{codim}(W_x + E).$$

By the commensurability condition, this is a finite integer. The E-Morse index $m_E(x, f)$ of a critical point x can be defined as the E-dimension of the negative eigenspace of $d^2 f(x)$.

The next step is to build a suitable generalized cohomology theory H^*_E, which may be called E-cohomology theory (a generalized cohomology theory is a theory which satisfies all the Eilenberg–Steenrod axioms, [9], except the dimension axiom; moreover, the functoriality and the homotopy invariance may hold for a restricted class of continuous maps and homotopies). We require that H^*_E detects infinite-dimensional spheres and distinguishes between them, according to the E-dimension. More precisely, we require that $H^q_E(B_m, \partial B_m) = \mathcal{A}$, $\mathcal{A}$ being the coefficient ring, if $q = m$, and 0 otherwise; here B_m is a closed ball in a subspace of E-dimension m.

Then our goal will be to prove that, for f in a certain class of functionals, the following generalized Morse relations hold:

$$\sum_{\substack{a<f(x)<b \\ x \text{ critical point}}} t^{m_E(x,f)} = P_E(f^b, f^a) + (1+t)Q(t) \tag{0.2}$$

where P_E is the E-Poincaré polynomial of the pair (f^b, f^a):

$$P_E(f^b, f^a) = \sum_{q\in\mathbb{Z}} \dim H^q_E(f^b, f^a)\, t^q.$$

We emphasize that, since the E-dimension can be also negative, P_E and Q are actually Laurent polynomials.

The idea of constructing a generalized cohomology to have a Morse theory for strongly indefinite functionals is due to Szulkin [17]. His construction was later modified and applied successfully in the study of Hamiltonian systems and wave equations in [12].

In Szulkin's approach, one fixes a flag of finite-dimensional linear subspaces of $E^\perp$ and defines the generalized cohomology of a set taking a direct limit of the usual cohomology over this flag. This construction, in a more complex way, was introduced by Gęba and Granas in a different framework [10].

Although the construction is quite simple, Szulkin's cohomology has some disadvantages. First of all it is functorial and homotopy invariant with respect to a very restricted class of maps and homotopies, essentially those which preserve the given flag.

Because of this fact, it is not even clear if Szulkin's cohomology is invariant with respect to arbitrary translations. What is more relevant, the gradient flow of the functional f is not an admissible homotopy. Since such a flow is essential to deform the sublevels of f, one has to approximate it with admissible homotopies.

Moreover, the dependence on the particular flag chosen makes it difficult to give a reasonable definition of the relative dimension and of the relative Morse index, such as the one we gave above. Finally, a theory which depends only on the subspace E should carry useful Alexander type duality properties and it could be a first step towards the extension of such a theory to Hilbert manifolds.

The material of this paper is arranged in the following way: in the first part we define our E-cohomology theory. We endow H with the product topology {weak on E}×{strong on $E^{\perp}$}. We call this topology $\mathcal{T}_E$. We define the E-cohomology theory of bounded $\mathcal{T}_E$-closed subsets of H, following more closely the Gęba–Granas original construction. For unbounded sets, we have to assume a nasty technical condition, which we call E-local compactness.

Then we show the functoriality of H_E^* with respect to $\mathcal{T}_E$-continuous maps of the form

$$\Phi(x) = x + K(x)$$

where K maps bounded sets into $\mathcal{T}_E$-precompact sets. These maps are called E-compact morphisms. The homotopy invariance with respect to analogous maps and all the other Eilenberg–Steenrod axioms are proved.

Notice that, since the $\mathcal{T}_E$-topology is weaker than the strong one, maps of the form Identity + Completely Continuous are admissible. Therefore the class of E-compact morphisms looks quite natural and it is analogous, for example, to the Leray–Schauder class of infinite-dimensional degree theory. It turns out that H_E^* coincides with Gęba and Granas' cohomology theory when $E = \{0\}$. When $E = H$, it coincides with the Alexander–Spanier cohomology theory with compact supports on the weak topology of H.

The typical functional one would like to study has the form

$$f(x) = \frac{1}{2}\langle Lx, x\rangle + b(x) \tag{0.3}$$

where L is self-adjoint and ∇b has some compactness property. Since its gradient flow has the form

$$\Phi(x,t) = e^{-tL}x + K(x,t)$$

where K has some compactness property, the class of E-compact morphisms is not large enough to include it.

Therefore in the second part of the paper we prove the functoriality of H_E^* with respect to maps of the form

$$\Phi(x) = Mx + K(x)$$

where M is a linear invertible operator such that $ME = E$ and K is as before. Such maps are called E-*morphisms*.

Notice that the group $\mathrm{GL}(E) \oplus \mathrm{GL}(E^{\perp})$ is contained in the class of E-morphisms and it is connected when both E and $E^{\perp}$ are infinite-dimensional (see [13]). For this reason we can prove the functoriality with respect to E-morphisms only taking $\mathbb{Z}_2$ as coefficient ring. This reminds a well known phenomenon: the same problem arises, for example, when one wants to generalize the Leray–Schauder degree theory to Fredholm maps; one defines actually only a $\mathbb{Z}_2$-valued degree (see [4]).

However, to prove the Morse relations we just need the functoriality with respect to maps of the form $\Phi(x) = Mx + K(x)$, where $M = e^{-tL}$ is a positive operator. In this particular case we can still work with arbitrary coefficients and so we will be able to prove the Morse relations with coefficients in any field.

The only difficult thing to prove in this second part is the homotopy invariance with respect to homotopies of the form

$$\Phi(x,t) = M_t x + K(x,t)$$

where both the linear part and the non-linear one are allowed to vary. To get the main idea surrounded by less technicalities, see Proposition 7.4, where a particular case is treated. This problem had already been solved in the simpler case of the Gęba–Granas cohomology theory in [1].

The reader who wants to avoid all the details of the construction of the E-cohomology theory can skip directly to the last part, reading only the very beginning of Parts 1 and 2, where the results are stated. This last part is devoted to Morse theory. First the concept of E-dimension is analyzed. The effect of replacing E by a commensurable space is also discussed.

Then the generalized Morse relations (0.2) are proved. In order to avoid excessive technicalities, useless at this level, we feel free to make rather strong assumptions on f. For example, we assume that the field ∇f is globally integrable, which is false in general, unless ∇f is globally Lipschitz.

Then we come back to the functionals of the form (0.3) and we see which assumptions on the function b are necessary to make our Morse theory work. The only unnatural hypothesis seems to be the E-local compactness of the sublevels, which involves a lower bound for the functional. We do not know how to avoid this condition.

Some final remarks and hints for developing this theory in more difficult situations conclude the paper.

I would like to express my gratitude to Vieri Benci who introduced me to this subject and encouraged me during the preparation of this paper. My gratitude goes also to Joseph Bernstein and Pietro Majer for many interesting discussions and useful suggestions.

1. The E-cohomology theory

Let E be a closed linear subspace of the Hilbert space H and let $\pi : H \to H/E$ be the projection onto the quotient space. H/E is a Hilbert space with the quotient norm

$$\|[x]\|_{H/E} = \inf_{y\in[x]} \|y\|.$$

We endow H with the weakest topology such that π and all the bounded linear functionals are continuous. This topology will be denoted by $\mathcal{T}_E$. Obviously $\mathcal{T}_E$ is stronger than the weak topology, but weaker than the Hilbert topology.

If F is a closed complement of E in H, i.e. $E \cap F = 0$ and $E + F = H$, then π restricted to F is an isomorphism of Hilbert spaces (also an isometry if F is the orthogonal complement of E). The topology $\mathcal{T}_E$ can be described in the following way: put the strong topology on F and the weak one on E and endow $H = E \oplus F$ with the product topology.

The topology $\mathcal{T}_E$ will be the only one used in Parts 1 and 2: closed sets, open sets, compact sets, continuous maps, compact maps will always be considered with respect to this topology.

A *pair* (X, A) is a couple of subsets of H such that $A \subset X$. A pair (X, A) is said to be *closed* if both X and A are closed in H. A pair (X, A) is called *bounded* if X (and therefore A) is bounded.

By $\Phi : (X, A) \to (Y, B)$ we denote a map from X to Y which takes A into B.

Definition 0.1. A continuous map $\Phi : (X, A) \to (Y, B)$ is an *E-compact morphism* if:

(1) it has the form
$$\Phi(x) = x + K(x)$$
where $K : X \to H$ maps bounded sets into precompact sets;

(2) $\Phi^{-1}(U)$ is bounded for every bounded U.

The next proposition characterizes the linear E-compact morphisms:

Proposition 0.1. *A bounded linear operator $T : H \to H$ is an E-compact morphism if and only if T is one-to-one, has closed rank and has the form*
$$T = I + K$$
where $\pi \circ K$ is compact. If T is invertible, then also its inverse is an E-compact morphism.

Proof. Assume that $T = I + K$ is an E-compact morphism. Since $T^{-1}(0)$ is bounded, T is one-to-one. For the same reason the linear operator $T^{-1} : T(H) \to H$ is bounded, and therefore continuous. Thus $T(H)$ is isomorphic to H and it must be closed. Let B be the unit ball in H. Then $K(B)$ is $\mathcal{T}_E$-precompact. Since $\pi : H \to H/E$ is $\mathcal{T}_E$-continuous, $\pi \circ K(B)$ is precompact.

Now assume that $T = I + K$ is one-to-one, has closed range and $\pi \circ K$ is compact. By the latter assumption, $\pi \circ K$ must be $\mathcal{T}_E$-continuous. Therefore also K, and thus T, are $\mathcal{T}_E$-continuous. By the open map theorem, $T^{-1} : T(H) \to H$ is bounded, and therefore $T^{-1}(U)$ is bounded for every bounded U.

If T is invertible, then its inverse must have the form $T^{-1} = I + H$ where $H = -K \circ T^{-1}$. Therefore $\pi \circ H = -\pi \circ K \circ T^{-1}$ is compact. □

Of course, if T is considered as a map from a bounded domain, one does not need the conditions on the rank and the injectivity.

Now consider the product space $H \times [0,1]$, endowed with the product topology. A subset Γ of $H \times [0,1]$ is called *bounded* if $\{x \mid (x,t) \in \Gamma$ for some $t \in [0,1]\}$ is bounded in H.

Definition 0.2. A continuous map $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ is an *E-compact homotopy* if:

(1) it has the form

$$\Psi_t(x) = \Psi(x,t) = x + K(x,t)$$

where $K : X \times [0,1] \to H$ maps bounded sets into precompact sets;

(2) $\Psi^{-1}(U)$ is bounded for every bounded U.

Two E-compact morphisms Φ_0 and Φ_1 from (X,A) to (Y,B) are called *E-compactly homotopic* if there exists an E-compact homotopy $\Psi : (X \times [0,1], A \times [0,1]) \to (Y,B)$ such that $\Psi(\,\cdot\,,0) = \Phi_0$ and $\Psi(\,\cdot\,,1) = \Phi_1$.

Definition 0.3. A closed set $X \subset H$ is *E-locally compact* if $X \cap \pi^{-1}(\alpha)$ is locally compact for every finite-dimensional linear subspace α of H/E.

For example, every closed bounded set is E-locally compact, because H induces the weak topology on $\pi^{-1}(\alpha)$. If E is infinite-dimensional, then H is not E-locally compact.

Definition 0.4. A pair (X,A) of subsets of H is an *E-pair* if it is closed and X is E-locally compact.

The aim of this first part is to construct a generalized cohomology theory which acts on the E-pairs and on the E-compact morphisms between them. More precisely, we will prove the following:

Theorem 0.2. *Let $\mathcal{A}$ be a ring. There exists a generalized cohomology theory H_E^*, with coefficients in $\mathcal{A}$, such that:*

(1) (Contravariant functoriality) *If $I : (X,A) \to (X,A)$ is the identity map, then $H_E^*(I)$ is the identity homomorphism on $H_E^*(X,A)$. If $\Phi : (X,A) \to (Y,B)$ and $\Phi' : (Y,B) \to (Z,C)$ are E-compact morphisms, then $H_E^*(\Phi' \circ \Phi) = H_E^*(\Phi) \circ H_E^*(\Phi')$.*

(2) (Homotopy invariance) *If two E-compact morphisms Φ and Φ' are E-compactly homotopic, then $H_E^*(\Phi) = H_E^*(\Phi')$.*

(3) (Strong excision) *If X and Y are closed E-locally compact subsets of H and $i : (X, X \cap Y) \hookrightarrow (X \cup Y, Y)$ is the inclusion map, then $H_E^*(i)$ is an isomorphism.*

(4) (Naturality of the coboundary) *For each E-pair (X,A) there exists a coboundary homomorphism $\delta_E^q(X,A) : H_E^q(A) \to H_E^{q+1}(X,A)$. If Φ :*

$(X,A) \to (Y,B)$ is an E-compact morphism, then the following diagram commutes:

$$\begin{array}{ccc} H^q_E(B) & \xrightarrow{H^q_E(\Phi|_A)} & H^q_E(A) \\ {\scriptstyle \delta^q_E(Y,B)}\downarrow & & \downarrow{\scriptstyle \delta^q_E(X,A)} \\ H^{q+1}_E(Y,B) & \xrightarrow{H^{q+1}_E(\Phi)} & H^{q+1}_E(X,A) \end{array}$$

(5) (Long exact sequence) *Given an E-pair (X,A) in H, let $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ be the inclusion maps. Then the following sequence of homomorphisms is exact:*

$$\ldots \to H^q_E(X) \xrightarrow{H^q_E(i)} H^q_E(A) \xrightarrow{\delta^q_E} H^{q+1}_E(X,A) \xrightarrow{H^{q+1}_E(j)} H^{q+1}_E(X) \to \ldots$$

(6) (Dimension property) *If F is any closed linear complement of E and S is the unit sphere in F, then*

$$H^q_E(S) = 0 \quad \forall q \in \mathbb{Z},\ q \neq -1, \quad H^{-1}_E(S) = \mathcal{A}.$$

The theory H^*_E will be called *E-cohomology.*

Remark 0.1. In Section 3 we will show, as a partial result, that H^*_E is functorial and homotopy invariant with respect to a class of maps called E-radial morphisms. This class is neither larger nor smaller than the class of E-compact morphisms. Functoriality and homotopy invariance with respect to this class will be useful in Part 3.

1. Alexander–Spanier cohomology with compact supports. In this section we review some useful properties of two ordinary cohomology theories. Chapter 6 of [16] is the standard reference for these topics.

By H^* we denote the Alexander–Spanier cohomology theory. By H^*_c we denote the Alexander–Spanier cohomology theory with compact supports. These theories coincide on topological pairs (X,A) such that $X \setminus A$ has compact closure in X.

H^* satisfies the Eilenberg–Steenrod axioms and moreover it has a strong excision property (see the more general Theorem 6.6.5 of [16]):

Proposition 1.1. *Let X and Y be two closed sets in a paracompact Hausdorff space. Let $i : (Y, X \cap Y) \hookrightarrow (X \cup Y, X)$ be the inclusion map. Then*

$$H^*(i) : H^*(X \cup Y, X) \to H^*(Y, X \cap Y)$$

is an isomorphism.

H^*_{c} satisfies the Eilenberg–Steenrod axioms in the following modified form (see Section 6.6 of [16]):

Proposition 1.2. *H^*_{c} acts on arbitrary topological pairs and on proper maps between them. The following properties hold:*

(1) (Contravariant functoriality) *If $I : (X,A) \to (X,A)$ is the identity map, then $H^*_{\mathrm{c}}(I)$ is the identity homomorphism on $H^*_{\mathrm{c}}(X,A)$. If $\Phi : (X,A) \to (Y,B)$ and $\Phi' : (Y,B) \to (Z,C)$ are proper maps, then $H^*_{\mathrm{c}}(\Phi' \circ \Phi) = H^*_{\mathrm{c}}(\Phi) \circ H^*_{\mathrm{c}}(\Phi')$.*

(2) (Homotopy invariance) *If two proper maps Φ and Φ' are homotopic via a proper homotopy, then $H^*_{\mathrm{c}}(\Phi) = H^*_{\mathrm{c}}(\Phi')$.*

(3) (Excision) *Let (X,A) be a topological pair and let U be open in X with $\overline{U} \subset \mathring{A}$. If $i : (X \setminus U, A \setminus U) \hookrightarrow (X,A)$ is the inclusion map, then $H^*_{\mathrm{c}}(i)$ is an isomorphism.*

(4) (Naturality of the coboundary) *If A is closed in X, then there exists a coboundary homomorphism $\delta^q_{\mathrm{c}}(X,A) : H^q_{\mathrm{c}}(A) \to H^{q+1}_{\mathrm{c}}(X,A)$. If $\Phi : (X,A) \to (Y,B)$ is a proper map, then the following diagram commutes:*

$$\begin{array}{ccc} H^q_{\mathrm{c}}(B) & \xrightarrow{H^q_{\mathrm{c}}(\Phi|_A)} & H^q_{\mathrm{c}}(A) \\ {\scriptstyle \delta^q_{\mathrm{c}}(Y,B)}\big\downarrow & & \big\downarrow{\scriptstyle \delta^q_{\mathrm{c}}(X,A)} \\ H^{q+1}_{\mathrm{c}}(Y,B) & \xrightarrow{H^{q+1}_{\mathrm{c}}(\Phi)} & H^{q+1}_{\mathrm{c}}(X,A) \end{array}$$

(5) (Long exact sequence) *Let A be closed in X and let $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ be the inclusion maps. Then the following sequence of homomorphisms is exact:*

$$\ldots \to H^q_{\mathrm{c}}(X) \overset{H^q_{\mathrm{c}}(i)}{\to} H^q_{\mathrm{c}}(A) \overset{\delta^q_{\mathrm{c}}}{\to} H^{q+1}_{\mathrm{c}}(X,A) \overset{H^{q+1}_{\mathrm{c}}(j)}{\to} H^{q+1}_{\mathrm{c}}(X) \to \ldots$$

(6) (Dimension property) *$H^0_c(\{point\}) = \mathcal{A}$ and $H^q_{\mathrm{c}}(\{point\}) = 0$ for $q \neq 0$.*

Moreover, H^*_{c} has the following continuity property:

Proposition 1.3. *Let $\{(X^m, A^m)\}$, $m \in \mathbb{N}$, be a sequence of compact Hausdorff pairs in some space, downward directed by inclusion, and let*

$$X = \bigcap_{m\in\mathbb{N}} X^m, \quad A = \bigcap_{m\in\mathbb{N}} A^m.$$

The inclusion maps $i^m : (X,A) \hookrightarrow (X^m, A^m)$ induce an isomorphism

$$\varinjlim_{m\in\mathbb{N}} H^*_{\mathrm{c}}(i^m) : \varinjlim_{m\in\mathbb{N}} H^*_{\mathrm{c}}(X^m, A^m) \to H^*_{\mathrm{c}}(X,A).$$

In fact, the same property holds for the Alexander–Spanier cohomology theory (Theorem 6.6.6 of [16]), and the two theories coincide on compact pairs.

The following property is peculiar of cohomology theories with compact supports:

Proposition 1.4. *Let X be a paracompact, locally compact Hausdorff space and let $A \subset X$ be closed. Let*

$$\mathcal{S}(X, A) = \{U \mid A \subset U \subset X,\ U \text{ is closed, } \overline{X \setminus U} \text{ is compact}\}.$$

Then $\mathcal{S}(X, A)$ is downward directed by inclusion. The inclusion maps induce an isomorphism

$$\varinjlim_{U \in \mathcal{S}(X,A)} H^*_{\mathrm{c}}(X, U) \cong H^*_{\mathrm{c}}(X, A). \tag{1.1}$$

Proof. Since X is paracompact, for every closed subset U,

$$H^*(X, U) \cong \varinjlim_{U \subset V \text{ open}} H^*(X, V)$$

by Corollary 6.6.3 of [16]. Therefore,

$$\varinjlim_{U \in \mathcal{S}(X,A)} H^*(X, U) \cong \varinjlim_{U \in \mathcal{S}(X,A)} \varinjlim_{U \subset V \text{ open}} H^*(X, V). \tag{1.2}$$

Since X is a locally compact Hausdorff space, by Theorem 6.6.16 of [16],

$$\varinjlim_{\substack{A \subset V \text{ open} \\ X \setminus V \text{ compact}}} H^*(X, V) \cong H^*_{\mathrm{c}}(X, A). \tag{1.3}$$

Now (1.2) and (1.3) imply the assertion if we can prove that for every open neighborhood V of A such that $X \setminus V$ is compact, there exists a set $U \in \mathcal{S}(X, A)$ such that $U \subset V$.

X is locally compact and Hausdorff, $X \setminus V$ is compact and $X \setminus A$ is open. By Theorem 5.18 of [11] there exists an open neighborhood Ω of $X \setminus V$ such that $\Omega \subset X \setminus A$ and $\overline{\Omega}$ is compact. Therefore $U = X \setminus \Omega$ is in $\mathcal{S}(X, A)$ and $U \subset V$.□

With some assumptions on the topological spaces, also H^*_{c} has the strong excision property:

Proposition 1.5. *Let X and Y be two closed subsets of a paracompact, locally compact Hausdorff space. If $i : (Y, X \cap Y) \hookrightarrow (X \cup Y, X)$ is the inclusion map, then $H^*_{\mathrm{c}}(i)$ is an isomorphism.*

Proof. Take $U \in \mathcal{S}(X \cup Y, X)$. Then $U \cap Y$ is in $\mathcal{S}(Y, X \cap Y)$. Let

$$i_U : (Y, U \cap Y) \hookrightarrow (X \cup Y, U)$$

be the inclusion map. Since $X \cup Y = U \cup Y$, $H^*(i_U)$ is an isomorphism, by Proposition 1.1. But $H^*_{\mathrm{c}}(i_U) = H^*_{\mathrm{c}}(i_U)$, because the theories H^* and H^*_{c} coincide on pairs (X, A) where $\overline{X \setminus A}$ is compact.

Notice that the family $\{U \cap Y \mid U \in \mathcal{S}(X \cup Y, X)\}$ is cofinal in $\mathcal{S}(Y, X \cap Y)$. Therefore, by Proposition 1.4,

$$H^*_c(i) = \varinjlim_{U \in \mathcal{S}(X \cup Y, X)} H^*_c(i_U)$$

and it must be an isomorphism, being the direct limit of isomorphisms. □

If a cohomology theory has the strong excision property, then a purely algebraic argument implies the existence of the Mayer–Vietoris homomorphism and the corresponding exact sequence (see Corollary 5.4.9 of [16] and Section IV.1 of [10]):

Proposition 1.6. *Let (X_1, A_1) and (X_2, A_2) be two closed pairs in a paracompact, locally compact, Hausdorff space. Then there exists a homomorphism $\Delta^q : H^q_c(X_1 \cap X_2, A_1 \cap A_2) \to H^{q+1}_c(X_1 \cup X_2, A_1 \cup A_2)$ such that the following sequence is exact:*

$$\begin{aligned} \ldots \to H^q_c(X_1, A_1) \oplus H^q_c(X_2, A_2) \to H^q_c(X_1 \cap X_2, A_1 \cap A_2) \xrightarrow{\Delta^q} \\ \to H^{q+1}_c(X_1 \cup X_2, A_1 \cup A_2) \to H^{q+1}_c(X_1, A_1) \oplus H^{q+1}_c(X_2, A_2) \to \ldots \end{aligned}$$

Moreover, Δ^q is functorial with respect to proper maps.

Finally, the Mayer–Vietoris homomorphism has the following property (see Section IV.1 of [10]):

Proposition 1.7. *If (X_1, A_1) and (X_2, A_2) are two closed pairs in a paracompact locally compact Hausdorff space, then the following diagram is commutative:*

$$\begin{array}{ccc} H^{q-1}_c(A_1 \cap A_2) & \xrightarrow{\delta^q_c} & H^q_c(X_1 \cap X_2, A_1 \cap A_2) \\ {\scriptstyle \Delta^{q-1}}\downarrow & & \downarrow{\scriptstyle \Delta^q} \\ H^q_c(A_1 \cup A_2) & \xrightarrow{\delta^{q+1}_c} & H^{q+1}_c(X_1 \cup X_2, A_1, A_2) \end{array}$$

2. The E-cohomology of an E-pair. The E-locally compact sets have the following property:

Proposition 2.1. *If the closed set $X \subset H$ is E-locally compact, then $X \cap \pi^{-1}(\alpha)$ is paracompact for every finite-dimensional linear subspace α of H/E.*

In fact, the topology of H induces the weak topology on $\pi^{-1}(\alpha)$. Therefore $X \cap \pi^{-1}(\alpha)$ is a countable union of compact sets and, being locally compact, it must be paracompact.

So all the properties of the Alexander–Spanier cohomology theory with compact supports, found in the previous section, hold for $X \cap \pi^{-1}(\alpha)$ whenever X is an E-locally compact closed set.

Let $\mathcal{V}$ be the set of all finite-dimensional linear subspaces of H/E. Then $\mathcal{V}$ is partially ordered by inclusion. Since for each α and β in $\mathcal{V}$ there exists $\gamma \in \mathcal{V}$ such that $\alpha \subset \gamma$ and $\beta \subset \gamma$, $(\mathcal{V}, \subset)$ is a directed set.

The dimension of $\alpha \in \mathcal{V}$ will be denoted by $d(\alpha)$.

Choose arbitrarily an orientation for each $\alpha \in \mathcal{V}$. If $\alpha \subset \beta$ and $d(\beta) = d(\alpha) + 1$, then α divides β in two half spaces. The orientations of α and β allow us to denote them uniquely as β^+_α and β^-_α. They satisfy

$$\beta^+_\alpha \cup \beta^-_\alpha = \beta, \quad \beta^+_\alpha \cap \beta^-_\alpha = \alpha.$$

If $X \subset H$, and α and β are as above, define

$$X_\alpha = X \cap \pi^{-1}(\alpha), \quad X_{\beta^+_\alpha} = X \cap \pi^{-1}(\beta^+_\alpha), \quad X_{\beta^-_\alpha} = X \cap \pi^{-1}(\beta^-_\alpha).$$

As before,

$$X_{\beta^+_\alpha} \cup X_{\beta^-_\alpha} = X_\beta, \quad X_{\beta^+_\alpha} \cap X_{\beta^-_\alpha} = X_\alpha.$$

Let (X, A) be an E-pair in H. By Proposition 1.6, for each $q \in \mathbb{Z}$ we have the relative Mayer–Vietoris homomorphism

$$\Delta^q_{\alpha\beta}(X, A) : H^{q+d(\alpha)}_c(X_\alpha, A_\alpha) \to H^{q+d(\beta)}_c(X_\beta, A_\beta).$$

If $\alpha = \alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_k = \beta$ are in $\mathcal{V}$ and $d(\alpha_{i+1}) = d(\alpha_i) + 1$ for $i = 0, \ldots, k-1$, define

$$\Delta^q_{\alpha\beta}(X, A) = \Delta^q_{\alpha_{k-1}\alpha_k}(X, A) \circ \ldots \circ \Delta^q_{\alpha_0\alpha_1}(X, A).$$

Proposition 2.2. *The definition of $\Delta^q_{\alpha\beta}(X, A)$ does not depend on the choice of α_i, $i = 1, \ldots, k-1$.*

The proof is completely analogous to the proof of Lemma IV.3.1 of [10].

Therefore $\{H^{q+d(\alpha)}_c(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X, A)\}$ is a direct system of $\mathcal{A}$-modules over the directed set $\mathcal{V}$.

Definition 2.1. Let $q \in \mathbb{Z}$; the *E-cohomology module* of the E-pair (X, A) of index q is the direct limit

$$H^q_E(X, A) = \varinjlim_{\alpha \in \mathcal{V}} \{H^{q+d(\alpha)}_c(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X, A)\}.$$

As usual, $H^*_E(X) = H^*_E(X, \emptyset)$.

Example 2.1. Let W be a closed subspace of H such that

$$\dim W \cap E = s < \infty, \quad \operatorname{codim}_H(W + E) = r < \infty.$$

Set

$$S = \{x \in W \mid \|x\| = 1\}.$$

Since $W \cap E$ is finite-dimensional, the topology chosen for H induces the strong topology on W and S is closed.

Let R be an r-dimensional complement of $W+E$ in H. Then π is one-to-one on R and $\varrho=\pi(R)$ has dimension r. Set

$$\mathcal{V}_\varrho=\{\alpha\in\mathcal{V}\mid\varrho\subset\alpha\}.$$

Then $\mathcal{V}_\varrho$ is a cofinal subset of $\mathcal{V}$, meaning that it is a directed subset such that for each $\alpha\in\mathcal{V}$ there exists $\beta\in\mathcal{V}_\varrho$ such that $\alpha\subset\beta$.

Since the limit of a direct system over a directed set is naturally isomorphic to the limit of the same direct system restricted to any cofinal subset, we get

$$H^q_E(S)=\varinjlim_{\alpha\in\mathcal{V}_\varrho}\{H_c^{q+d(\alpha)}(S_\alpha);\Delta^q_{\alpha\beta}(S)\}.$$

Let $\alpha\in\mathcal{V}_\varrho$. It is easy to prove that the restriction of π from $W\cap\pi^{-1}(\alpha)$ to α has kernel $W\cap E$ and image complementary to ϱ in α. Thus

$$\dim W\cap\pi^{-1}(\alpha)=\dim\pi(W\cap\pi^1(\alpha))+\dim W\cap E=d(\alpha)-r+s.$$

Therefore S_α is the unit sphere in a Euclidean space of dimension $d(\alpha)-r+s$. So

$$H_c^{q+d(\alpha)}(S_\alpha)=\begin{cases}\mathcal{A} & \text{if } q+d(\alpha)=d(\alpha)-r+s-1 \text{ or } q+d(\alpha)=0,\\ 0 & \text{otherwise.}\end{cases}$$

Taking the direct limit, $H^q_E(S)=0$ if $q\neq s-r-1$.

If $\alpha\subset\beta$ are in $\mathcal{V}_\varrho$ and $d(\beta)=d(\alpha)+1$, then the Mayer–Vietoris homomorphism $\Delta^{s-r-1}_{\alpha\beta}(S)$ is an isomorphism. Therefore $H^{s-r-1}_E(S)=\mathcal{A}$. To sum up,

$$H^q_E(S)=\begin{cases}\mathcal{A} & \text{if } q=s-r-1,\\ 0 & \text{otherwise.}\end{cases}$$

The above example proves the dimension axiom (6) of Theorem 0.2.

Example 2.2. With W as in the above example, let V be a closed complement of W in H. Denote by $B_W(\varrho)$ the open ball in W of radius ϱ, centered at 0, and by $\partial B_W(\varrho)$ its relative boundary in W. Set

$$Q=Q(\varrho_+,\varrho_-)=\overline{B_V(\varrho_+)}\oplus\overline{B_W(\varrho_-)},\quad \partial_W Q=\overline{B_V(\varrho_+)}\oplus\partial B_W(\varrho_-).$$

Both Q and $\partial_W Q$ are closed. Arguing as in the previous example, it is easy to show that

$$H^q_E(Q,\partial_W Q)=\begin{cases}\mathcal{A} & \text{if } q=s-r,\\ 0 & \text{otherwise.}\end{cases}$$

3. E-radial morphisms and E-finite morphisms

Definition 3.1. A continuous map $\Phi : (X,A) \to (Y,B)$ is an *E-radial morphism* if $\Phi^{-1}(U)$ is bounded for every bounded U and there exists $\alpha_0 \in \mathcal{V}$ such that:

(1) $\Phi(X_\alpha) \subset \pi^{-1}(\alpha)$ for every $\alpha \in \mathcal{V}$ with $\alpha_0 \subset \alpha$;
(2) $\Phi(X_{\beta_\alpha^+}) \subset \pi^{-1}(\beta_\alpha^+)$ and $\Phi(X_{\beta_\alpha^-}) \subset \pi^{-1}(\beta_\alpha^-)$ for each $\alpha \subset \beta \in \mathcal{V}$ with $\alpha_0 \subset \alpha$ and $d(\beta) = d(\alpha) + 1$.

We then say that Φ is an E-radial morphism *with respect to* α_0.

The composition of two E-radial morphisms is again an E-radial morphism. Therefore we have a category whose objects are the E-pairs in H and whose morphisms are the E-radial morphisms. The purpose of this section is to extend the definition of H_E^* to the E-radial morphisms, so to construct a contravariant functor on this category.

Let $\Phi : (X,A) \to (Y,B)$ be an E-radial morphism with respect to α_0 between E-pairs. Set

$$\mathcal{V}_{\alpha_0} = \{\beta \in \mathcal{V} \mid \alpha_0 \subset \beta\}.$$

Let $\beta \in \mathcal{V}_{\alpha_0}$. By property (1) of Definition 3.1, we can define

$$\Phi_\beta = \Phi|_{X_\beta} : (X_\beta, A_\beta) \to (Y_\beta, B_\beta).$$

Let $q \in \mathbb{Z}$. Since (X_β, A_β) is a closed pair, Φ_β is a proper map and thus it induces homomorphisms

$$H_{\mathrm{c}}^{q+d(\beta)}(\Phi_\beta) : H_{\mathrm{c}}^{q+d(\beta)}(Y_\beta, B_\beta) \to H_{\mathrm{c}}^{q+d(\beta)}(X_\beta, A_\beta).$$

Now let $\alpha \subset \beta$ be in $\mathcal{V}_{\alpha_0}$, with $d(\beta) = d(\alpha) + 1$. By property (2) of Definition 3.1, Φ_β maps $X_{\beta_\alpha^+}$ into $Y_{\beta_\alpha^+}$ and $X_{\beta_\alpha^-}$ into $Y_{\beta_\alpha^-}$.

By the functoriality of the Mayer–Vietoris homomorphism, stated in Proposition 1.6, the homomorphisms $\{H_{\mathrm{c}}^{q+d(\beta)}(\Phi_\beta)\}$ form a direct system of homomorphisms from the direct system

$$\{H_{\mathrm{c}}^{q+d(\beta)}(Y_\beta, B_\beta); \Delta_{\alpha\beta}^q(Y,B)\}$$

of $\mathcal{A}$-modules to the direct system

$$\{H_{\mathrm{c}}^{q+d(\beta)}(X_\beta, A_\beta); \Delta_{\alpha\beta}^q(X,A)\}$$

of $\mathcal{A}$-modules over the directed set $\mathcal{V}_{\alpha_0}$. Therefore we can define $H_E^q(\Phi)$ as the direct limit of this system:

$$H_E^q(\Phi) = \varinjlim_{\beta \in \mathcal{V}_{\alpha_0}} H_{\mathrm{c}}^{q+d(\beta)}(\Phi_\beta) : H_E^q(Y,B) \to H_E^q(X,A).$$

We now study the functorial properties of H_E^*.

Proposition 3.1. *Assume that* $\Phi : (X, A) \to (Y, B)$ *and* $\Phi' : (Y, B) \to (Z, C)$ *are* E*-radial morphisms of* E*-pairs. Then:*

(1) *if* $I : (X, A) \to (X, A)$ *is the identity map, then* $H^q_E(I)$ *is the identity homomorphism on* $H^q_E(X, A)$ *for each* $q \in \mathbb{Z}$*;*
(2) $H^*_E(\Phi' \circ \Phi) = H^*_E(\Phi) \circ H^*_E(\Phi')$.

Proof. Assertion (1) is trivial.

Assume that Φ is an E-radial morphism with respect to α_0 and that Φ' is an E-radial morphism with respect to α'_0. Set $\gamma_0 = \alpha_0 + \alpha'_0$. The set $\mathcal{V}_{\gamma_0} = \{\gamma \in \mathcal{V} \mid \gamma_0 \subset \gamma\}$ is a cofinal subset of both $\mathcal{V}_{\alpha_0}$ and $\mathcal{V}_{\alpha'_0}$. If $\gamma \in \mathcal{V}_{\gamma_0}$, then $\Phi(X_\gamma) \subset Y_\gamma$ and $\Phi'(Y_\gamma) \subset Z_\gamma$. Therefore $\Phi' \circ \Phi$ is an E-radial morphism with respect to γ_0 and

$$H^{q+d(\gamma)}_{\mathrm{c}}((\Phi' \circ \Phi)_\gamma) = H^{q+d(\gamma)}_{\mathrm{c}}(\Phi'_\gamma \circ \Phi_\gamma) = H^{q+d(\gamma)}_{\mathrm{c}}(\Phi_\gamma) \circ H^{q+d(\gamma)}_{\mathrm{c}}(\Phi'_\gamma).$$

Taking the direct limit over $\mathcal{V}_{\gamma_0}$, we get (2). □

Definition 3.2. A continuous map $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ is an *E-radial homotopy* if $\Psi^{-1}(U)$ is bounded for every bounded U and there exists $\alpha_0 \in \mathcal{V}$ such that:

(1) $\Psi(X_\alpha \times [0,1]) \subset \pi^{-1}(\alpha)$ for every $\alpha \in \mathcal{V}$ with $\alpha_0 \subset \alpha$;
(2) $\Psi(X_{\beta^+_\alpha} \times [0,1]) \subset \pi^{-1}(\beta^+_\alpha)$ and $\Psi(X_{\beta^-_\alpha} \times [0,1]) \subset \pi^{-1}(\beta^-_\alpha)$ for each $\alpha \subset \beta \in \mathcal{V}$ with $\alpha_0 \subset \alpha$ and $d(\beta) = d(\alpha) + 1$.

Two E-radial morphisms Φ_0 and Φ_1 from (X, A) to (Y, B) are called *E-radially homotopic* if there exists an E-radial homotopy $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ such that $\Psi(\,\cdot\,, 0) = \Phi_0$ and $\Psi(\,\cdot\,, 1) = \Phi_1$.

Proposition 3.2. *If the* E*-radial morphisms* Φ_0, $\Phi_1 : (X, A) \to (Y, B)$ *of* E*-pairs are* E*-radially homotopic, then* $H^*_E(\Phi_0) = H^*_E(\Phi_1)$*.*

Proof. Let $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ be an E-radial homotopy with respect to α_0 such that $\Psi(\,\cdot\,, 0) = \Phi_0$ and $\Psi(\,\cdot\,, 1) = \Phi_1$. Set $\mathcal{V}_{\alpha_0} = \{\beta \in \mathcal{V} \mid \alpha_0 \subset \beta\}$.

If $\beta \in \mathcal{V}_{\alpha_0}$, then $\Psi_{t\beta} = \Psi_t|_{X_\beta}$ maps (X_β, A_β) into (Y_β, B_β) for each $t \in [0,1]$. Then the continuous maps $\Phi_{0\beta}$ and $\Phi_{1\beta}$ are homotopic via a proper homotopy and

$$H^{q+d(\beta)}_{\mathrm{c}}(\Phi_{0\beta}) = H^{q+d(\beta)}_{\mathrm{c}}(\Phi_{1\beta}).$$

Taking the direct limit over $\mathcal{V}_{\alpha_0}$, we find $H^q_E(\Phi_0) = H^q_E(\Phi_1)$. □

An interesting class of E-radial morphisms is formed by the E-finite morphisms:

Definition 3.3. A map $\Phi : (X, A) \to (Y, B)$ is an *E-finite morphism* if it is an E-compact morphism of the form

$$\Phi(x) = x + R(x)$$

where $\pi \circ R(X) \subset \alpha_0$, with α_0 a finite-dimensional linear subspace of H/E.

Definition 3.4. A map $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ is an *E-finite homotopy* if it is an E-compact homotopy of the form

$$\Psi_t(x) = \Psi(x,t) = x + R(x,t)$$

where $\pi \circ R(X \times [0,1]) \subset \alpha_0$, with α_0 a finite-dimensional linear subspace of H/E. Two E-finite morphisms Φ_0 and Φ_1 from (X, A) to (Y, B) are called *E-finitely homotopic* if there exists an E-finite homotopy $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ such that $\Psi(\,\cdot\,, 0) = \Phi_0$ and $\Psi(\,\cdot\,, 1) = \Phi_1$.

Notice that every inclusion map is an E-finite morphism.

We can now prove the strong excision property for H^*_E, which is assertion (3) of Theorem 0.2:

Proposition 3.3. *Let X and Y be two closed E-locally compact sets. Let $i : (X, X \cap Y) \hookrightarrow (X \cup Y, Y)$ be the inclusion map. Then*

$$H^*_E(i) : H^*_E(X \cup Y, Y) \to H^*_E(X, X \cap Y)$$

is an isomorphism.

Proof. By Proposition 2.1, $X_\alpha \cup Y_\alpha$ is a paracompact locally compact Hausdorff space; by Proposition 1.5,

$$H^*_c(i_\alpha) : H^*_c(X_\alpha \cup Y_\alpha, X_\alpha) \to H^*_c(Y_\alpha, X_\alpha \cap Y_\alpha) \tag{3.1}$$

is an isomorphism. Therefore $H^*_E(i)$ is an isomorphism, being the direct limit of isomorphisms. □

4. Approximation features. Arguing as in [10], we will define the homomorphisms induced by an E-compact morphism by approximating it with E-finite morphisms. In order to do this, we must prove some continuity property for H^*_E.

Definition 4.1. Let (X, A) be a bounded closed pair. An *approximating sequence* for (X, A) is a sequence $\{(U^m, V^m)\}$ of bounded closed pairs such that:

(1) $U^m \subset U^n$ and $V^m \subset V^n$ if $n \le m$;
(2) $\bigcap_{m \in \mathbb{N}} U^m = X$ and $\bigcap_{m \in \mathbb{N}} V^m = A$.

If F is a closed complement of E and $B_F(r)$ is the open ball in F, centered at 0, of radius r, then

$$\{(\overline{X + B_F(1/m)}, \overline{A + B_F(1/m)})\}$$

is an approximating sequence for the bounded closed pair (X, A): let $Y^m = X + B_F(1/m)$; let $y \in \bigcap_m Y^m$, $y = x_m + f_m$, with $x_m \in X$ and $f_m \in B_F(1/m)$; since $f_m \to 0$, it follows that $x_m \to y$ and y must belong to X. Therefore $\bigcap_m \overline{X + B_F(1/m)} = \bigcap_m \overline{Y^m} = \overline{\bigcap_m Y^m} = \overline{X} = X$.

Given an approximating sequence $\{(U^m, V^m)\}$ for a bounded closed pair (X, A), consider the inclusion maps

$$i^m : (X, A) \hookrightarrow (U^m, V^m), \quad j^{m,n} : (U^m, V^m) \hookrightarrow (U^n, V^n) \quad \text{if } n \leq m.$$

Since every inclusion map is an E-finite morphism, we can consider the induced homomorphisms

$$\begin{aligned} H_E^*(i^m) &: H_E^*(U^m, V^m) \to H_E^*(X, A), \\ H_E^*(j^{m,n}) &: H_E^*(U^n, V^n) \to H_E^*(U^m, V^m) \quad \text{if } n \leq m. \end{aligned}$$

By the functoriality of H_E^*, $\{H_E^*(i^m)\}$ is a direct system of homomorphisms from the direct system $\{H_E^*(U^m, V^m); H_E^*(j^{m,n})\}$ of groups to the group $H_E^*(X, A)$. The following continuity property holds:

Proposition 4.1. *Let (X, A) be a bounded closed pair. Then the direct limit*

$$\varinjlim_{m\in\mathbb{N}} H_E^*(i^m) : \varinjlim_{m\in\mathbb{N}} \{H_E^*(U^m, V^m); H_E^*(j^{m,n})\} \to H_E^*(X, A)$$

of the above system is an isomorphism.

Proof. Let $\alpha \in \mathcal{V}$ and let $i_\alpha^m : (X_\alpha, A_\alpha) \hookrightarrow (U_\alpha^m, V_\alpha^m)$ and $j_\alpha^{m,n} : (U_\alpha^m, V_\alpha^m) \hookrightarrow (U_\alpha^n, V_\alpha^n)$, for $n \leq m$, be the inclusion maps.

Since α is finite-dimensional, the topology induced on $\pi^{-1}(\alpha)$ coincides with the weak topology. U_α^m, V_α^m, X_α and A_α are weakly closed and bounded and therefore compact. By the continuity property of H_c^* stated in Proposition 1.3, the following homomorphism is an isomorphism:

$$\begin{aligned} &\varinjlim_{m\in\mathbb{N}} H_c^{*+d(\alpha)}(i_\alpha^m) : \\ &\qquad \varinjlim_{m\in\mathbb{N}} \{H_c^{*+d(\alpha)}(U_\alpha^m, V_\alpha^m); H_c^{*+d(\alpha)}(j_\alpha^{m,n})\} \to H_c^{*+d(\alpha)}(X_\alpha, A_\alpha). \end{aligned}$$

By Lemma IV.5.2 of [10], direct limits of abelian groups commute; therefore

$$\varinjlim_{m\in\mathbb{N}} H_E^*(i^m) = \varinjlim_{m\in\mathbb{N}} \varinjlim_{\alpha\in\mathcal{V}} H_c^{*+d(\alpha)}(i_\alpha^m) = \varinjlim_{\alpha\in\mathcal{V}} \varinjlim_{m\in\mathbb{N}} H_c^{*+d(\alpha)}(i_\alpha^m).$$

Thus $\varinjlim_{m\in\mathbb{N}} H_E^*(i^m)$ is an isomorphism, being a direct limit of isomorphisms.□

Now assume that (X, A) is any E-pair and set

$$\mathcal{T}(X, A) = \{S \mid A \subset S \subset X,\ S \text{ is closed, } X \setminus S \text{ is bounded}\}.$$

Then $\mathcal{T}(X, A)$ is downward directed by inclusion. If $i_S : (X, A) \hookrightarrow (X, S)$ is the inclusion map, then $H^*_E(i_S)$ is a direct system of homomorphisms from the direct system $\{H^*_E(X, S)\}$ of groups to $H^*_E(X, A)$, over the directed set $\mathcal{T}(X, A)$.

Proposition 4.2. *Assume that* (X, A) *is an* E*-pair. Then the direct limit*

$$\varinjlim_{S \in \mathcal{T}(X,A)} H^*_E(i_S) : \varinjlim_{S \in \mathcal{T}(X,A)} H^*_E(X, S) \to H^*_E(X, A)$$

of the above system is an isomorphism.

Proof. This follows immediately from Proposition 1.4 and from the possibility of changing the order of two direct limits. □

A pair (X, A) will be called *cobounding* if $X \setminus A$ is bounded.

Definition 4.2. Let (X, A) and (Y, B) be two cobounding closed pairs. Let $\Phi : (X, A) \to (Y, B)$ be an E-compact morphism. An *approximating system* for Φ is given by:

(a) two bounded closed sets U, V such that:
  (a1) $X \setminus A \subset U \subset X$, $Y \setminus B \subset V \subset Y$;
  (a2) $\Phi(U) \subset V$;

(b) an approximating sequence $\{(Y^m, B^m)\}$ for $(V, B \cap V)$; denote by $i^m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$ and $j^{m,n} : (Y^m, B^m) \hookrightarrow (Y^n, B^n)$, for $n \leq m$, the inclusion maps;

(c) a sequence of E-finite morphisms

$$\Phi^m : (U, A \cap U) \to (Y^m, B^m), \quad m \in \mathbb{N},$$

  such that, if $\widetilde{\Phi} = \Phi|_{(U, A\cap U)} : (U, A \cap U) \to (V, B \cap V)$, then:
  (c1) Φ^m is E-compactly homotopic to $i^m \circ \widetilde{\Phi}$;
  (c2) Φ^n is E-finitely homotopic to $j^{m,n} \circ \Phi^m$.

Denote by P_E the orthogonal projection onto E and let $E^\perp$ be the orthogonal complement of E. We need a lemma in order to prove the existence of approximating systems:

Lemma 4.3. *Assume that* Φ *and* Φ' *are* E*-compact morphisms* (*resp.* E*-finite morphisms*) *from* (X, A) *to* (Y, B) *such that:*

(1) $P_E \circ \Phi = P_E \circ \Phi'$;
(2) $\|\Phi(x) - \Phi'(x)\| \leq \mathrm{dist}(\Phi(x), (H \setminus Y) \cap (\Phi(x) + E^\perp))$, $\forall x \in X$;
(3) $\|\Phi(x) - \Phi'(x)\| \leq \mathrm{dist}(\Phi(x), (H \setminus B) \cap (\Phi(x) + E^\perp))$ $\forall x \in A$.

Then Φ and Φ' are E-compactly (resp. E-finitely) homotopic by means of an E-compact (resp. E-finite) homotopy $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ such that

$$P_F \circ \Psi(x,t) = P_F \circ \Phi(x) = P_F \circ \Phi'(x), \quad \forall x \in X, \ \forall t \in [0,1].$$

PROOF. Set

$$\Psi(x,t) = t\Phi'(x) + (1-t)\Phi(x).$$

By hypothesis, if $x \in X$ and $t \in [0,1]$, then $\Psi(x,t) \in \Phi(x) + E^{\perp}$ and

$$\|\Psi(x,t) - \Phi(x)\| = t\|\Phi'(x) - \Phi(x)\| \le t \cdot \operatorname{dist}(\Phi(x), (H \setminus Y) \cap (\Phi(x) + E^{\perp})).$$

Thus $\Psi(x,t)$ must lie in Y. In the same way, $\Psi(x,t) \in B$ if $x \in A$. Therefore Ψ is the required E-compact (resp. E-finite) homotopy between Φ and Φ'. □

PROPOSITION 4.4. *Let (X, A) and (Y, B) be two cobounding closed pairs. Let*

$$\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$$

be an E-compact homotopy. Let U, V be two bounded closed sets such that:

(1) *$X \setminus A \subset U \subset X$ and $Y \setminus B \subset V \subset Y$;*
(2) *$\Psi(U \times [0,1]) \subset V$.*

Then for each $m \in \mathbb{N}$ there exists an E-finite homotopy

$$\Psi^m : (U \times [0,1], (A \cap U) \times [0,1]) \to (Y^m, B^m),$$
$$Y^m = \overline{V + B_{E^{\perp}}(1/m)}, \quad B^m = \overline{(B \cap V) + B_{E^{\perp}}(1/m)},$$

such that $\{\Psi^m(\,\cdot\,, t)\}$ is an approximating system for $\Psi(\,\cdot\,, t)$ for every $t \in [0,1]$. Moreover,

$$\begin{aligned} &\|\Psi^m(x,t) - \Psi(x,t)\| < 1/(3m), \quad \forall (x,t) \in U \times [0,1], \\ &P_E \circ \Psi^m(x,t) = P_E \circ \Psi(x,t), \qquad \forall (x,t) \in U \times [0,1]. \end{aligned} \tag{4.1}$$

PROOF. Assume that $\Psi(x,t) = x + K(x,t)$. Set $K_E = P_E \circ K$ and $K_{E^{\perp}} = P_{E^{\perp}} \circ K$, so that

$$K(x,t) = K_E(x,t) + K_{E^{\perp}}(x,t).$$

Remember that $\mathcal{T}_E$ induces the strong topology on $E^{\perp}$ and that $K_{E^{\perp}}$ is a compact map.

We are going to show that there exists a sequence of continuous maps $K^m_{E^{\perp}} : U \times [0,1] \to W_m$, where W_m is a finite-dimensional linear subspace of $E^{\perp}$, such that

$$\|K^m_{E^{\perp}}(x,t) - K_{E^{\perp}}(x,t)\| < 1/(3m), \quad \forall (x,t) \in U \times [0,1], \tag{4.2}$$

Since $K_{E^\perp}(U \times [0,1])$ is precompact in the complete metric space $E^\perp$, it is totally bounded and there exist $x_1, \ldots, x_{N(m)}$ in $E^\perp$ such that

$$K_{E^\perp}(U \times [0,1]) \subset \bigcup_{i=1}^{N(m)} B_{E^\perp}(x_i, 1/(3m)). \tag{4.3}$$

Let W_m be the linear space generated by $\{x_1, \ldots, x_{N(m)}\}$. Let $P_m : E \to W_m$ be the orthogonal projection onto W_m. Set

$$K^m_{E^\perp}(x,t) = P_m \circ K_E(x,t), \quad K^m(x,t) = K_E(x,t) + K^m_{E^\perp}(x,t).$$

Let $(x,t) \in U \times [0,1]$. By (4.3) there exists $j \le N(m)$ such that $K_{E^\perp}(x,t) \in B_{E^\perp}(x_j, 1/(3m))$. Therefore,

$$\|P_m \circ K_{E^\perp}(x,t) - K_{E^\perp}(x,t)\| \le \|K_{E^\perp}(x,t) - x_j\| < 1/(3m)$$

and $K^m_{E^\perp}$ satisfies (4.2).

Now set

$$\Psi^m(x,t) = x + K^m(x,t)$$

By (4.2), Ψ^m maps $(U \times [0,1], (A \cap U) \times [0,1])$ into (Y^m, B^m). Moreover, we have $\pi \circ K^m(U \times [0,1]) \subset \pi(W_m)$ and Ψ^m is an E-finite morphism. By (4.2),

$$\begin{aligned}
&\|\Psi^m(x,t) - \Psi(x,t)\| < 1/m \le \operatorname{dist}(\Psi(x,t), (H \setminus Y^m) \cap (\Psi(x,t) + E^\perp)), \\
&\hspace{20em} \forall (x,t) \in U \times [0,1], \\
&\|\Psi^m(x,t) - \Psi(x,t)\| < 1/m \le \operatorname{dist}(\Psi(x,t), (H \setminus B^m) \cap (\Psi(x,t) + E^\perp)), \\
&\hspace{17em} \forall (x,t) \in (A \cap U) \times [0,1].
\end{aligned}$$

Set

$$\widetilde{\Psi} = \Psi|_{(U \times [0,1], A \cap U \times [0,1])} : (U \times [0,1], (A \cap U) \times [0,1]) \to (V, B \cap V).$$

By Lemma 4.3, $\Psi^m(\,\cdot\,, t)$ and $i^m \circ \widetilde{\Psi}(\,\cdot\,, t)$ are E-compactly homotopic. Again, by (4.2), if $n < m$ then

$$\begin{aligned}
&\|\Psi^m(x,t) - \Psi^n(x,t)\| \\
&\qquad < \frac{1}{3m} + \frac{1}{3n} \le \frac{1}{n} - \frac{1}{3m} \\
&\qquad \le \operatorname{dist}(\Psi^m(x,t), (H \setminus Y^n) \cap (\Psi^m(x,t) + E^\perp)), \quad \forall (x,t) \in U \times [0,1], \\
&\|\Psi^m(x,t) - \Psi^n(x,t)\| \\
&\quad < \frac{1}{3m} + \frac{1}{3n} \le \frac{1}{n} - \frac{1}{3m} \\
&\quad \le \operatorname{dist}(\Psi^m(x,t), (H \setminus B^n) \cap (\Psi^m(x,t) + E^\perp)), \quad \forall (x,t) \in (A \cap U) \times [0,1].
\end{aligned}$$

By Lemma 4.3, Ψ^n and $j^{m,n} \circ \Psi^m$ are E-finitely homotopic. Therefore $\{\Psi^m(\,\cdot\,, t)\}$ is an approximating system for $\Psi(\,\cdot\,, t)$. $\square$

Corollary 4.5. *Let (X, A) and (Y, B) be cobounding closed pairs. Let $\Phi : (X, A) \to (Y, B)$ be an E-compact morphism. Then there exists an approximating system for Φ.*

Proof. Let $U = \overline{X \setminus A}$. Since Φ maps bounded closed sets into bounded closed sets, $V = \Phi(U) \cup \overline{Y \setminus B}$ is bounded and closed. Apply Proposition 4.4 with these U, V and with $\Psi(\,\cdot\,, t) = \Phi(\,\cdot\,)$ for each $t \in [0, 1]$. □

5. E-compact morphisms. Let (X, A) and (Y, B) be two cobounding E-pairs. Let $\Phi : (X, A) \to (Y, B)$ be an E-compact morphism. Consider an approximating system $\{\Phi^m : (U, A \cap U) \to (Y^m, B^m)\}$ for Φ (notations as in Definition 4.2).

Since Φ^m is an E-finite morphism, we can consider the induced homomorphism

$$H^*_E(\Phi^m) : H^*_E(Y^m, B^m) \to H^*_E(U, A \cap U).$$

By property (c2) of Definition 4.2, $\{H^*_E(\Phi^m)\}$ is a direct system of homomorphisms from the direct system $\{H^*_E(Y^m, B^m); H^*_E(j^{m,n})\}$ of $\mathcal{A}$-modules to the $\mathcal{A}$-module $H^*_E(U, A \cap U)$. Consider the direct limit of this system:

$$\varinjlim_{m\in\mathbb{N}} H^*_E(\Phi^m) : \varinjlim_{m\in\mathbb{N}} \{H^*_E(Y^m, B^m); H^*_E(j^{m,n})\} \to H^*_E(U, A \cap U).$$

Since $(V, B \cap V)$ is a bounded closed pair, the domain of this homomorphism is isomorphic to $H^*_E(V, B \cap V)$, via the isomorphism

$$\varinjlim_{m\in\mathbb{N}} H^*_E(j_m) : \varinjlim_{m\in\mathbb{N}} H^*_E(Y^m, B^m) \to H^*_E(V, B \cap V)$$

where $j_m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$ are the inclusion maps.

Since X and Y are E-locally compact, the strong excision property stated in Proposition 3.3 holds and the inclusion maps induce isomorphisms

$$H^*_E(X, A) \cong H^*_E(U, A \cap U), \quad H^*_E(Y, B) \cong H^*_E(V, B \cap V).$$

Therefore we can define the homomorphism

$$H^*_E(\Phi) : H^*_E(Y, B) \to H^*_E(X, A)$$

as

$$H^*_E(\Phi) = H^*_E(i)^{-1} \circ \varinjlim_{m\in\mathbb{N}} H^*_E(\Phi^m) \circ [\varinjlim_{m\in\mathbb{N}} H^*_E(j_m)]^{-1} \circ H^*_E(j)$$

where $i : (U, A \cap U) \hookrightarrow (X, A)$ and $j : (V, B \cap V) \hookrightarrow (Y, B)$ are the inclusion maps.

In order to prove that this is a good definition, we must check that it does not depend on the choice of the approximating system for Φ.

Proposition 5.1. *The definition of $H^q_E(\Phi)$ does not depend on the choice of the approximating system for Φ.*

Proof. Consider two approximating systems for Φ.

1. First we assume that the approximating systems share the same U and V, and the same approximating sequence $\{(Y^m, B^m)\}$ of $(V, B \cap V)$:

$$\Phi^m : (U, A \cap U) \to (Y^m, B^m), \quad \Phi'^m : (U, A \cap U) \to (Y^m, B^m).$$

Let $\Psi^m : (U \times [0,1], (A \cap U) \times [0,1]) \to (Y^m, B^m)$ be an E-compact homotopy between Φ^m and Φ'^m. For $r \in \mathbb{N}$ let

$$Z^m_r = \overline{Y^m + B_{E^\perp}(1/r)}, \quad W^m_r = \overline{B^m + B_{E^\perp}(1/r)}.$$

Apply Proposition 4.4 to find an E-finite homotopy

$$\Psi^m_r : (U \times [0,1], (A \cap U) \times [0,1]) \to (Z^m_r, W^m_r)$$

such that $\{\Psi^m_r(\,\cdot\,, t)\}_{r \in \mathbb{N}}$ is an approximating system for $\Psi^m(\,\cdot\,, t)$ for each $t \in [0,1]$, and

$$\|\Psi^m_r(x,t) - \Psi^m(x,t)\| < 1/(3r), \quad \forall (x,t) \in U \times [0,1]. \tag{5.1}$$

Set $\widetilde{Z}^m_r = \overline{Z^m_r + B_{E^\perp}(1/r)}$, $\widetilde{W}^m_r = \overline{W^m_r + B_{E^\perp}(1/r)}$ and let

$$h^m_r : (Z^m_r, W^m_r) \hookrightarrow (\widetilde{Z}^m_r, \widetilde{W}^m_r), \quad f^m_r : (F^m, G^m) \hookrightarrow (\widetilde{Z}^m_r, \widetilde{W}^m_r)$$

be the inclusion maps.

By (5.1) we can apply Lemma 4.3: the E-finite morphisms $f^m_r \circ \Phi^m$ and $h^m_r \circ \Psi^m_r(\,\cdot\,, 0)$ from $(U, A \cap U)$ to $(\widetilde{Z}^m_r, \widetilde{W}^m_r)$ are E-finitely homotopic. For the same reason $f^m_r \circ \Phi'^m$ and $h^m_r \circ \Psi^m_r(\,\cdot\,, 1)$ are E-finitely homotopic. Therefore,

$$\begin{aligned} H^*_E(\Phi^m) \circ H^*_E(f^m_r) &= H^*_E(f^m_r \circ \Phi^m) \\ &= H^*_E(h^m_r \circ \Psi^m_r(\,\cdot\,, 0)) = H^*_E(h^m_r \circ \Psi^m_r(\,\cdot\,, 1)) \\ &= H^*_E(f^m_r \circ \Phi'^m) = H^*_E(\Phi'^m) \circ H^*_E(f^m_r). \end{aligned}$$

By Proposition 4.1, $\varinjlim_{r \in \mathbb{N}} H^*_E(f^m_r)$ is an isomorphism and therefore

$$H^*_E(\Phi^m) = H^*_E(\Phi'^m), \quad \forall m \in \mathbb{N}.$$

2. Now we only assume that U and V are the same for the two approximating systems. Thus we have two approximating sequences $\{(Y^m, B^m)\}$ and $\{(Y'^m, B'^m)\}$ for $(V, B \cap V)$, and two sequences of E-finite morphisms:

$$\Phi^m : (U, A \cap U) \to (Y^m, B^m), \quad \Phi'^m : (U, A \cap U) \to (Y'^m, B'^m).$$

Denote by

$$i^m : (V, B \cap V) \hookrightarrow (Y^m, B^m), \quad i'^m : (V, B \cap V) \hookrightarrow (Y'^m, B'^m)$$

the inclusion maps. The sequence

$$(\widetilde{Y}^m, \widetilde{B}^m) = (Y^m \cup Y'^m, B^m \cup B'^m), \quad m \in \mathbb{N},$$

is an approximating sequence for $(V, B \cap V)$. Let

$$u_m : (Y^m, B^m) \hookrightarrow (\widetilde{Y}^m, \widetilde{B}^m), \quad u'_m : (Y'^m, B'^m) \hookrightarrow (\widetilde{Y}^m, \widetilde{B}^m)$$
$$z_m : (V, B \cap V) \hookrightarrow (\widetilde{Y}^m, \widetilde{B}^m)$$

denote the inclusion maps. By our previous argument,

$$H^*_E(\Phi^m) \circ H^*_E(u_m) = H^*_E(u_m \circ \Phi^m) = H^*_E(u'_m \circ \Phi'^m) = H^*_E(\Phi'^m) \circ H^*_E(u'_m).$$

Moreover,

$$H^*_E(i^m) \circ H^*_E(u_m) = H^*_E(z_m) = H^*_E(i'^m) \circ H^*_E(u'_m).$$

By Proposition 4.1,

$$\varinjlim_{m\in\mathbb{N}} H^*_E(u_m) = \varinjlim_{m\in\mathbb{N}} H^*_E(u'_m) = \mathrm{Id}_{H^*_E(V, B\cap V)}\,.$$

Therefore,

$$\varinjlim_{m\in\mathbb{N}} H^*_E(\Phi^m) = \varinjlim_{m\in\mathbb{N}} H^*_E(\Phi'^m).$$

3. Finally, consider the general case. The smallest U and V which can be chosen are

$$U = \overline{X \setminus A}, \quad V = \Phi(U) \cup \overline{Y \setminus B}.$$

Let U' and V' be another choice. Apply twice Proposition 4.4 to find an approximating sequence $\{(Y^m, B^m)\}$ for $(V, B \cap V)$, an approximating sequence $\{(Y'^m, B'^m)\}$ for $(V', B \cap V')$ and two sequences of E-finite morphisms

$$\Phi^m : (U, A \cap U) \to (Y^m, B^m), \quad \Phi'^m : (U', B \cap V') \to (Y'^m, B'^m)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
(U, A\cap U) & \xrightarrow{\Phi^m} & (Y^m, B^m) \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j^m} \\
(U', A\cap U') & \xrightarrow{\Phi'^m} & (Y'^m, B'^m)
\end{array}$$

where i and j^m are the inclusion mappings. Taking the limit over $\mathbb{N}$ and then using the strong excision property given by Proposition 3.3, we conclude the proof. $\square$

Thus we have defined $H^*_E(\Phi)$ for every E-compact morphism $\Phi : (X, A) \to (Y, B)$ where (X, A) and (Y, B) are cobounding E-pairs. H^*_E is invariant under E-compact homotopies:

PROPOSITION 5.2. *Assume that (X, A) and (Y, B) are cobounding E-pairs. If the E-compact morphisms Φ_0 and Φ_1 from (X, A) to (Y, B) are E-compactly homotopic, then $H^*_E(\Phi_0) = H^*_E(\Phi_1)$.*

PROOF. Let $\Psi : (X \times [0,1], A \times [0,1]) \to (Y, B)$ be an E-compact homotopy between Φ_0 and Φ_1. Choose $U = \overline{X \setminus A}$ and $V = \Psi(U \times [0,1]) \cup \overline{Y \setminus B}$. By Proposition 4.4 we can find a sequence

$$\Psi^m : (U \times [0,1], (A \cap U) \times [0,1]) \to (Y^m, B^m)$$

of E-finite homotopies such that $\Psi^m(\,\cdot\,, 0)$ is an approximating system for Φ_0 and $\Psi^m(\,\cdot\,, 1)$ is an approximating system for Φ_1. By Proposition 3.2, $H^*_E(\Psi^m(\,\cdot\,, 0)) = H^*_E(\Psi^m(\,\cdot\,, 1))$. Taking the direct limit over $\mathbb{N}$ we find $H^*_E(\Phi_0) = H^*_E(\Phi_1)$. $\square$

Proving the functoriality of H^*_E is a bit more difficult:

PROPOSITION 5.3. *Assume that (X, A), (Y, B) and (Z, C) are cobounding E-pairs. Assume that $\Phi : (X, A) \to (Y, B)$ and $\Psi : (Y, B) \to (Z, C)$ are E-compact morphisms. Then:*

1. *if $I : (X, A) \to (X, A)$ is the identity map, then $H^*_E(I)$ is the identity homomorphism on $H^*_E(X, A)$;*
2. *$H^*_E(\Psi \circ \Phi) = H^*_E(\Phi) \circ H^*_E(\Psi)$.*

PROOF. (1) is trivial. We prove (2). Set $U = \overline{X \setminus A}$, $V = \Phi(U) \cup \overline{Y \setminus B}$ and $W = \Psi(V) \cup \overline{Z \setminus C}$. Set

$$\widetilde{\Phi} = \Phi|_{(U, A \cap U)} : (U, A \cap U) \to (V, B \cap V),$$
$$\widetilde{\Psi} = \Psi|_{(V, B \cap V)} : (V, B \cap V) \to (W, C \cap W).$$

By our excision argument, we must show that

$$H^*_E(\widetilde{\Psi} \circ \widetilde{\Phi}) = H^*_E(\widetilde{\Phi}) \circ H^*_E(\widetilde{\Psi}).$$

1. First we assume that $\widetilde{\Psi}$ is an E-finite morphism:

$$\widetilde{\Psi}(x) = x + \widetilde{R}(x)$$

where $\widetilde{R} : V \to \pi^{-1}(\alpha_0)$ is continuous and has precompact image.

By Dugundji's generalization of Tietze's Theorem [8] we can find a continuous extension $\overline{R} : H \to \pi^{-1}(\alpha_0)$ of $\widetilde{R}$ such that $\overline{R}(H) \subset \overline{\mathrm{conv}}(\widetilde{R}(V))$, and thus $\overline{R}(H)$ is precompact. Therefore $\overline{\Psi}(x) = x + \overline{R}(x)$ is an E-finite morphism on H.

Let $\Phi^m : (U, A \cap U) \to (Y^m, B^m)$ be an approximating system for Φ. Set

$$Z^m = \overline{\Psi}(Y^m) \cup W, \quad C^m = \overline{\Psi}(B^m) \cup (C \cap W)$$

Then $\{(Z^m, C^m)\}$ is an approximating sequence for $(W, C \cap W)$: if $z \in \bigcap_m Z^m \setminus W$ then $z = \overline{\Psi}(y_m)$ for some sequence $y_m \in Y^m$. Since $\overline{\Psi}$ is proper, there exists a subsequence y_{m_k} converging to some $y \in \bigcap_k Y^{m_k} = V$. Therefore $\overline{\Psi}(y) = z$ and $\bigcap_m Z^m = W$. With the same argument we find that $\bigcap_m C^m = C$.

Let $i^m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$ and $j^m : (W, C \cap W) \hookrightarrow (Z^m, C^m)$ be the inclusion maps. Set

$$\Psi^m = j^m \circ \widetilde{\Psi} : (V, B \cap V) \to (Z^m, C^m),$$
$$\overline{\Psi}^m = \overline{\Psi}|_{Y^m, B^m} : (Y^m, B^m) \to (Z^m, C^m).$$

Since $\Psi^m \circ \widetilde{\Phi}$ is E-compactly homotopic to $\overline{\Psi}^m \circ \Phi^m$, by Proposition 5.2,

$$H_E^*(\Psi^m \circ \widetilde{\Phi}) = H_E^*(\overline{\Psi}^m \circ \Phi^m) = H_E^*(\Phi^m) \circ H_E^*(\overline{\Psi}^m).$$

Therefore, by Proposition 4.1,

$$\begin{aligned}\varinjlim_{m\in\mathbb{N}} H_E^*(\Psi^m \circ \widetilde{\Phi}) &= \varinjlim_{m\in\mathbb{N}} H_E^*(\Phi^m) \circ \varinjlim_{m\in\mathbb{N}} H_E^*(\overline{\Psi}^m) \\ &= \varinjlim_{m\in\mathbb{N}} H_E^*(\Phi^m) \circ [\varinjlim_{m\in\mathbb{N}} H_E^*(i^m)]^{-1} \circ \varinjlim_{m\in\mathbb{N}} H_E^*(\Psi^m).\end{aligned}$$

Since $\{\Psi^m \circ \widetilde{\Phi}\}$ is an approximating system for $\widetilde{\Psi} \circ \widetilde{\Phi}$,

$$\begin{aligned}H_E^*(\widetilde{\Psi} \circ \widetilde{\Phi}) &= \varinjlim_{m\in\mathbb{N}} H_E^*(\Psi^m \circ \widetilde{\Phi}) \circ [\varinjlim_{m\in\mathbb{N}} H_E^*(j^m)]^{-1} \\ &= H_E^*(\widetilde{\Phi}) \circ \varinjlim_{m\in\mathbb{N}} H_E^*(\Psi^m) \circ [\varinjlim_{m\in\mathbb{N}} H_E^*(j^m)]^{-1} = H_E^*(\widetilde{\Phi}) \circ H_E^*(\widetilde{\Psi}).\end{aligned}$$

2. We now pass to the general case. Let

$$\Psi^m : (V, B \cap V) \to (Z^m, C^m), \quad \Theta^m : (U, A \cap U) \to (Z^m, C^m)$$

be two approximating systems for $\widetilde{\Psi}$ and $\widetilde{\Psi} \circ \widetilde{\Phi}$, constructed as in Proposition 4.4. By 4.1 we can apply Lemma 4.3: $\Psi^m \circ \widetilde{\Phi}$ and Θ^m are E-compactly homotopic. By Proposition 5.2, $H_E^*(\Psi^m \circ \widetilde{\Phi}) = H_E^*(\Theta^m)$. By the first part of this proof, $H_E^*(\Psi^m \circ \widetilde{\Phi}) = H_E^*(\widetilde{\Phi}) \circ H_E^*(\Psi^m)$. Taking the direct limit over $m \in \mathbb{N}$ we conclude the proof. □

Now let (X, A) and (Y, B) be arbitrary E-pairs. Let $\Phi : (X, A) \to (Y, B)$ be an E-compact morphism. Recall that

$$\mathcal{T}(X, A) = \{S \mid A \subset S \subset X,\ S \text{ is closed, } X \setminus S \text{ is bounded}\}.$$

For each $T \in \mathcal{T}(Y, B)$, $\Phi^{-1}(T)$ is in $\mathcal{T}(X, A)$: in fact, $A \subset \Phi^{-1}(B) \subset \Phi^{-1}(T)$ and $X \setminus \Phi^{-1}(T) = \Phi^{-1}(Y \setminus T)$ is bounded by property (2) of Definition 0.1.

Let $i_T : (X, A) \hookrightarrow (X, \Phi^{-1}(T))$ be the inclusion map. Let Φ_T be Φ seen as a map from $(X, \Phi^{-1}(T))$ to (Y, T). Then we have the homomorphisms

$$H_E^*(Y, T) \overset{H_E^*(\Phi_T)}{\longrightarrow} H_E^*(X, \Phi^{-1}(T)) \overset{H_E^*(i_T)}{\longrightarrow} H_E^*(X, A).$$

Taking the direct limit over $\mathcal{T}(Y,B)$ and using Proposition 4.2, we can define

$$H_E^*(\Phi) = \varinjlim_{T\in\mathcal{T}(Y,B)} H_E^*(i_T)\circ H_E^*(\Phi_T) : H_E^*(Y,B)\to H_E^*(X,A).$$

The following proposition proves assertion (2) of Theorem 0.2:

PROPOSITION 5.4. *Assume that (X,A) and (Y,B) are E-pairs. If the E-compact morphisms Φ_0 and Φ_1 from (X,A) to (Y,B) are E-compactly homotopic, then $H_E^*(\Phi_0)=H_E^*(\Phi_1)$.*

PROOF. Let Φ_t, $t\in[0,1]$, be an E-compact homotopy between Φ_0 and Φ_1. For $T\in\mathcal{T}(Y,B)$ set

$$S = \bigcap_{t\in[0,1]} \Phi_t^{-1}(T).$$

Since $A\subset\Phi_t^{-1}(B)\subset\Phi_t^{-1}(T)$ for every t, A is a subset of S. Moreover,

$$X\setminus S = \bigcup_{t\in[0,1]} \Phi_t^{-1}(X\setminus T)$$

is bounded by condition (2) in the definition of E-compact homotopy (see Definition 0.2). Therefore $S\in\mathcal{T}(X,A)$.

Let $\widetilde{\Phi}_{tT}$ be Φ_t seen as a map from (X,S) to (Y,T) and let

$$i_T^t : (X,A)\hookrightarrow(X,\Phi_t^{-1}(T)),\quad j_T^t : (X,A)\hookrightarrow(X,\Phi_t^{-1}(T)),$$
$$k_T : (X,A)\hookrightarrow(X,S)$$

be the inclusion maps. By Proposition 5.3, $H_E^*(\widetilde{\Phi}_{tT})=H_E^*(j_T^t)\circ H_E^*(\Phi_{tT})$. By Proposition 3.1, $H_E^*(i_T^t)=H_E^*(k_T)\circ H_E^*(j_T^t)$. Therefore the following diagram commutes, for every $t\in[0,1]$:

$$\begin{array}{ccc}
H_E^*(Y,T) & \xrightarrow{H_E^*(\widetilde{\Phi}_{tT})} & H_E^*(X,S)\\
{\scriptstyle H_E^*(\Phi_{tT})}\downarrow & & \downarrow{\scriptstyle H_E^*(k_T)}\\
H_E^*(X,\Phi_t^{-1}(T)) & \xrightarrow{H_E^*(i_T^t)} & H_E^*(X,A)
\end{array}$$

By Proposition 5.2, the assertion is true for E-compact homotopies between cobounding pairs and therefore $H_E^*(\widetilde{\Phi}_{0T})=H_E^*(\widetilde{\Phi}_{1T})$. Thus

$$H_E^*(i_T^0)\circ H_E^*(\Phi_{0T}) = H_E^*(i_T^1)\circ H_E^*(\Phi_{1T})$$

and the assertion follows. □

Now it is easy to prove assertion (1) of Theorem 0.2:

Proposition 5.5. *Assume that (X,A), (Y,B) and (Z,C) are E-pairs. Assume that $\Phi : (X,A) \to (Y,B)$ and $\Psi : (Y,B) \to (Z,C)$ are E-compact morphisms. Then:*

(1) *if $I : (X,A) \to (X,A)$ is the identity map, then $H^*_E(I)$ is the identity homomorphism on $H^*_E(X,A)$;*
(2) $H^*_E(\Psi \circ \Phi) = H^*_E(\Phi) \circ H^*_E(\Psi)$.

Proof. (1) is trivial. We prove (2).

If $W \in \mathcal{T}(Z,C)$, then $\Psi^{-1}(W)$ is in $\mathcal{T}(Y,B)$ and $\Phi^{-1}(\Psi^{-1}(W))$ is in $\mathcal{T}(X,A)$. By the definition of direct limit the following diagram commutes:

$$\begin{tikzcd}
H^*_E(Z,W) \arrow[r, "H^*_E(\Psi_W)"] \arrow[d] & H^*_E(Y,\Psi^{-1}(W)) \arrow[r, "H^*_E(\Phi_{\Psi^{-1}(W)})"] \arrow[d] & H^*_E(\Phi^{-1}(\Psi^{-1}(W))) \arrow[d] \\
H^*_E(Z,C) \arrow[r, "H^*_E(\Psi)"] & H^*_E(Y,B) \arrow[r, "H^*_E(\Phi)"] & H^*_E(X,A)
\end{tikzcd}$$

where the vertical arrows are induced by the inclusion maps. By Proposition 5.3, the functoriality holds for E-compact morphisms between cobounding pairs and therefore the composition of the upper homomorphisms in the diagram equals $H^*_E((\Psi \circ \Phi)_W)$. By the unicity property of direct limits, $H^*_E(\Psi \circ \Phi) = H^*_E(\Phi) \circ H^*_E(\Psi)$. □

6. The E-coboundary homomorphism. Let (X,A) be an E-pair. Since A is closed in X, for each $\alpha \in \mathcal{V}$ we have the coboundary homomorphism

$$\delta_{\mathrm{c}}^{q+d(\alpha)} : H_{\mathrm{c}}^{q+d(\alpha)}(A_\alpha) \to H_{\mathrm{c}}^{q+1+d(\alpha)}(X_\alpha, A_\alpha)$$

Set $\partial^q_\alpha = (-1)^{d(\alpha)}\delta_{\mathrm{c}}^{q+d(\alpha)}$. By Proposition 1.7, $\{\partial^q_\alpha\}$ is a direct system of homomorphisms from the direct system

$$\{H_{\mathrm{c}}^{q+d(\alpha)}(A_\alpha); \Delta^q_{\alpha\beta}(A)\}$$

to the direct system

$$\{H_{\mathrm{c}}^{q+1+d(\alpha)}(X_\alpha, A_\alpha); \Delta^{q+1}_{\alpha\beta}(X,A)\}$$

over the directed set $\mathcal{V}$. We define $\delta^q_E(X,A)$ as the direct limit of this system:

$$\delta^q_E(X,A) = \varinjlim_{\alpha\in\mathcal{V}} \{\partial^q_\alpha\} : H^q_E(A) \to H^q_E(X,A)$$

Proposition 6.1. *Given an E-pair (X,A), let $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X,A)$ be the inclusion maps. Then the following sequence of homomorphisms is exact:*

$$\ldots \to H^q_E(X) \overset{H^q_E(i)}{\to} H^q_E(A) \overset{\delta^q_E}{\to} H^{q+1}_E(X,A) \overset{H^{q+1}_E(j)}{\to} H^{q+1}_E(X) \to \ldots$$

PROOF. By the exactness of the Alexander–Spanier cohomology with compact supports, the following sequence is exact:

$$\ldots \to H_c^{q+d(\alpha)}(X_\alpha) \xrightarrow{H_c^{q+d(\alpha)}(i_\alpha)} H_c^{q+d(\alpha)}(A_\alpha) \xrightarrow{\partial_\alpha^q} H_c^{q+1+d(\alpha)}(X_\alpha, A_\alpha) \xrightarrow{H_c^{q+1+d(\alpha)}(j_\alpha)} H_c^{q+1+d(\alpha)}(X_\alpha) \to \ldots$$

Since the direct limit takes exact sequences into exact sequences, the assertion follows. □

The above proposition proves assertion (5) of Theorem 0.2.

PROPOSITION 6.2. *Let (X,A) and (Y,B) be E-pairs. If $\Phi : (X,A) \to (Y,B)$ is an E-finite morphism, then the following diagram commutes:*

$$\begin{array}{ccc} H_E^q(B) & \xrightarrow{H_E^q(\Phi|_A)} & H_E^q(A) \\ {\scriptstyle \delta_E^q(Y,B)}\downarrow & & \downarrow{\scriptstyle \delta_E^q(X,A)} \\ H_E^{q+1}(Y,B) & \xrightarrow{H_E^{q+1}(\Phi)} & H_E^{q+1}(X,A) \end{array}$$

PROOF. If $\Phi(x) = x + R(x)$ with $\pi \circ R(X) \subset \alpha_0$, and $\alpha_0 \subset \alpha$, then by the naturality of the coboundary for H_c^*, the following diagram commutes:

$$\begin{array}{ccc} H_c^{q+d(\alpha)}(B_\alpha) & \xrightarrow{H_c^{q+d(\alpha)}((\Phi|_A)_\alpha)} & H^{q+d(\alpha)}{}_c(A_\alpha) \\ {\scriptstyle (-1)^{d(\alpha)}\delta_c^q(Y_\alpha,B_\alpha)}\downarrow & & \downarrow{\scriptstyle (-1)^{d(\alpha)}\delta_c^q(X_\alpha,A_\alpha)} \\ H_c^{q+d(\alpha)+1}(Y_\alpha,B_\alpha) & \xrightarrow{H_c^{q+d(\alpha)+1}(\Phi_\alpha)} & H_c^{q+d(\alpha)+1}(X_\alpha,A_\alpha) \end{array}$$

Taking the direct limit over $\alpha \in \mathcal{V}$, we get the assertion. □

PROPOSITION 6.3. *Let (X,A) and (Y,B) be closed pairs with X and Y E-locally compact. If $\Phi : (X,A) \to (Y,B)$ is an E-compact morphism, then the following diagram commutes:*

$$\begin{array}{ccc} H_E^q(B) & \xrightarrow{H_E^q(\Phi|_A)} & H_E^q(A) \\ {\scriptstyle \delta_E^q(Y,B)}\downarrow & & \downarrow{\scriptstyle \delta_E^q(X,A)} \\ H_E^{q+1}(Y,B) & \xrightarrow{H_E^{q+1}(\Phi)} & H_E^{q+1}(X,A) \end{array}$$

PROOF. Set $U = \overline{X \setminus A}$ and $V = \overline{Y \setminus B} \cup \Phi(U)$. We must prove the conclusion for the restriction $\widetilde{\Phi} : (U, A \cap U) \to (V, B \cap V)$.

Let $\Phi^m : (U, A \cap U) \to (V, B \cap V)$ be an approximating system for $\widetilde{\Phi}$. By Proposition 6.2, the following diagram commutes:

$$\begin{tikzcd}
H_E^q(B^m) \arrow[r, "H_E^q(\Phi^m|_{A\cap U})"] \arrow[d, "\delta_E^q(Y^m,B^m)"'] & H_E^q(A \cap U) \arrow[d, "\delta_E^q(U,A\cap U)"] \\
H_E^{q+1}(Y^m, B^m) \arrow[r, "H_E^{q+1}(\Phi^m)"] & H_E^{q+1}(U, A \cap U)
\end{tikzcd}$$

By Proposition 6.2 applied to the maps $i_m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$ and by Proposition 4.1, the direct limit of the left vertical arrow is $\delta_E^q(V, B \cap V)$. Therefore,

$$\begin{tikzcd}
H_E^q(B \cap V) \arrow[r, "H_E^q(\widetilde{\Phi}|_{A\cap U})"] \arrow[d, "\delta_E^q(V,B\cap V)"'] & H_E^q(A \cap U) \arrow[d, "\delta_E^q(U,A\cap U)"] \\
H_E^{q+1}(V, B \cap V) \arrow[r, "H_E^{q+1}(\widetilde{\Phi})"] & H_E^{q+1}(U, A \cap U)
\end{tikzcd}$$

The statement now follows from the excision property of Proposition 3.3 and from Proposition 6.2 applied to the inclusions $(V, B \cap V) \hookrightarrow (Y, B)$ and $(U, A \cap U) \hookrightarrow (X, A)$. ☐

The above proposition proves assertion (4) of Theorem 0.2, which is now completely proved.

2. Extension to E-morphisms

In this second part we want to extend the E-cohomology theory to a wider class of maps and homotopies, so as to include the gradient flows of suitable functionals. These maps will be called E-morphisms:

Definition 6.1. A continuous map $\Phi : (X, A) \to (Y, B)$ is an *E-morphism* if:

(1) it has the form
$$\Phi(x) = Lx + K(x)$$
where $K : X \to H$ maps bounded sets into precompact sets and L is a linear automorphism of H such that $LE = E$;

(2) $\Phi^{-1}(U)$ is bounded for every bounded U.

Definition 6.2. A continuous map $\Psi : (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ is an *E-homotopy* if:

(1) it has the form
$$\Psi(x, t) = L_t x + K(x, t)$$

where the continuous map $K : X \times [0,1] \to H$ maps bounded sets into precompact sets and L_t is a linear automorphism of H such that $L_t E = E$;

(2) $\Psi^{-1}(U)$ is bounded for every bounded U.

Definition 6.3. An E-morphism $\Phi(x) = Lx + K(x)$ is called *positive* if L is a positive operator. An E-homotopy $\Psi(x,t) = L_t x + K(x,t)$ is called *positive* if every L_t is a positive operator.

Two [positive] E-morphisms Φ_0 and Φ_1 from (X,A) to (Y,B) are said to be [*positively*] *E-homotopic* if there exists a [positive] E-homotopy $\Psi : (X \times [0,1], A \times [0,1]) \to (Y,B)$ such that $\Psi(\,\cdot\,,0) = \Phi_0$ and $\Psi(\,\cdot\,,1) = \Phi_1$.

The following theorem summarizes the main result of this part:

Theorem 6.4. *The E-cohomology theory with $\mathbb{Z}_2$ coefficients* [*with arbitrary coefficients*] *can be extended to* [*positive*] *E-morphisms and* [*positive*] *E-homotopies between E-pairs. More precisely, assertions* (1), (2) *and* (4) *of Theorem* 0.2 *can be generalized in the following way:*

(1) (Contravariant functoriality) *If* $\Phi : (X,A) \to (Y,B)$ *and* $\Phi' : (Y,B) \to (Z,C)$ *are* [*positive*] *E-morphisms, then* $H_E^*(\Phi' \circ \Phi) = H_E^*(\Phi) \circ H_E^*(\Phi')$.

(2) (Homotopy invariance) *If two* [*positive*] *E-morphisms* Φ *and* Φ' *are* [*positively*] *E-homotopic, then* $H_E^*(\Phi) = H_E^*(\Phi')$.

(4) (Naturality of the coboundary) *If* $\Phi : (X,A) \to (Y,B)$ *is a* [*positive*] *E-morphism, then the following diagram commutes:*

$$\begin{array}{ccc} H_E^q(B) & \xrightarrow{H_E^q(\Phi|_A)} & H_E^q(A) \\ {\scriptstyle \delta_E^q(Y,B)}\downarrow & & \downarrow{\scriptstyle \delta_E^q(X,A)} \\ H_E^{q+1}(Y,B) & \xrightarrow{H_E^{q+1}(\Phi)} & H_E^{q+1}(X,A) \end{array}$$

The following sections are devoted to the proof of this theorem.

7. E-isomorphisms

Definition 7.1. A map $L : (X,A) \to (Y,B)$ is an *E-isomorphism* if it is the restriction of a linear invertible automorphism of H, also denoted by L, such that $LE = E$.

Notice that the E-isomorphisms are not assumed to be onto (Y,B).

Let (X,A) and (Y,B) be two E-pairs. Let $L : (X,A) \to (Y,B)$ be an E-isomorphism. Since $LE = E$, L induces a linear isomorphism $\widetilde{L} : H/E \to H/E$. If $\alpha \in \mathcal{V}$, then $L_\alpha = L|_{\pi^{-1}(\alpha)}$ maps (X_α, A_α) into $(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha})$. Therefore the proper map L_α induces homomorphisms

$$H_c^{q+d(\alpha)}(L_\alpha) : H_c^{q+d(\alpha)}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}) \to H_c^{q+d(\alpha)}(X_\alpha, A_\alpha).$$

If $\alpha \subset \beta$ and $d(\beta) = d(\alpha) + 1$, then $\widetilde{L}$ maps β^+_α isomorphically onto either $(\widetilde{L}\beta)^+_{\widetilde{L}\alpha}$ or $(\widetilde{L}\beta)^-_{\widetilde{L}\alpha}$. Therefore L_β maps $(X_{\beta^+_\alpha}, A_{\beta^+_\alpha})$ into either $(Y_{\widetilde{L}\beta^+_{\widetilde{L}\alpha}}, B_{\widetilde{L}\beta^+_{\widetilde{L}\alpha}})$ or $(Y_{\widetilde{L}\beta^-_{\widetilde{L}\alpha}}, B_{\widetilde{L}\beta^-_{\widetilde{L}\alpha}})$.

In the first case, by the functoriality of the Mayer–Vietoris homomorphism (Proposition 1.6), the following diagram commutes:

$$\begin{array}{ccc} H^{q+d(\widetilde{L}\alpha)}_{\mathrm{c}}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}) & \xrightarrow{H^{q+d(\alpha)}_{\mathrm{c}}(L_\alpha)} & H^{q+d(\alpha)}_{\mathrm{c}}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\widetilde{L}\alpha\widetilde{L}\beta}(Y,B)}\big\downarrow & & \big\downarrow{\scriptstyle \Delta^q_{\alpha\beta}(X,A)} \\ H^{q+d(\widetilde{L}\beta)}_{\mathrm{c}}(Y_{\widetilde{L}\beta}, B_{\widetilde{L}\beta}) & \xrightarrow{H^{q+d(\beta)}_{\mathrm{c}}(L_\beta)} & H^{q+d(\beta)}_{\mathrm{c}}(X_\beta, A_\beta) \end{array} \tag{7.1}$$

In the second case, since exchanging the roles of the two sets changes the sign of the Mayer–Vietoris homomorphism, the same diagram anti-commutes.

If we choose $\mathbb{Z}_2$ coefficients, then diagram (7.1) commutes in both cases.

If L is positive, then $\widetilde{L}$ is also positive and $\widetilde{L}\beta^+_\alpha = (\widetilde{L}\beta)^+_{\widetilde{L}\alpha}$. In this case, diagram (7.1) commutes for arbitrary coefficients.

Therefore $\{H^{q+d(\alpha)}_{\mathrm{c}}(L_\alpha)\}$ is a direct system of homomorphisms from the direct system

$$\{H^{q+d(\widetilde{L}\alpha)}_{\mathrm{c}}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}); \Delta^q_{\widetilde{L}\alpha\widetilde{L}\beta}(Y,B)\}$$

of $\mathbb{Z}_2$-vector spaces [or $\mathcal{A}$-modules] to the direct system

$$\{H^{q+d(\alpha)}_{\mathrm{c}}(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X,A)\}$$

of $\mathbb{Z}_2$-vector spaces [or $\mathcal{A}$-modules], over the directed set $\mathcal{V}$. We define $\widetilde{G}^q_E(L)$ as the direct limit of this system:

$$\widetilde{G}^q_E(L) = \varinjlim_{\alpha\in\mathcal{V}} H^{q+d(\alpha)}_{\mathrm{c}}(L_\alpha) : \\ \varinjlim_{\alpha\in\mathcal{V}} \{H^{q+d(\widetilde{L}\alpha)}_{\mathrm{c}}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}); \Delta^q_{\widetilde{L}\alpha\widetilde{L}\beta}(Y,B)\} \to H^q_E(X,A).$$

Since $\widetilde{L}$ acts on $\mathcal{V}$ as an order preserving bijection, there exists a natural isomorphism

$$\widehat{L}^q(Y,B) : H^q_E(Y,B) \to \varinjlim_{\alpha\in\mathcal{V}} \{H^{q+d(\widetilde{L}\alpha)}_{\mathrm{c}}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}); \Delta^q_{\widetilde{L}\alpha\widetilde{L}\beta}(Y,B)\}.$$

We define

$$G^q_E(L) = \widetilde{G}^q_E(L) \circ \widehat{L}^q(Y,B) : H^q_E(Y,B) \to H^q_E(X,A).$$

It is trivial to show that the definition of $G^q_E(L)$ does not depend on the linear extension of the E-isomorphism L. Moreover, G^*_E is a contravariant functor:

Proposition 7.1. *Assume that* $L : (X, A) \to (Y, B)$ *and* $L' : (Y, B) \to (Z, C)$ *are* E*-isomorphisms of* E*-pairs. Then:*

(1) *if* $I : (X, A) \to (X, A)$ *is the identity map, then* $G_E^*(I)$ *is the identity homomorphism on* $H_E^*(X, A)$*;*
(2) $G_E^*(L' \circ L) = G_E^*(L) \circ G_E^*(L')$.

Proof. Assertion (1) is trivial. We prove (2):

$$\begin{aligned}\widetilde{G}_E^q(L' \circ L) &= \varinjlim_{\alpha \in \mathcal{V}} H_c^{q+d(\alpha)}((L' \circ L)_\alpha) = \varinjlim_{\alpha \in \mathcal{V}} H_c^{q+d(\alpha)}(L'_{\widetilde{L}\alpha} \circ L_\alpha) \\ &= \varinjlim_{\alpha \in \mathcal{V}} H_c^{q+d(\alpha)}(L_\alpha) \circ \varinjlim_{\alpha \in \mathcal{V}} H_c^{q+d(\alpha)}(L'_{\widetilde{L}\alpha}).\end{aligned}$$

Applying the order preserving bijection L on $\mathcal{V}$ gives

$$\varinjlim_{\alpha \in \mathcal{V}} H_c^{q+d(\alpha)}(L'_{\widetilde{L}\alpha}) = \widehat{L}^q(Y, B) \circ \varinjlim_{\alpha \in \mathcal{V}} H_c^{q+d(\alpha)}(L'_\alpha) \circ \widehat{L}^q(Z, C)^{-1}.$$

Therefore,

$$\widetilde{G}_E^q(L' \circ L) = \widetilde{G}_E^q(L) \circ \widehat{L}^q(Y, B) \circ \widetilde{G}_E^q(L') \circ \widehat{L}^q(Z, C)^{-1}.$$

Since $\widehat{L' \circ L}^q(Z, C) = \widehat{L}^q(Z, C) \circ \widehat{L'}^q(Z, C)$, we get

$$G_E^q(L' \circ L) = \widetilde{G}_E^q(L' \circ L) \circ \widehat{L' \circ L}^q(Z, C) = G_E^q(L) \circ G_E^q(L'). \qquad \square$$

Definition 7.2. An *E-isotopy* is a continuous map $\mathcal{L} : (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ of the form

$$\mathcal{L}(x, t) = L_t x$$

where each L_t is an E-isomorphism. Two E-isomorphisms L_0 and L_1 from (X, A) to (Y, B) are called *E-isotopic* if there exists an E-isotopy $\mathcal{L} : (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ such that $\mathcal{L}(\,\cdot\,, 0) = L_0$ and $\mathcal{L}(\,\cdot\,, 1) = L_1$.

Now we want to prove that $G_E^*(L_0) = G_E^*(L_1)$ if L_0 and L_1 are E-isotopic. This task turns out to be more difficult than one may think. We need two lemmas, the first about the possibility of extending a homotopy, the second about direct limits:

Lemma 7.2. *Let* X *be a normal space and let* X' *be closed in* X*. Let* W *be a Banach space, with norm* $\|\cdot\|$*, and let* α *be a finite-dimensional linear subspace of* W*. Let* $\Theta : X \times [0, 1] \to W$ *be a continuous map. Assume that* $\varphi : X \to \alpha$ *and* $\Psi : X' \times [0, 1] \to \alpha$ *are continuous maps such that* $\Psi|_{X' \times \{1\}} = \varphi|_{X'}$ *and*

$$\begin{aligned}\|\Psi(x, t) - \Theta(x, t)\| &< a, \quad \forall (x, t) \in X' \times [0, 1], \\ \|\varphi(x) - \Theta(x, 1)\| &< a, \quad \forall x \in X.\end{aligned}$$

Then there exist a continuous map $\Phi : X \times [0,1] \to \alpha$ *and a continuous function* $\mu : X \times [0,1] \to [0,1]$ *such that:*

(1) $\mu(x,t) = t$ *when* $x \in X'$, $\mu(x,1) = 1$ *for every* $x \in X$*;*
(2) $\Phi|_{X' \times [0,1]} = \Psi$ *and* $\Phi|_{X \times \{1\}} = \varphi$*;*
(3) $\|\Phi(x,t) - \Theta(x, \mu(x,t))\| < a$ *for every* $(x,t) \in X \times [0,1]$.

PROOF. Set $T = (X' \times [0,1]) \cup (X \times \{1\})$ and let $\widetilde{\Psi} : T \to \alpha$ be defined by

$$\widetilde{\Psi}(x,t) = \begin{cases} \Psi(x,t) & \text{if } 0 \le t < 1, \\ \varphi(x) & \text{if } t = 1. \end{cases}$$

By Tietze's Theorem, there exists a continuous map $\widetilde{\Phi} : X \times [0,1] \to \alpha$ which extends $\widetilde{\Psi}$. Set

$$Y = \{x \in X \mid \exists t \in [0,1] \text{ such that } \|\widetilde{\Phi}(x,t) - \Theta(x,t)\| \ge a\}.$$

Then Y is closed and disjoint from X'. Since X is normal, we can find a continuous function $\lambda : X \to [0,1]$ such that $\lambda(Y) = 0$ and $\lambda(X') = 1$. Define

$$\mu(x,t) = 1 - \lambda(x)(1-t), \quad \Phi(x,t) = \widetilde{\Phi}(x, \mu(x,t)).$$

It is easy to verify that Φ and μ satisfy the required conditions. $\square$

LEMMA 7.3. *Let* $L_0, L_1 : (X,A) \to (Y,B)$ *be two E-isomorphisms of E-pairs. Assume that for each* $\alpha \in \mathcal{V}$ *there exists* $\gamma \in \mathcal{V}$ *with* $\alpha \subset \gamma$ *and* $\widetilde{L}_0\alpha \subset \widetilde{L}_1\gamma$ *such that the following diagram commutes:*

$$\begin{array}{ccc} H_c^{q+d(\alpha)}(Y_{\widetilde{L}_0\alpha}, B_{\widetilde{L}_0\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(L_{0\alpha})} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\widetilde{L}_0\alpha\widetilde{L}_1\gamma}(Y,B)}\downarrow & & \downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\ H_c^{q+d(\gamma)}(Y_{\widetilde{L}_1\gamma}, B_{\widetilde{L}_1\gamma}) & \xrightarrow{H_c^{q+d(\gamma)}(L_{1\gamma})} & H_c^{q+d(\gamma)}(X_\gamma, A_\gamma) \end{array} \tag{7.2}$$

Then $G_E^q(L_0) = G_E^q(L_1)$.

PROOF. We recall that, by the definition of direct limit, $H_E^q(X,A)$ is the direct sum of all $H_c^{q+d(\alpha)}(X_\alpha, A_\alpha)$, $\alpha \in \mathcal{V}$, factored by the following equivalence relation: $\eta \in H_c^{q+d(\alpha)}(X_\alpha, A_\alpha)$ is equivalent to $\zeta \in H_c^{q+d(\beta)}(X_\beta, A_\beta)$ if there exists $\gamma \in \mathcal{V}$ with $\alpha, \beta \subset \gamma$ such that

$$\Delta^q_{\alpha\gamma}(X,A)\eta = \Delta^q_{\beta\gamma}(X,A)\zeta.$$

Choose $\Xi \in H_E^q(Y,B)$; it is the equivalence class of some $\xi \in H_c^{q+d(\beta)}(Y_\beta, B_\beta)$. Set $\alpha = L_0^{-1}\beta$ and $\overline{\alpha} = L_1^{-1}\beta$. Then $G_E^q(L_0)\Xi$ is represented by

$$H_c^{q+d(\alpha)}(L_{0\alpha})\xi \in H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \tag{7.3}$$

while $G_E^q(L_1)\Xi$ is represented by

$$H_c^{q+d(\overline{\alpha})}(L_{1\overline{\alpha}})\xi \in H_c^{q+d(\overline{\alpha})}(X_{\overline{\alpha}}, A_{\overline{\alpha}}). \tag{7.4}$$

To prove that (7.3) and (7.4) are equivalent in $H_E^q(X,A)$, we must find $\gamma \in \mathcal{V}$ with $\alpha, \overline{\alpha} \subset \gamma$ such that

$$\Delta^q_{\alpha\gamma}(X,A) \circ H_c^{q+d(\alpha)}(L_{0\gamma})\xi = \Delta^q_{\overline{\alpha}\gamma} \circ H_c^{q+d(\overline{\alpha})}(L_{1\overline{\alpha}})\xi. \tag{7.5}$$

Since $\{H_c^{q+d(\omega)}(L_{1\omega})\}_{\omega\in\mathcal{V}}$ is a direct system of homomorphisms,

$$\begin{aligned}\Delta^q_{\overline{\alpha}\gamma} \circ H_c^{q+d(\overline{\alpha})}(L_{1\overline{\alpha}}) &= H_c^{q+d(\gamma)}(L_{1\gamma}) \circ \Delta^q_{\widetilde{L}_1\overline{\alpha}\widetilde{L}_1\gamma}(Y,B) \\ &= H_c^{q+d(\gamma)}(L_{1\gamma}) \circ \Delta^q_{\widetilde{L}_0\alpha\widetilde{L}_1\gamma}(Y,B).\end{aligned}$$

If we choose γ as in the hypotheses, the above relation and the commutativity of diagram (7.2) imply (7.5). $\square$

Now we are ready to prove the invariance of G_E^* with respect to E-isotopies. We just need this fact in the case of bounded pairs. A more general statement will be proved later (see Proposition 10.1).

Proposition 7.4. *Assume that (X,A) and (Y,B) are bounded closed pairs. If the E-isomorphisms L_0 and L_1 from (X,A) to (Y,B) are E-isotopic, then $G_E^*(L_0) = G_E^*(L_1)$.*

Proof. Let $Y^m = \overline{Y + B_{E^\perp}(1/m)}$, $B^m = \overline{B + B_{E^\perp}(1/m)}$ and let $i_m : (Y,B) \hookrightarrow (Y^m, B^m)$ be the inclusion map (recall that $B_{E^\perp}(r)$ is the open ball in $E^\perp$ of radius r).

Let $L(\,\cdot\,,t) = L_t : (X,A) \to (Y,B)$ be an E-isotopy between L_0 and L_1 and let $L_t^m = i_m \circ L_t : (X,A) \to (Y^m, B^m)$. Let $\alpha \in \mathcal{V}$ and set

$$\Gamma(\alpha) = L(X_\alpha \times [0,1]).$$

Then $\Gamma(\alpha)$ is a compact subset of Y. Therefore we can find a finite-dimensional linear subspace $\widetilde{W}$ of $E^\perp$ such that:

(i) $\widetilde{L}_0\alpha + \widetilde{L}_1\alpha \subset \widetilde{\gamma} = \pi(\widetilde{W})$;

(ii) if $P_{E\oplus\widetilde{W}} : H \to H$ is the orthogonal projection onto $E \oplus \widetilde{W}$, then

$$\|P_{E\oplus\widetilde{W}} \circ L(x,t) - L(x,t)\| < 1/m, \quad \forall x \in X_\alpha,\ \forall t \in [0,1].$$

Set $\gamma = \widetilde{L}_1{}^{-1}(\widetilde{\gamma})$. Then $\alpha \subset \gamma$ and $\widetilde{L}_0\alpha \subset \widetilde{L}_1\gamma$. Write $L(x,t) = L_E^t x + L_{E^\perp}^t x$, where $L_E^t = P_E \circ L_t$ and $L_{E^\perp}^t = P_{E^\perp} \circ L_t$. Define the following maps:

$$\begin{gathered}\varphi : X_\gamma \ni x \mapsto L^1_{E^\perp} x \in \widetilde{W}, \\ \Psi : X_\alpha \times [0,1] \ni (x,t) \mapsto P_{\widetilde{W}} \circ L^t_{E^\perp} x \in \widetilde{W}, \\ \Theta : X_\gamma \times [0,1] \ni (x,t) \mapsto L^t_{E^\perp} x \in E^\perp.\end{gathered}$$

Clearly, $\varphi(x) = \Theta(x,1)$ for every $x \in X_\gamma$ and, by (ii),

$$\|\Psi(x,t) - \Theta(x,t)\| < 1/m, \quad \forall (x,t) \in X_\alpha \times [0,1].$$

Since X_γ is compact Hausdorff, and thus normal, we can apply Lemma 7.2: there exist a map $\Phi : X_\gamma \times [0,1] \to \widetilde{W}$ and a function $\mu : X_\gamma \times [0,1] \to [0,1]$ such that:

(1) $\mu(x,t) = t$ for $x \in X_\alpha$, and $\mu(x,1) = 1$ for every $x \in X_\gamma$;
(2) $\Phi|_{X_\alpha \times [0,1]}(x,t) = P_{\widetilde{W}} \circ L^t_{E^\perp} x$ and $\Phi|_{X_\gamma \times \{1\}} = L^1_{E^\perp}$;
(3) $\|\Phi(x,t) - L^{\mu(x,t)}_{E^\perp} x\| < 1/m$.

Define $\overline{\Phi} : (X_\gamma \times [0,1], A_\gamma \times [0,1]) \to (Y^m_{\widetilde{\gamma}}, B^m_{\widetilde{\gamma}})$ by

$$\overline{\Phi}(x,t) = \Phi(x,t) + L^{\mu(x,t)}_E x.$$

Then $\overline{\Phi}$ is well defined by (3). Since $L^m_{0\alpha} : (X_\alpha, A_\alpha) \to (Y^m_{\widetilde{L}_0\alpha}, B^m_{\widetilde{L}_0\alpha})$ is a restriction of $\overline{\Phi}_0 = \overline{\Phi}(\,\cdot\,,0) : (X_\gamma, A_\gamma) \to (Y^m_{\widetilde{\gamma}}, B^m_{\widetilde{\gamma}})$, by the functoriality of the Mayer–Vietoris homomorphism, the following diagram commutes:

$$\begin{array}{ccc}
H^{q+d(\alpha)}_c(Y^m_{\widetilde{L}_0\alpha}, B^m_{\widetilde{L}_0\alpha}) & \xrightarrow{H^{q+d(\alpha)}_c(L^m_{0\alpha})} & H^{q+d(\alpha)}_c(X_\alpha, A_\alpha) \\
{\scriptstyle \Delta^q_{\widetilde{L}_0\alpha\widetilde{L}_1\gamma}(Y^m,B^m)}\big\downarrow & & \big\downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\
H^{q+d(\gamma)}_c(Y^m_{\widetilde{L}_1\gamma}, B^m_{\widetilde{L}_1\gamma}) & \xrightarrow{H^{q+d(\gamma)}_c(\overline{\Phi}_0)} & H^{q+d(\gamma)}_c(X_\gamma, A_\gamma)
\end{array}$$

Finally, since $\overline{\Phi}_0$ is homotopic to $\overline{\Phi}_1 = \overline{\Phi}(\,\cdot\,,1) = L^m_{1\gamma}$ via the proper homotopy $\overline{\Phi}$, we get

$$H^{q+d(\gamma)}_c(\overline{\Phi}_0) = H^{q+d(\gamma)}_c(L^m_{1\gamma}).$$

Thus we have proved the following fact: for each $\alpha \in \mathcal{V}$ there exists $\gamma \in \mathcal{V}$ such that $\alpha \subset \gamma$, $\widetilde{L}_0\alpha \subset \widetilde{L}_1\gamma$, and the following diagram commutes:

$$\begin{array}{ccc}
H^{q+d(\alpha)}_c(Y^m_{\widetilde{L}_0\alpha}, B^m_{\widetilde{L}_0\alpha}) & \xrightarrow{H^{q+d(\alpha)}_c(L^m_{0\alpha})} & H^{q+d(\alpha)}_c(X_\alpha, A_\alpha) \\
{\scriptstyle \Delta^q_{\widetilde{L}_0\alpha\widetilde{L}_1\gamma}(Y^m,B^m)}\big\downarrow & & \big\downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\
H^{q+d(\gamma)}_c(Y^m_{\widetilde{L}_1\gamma}, B^m_{\widetilde{L}_1\gamma}) & \xrightarrow{H^{q+d(\gamma)}_c(L^m_{1\gamma})} & H^{q+d(\gamma)}_c(X_\gamma, A_\gamma)
\end{array}$$

By Lemma 7.3, this implies $G^q_E(L^m_0) = G^q_E(L^m_1)$. Then, by the functoriality of G^*_E, $G^q_E(L_0) \circ G^q_E(i_m) = G^q_E(L_1) \circ G^q_E(i_m)$. It is trivial to show that $G^q_E(i_m) = H^q_E(i_m)$. Therefore, by the continuity property of H^*_E stated in Proposition 4.1, $G^q_E(L_0) = G^q_E(L_1)$. $\square$

8. Comparison between H^*_E and G^*_E. Now we want to prove that the functors H^*_E and G^*_E coincide on maps which are both E-compact morphisms and E-isomorphisms. We need an approximation lemma:

Lemma 8.1. *Let (X,A) and (Y,B) be two bounded closed pairs. Assume that $L : (X,A) \to (Y,B)$ is an E-compact morphism and an E-isomorphism. Then for each $m \in \mathbb{N}$ there exists an E-isomorphism*

$$L_m : (X,A) \to (Y^m, B^m) = (\overline{Y + B_{E^\perp}(1/m)}, \overline{B + B_{E^\perp}(1/m)})$$

such that $\{L_m\}$ is an approximating system for L with L_m and $i_m \circ L$ E-isotopic ($i_m : (Y,B) \hookrightarrow (Y^m, B^m)$ is the inclusion map).

Proof. Write $Lx = x + Kx$, where K is a compact operator such that $KE = E$.

Since the set of linear automorphisms of H is open in the space of all bounded operators endowed with the usual norm $\|\cdot\|_{L(H,H)}$, we can find $\varepsilon > 0$ such that $\|M - L\|_{L(H,H)} < \varepsilon$ implies that M is invertible.

Let $a = \sup\{\|x\| \mid x \in X\}$. For each $m \in \mathbb{N}$ find a bounded linear operator $R_m : H \to E^\perp$ with finite rank such that

$$\|P_{E^\perp} \circ K - R_m\|_{L(H,H)} < \min\{1/(ma), \varepsilon\}.$$

Set $L_m x = x + P_E \circ Kx + R_m x$. Since $L_m E = LE = E$ and

$$\|P_{E^\perp} \circ L_m x - P_{E^\perp} \circ Lx\| \leq \|R_m - P_{E^\perp} \circ K\|_{L(H,H)} \|x\| \leq 1/m,$$

L_m maps (X,A) into (Y^m, B^m) and it is an E-finite morphism and an E-isomorphism. Set

$$L_m^t x = x + P_E \circ Kx + tR_m x + (1-t) P_{E^\perp} \circ Kx.$$

It is easy to verify that:

(1) $L_m^0 = L$, $L_m^1 = L_m$;
(2) L_m^t maps (X,A) into (Y^m, B^m) for each $t \in [0,1]$;
(3) L_m^t is invertible for each $t \in [0,1]$ and $L_m^t E = E$.

These three facts imply that $\{L_m\}$ is the required approximating system for L and that L_m is E-isotopic to $i_m \circ L$. □

Proposition 8.2. *Assume that $L : (X,A) \to (Y,B)$ is an E-isomorphism and an E-compact morphism of E-pairs. Then $G_E^*(L) = H_E^*(L)$.*

Proof. Notice that the assertion is trivially true if L is an E-finite morphism. We prove the general case in three steps.

1. First assume that (X,A) and (Y,B) are bounded closed pairs. Using Lemma 8.1, we can construct an approximating sequence $\{(Y^m, B^m)\}$ for (Y,B) and an approximating system $\{L_m : (X,A) \to (Y^m, B^m)\}$ for L such that each L_m is an E-isomorphism and $i_m \circ L$ is E-isotopic to L_m, where $i_m : (Y,B) \hookrightarrow$

(Y^m, B^m) is the inclusion map. Therefore, by Proposition 7.4,

$$H_E^*(L) \circ \varinjlim_{m\in\mathbb{N}} H_E^*(i_m) = \varinjlim_{m\in\mathbb{N}} H_E^*(L_m) = \varinjlim_{m\in\mathbb{N}} G_E^*(L_m) = \varinjlim_{m\in\mathbb{N}} G_E^*(i_m \circ L)$$
$$= \varinjlim_{m\in\mathbb{N}} G_E^*(L) \circ G_E^*(i_m) = G_E^*(L) \circ \varinjlim_{m\in\mathbb{N}} H_E^*(i_m).$$

By Proposition 4.1, $H_E^*(L) = G_E^*(L)$.

2. Assume now that (X, A) and (Y, B) are cobounding closed pairs. Since G_E^* coincides with H_E^* on inclusion maps, the strong excision property stated in Theorem 0.2(3) also holds for G_E^*. Therefore the statement follows from step 1.

3. Assume now that (X, A) and (Y, B) are arbitrary E-pairs. Since G_E^* coincides with H_E^* on inclusion maps, Proposition 4.2 also holds for G_E^*. Therefore the assertion follows from step 2. □

The following lemma will be useful in the next section:

Lemma 8.3. *Let $\Phi : (X, A) \to (Y, B)$ be an E-compact morphism of E-pairs and let L be an E-isomorphism. Then $L \circ \Phi \circ L^{-1} : (LX, LA) \to (LY, LB)$ is an E-compact morphism and*

$$H_E^*(L \circ \Phi \circ L^{-1}) = G_E^*(L^{-1}) \circ H_E^*(\Phi) \circ G_E^*(L). \tag{8.1}$$

Proof. Arguing as in the previous proposition, it is enough to prove the assertion when (X, A) and (Y, B) are bounded closed pairs. Assume that Φ has the form $\Phi(x) = x + K(x)$. Then

$$L \circ \Phi \circ L^{-1}(x) = x + L \circ K \circ L^{-1}(x) = x + K'(x)$$

and $L \circ \Phi \circ L^{-1}$ is an E-compact morphism.

The remaining part of the conclusion is trivial if Φ is an E-finite morphism. In the general case, consider an approximating system $\{\Phi_m : (X, A) \to (Y^m, B^m)\}$ for Φ. Then $\{L \circ \Phi_m \circ L^{-1} : (LX, LA) \to (LY^m, LB^m)\}$ is an approximating system for $L \circ \Phi \circ L^{-1}$. Since (8.1) holds for E-finite morphisms, we get

$$H_E^*(L \circ \Phi_m \circ L^{-1}) = G_E^*(L^{-1}) \circ H_E^*(\Phi_m) \circ G_E^*(L), \quad \forall m \in \mathbb{N}.$$

Taking the direct limit over $m \in \mathbb{N}$, we conclude the proof. □

9. E-morphisms. Notice that every E-morphism $\Phi : (X, A) \to (Y, B)$ can be written as $\Phi = \Psi \circ L$ where $L : (X, A) \to (LX, LA)$ is an E-isomorphism and $\Psi : (LX, LA) \to (Y, B)$ is an E-compact morphism. However, this decomposition is not unique.

If $\Phi = \Psi \circ L$ as above, we define

$$H^*_E(\Phi) = G^*_E(L) \circ H^*_E(\Psi).$$

The next result implies that this is a good definition:

Proposition 9.1. *Assume that $\Phi = \Psi \circ L = M \circ \Psi' \circ M'$ is an E-morphism, with Ψ and Ψ' E-compact morphisms, and L, M and M' E-isomorphisms. Then*

$$G^*_E(L) \circ H^*_E(\Psi) = G^*_E(M') \circ H^*_E(\Psi') \circ G^*_E(M).$$

Proof. If $\Psi'(x) = s + K'(x)$ then

$$\Psi = (I + M \circ K' \circ M^{-1}) \circ M \circ M' \circ L^{-1}.$$

Therefore $M \circ M' \circ L^{-1}$ is also an E-compact morphism. By Proposition 8.2 and Lemma 8.3,

$$\begin{aligned} H^*_E(\Psi) &= H^*_E(M \circ M' \circ L^{-1}) \circ H^*_E(I + M \circ K' \circ M^{-1}) \\ &= G^*_E(M \circ M' \circ L^{-1}) \circ H^*_E(M \circ \Psi' \circ M^{-1}) \\ &= G^*_E(L^{-1}) \circ G^*_E(M') \circ G^*_E(M) \circ G^*_E(M^{-1}) \circ H^*_E(\Psi') \circ G^*_E(M) \\ &= G^*_E(L)^{-1} \circ G^*_E(M') \circ H^*_E(\Psi') \circ G^*_E(M). \qquad \square \end{aligned}$$

This proposition allows us to write an E-morphism Φ as an arbitrary composition of E-compact morphisms and E-isomorphisms, and then to compute $H^*_E(\Phi)$ contravariantly, computing H^*_E on the E-compact morphisms and G^*_E on the E-isomorphisms.

The next proposition proves assertion (1) of Theorem 6.4:

Proposition 9.2. *If $\Phi : (X, A) \to (Y, B)$ and $\Phi' : (Y, B) \to (Z, C)$ are E-morphisms of E-pairs, then $H^*_E(\Phi' \circ \Phi) = H^*_E(\Phi) \circ H^*_E(\Phi')$.*

Proof. Write $\Phi = \Psi \circ L$ and $\Phi' = L' \circ \Psi'$ where Ψ and Ψ' are E-compact morphisms, and L and L' are E-isomorphisms. Then

$$\begin{aligned} H^*_E(\Phi' \circ \Phi) &= H^*_E(L' \circ \Psi' \circ \Psi \circ L) = G^*_E(L) \circ H^*_E(\Psi' \circ \Psi) \circ G^*_E(L') \\ &= G^*_E(L) \circ H^*_E(\Psi) \circ H^*_E(\Psi') \circ G^*_E(L') = H^*_E(\Phi) \circ H^*_E(\Phi'). \qquad \square \end{aligned}$$

10. E-homotopies. We prove assertion (2) of Theorem 6.4:

Proposition 10.1. *Let $\Phi_0, \Phi_1 : (X, A) \to (Y, B)$ be two E-morphisms of E-pairs. If Φ_0 and Φ_1 are E-homotopic, then $H^*_E(\Phi_0) = H^*_E(\Phi_1)$.*

Proof. Arguing as in the proof of Proposition 8.2, it is enough to consider the case where (X, A) and (Y, B) bounded. Let $\Phi : (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ be an E-homotopy between Φ_0 and Φ_1. The proof is divided into two steps.

1. First assume that Φ has the special form

$$\Phi_t = \Phi(\,\cdot\,,t) = \Psi_t \circ L_t \tag{10.1}$$

where $L_t : (X,A) \to (Z_t, C_t) = (L_t X, L_t A)$ is an E-isomorphism, $L.\cdot : X \times [0,1] \to H$ is continuous, and $\Psi_t : (Z_t, C_t) \to (Y,B)$ has the form

$$\Psi_t(x) = x + R(x,t)$$

where

$$R : \Omega = \{(x,t) \in H \times [0,1] \mid x \in Z_t\} \to E$$

is continuous and $\pi \circ R(\Omega) \subset \alpha_0 \in \mathcal{V}$. Such E-homotopies will be called *special E-homotopies.*

Let $(Y^m, B^m) = (\overline{Y + B_{E^\perp}(1/m)}, \overline{B + B_{E^\perp}(1/m)})$, let $i_m : (Y,B) \hookrightarrow (Y^m, B^m)$ be the inclusion maps and let $\Phi^m = i_m \circ \Phi$. Let $\beta = \widetilde{L}_0^{-1}\alpha_0 + \widetilde{L}_1^{-1}\alpha_0$. Now we are going to prove that for each α in $\mathcal{V}_\beta = \{\alpha \in \mathcal{V} \mid \beta \subset \alpha\}$, there exists $\gamma \in \mathcal{V}_\beta$ with $\alpha \subset \gamma$ and $\widetilde{L}_0\alpha \subset \widetilde{L}_1\gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} H_c^{q+d(\alpha)}(Y^m_{\widetilde{L}_0\alpha}, B^m_{\widetilde{L}_0\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(\Phi^m_{0\alpha})} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\widetilde{L}_0\alpha\widetilde{L}_1\gamma}(Y^m,B^m)}\downarrow & & \downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\ H_c^{q+d(\gamma)}(Y^m_{\widetilde{L}_1\gamma}, B^m_{\widetilde{L}_1\gamma}) & \xrightarrow{H_c^{q+d(\gamma)}(\Phi^m_{1\gamma})} & H_c^{q+d(\gamma)}(X_\gamma, A_\gamma) \end{array} \tag{10.2}$$

Choose $\alpha \in \mathcal{V}_\beta$ and set

$$\Gamma(\alpha) = \Phi(X_\alpha \times [0,1]).$$

Then $\Gamma(\alpha)$ is a compact subset of Y. Therefore we can find a finite-dimensional linear subspace $\widetilde{W}$ of $E^\perp$ such that:

(i) $\alpha_0 + \widetilde{L}_0\alpha + \widetilde{L}_1\alpha \subset \widetilde{\gamma} = \pi(\widetilde{W})$;
(ii) if $P_{E\oplus\widetilde{W}} : H \to H$ is the orthogonal projection onto $E \oplus \widetilde{W}$, then

$$\|P_{E\oplus\widetilde{W}}y - y\| < 1/m, \quad \forall y \in \Gamma(\alpha).$$

Set $\gamma = \widetilde{L}_1^{-1}\widetilde{\gamma}$. Then $\beta \subset \alpha \subset \gamma$ and $\widetilde{L}_0\alpha \subset \widetilde{L}_1\gamma$. Write $\Phi(x,t) = P_E \circ \Phi(x,t) + P_{E^\perp} \circ \Phi(x,t)$. Define the following maps:

$$\begin{gathered} \varphi : X_\gamma \ni x \mapsto P_{E^\perp} \circ \Phi(x,1) \in \widetilde{W}, \\ \Xi : X_\alpha \times [0,1] \ni (x,t) \mapsto P_{\widetilde{W}} \circ \Phi(x,t) \in \widetilde{W}, \\ \Theta : X_\gamma \times [0,1] \ni (x,t) \mapsto P_{E^\perp} \circ \Phi(x,t) \in E^\perp. \end{gathered}$$

Clearly, $\varphi(x) = \Theta(x,1)$ and by (ii),

$$\|\Xi(x,t) - \Theta(x,t)\| < 1/m, \quad \forall (x,t) \in X_\alpha \times [0,1].$$

Since X_γ is a normal space, we can apply Lemma 7.2: there exist a map $\overline{\Xi}$: $X_\gamma \times [0,1] \to \widetilde{W}$ and a function $\mu : X_\gamma \times [0,1] \to [0,1]$ such that:

(1) $\mu(x,t) = t$ when $x \in X_\alpha$, and $\mu(x,1) = 1$ for every $x \in X_\gamma$;
(2) $\overline{\Xi}|_{X_\alpha \times [0,1]}(x,t) = P_{\widetilde{W}} \circ P_{E^\perp} \circ \Phi(x,t)$ and $\overline{\Xi}|_{X_\gamma \times \{1\}} = P_{E^\perp} \circ \Phi_1$;
(3) $\|\overline{\Xi}(x,t) - P_{E^\perp} \circ \Phi(x,\mu(x,t))\| < 1/m$ for every $(x,t) \in X_\gamma \times [0,1]$.

Define $\overline{\Phi} : (X_\gamma \times [0,1], A_\gamma \times [0,1]) \to (Y^m_{\widetilde{\gamma}}, B^m_{\widetilde{\gamma}})$ by

$$\overline{\Phi}(x,t) = \overline{\Xi}(x,t) + P_E \circ \Phi(x,\mu(x,t)).$$

Then $\overline{\Phi}$ is well defined by (3). Since $\Phi^m_{0\alpha} : (X_\alpha, A_\alpha) \to (Y^m_{\widetilde{L}_0\alpha}, B^m_{\widetilde{L}_0\alpha})$ is a restriction of $\overline{\Phi}_0 = \overline{\Phi}(\,\cdot\,,0) : (X_\gamma, A_\gamma) \to (Y^m_{\widetilde{\gamma}}, B^m_{\widetilde{\gamma}})$, by the functoriality of the Mayer–Vietoris homomorphism, the following diagram commutes:

$$\begin{array}{ccc}
H_c^{q+d(\alpha)}(Y^m_{\widetilde{L}_0\alpha}, B^m_{\widetilde{L}_0\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(\Phi^m_{0\alpha})} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\
{\scriptstyle \Delta^q_{\widetilde{L}_0\alpha\widetilde{L}_1\gamma}(Y^m,B^m)}\big\downarrow & & \big\downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\
H_c^{q+d(\gamma)}(Y^m_{\widetilde{L}_1\gamma}, B^m_{\widetilde{L}_1\gamma}) & \xrightarrow{H_c^{q+d(\gamma)}(\overline{\Phi}_0)} & H_c^{q+d(\gamma)}(X_\gamma, A_\gamma)
\end{array}$$

Since $\overline{\Phi}_0$ is homotopic to $\overline{\Phi}_1 = \Phi^m_{1\gamma}$ via the proper homotopy $\overline{\Phi}$, we get

$$H_c^{q+d(\gamma)}(\overline{\Phi}_0) = H_c^{q+d(\gamma)}(\Phi^m_{1\gamma}).$$

Therefore diagram (10.2) commutes.

Arguing as in Lemma 7.2, it is easy to show that the commutativity of (10.2) implies

$$\text{(10.3)}\quad \varinjlim_{\alpha\in\mathcal{V}_\beta} H_c^{q+d(\alpha)}(\Phi^m_{0\ \alpha}) \circ \widehat{L}^q_0(Y^m, B^m) = \varinjlim_{\alpha\in\mathcal{V}_\beta} H_c^{q+d(\alpha)}(\Phi^m_{1\alpha}) \circ \widehat{L}^q_1(Y^m, B^m).$$

We develop the left-hand side of this equality:

$$\begin{aligned}
&= \varinjlim_{\alpha\in\mathcal{V}_\beta} H_c^{q+d(\alpha)}(L_{0\alpha}) \circ H_c^{q+d(\alpha)}((i_m \circ \Psi_0)_{\widetilde{L}_0\alpha}) \circ \widehat{L}^q_0(Y^m, B^m) \\
&= \widetilde{G}^q_E(L_0) \circ \widehat{L}^q_0(Z_0, C_0) \circ H^q_E(i_m \circ \Phi_0) \circ \widehat{L}^q_0(Y^m, B^m)^{-1} \circ \widehat{L}^q_0(Y^m, B^m) \\
&= G^q_E(L_0) \circ H^q_E(i_m \circ \Phi_0) = H^q_E(\Phi^m_0).
\end{aligned}$$

In the same way, the right-hand side of (10.3) is equal to $H^q_E(\Phi^m_1)$. Therefore,

$$H^q_E(\Phi_0) \circ H^q_E(i_m) = H^q_E(\Phi_1) \circ H^q_E(i_m).$$

Taking the direct limit over $m \in \mathbb{N}$, by Proposition 4.1 we get $H^q_E(\Phi_0) = H^q_E(\Phi_1)$. This proves the assertion in the case of a special E-homotopy.

2. Notice that every E-homotopy can be written in the form (10.1), but with $\Psi_t(x) = x + K(x,t)$ and $K(\Omega)$ precompact.

Set $\Omega' = \{(x,t) \in H \times [0,1] \mid x \in C_t\}$. Let $r \in \mathbb{N}$. By Proposition 4.4, in a slightly modified form, we can find maps

$$\Psi^r : (\Omega, \Omega') \to (Y^r, B^r) \quad \text{of the form} \quad \Psi^r(x,t) = x + R^r(x,t)$$

where $\pi \circ R^r(\Omega) \subset \alpha_0 \in \mathcal{V}$ and $\{\Psi^r(\,\cdot\,,t)\}_{r\in\mathbb{N}}$ is an approximating system for $\Psi(\,\cdot\,,t)$, for each $t \in [0,1]$. Let $\Phi^r(x,t) = \Psi^r(L_t x, t)$. Then Φ^r is a special E-homotopy and therefore, by step 1,

$$H_E^*(\Phi_0^r) = H_E^*(\Phi_1^r), \quad \forall r \in \mathbb{N}.$$

Thus

$$G_E^*(L_0) \circ H_E^*(\Psi_0^r) = G_E^*(L_1) \circ H_E^*(\Psi_1^r).$$

Taking the direct limit over $r \in \mathbb{N}$, we get $G_E^*(L_0) \circ H_E^*(\Psi_0) = G_E^*(L_1) \circ H_E^*(\Psi_1)$ and therefore $H_E^*(\Phi_0) = H_E^*(\Phi_1)$. □

11. Naturality of the coboundary. The coboundary operator δ_E^* is natural with respect to the functor G_E^*:

Proposition 11.1. *Let $L : (X,A) \to (Y,B)$ be an E-isomorphism. Then the following diagram is commutative:*

$$\begin{tikzcd}
H_E^q(B) \arrow[r, "G_E^q(L|_A)"] \arrow[d, "\delta_E^q(Y,B)"'] & H_E^q(A) \arrow[d, "\delta_E^q(X,A)"] \\
H_E^{q+1}(Y,B) \arrow[r, "G_E^{q+1}(L)"] & H_E^{q+1}(X,A)
\end{tikzcd}$$

Proof. The diagram

$$\begin{tikzcd}
H_c^{q+d(\alpha)}(B_{\widetilde{L}\alpha}) \arrow[r, "H_c^{q+d(\alpha)}(L|_{A_\alpha})"] \arrow[d, "\delta_c^{q+d(\alpha)}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha})"'] & H_c^{q+d(\alpha)}(A_\alpha) \arrow[d, "\delta_c^{q+d(\alpha)}(X_\alpha, A_\alpha)"] \\
H_c^{q+d(\alpha)+1}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}) \arrow[r, "H_c^{q+d(\alpha)+1}(L_\alpha)"] & H_c^{q+d(\alpha)+1}(X_\alpha, A_\alpha)
\end{tikzcd}$$

is commutative for each $\alpha \in \mathcal{V}$. Taking the direct limit over $\mathcal{V}$ of these systems of homomorphisms, we find that the following diagram commutes:

$$\begin{tikzcd}
\varinjlim_{\alpha\in\mathcal{V}} H_c^{q+d(\alpha)}(B_{\widetilde{L}\alpha}) \arrow[r, "\widetilde{G}_E^q(L|_A)"] \arrow[d, "\varinjlim_{\alpha\in\mathcal{V}} \delta_E^{q+d(\alpha)}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha})"'] & H_E^q(A) \arrow[d, "\delta_E^q(X,A)"] \\
\varinjlim_{\alpha\in\mathcal{V}} H_c^{q+d(\alpha)+1}(Y_{\widetilde{L}\alpha}, B_{\widetilde{L}\alpha}) \arrow[r, "\widetilde{G}_E^{q+1}(L)"] & H_E^{q+1}(X,A)
\end{tikzcd}$$

Since $\widetilde{L}$ acts as an order preserving bijection on $\mathcal{V}$, the following diagram is commutative:

$$\begin{array}{ccc}
\varinjlim_{\alpha\in\mathcal{V}} H_c^{q+d(\alpha)}(B_\alpha) & \xrightarrow{\widehat{L}^q(B)} & \varinjlim_{\alpha\in\mathcal{V}} H_c^{q+d(\alpha)}(B_{\widetilde{L}\alpha}) \\
{\scriptstyle \varinjlim_{\alpha\in\mathcal{V}} \delta_c^{q+d(\alpha)}(Y_\alpha,B_\alpha)}\Big\downarrow & & \Big\downarrow{\scriptstyle \varinjlim_{\alpha\in\mathcal{V}} \delta_c^{q+d(\alpha)}(Y_{\widetilde{L}\alpha},B_{\widetilde{L}\alpha})} \\
\varinjlim_{\alpha\in\mathcal{V}} H_c^{q+d(\alpha)+1}(Y_\alpha,B_\alpha) & \xrightarrow{\widehat{L}^{q+1}(Y,B)} & \varinjlim_{\alpha\in\mathcal{V}} H_c^{q+d(\alpha)+1}(Y_{\widetilde{L}\alpha},B_{\widetilde{L}\alpha})
\end{array}$$

The conclusion follows from the commutativity of the above two diagrams. $\square$

Assertion (4) of Theorem 6.4 follows from this proposition and from assertion (4) of Theorem 0.2.

3. Morse theory

12. Dimension theory. We recall the following definition:

Definition 12.1. Two closed subspaces E and E' of a Hilbert space H are called *commensurable* if $\pi|_{E'} : E' \to H/E$ and $\pi'|_E : E \to H/E'$ are compact. Here H/E and H/E' are given the Hilbert topology induced by H, and π, π' are the quotient projections.

Commensurability is an equivalence relation. If $E' = V \oplus W$, where V is a subspace of E with finite codimension and W has finite dimension, then E' and E are commensurable: in fact, the projections involved have finite rank.

If P_E is the orthogonal projection onto E, the commensurability of E and E' can be rewritten in the following way: both $P_{E^\perp}$ restricted to E' and $P_{E'^\perp}$ restricted to E are compact. As a consequence, both $E^\perp \cap E'$ and $E \cap E'^\perp$ must be finite-dimensional. Therefore we can define a relative dimension:

$$D(E,E') = \dim E \cap E'^\perp - \dim E' \cap E^\perp.$$

The function D is anti-symmetric: $D(E,E') = -D(E',E)$. Moreover, if E and E' are commensurable, then also $E^\perp$ and $E'^\perp$ are commensurable and $D(E^\perp, E'^\perp) = -D(E,E')$.

Now we fix a closed subspace E of H and we take a closed subspace W commensurable with $E^\perp$. We can define the *E-dimension* of W by

$$E\text{-}\dim W = D(W,E^\perp) = \dim W \cap E - \operatorname{codim}_H(W+E).$$

Here we have used the fact that $(W+E)^\perp = W^\perp \cap E^\perp$.

From the discussion of Example 2.1 we know that, if W and $E^\perp$ are commensurable, then $\mathcal{T}_E$ induces the strong topology on W. Moreover, if S is the unit sphere in W then

$$H^q_E(S) = \begin{cases} \mathcal{A} & \text{if } q = E\text{-}\dim W - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We would like to see what happens to our cohomology theory H^*_E when we change E to a commensurable space E'. The topology $\mathcal{T}_E$ and the class of E-compact morphisms do not change:

Proposition 12.1. *Assume that E and E' are commensurable. Then:*

(1) *the topologies $\mathcal{T}_E$ and $\mathcal{T}_{E'}$ coincide;*
(2) *a map Φ is an E-compact morphism if and only if it is an E'-compact morphism.*

Proof. To prove the first assertion it is enough to show that the quotient projection $\pi' : H \to H/E'$ is $\mathcal{T}_E$-continuous. Now, $H = E \oplus E^\perp$ and the topology $\mathcal{T}_E$ coincides with the product topology $\{\text{weak on } E\} \times \{\text{strong on } E^\perp\}$; $\pi'|_{E^\perp}$ is obviously continuous and $\pi'|_E$ is also continuous when E has the weak topology because it is compact.

The second assertion follows readily from the first one, because the definition of E-compact morphisms involves only the topology $\mathcal{T}_E$. □

Let $\mathcal{E}$ be an equivalence class of the commensurability relation. By the above proposition we can denote by $\mathcal{T}_\mathcal{E}$ the topology $\mathcal{T}_E$ for some $E \in \mathcal{E}$. An *$\mathcal{E}$-compact morphism* will be an E-compact morphism for $E \in \mathcal{E}$.

However, the property of a set of being E-locally compact is not invariant under changing E to a commensurable space E': take E' commensurable with E such that $E \cap E' = 0$. If α is a finite-dimensional subspace of H/E and $\pi : H \to H/E$ is the quotient projection, then π restricted to $E' \cap \pi^{-1}(\alpha)$ is injective. Therefore $\dim E' \cap \pi^{-1}(\alpha) \leq d(\alpha) < \infty$ and $E' \cap \pi^{-1}(\alpha)$ is locally compact. So E' is E-locally compact but it fails to be E'-locally compact if it has infinite dimension.

For this reason we introduce a smaller class of sets:

Definition 12.2. Let $\mathcal{E}$ be an equivalence class of the commensurability relation. A $\mathcal{T}_\mathcal{E}$-closed set X is called *$\mathcal{E}$-locally compact* if $X \cap E$ is locally compact for every $E \in \mathcal{E}$.

If α is a finite-dimensional subspace of H/E and $\pi : H \to H/E$ is the quotient projection, then $\pi^{-1}(\alpha)$ is commensurable with E. Therefore an $\mathcal{E}$-locally compact set is E-locally compact for every $E \in \mathcal{E}$. All the bounded $\mathcal{T}_\mathcal{E}$-closed sets are $\mathcal{E}$-locally compact.

THEOREM 12.2. *Assume that E and E' belong to the same commensurability class $\mathcal{E}$. Then the cohomology theory $H^*_{E'}$ coincides with the theory $H_E^{*+D(E,E')}$ on the subcategory where the objects are only the $\mathcal{E}$-locally compact pairs and the morphisms are the $\mathcal{E}$-compact morphisms.*

PROOF. Assume first that E' has the form $S \oplus V$, where S is a subspace of $E^\perp$ of dimension s and V is a subspace of E of codimension r. In this case

$$D(E, E') = \dim E \cap (S \oplus V)^\perp - \dim E^\perp \cap (S \oplus V) = r - s.$$

Denote by

$$\pi_E : H \to H/E, \quad \pi_{S\oplus V} : H \to H/(S \oplus V), \quad \pi_{S\oplus E} : H \to H/(S \oplus E)$$

the quotient projections.

Now, $H/(S \oplus E)$ can be considered a subspace of $H/(S \oplus V)$ whose complementary subspace is $\varrho = \pi_{S\oplus V}(E)$, which has dimension r. It can also be considered a subspace of H/E whose complementary subspace is $\sigma = \pi_E(S)$, which has dimension s.

Denote by $\mathcal{V}(Y)$ the set of finite-dimensional linear subspaces of H/Y. If $\alpha \in \mathcal{V}(Y)$ denote by $\mathcal{V}_\alpha(Y)$ the cofinal subset of all $\beta \in \mathcal{V}(Y)$ containing α. If X is $\mathcal{E}$-locally compact, then

$$\begin{aligned} H^q_{S\oplus V}(X) &= \varinjlim_{\alpha \in \mathcal{V}_\varrho(S\oplus V)} H_c^{q+d(\alpha)}(X \cap \pi^{-1}_{S\oplus V}(\alpha)) \\ &= \varinjlim_{\beta \in \mathcal{V}(S\oplus E)} H_c^{q+d(\beta)+r}(X \cap \pi^{-1}_{S\oplus V}(\varrho \oplus \beta)). \end{aligned}$$

Notice that $\pi^{-1}_{S\oplus V}(\varrho \oplus \beta) = \pi_E^{-1}(\sigma \oplus \beta)$ for every $\beta \in \mathcal{V}(S \oplus E)$. Therefore,

$$\begin{aligned} H^q_{S\oplus V}(X) &= \varinjlim_{\beta \in \mathcal{V}(S\oplus E)} H_c^{q+d(\beta)+r}(X \cap \pi_E^{-1}(\sigma \oplus \beta)) \\ &= \varinjlim_{\alpha \in \mathcal{V}_\sigma(E)} H_c^{q+d(\alpha)-s+r}(X \cap \pi_E^{-1}(\alpha)) \\ &= H_E^{q-s+r}(X) = H_E^{q+D(E,E')}(X). \end{aligned}$$

We want to reduce the general case to the above one, by proving that if E' is commensurable with E, then it can be mapped onto a subspace of the form $S \oplus V$ by means of an invertible linear map of the form Identity + Compact. Set

$$S = E' \cap E^\perp, \quad R = E'^\perp \cap E.$$

Since E and E' are commensurable, it follows that S has finite dimension s, R has finite dimension r and $D(E, E') = r - s$.

Let V be the orthogonal complement of R in E and let W be the orthogonal complement of S in $E^\perp$. Then H splits as

$$H = E \oplus E^\perp = R \oplus V \oplus S \oplus W.$$

Let V' be the orthogonal complement of S in E' and let W' be the orthogonal complement of R in ${E'}^{\perp}$. Then H splits as

$$H = E' \oplus {E'}^{\perp} = S \oplus V' \oplus R \oplus W'.$$

Now, P_E restricted to V' is one-to-one. Moreover, $P_E(V') \subset V$: if $v' \in V'$ and $x \in R$, then $\langle P_E v', x\rangle = \langle v', x\rangle = 0$, because R is orthogonal to E'. So $P_E v'$ belongs to the orthogonal complement of R in E, which is V.

In the same way $P_{E^\perp}$ restricted to W' is one-to-one and $P_{E^\perp}(W') \subset W$. We can define a bounded linear operator T on H in the following way: $T = P_E$ on V', $T = P_{E^\perp}$ on W' and $T = I$ on $R \oplus S$. Then T is one-to-one and $T = I - K$ where K is defined as follows: $K = P_{E^\perp}$ on V', $K = P_E$ on W' and $K = 0$ on $R \oplus S$.

By the commensurability of E and E', $P_{E^\perp}$ is compact on E', and so also on $V' \subset E'$. Since also $E^\perp$ and ${E'}^{\perp}$ are commensurable, P_E is compact on ${E'}^{\perp}$, and so also on $W' \subset {E'}^{\perp}$. Therefore K is compact. Thus T is a compact perturbation of the identity and, being one-to-one, it must be onto: so $P_E(V') = V$ and $P_{E^\perp}(W') = W$.

By Proposition 0.1, T is an invertible $\mathcal{E}$-compact morphism and $TE' = S \oplus V$. Therefore, by our previous argument,

$$H^q_{E'}(X) = H^q_{S\oplus V}(T(X)) = H^q_{S\oplus V}(X) = H_E^{q+D(E,E')}(X).$$

A similar argument applies to the $\mathcal{E}$-compact morphisms. $\square$

13. The Morse index. Assume that $f : H \to \mathbb{R}$ is a function of class C^2. Let x be a critical point of f, that is, $df(x) = 0$. Let $d^2 f(x)$ be the second order differential of f at x, thought of as a symmetric bilinear form. Let $D^2 f(x)$ be the associated self-adjoint operator, defined by the relation

$$d^2 f(x)[u, v] = \langle D^2 f(x) u, v\rangle, \quad \forall u, v \in H.$$

By the spectral representation of self-adjoint operators,

$$D^2 f(x) = \int_{-\infty}^{\infty} \lambda \, dP_\lambda$$

where $\{P_\lambda \mid \lambda \in \mathbb{R}\}$ is a partition of the identity (see, for example, Chapter 6 of [3]). If $D^2 f(x)$ is invertible, its spectrum is bounded away from zero and H splits into two closed $D^2 f(x)$-invariant orthogonal subspaces:

$$H = V \oplus W, \quad V = \int_0^{\infty} dP_\lambda(H), \quad W = \int_{-\infty}^{0} dP_\lambda(H).$$

$D^2 f(x)$ is positive on V and negative on W; V and W are called the *positive* and *negative eigenspaces* of $d^2 f(x)$.

We will assume that V and E are commensurable. So also W and $E^{\perp}$ are commensurable and the following definition makes sense:

Definition 13.1. If x is a critical point of $f \in C^2(H)$, then the *E-Morse index* of x is the E-dimension of the negative eigenspace of $D^2 f(x)$:

$$m_E(x, f) = E\text{-}\dim W.$$

If $E = H$ one finds the usual definition of the Morse index of a critical point. If $E = \{0\}$ one finds the definition of the Morse co-index, that is, the dimension of the positive eigenspace of $D^2 f(x)$.

14. The Morse relations. In order to prove the Morse relations, we assume $\mathcal{A}$ to be a field, so that H_E^* is a functor from E-pairs and positive E-morphisms to $\mathcal{A}$-vector spaces and $\mathcal{A}$-linear maps.

The function $f : H \to \mathbb{R}$ is assumed to satisfy the following conditions:

(1) $f \in C^2(H)$;
(2) f satisfies the Palais–Smale condition: if x_n has the property that $f(x_n)$ is bounded and $\nabla f(x_n)$ converges to 0, then $\{x_n\}$ is precompact;
(4) the gradient flow defined by

$$\frac{\partial}{\partial t}\Phi(x, t) = -\nabla f(\Phi(x, t)), \quad \Phi(x, 0) = x,$$

exists for every $t \in \mathbb{R}$;
(4) the map Φ defined above is a positive E-homotopy;
(5) f is a *Morse function*: for each critical point x of f the bilinear form $d^2 f(x)$ is strongly non-degenerate, meaning that the associated linear operator is invertible;
(6) for every critical point x of f, the positive eigenspace of $D^2 f(x)$ is commensurable with E;
(7) each sublevel of f, $\{x \in H \mid f(x) \leq a\}$, is $\mathcal{T}_E$-closed and E-locally compact.

We introduce the following notations:

$$K = \{x \in H \mid df(x) = 0\}, \quad K_c = K \cap f^{-1}(c), \quad f^a = \{x \in H \mid f(x) \leq a\}.$$

By (7), (f^b, f^a) is an E-pair for every $a, b \in \mathbb{R}$.

By (5), K is a discrete subset of H with the strong topology. By (2) and (5), $K \cap f^{-1}([a, b])$ consists of finitely many points, for each $a < b$.

We recall that the Palais–Smale condition enables one to prove the following familiar deformation lemma:

LEMMA 14.1. *If $[a,b]$ contains no critical levels for f, then there exists $T>0$ such that $\Phi_T(f^b)\subset f^a$. If $[a,b]$ contains only one critical level $c\in\,]a,b[$, then for every neighborhood U of K_c there exists $\overline{a}$ with $a\le\overline{a}<c$ and $T>0$ such that $\Phi_T(f^b)\subset f^{\overline{a}}\cup U$.*

Now let x_0 be a critical point of f. We can assume that $x_0=0$ and $f(0)=0$. Let V and W be the positive and negative eigenspaces of $D^2f(x_0)$, respectively.

LEMMA 14.2. *Let $0\in\,]a,b[$ be the only critical level in $[a,b]$. Let $x_0=0$ be the only critical point at level 0. For each $\varepsilon>0$ there exist ε_-, ε_+ and $\overline{a}$ with $0<\varepsilon_-\le\varepsilon$, $0<\varepsilon_+\le\varepsilon$ and $a\le\overline{a}<0$ such that:*

(1) *if we set $Q=Q(\varepsilon_+,\varepsilon_-)=\overline{B_V(\varepsilon_+)}\oplus\overline{B_W(\varepsilon_-)}$ then*
$$\partial_W Q=\overline{B_V(\varepsilon_+)}\oplus\partial B_W(\varepsilon_-)\subset f^{\overline{a}};$$
(2) *for each $x^+\in\overline{B_V(\varepsilon_+)}$ the set*
$$(Q\setminus f^{\overline{a}})\cap(x^+\oplus W)$$
is star-shaped with respect to x^+;

(3) *$\Phi_t(Q\cup f^{\overline{a}})\subset Q\cup f^{\overline{a}}$ for every $t\ge0$ and there exists $T>0$ such that $\Phi_T(f^b)\subset Q\cup f^{\overline{a}}$;*

(4) *there exists δ with $0<\delta<\varepsilon_-$ such that*
$$\overline{B_V(\varepsilon_+)}\oplus\overline{B_W(\delta)}\subset Q\setminus f^{\overline{a}}.$$

PROOF. We just sketch the proof. Set
$$C=\|d^2f(0)\|=\sup_{\substack{v,w\in H\\ \|v\|=\|w\|\le1}}|d^2f(0)[v,w]|.$$
Since the self-adjoint operator corresponding to $d^2f(0)$ is invertible (by assumption (5)), we can find positive numbers λ_+ and λ_- such that

$$d^2f(0)[v,v]\ge\lambda_+\|v\|^2,\qquad\forall v\in V,\tag{14.1}$$
$$d^2f(0)[v,v]\le-\lambda_-\|v\|^2,\quad\forall v\in W.\tag{14.2}$$

Since the set of strictly positive symmetric forms is open and $f\in C^2(H)$, if ε_+ and ε_- are small enough then

$$d^2f(x)>0\quad\text{on } V,\quad\forall x\in Q(\varepsilon_+,\varepsilon_-),\tag{14.3}$$
$$d^2f(x)<0\quad\text{on } W,\quad\forall x\in Q(\varepsilon_+,\varepsilon_-).\tag{14.4}$$

If $x\in\partial_W Q(\varepsilon_+,\varepsilon_-)$ then

$$f(x)=\frac12 d^2f(0)[x,x]+o(\|x\|^2)\le\frac{C}{2}\varepsilon_+^2-\frac{\lambda_-}{2}\varepsilon_-^2+o(\varepsilon_+^2+\varepsilon_-^2).\tag{14.5}$$

Thus, if ε_+ and ε_- are small enough and

$$\varepsilon_-^2 = (C/\lambda_- + 1)\varepsilon_+^2 \tag{14.6}$$

then we can find $\overline{a}$ with $a \leq \overline{a} < 0$ such that

$$f(x) < \overline{a}, \quad \forall x \in \partial_W Q(\varepsilon_+, \varepsilon_-), \tag{14.7}$$

and assertion (1) is proved.

By (14.4), f is concave on $(x^+ \oplus W) \cap Q$. Therefore the set

$$\{y \in (x^+ \oplus W) \cap Q \mid f(y) > \overline{a}\} \tag{14.8}$$

is convex for every $x^+ \in \overline{B_V(\varepsilon_+)}$. From (14.6), $f(x^+) \geq f(0) = 0$ and thus x_+ belongs to the set (14.8), which must be star-shaped with respect to x^+. This proves assertion (2).

Assertion (3) is an immediate consequence of the Taylor formula.

To prove (4) it is enough to prove that, if a flow line of Φ_t enters Q, then it can exit only through $f^{\overline{a}}$. The remaining part of the statement follows from this fact and from a standard application of the Palais–Smale condition.

So assume that $y = \Phi_{t_0}(x)$ belongs to the boundary of Q. If $y \in \partial_W Q$, then $f(y) < \overline{a}$ by (14.7) and there is nothing to prove. Otherwise y must belong to $\partial B_V(\varepsilon_+) \oplus \overline{B_W(\varepsilon_-)}$. Set

$$h(t) = \|P_V \Phi(x,t)\|^2$$

where $P_V : H \to H$ is the orthogonal projection onto V. Then

$$\begin{aligned} h'(t_0) &= -2\langle P_V \nabla f(y), P_V y\rangle \\ &= -2\langle D^2 f(0) y + o(\|y\|), P_V y\rangle \leq -2\lambda_+ \varepsilon_+^2 + o(\varepsilon_+^2 + \varepsilon_-^2) \end{aligned} \tag{14.9}$$

where $D^2 f(0)$ is the self-adjoint operator corresponding to the symmetric form $d^2 f(0)$.

By (14.9), $h'(t_0)$ is negative provided ε_+ and ε_- are small enough: this means that $\Phi(x,t)$ enters Q at time $t = t_0$. □

Let (X, Y, A) be a triplet of $\mathcal{T}_E$-closed and E-locally compact subsets of H. The following two lemmas are immediate consequences of the homotopy invariance of H_E^*:

Lemma 14.3. *If there exists an E-homotopy or an E-radial homotopy* (see Definition 3.2)

$$\Psi : (X \times [0,1], A \times [0,1]) \to (X, A)$$

such that $\Psi_0 = \mathrm{id}$, $\Psi_1(X) \subset Y$ and $\Psi_t(Y) \subset Y$ for every $t \in [0,1]$, then

$$H_E^*(X, A) \cong H_E^*(Y, A),$$

the isomorphism being induced by the inclusion map.

Lemma 14.4. *If there exists an E-homotopy or an E-radial homotopy*

$$\Psi : (X \times [0,1], Y \times [0,1]) \to (X, Y)$$

such that $\Psi_0 = \mathrm{id}$, $\Psi_1(Y) \subset A$ *and* $\Psi_t(A) \subset A$ *for every* $t \in [0,1]$, *then*

$$H^*_E(X, Y) \cong H^*_E(X, A),$$

the isomorphism being induced by the inclusion map.

Now we can use Lemmas 14.1 and 14.2 to compute the E-cohomology of a pair of sublevels of f.

Proposition 14.5. *If* $[a, b]$ *contains no critical levels, then* $H^*_E(f^b, f^a) = 0$.

Proof. Choose $T > 0$ which satisfies the first statement of Lemma 14.1. Then, by property (4),

$$\Phi : (f^b \times [0,T], f^a \times [0,T]) \to (f^b, f^a)$$

is a positive E-homotopy such that $\Phi_0 = \mathrm{id}$ and $\Phi_T(f^b) \subset f^a$. By Lemma 14.3,

$$H^*_E(f^b, f^a) \cong H^*_E(f^a, f^a) = 0. \qquad \square$$

Proposition 14.6. *Let* $\overline{a} < c < \overline{b}$. *If* $K_c = \{x_0\}$, *then there exist* a *and* b *with* $\overline{a} < a < c < b < \overline{b}$ *such that*

$$H^q_E(f^b, f^a) = \begin{cases} \mathcal{A} & \text{if } q = m_E(f, x_0), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By a translation and adding a constant to f, we can assume that $x_0 = 0$ and $f(0) = c = 0$.

Let V and W be the positive and negative eigenspaces of $D^2 f(0)$. By assumption (6), V and E are commensurable. So also W and $E^\perp$ are commensurable and the E-Morse index of x_0 is

$$m_E(x_0, f) = E\text{-}\dim W = D(W, E^\perp) = -D(V, E).$$

Then, by Theorem 12.2,

$$H^q_E(X, A) = H^{q+D(V,E)}_V(X, A) = H^{q-m_E(x_0,f)}_V(X, A), \tag{14.10}$$

at least for any bounded $\mathcal{T}_E$-closed pair (X, A).

Choose Q, $\partial_V D$, $\overline{a}$, T and δ as in Lemma 14.2. Then

$$\Phi : (f^b \times [0,T], f^{\overline{a}} \times [0,T]) \to (f^b, f^{\overline{a}})$$

is a positive E-homotopy such that $\Phi_0 = \mathrm{id}$, $\Phi_T(f^b) \subset Q \cup f^{\overline{a}}$ and $\Phi_t(Q \cup f^{\overline{a}}) \subset Q \cup f^{\overline{a}}$. Then, by Lemma 14.3,

$$H^*_E(f^b, f^{\overline{a}}) \cong H^*_E(f^{\overline{a}} \cup Q, f^{\overline{a}}). \tag{14.11}$$

By the strong excision property of H^*_E,

$$H^*_E(f^{\overline{a}} \cup Q, f^{\overline{a}}) \cong H^*_E(Q, Q \cap f^{\overline{a}}). \tag{14.12}$$

Let $\varrho : [0, \varepsilon_-] \times [0,1] \to [0, \varepsilon_-]$ be a smooth function such that:

(1) $\varrho_0(y) = \varrho(y, 0) = y$ for every $y \in [0, \varepsilon_-]$;
(2) $\varrho_1(y) = \varrho(y, 1) = \varepsilon_-$ for every $y \in [\delta, \varepsilon_-]$;
(3) $\varrho_t(0) = \varrho(0, t) = 0$ and $\varrho_t(\varepsilon_-) = \varrho(\varepsilon_-, t) = \varepsilon_-$ for every $t \in [0,1]$.

Set

$$R(x,t) = P_V x + \frac{\varrho_t(\|P_W x\|)}{\|P_W x\|} P_W x$$

where P_V and P_W are the orthogonal projections onto V and W, respectively. By Lemma 14.2(2), (3), we can consider R as a map

$$R : (Q \times [0,1], f^{\overline{a}} \cap Q \times [0,1]) \to (Q, f^{\overline{a}} \cap Q).$$

Clearly, $R_0 = \mathrm{id}$ and $R_1(f^{\overline{a}} \cap Q) = \partial_W Q$. Moreover, $R_t(\partial_W Q) \subset \partial_W Q$ for each $t \in [0,1]$. It is easy to see that R is $\mathcal{T}_V$-continuous and it is a V-radial homotopy. By Lemma 14.4,

$$H^*_V(Q, f^{\overline{a}} \cap Q) \cong H^*_V(Q, \partial_W Q). \tag{14.13}$$

By (14.10)–(14.13) and the result of Example 2.2,

$$H^q_E(f^b, f^{\overline{a}}) \cong H_V^{q - m_E(x_0, f)}(Q, \partial_W Q) = \begin{cases} \mathcal{A} & \text{if } q - m_E(x_0, f) = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

It is easy to generalize Lemma 14.2 and Proposition 14.6 to the case of more critical points at the same level. We obtain:

Proposition 14.7. *Let $\overline{a} < c < \overline{b}$. If $K_c = \{x_1, \ldots, x_s\}$, then there exist a and b with $\overline{a} < a < c < b < \overline{b}$ such that*

$$P_E(f^b, f^a) = \sum_{i=1}^{s} t^{m_E(f, x_i)}.$$

The last ingredient to prove Morse relations is a well known property of any cohomology theory; we need it in the following form:

Lemma 14.8. *Assume that $X_0 \subset X_1 \subset \ldots \subset X_n$ are $\mathcal{T}_E$-closed and E-locally compact subsets of H such that the graded linear space $H^*_E(X_{i+1}, X_i)$ is finitely generated for each $i = 0, \ldots, n-1$. Then there exists a Laurent polynomial Q with non-negative integer coefficients such that*

$$\sum_{i=0}^{n} P_E(X_{i+1}, X_i) = P_E(X_n, X_0) + (1+t)Q(t).$$

Proof. We argue by induction on n. The assertion is true if $n = 1$: in this case $Q = 0$. Assume that it holds for $n = m$:

$$\sum_{i=0}^{m-1} P_E(X_{i+1}, X_i) = P_E(X_m, X_0) + (1+t)Q_m(t). \tag{14.14}$$

Write the long exact sequence for the triplet (X_{m+1}, X_m, X_0):

$$\begin{aligned} \ldots \to H_E^q(X_{m+1}, X_m) \to H_E^q(X_{m+1}, X_0) \to H_E^q(X_m, X_0)& \\ \to H_E^{q+1}(X_{m+1}, X_m) \to \ldots& \end{aligned}$$

Since the above sequence is exact, we have, for all $q \in \mathbb{Z}$,

$$\begin{aligned} H_E^q(X_{m+1}, X_m) &= A_q \oplus B_q, \\ H_E^q(X_{m+1}, X_0) &= B_q \oplus C_q, \\ H_E^q(X_m, X_0) &= C_q \oplus A_{q+1}, \end{aligned}$$

where A_q, B_q and C_q are finite-dimensional $\mathcal{A}$-vector spaces. Set

$$a_q = \dim_{\mathcal{A}} A_q, \quad b_q = \dim_{\mathcal{A}} B_q, \quad c_q = \dim_{\mathcal{A}} C_q.$$

Then

$$\begin{aligned} P_E(X_m, X_0) + P_E(X_{m+1}, X_m) &= \sum_{q\in\mathbb{Z}} (c_q + a_{q+1})t^q + \sum_{q\in\mathbb{Z}} (a_q + b_q)t^q \\ &= \sum_{q\in\mathbb{Z}} (c_q + b_q)t^q + \sum_{q\in\mathbb{Z}} a_{q+1}t^q + \sum_{q\in\mathbb{Z}} a_q t^q \\ &= P_E(X_{m+1}, X_0) + (1+t)\sum_{q\in\mathbb{Z}} a_{q+1}t^q. \end{aligned}$$

Then, by (14.14),

$$\begin{aligned} \sum_{i=0}^{m} P_E(X_{i+1}, X_i) &= P_E(X_m, X_0) + P_E(X_{m+1}, X_m) + (1+t)Q_m(t) \\ &= P_E(X_{m+1}, X_0) + (1+t)\Big[Q_m(t) + \sum_{q\in\mathbb{Z}} a_{q+1}t^q\Big]. \qquad \square \end{aligned}$$

Finally, we can prove the Morse relations:

Theorem 14.9. *Assume that f satisfies the conditions* (1)–(7). *Let a and b be regular values for f. Then there exists a Laurent polynomial Q with non-negative integer coefficients such that*

$$\sum_{x\in K\cap f^{-1}([a,b])} t^{m_E(f,x)} = P_E(f^b, f^a) + (1+t)Q(t).$$

Proof. Let $c_1, \dots, c_k$ be the critical levels of f in $[a,b]$. We can assume that $a = c_0 < c_1 < \dots < c_k < c_{k+1} = b$. For each $j = 1, \dots, k$ choose regular levels a_j, b_j with

$$c_{j-1} < a_j < c_j < b_j < c_{j+1}$$

which satisfy the assumptions of Proposition 14.7. Then

$$P_E(f^{b_j}, f^{a_j}) = \sum_{x \in K_{c_j}} t^{m_E(f,x)}. \tag{14.15}$$

There are no critical levels in $[a, a_1]$ and $[b_j, a_{j+1}]$, $j = 1, \dots, k$. Therefore, by Proposition 14.5,

$$P_E(f^{a_1}, f^a) = 0, \quad P_E(f^{b_j}, f^{a_{j+1}}) = 0, \quad \forall j = 1, \dots, k. \tag{14.16}$$

So we have subdivided the interval $[a,b]$ in

$$d_0 = a < d_1 = a_1 < d_2 = c_1 < d_3 = b_1 < \dots < d_n = b_k < d_{n+1} = b$$

and, by (14.15) and (14.16), we know $P_E(f^{d_{i+1}}, f^{d_i})$ for each $i = 0, \dots, n$. By Lemma 14.8 there exists a Laurent polynomial Q with non-negative integer coefficients such that

$$\sum_{i=0}^{n} P_E(f^{d_{i+1}}, f^{d_i}) = P_E(f^b, f^a) + (1+t)Q(t),$$

and by (14.15) and (14.16) the left-hand side of this equality coincides with

$$\sum_{x \in K \cap f^{-1}([a,b])} t^{m_E(f,x)}. \qquad \square$$

15. Testing the hypotheses. Assume that $f : H \to \mathbb{R}$ is a function of class C^2 of the form

$$f(x) = \tfrac{1}{2}\langle Lx, x\rangle + b(x) \tag{15.1}$$

where L is a self-adjoint invertible linear operator. We are going to exhibit conditions on b allowing Theorem 14.9 to be applied to functions of this form.

It seems natural to take as E the positive eigenspace of L.

The first thing we need is a globally defined gradient flow, that is, a map $\Phi : H \times \mathbb{R} \to H$ which solves the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(x,t) = -\nabla f(\Phi(x,t)) = -L\Phi(x,t) - \nabla b(\Phi(x,t)), \\ \Phi(x,0) = x. \end{cases} \tag{15.2}$$

We assume that ∇b is globally Lipschitz, so that (15.2) has a global solution, by standard arguments.

Next we need Φ to be a positive E-homotopy.

Proposition 15.1. *Assume that ∇b is globally Lipschitz, $\mathcal{T}_E$-continuous and ∇b maps bounded sets into $\mathcal{T}_E$-precompact sets. Then $\Phi|_{H\times[-T,T]}$ is a positive E-homotopy for every $T>0$.*

Proof. Set

$$\Phi_0(x,t)=x,\quad \Phi_n(x,t)=x-\int_0^t \nabla f(\Phi_{n-1}(x,s))\,ds,\quad n\geq 1.$$

It is a standard fact in the theory of ordinary differential equations that, since $\nabla f = L+\nabla b$ is globally Lipschitz, Φ_n converges to Φ uniformly on bounded subsets of $H\times\mathbb{R}$.

Since ∇f maps bounded sets into bounded sets, so does Φ_n. Therefore also Φ maps bounded sets into bounded sets.

Since $LE=E$, L is $\mathcal{T}_E$-continuous; hence so are $\nabla f=L+\nabla b$ and Φ_n. To show that Φ is also $\mathcal{T}_E$-continuous, we must show that both $\pi\circ\Phi : H\times\mathbb{R}\to H/E$ and $g_y(x,t)=\langle\Phi(x,t),y)\rangle$ are $\mathcal{T}_E$-continuous for every $y\in H$.

Both $\pi\circ\Phi_n$ and $g_y^n(x,t)=\langle\Phi_n(x,t),y\rangle$, $y\in H$, are $\mathcal{T}_E$-continuous. Moreover, $\pi\circ\Phi_n$ and g_y^n converge uniformly to $\pi\circ\Phi$ and g_y, respectively. Therefore $\pi\circ\Phi$ and g_y are $\mathcal{T}_E$-continuous for every $y\in H$.

Since Φ solves the non-homogeneous equation

$$\frac{\partial}{\partial t}\Phi(x,t)+L\Phi(x,t)=-\nabla b(\Phi(x,t))$$

it can be represented as

$$\Phi(x,t)=e^{-tL}x-\int_0^t e^{(s-t)L}\nabla b(\Phi(x,s))\,ds.$$

Since $LE=E$, also $e^{-tL}E=E$. Therefore e^{-tL} is a positive E-isotopy. Set

$$K(x,t)=-\int_0^t e^{(s-t)L}\nabla b(\Phi(x,s))\,ds.$$

If $X\subset H$ is bounded and $T>0$, then $\Phi(X\times[-T,T])$ is bounded, as we showed before. Therefore $\nabla b(\Phi(X\times[-T,T]))$ is $\mathcal{T}_E$-precompact. Since $e^{(s-t)L}$ is $\mathcal{T}_E$-continuous, we conclude that $K(X\times[-T,T])$ is $\mathcal{T}_E$-precompact.

Finally, since $\Phi_t=\Phi(\,\cdot\,,t)$ is a diffeomorphism and $\Phi_t^{-1}=\Phi_{-t}$,

$$(\Phi|_{H\times[-T,T]})^{-1}(X)=\Phi(X\times[-T,T])$$

must be bounded for every bounded $X\subset H$. This concludes the proof. □

By our choice of E, $\frac{1}{2}\langle Lx,x\rangle$ is $\mathcal{T}_E$-lower semicontinuous, being strongly continuous and convex on E. Hence:

Proposition 15.2. *If b is $\mathcal{T}_E$-lower semicontinuous, then f^a is $\mathcal{T}_E$-closed for every $a \in \mathbb{R}$.*

Since every bounded and $\mathcal{T}_E$-closed set is E-locally compact, the condition on the E-local compactness of the sublevels of f is only a condition on the growth of f at infinity.

Notice that if $f \geq g$ and g has E-locally compact sublevels, the same happens to f. Therefore it seems useful to find a class of functions with this property, to be compared with f.

Lemma 15.3. *Let V be a linear subspace of E with finite codimension and let W be its orthogonal complement in H. If σ, θ, λ_+ and λ_- are positive constants, then the function*

$$g(x) = \lambda_+ \|P_V x\|^\sigma - \lambda_- \|P_W x\|^\theta$$

has E-locally compact sublevels.

Proof. Let $\pi_E : H \to H/E$ and $\pi_V : H \to H/V$ be the quotient projections. Since $V \subset E$, H/E can be considered a subspace of H/V. Since V has finite codimension in E, the space $\pi_V(E)$, which is complementary to H/E in H/V, is finite-dimensional.

If α is a finite-dimensional subspace of H/E, then

$$\pi_E^{-1}(\alpha) = \pi_V^{-1}(\alpha + \pi_V(E)).$$

Therefore E-local compactness is equivalent to V-local compactness, and it is enough to show that g has V-locally compact sublevels.

Notice that g is $\mathcal{T}_V$-lower semicontinuous, and thus its sublevels are $\mathcal{T}_V$-closed.

Let Y be a finite-dimensional linear subspace of W. We must show that for every $a \in \mathbb{R}$,

$$g^a \cap (V \oplus Y) = \{x \in V \oplus Y \mid g(x) \leq a\}$$

is weakly locally compact.

Since Y has finite dimension, the function

$$h : V \oplus Y \to \mathbb{R}, \quad h(x) = \|P_Y x\|,$$

is weakly continuous. Therefore the set

$$U_R = \{x \in V \oplus Y \mid \|P_Y x\| < R\}$$

is weakly open in $V \oplus Y$ for every R. If $x \in g^a \cap U_R$ then

$$\|P_V x\|^\sigma \leq \frac{\lambda_-}{\lambda_+} \|P_Y x\|^\theta + a < \frac{\lambda_-}{\lambda_+} R^\theta + a$$

and thus $g^a \cap U_R$ is bounded. Therefore the weak closure of $g^a \cap U_R$ in $V \oplus Y$ is weakly compact and $g^a \cap (V \oplus Y)$ is weakly locally compact. □

For example, the following growth condition on b guarantees the E-local compactness of the sublevels of f:

Proposition 15.4. *Take $\lambda > 0$ such that $E_\lambda = \int_0^\lambda dP_\nu(E)$ is finite-dimensional, where $L = \int \nu \, dP_\nu$ is the spectral decomposition of L. If there exist $\mu < \lambda$ and $C > 0$ such that*

$$b(x) \geq -\frac{\mu}{2}\|x\|^2 - C, \quad \forall x \in H,$$

then f has E-locally compact sublevels.

Proof. Set $W = E_\lambda \oplus F$ and let V be its orthogonal complement. Then both V and W are L-invariant and

$$\begin{aligned} f(x) &= \frac{1}{2}\langle LP_V x, P_V x\rangle + \frac{1}{2}\langle LP_W x, P_W x\rangle + b(x) \\ &\geq \frac{\lambda}{2}\|P_V x\|^2 - \frac{\|L\|}{2}\|P_W x\|^2 - \frac{\mu}{2}\|x\|^2 - C \\ &= \frac{\lambda - \mu}{2}\|P_V x\|^2 - \frac{\|L\| + \mu}{2}\|P_W x\|^2 - C. \end{aligned}$$

Since V has finite codimension in E, by Lemma 15.3, f has E-locally compact sublevels. □

Finally, we need the fact that the positive eigenspace of $D^2 f(x)$ is commensurable with E, for every critical point x. In our case:

$$D^2 f(x) = L + D^2 b(x), \quad \forall x \in H.$$

Proposition 15.5. *Assume that $D^2 f(x)$ is invertible and that $D^2 b(x)$ is compact. Then the positive eigenspace of $D^2 f(x)$ is commensurable with E.*

Proof. Set $K = D^2 b(x)$. Since the problem is symmetric, it is enough to prove that the quotient projection $\pi : H \to H/E$ restricted to the positive eigenspace of $L + K$ is compact.

If T is an invertible self-adjoint operator, one can define its positive part T^+ and its negative part T^-. These are bounded positive operators such that

$$T = T^+ - T^-, \quad |T| = T^+ + T^-,$$

where $|T| = \sqrt{T^2}$ is the modulus of T. The positive eigenspace of T is $\operatorname{Ker} T^- = T^+(H)$, while the negative eigenspace of T is $\operatorname{Ker} T^+ = T^-(H)$. Therefore $E = \operatorname{Ker} L^- = L^+(H)$.

Since $H = E \oplus L^-(H)$, $\pi|_{L^-(H)}$ is an isomorphism and the following diagram commutes:

$$\begin{tikzcd} & L^-(H) \arrow[d, "\pi|_{L^-(H)}"] \\ H \arrow[ur, "L^-"] \arrow[r, "\pi"] & H/E \end{tikzcd}$$

So we must show that L^- restricted to $(L+K)^+(H)$ is compact. Since $(L+K)^+$ is invertible on $(L+K)^+(H)$ we can show, equivalently, that $L^- \circ (L+K)^+$ is compact on H. Since $(L+K)^+ = \frac{1}{2}(|L+K| + L + K)$ we have

$$L^- \circ (L+K)^+ = \tfrac{1}{2} L^- \circ (|L+K| + L + K) = \tfrac{1}{2} L^- \circ (|L+K| - |L| + K)$$

where we have used the fact that $L^- \circ L = -(L^-)^2 = -L^- \circ |L|$. Therefore it is enough to prove that $|L+K| - |L|$ is compact, whenever K is compact.

This follows from this general fact: if L is a bounded self-adjoint operator, K is a compact self-adjoint operator and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $h(L+K) - h(L)$ is compact. In our case $h(s) = |s|$.

To prove the last assertion, notice that

$$(L+K)^m - L^m = L^{m-1} \circ K + L \circ K \circ L^{m-2} + \ldots + K^m$$

is compact for every $m \in \mathbb{N}$. Therefore $p(L+K) - p(L)$ is compact for every polynomial p. Now choose a sequence of polynomials p_n which converges uniformly to h on a bounded set containing both the spectrum of L and the spectrum of $L+K$. Then $p_n(L+K)$ converges to $h(L+K)$ and $p_n(L)$ converges to $h(L)$ in the operator norm. So $h(L+K) - h(L)$ is compact, being a limit in the operator norm of compact operators. □

Looking back at all the results of this section, we can state the following result, which gives sufficient conditions on b to apply the E-Morse theory.

Corollary 15.6. *Assume that*

$$f(x) = \tfrac{1}{2}\langle Lx, x\rangle + b(x)$$

is a C^2 function which satisfies the Palais–Smale condition and which has only strongly non-degenerate critical points. Assume that the linear operator L is self-adjoint and invertible and let E be its positive eigenspace. Assume that b is weakly continuous, ∇b is Lipschitz and completely continuous and $D^2 b(x)$ is a compact operator for every $x \in H$. Assume, moreover, that the following lower estimate holds: there exist $0 < p < 2$ and $C > 0$ such that

$$b(x) \geq -C\|x\|^p - C, \quad \forall x \in H. \tag{15.3}$$

Then the conditions (1)–(7) *of Section* 14 *hold and Theorem* 14.9 *can be applied.*

Proof. Conditions (1), (2) and (5) hold. Since ∇b is Lipschitz, so is ∇f and (3) holds.

Recall that a map $\Phi : H \to H$ is completely continuous if it is continuous from the strong topology of H to the weak one. In particular, ∇b is $\mathcal{T}_E$-continuous and it maps bounded sets into $\mathcal{T}_E$-precompact sets. So, by Proposition 15.1, condition (4) holds.

Being weakly continuous, b is a fortiori $\mathcal{T}_E$-lower semicontinuous and by Proposition 15.2, f^a is $\mathcal{T}_E$-closed for every $a \in \mathbb{R}$. By (15.3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $b(x) \geq -\varepsilon \|x\|^2 - C_\varepsilon$ for all $x \in H$, and Proposition 15.4 can be applied; so condition (7) holds.

Finally, condition (6) follows from Proposition 15.5. □

16. Final remarks. In the previous section we saw that the E-local compactness of the sublevels of f involves a lower bound which is quite unnatural. The E-local compactness was introduced in Part 1 in order to use the Alexander–Spanier cohomology with compact supports as the starting point of the whole theory. Such a cohomology was necessary to have Proposition 4.2, which allows one to extend the E-cohomology theory from cobounding pairs to generic unbounded pairs.

Another approach is possible: one can construct an E-cohomology theory using the normal Alexander–Spanier cohomology and then use the formula of Proposition 4.2 as a definition for the E-cohomology of a generic unbounded pair (in a similar way a cohomology with compact supports is obtained from a usual cohomology in [7]). In this way one finds an E-cohomology theory for arbitrary $\mathcal{T}_E$-closed pairs. However, one needs E-local compactness to define a coboundary homomorphism and to have the long exact sequence. Since the exactness of the long sequence is necessary to pass from the local results of Morse theory to the global Morse relations (see Proposition 14.7), this approach does not improve the final result.

Another strong hypothesis we made on f was asking ∇f to be Lipschitz: if this is not the case, one may not be able to integrate the field $-\nabla f$ globally. However, it is not necessary to have a true gradient flow to deform the sublevels: it should be possible to build an E-homotopy which achieves this purpose also in more general situations.

Another approach could be the following: notice that the field $Y = -(1 + \|\nabla f\|)^{-1}\nabla f$ is always globally integrable, being bounded. If f has the form (15.1), then the flow determined by Y has the form

$$\Phi(x,t) = e^{-t\theta(x)L}x + K(x,t) \tag{16.1}$$

where θ maps H into $[0,1]$ and K has good compactness properties (a more general result is proved in [15]). Therefore one could try to prove the functoriality and the homotopy invariance of H^*_E with respect to maps and homotopies of the form (16.1).

Finally, the condition that f should be a Morse function can be eliminated, developing a suitable generalization of Conley's approach to Morse theory (see [6]). Szulkin uses such an approach in his paper [17].

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Manuscript received November 8, 1996

Alberto Abbondandolo
Scuola Normale Superiore
Piazza dei Cavalieri 7
Pisa, ITALY

E-mail address: abbo@sns.it