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INFINITELY MANY ENTIRE SOLUTIONS OF AN ELLIPTIC SYSTEM WITH SYMMETRY

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1. Introduction

In [6], we have considered the existence of at least one nontrivial solution for the following elliptic system on $\mathbb{R}^N$:

$$\text{(ES)} \qquad -\Delta u = \frac{\partial H}{\partial v}(x,u,v), \quad -\Delta v = \frac{\partial H}{\partial u}(x,u,v)$$

such that $u, v \in W^{1,2}(\mathbb{R}^N)$, where $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ has the form of

$$H(x,u,v) = -q(x)uv + \overline{H}(x,u,v) \tag{1.1}$$

and satisfies, with $(u,v) \in \mathbb{R}^2$ denoted by z and $(u^2+v^2)^{1/2}$ by $|z|$, the following conditions:

(Q) $q \in C(\mathbb{R}^N)$ and $q(x) \to \infty$ as $|x| \to \infty$;
(H_1) there is an $\mu > 2$ such that

$$0 < \mu \overline{H}(x,z) \le \overline{H}_z(x,z)z$$

for all $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^2 \setminus \{0\}$, where $\overline{H}_z(x,z) = \nabla_z \overline{H}(x,z)$;
(H_2) $0 < \underline{b} \equiv \inf_{x\in\mathbb{R}^N, |z|=1} \overline{H}(x,z)$;
(H_3) $|\overline{H}_z(x,z)| = o(|z|)$ as $|z| \to 0$ uniformly in $x \in \mathbb{R}^N$;

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(H$_4$) there are $0 \le a_1 \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $a_2 > 0$ such that

$$|\overline{H}_z(x,z)|^\gamma \le a_1(x) + a_2 \overline{H}_z(x,z)z, \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $\gamma > 1, \mu \le \frac{\gamma}{\gamma-1} \equiv \overline{\gamma} < \overline{N} \equiv \frac{2N}{N-2}$ if $N > 2$ and $\overline{\gamma} < \infty$ if $N = 1, 2$.

In [6] we also proved that (ES) has at least one nontrivial solution if H has the form of (1.1) and satisfies, roughly, the following:

(Q$_\alpha$) $q \in C(\mathbb{R}^N)$ and there is an $\alpha < 2$ such that $q(x)|x|^{\alpha-2} \to \infty$ as $|x| \to \infty$;

(H$_5$) $\overline{H}(x,0) \equiv 0$, and there is $1 < \beta \in \left(\frac{2N}{2-\alpha+N}, 2\right)$ such that

$$0 < \overline{H}_z(x,z)z \le \beta \overline{H}(x,z), \quad \forall x \in \mathbb{R}^N \text{ and } z \in \mathbb{R}^2 \setminus \{0\};$$

(H$_6$) there is $a_3 > 0$ such that

$$a_3 |z|^\beta \le \overline{H}(x,z), \quad \forall x \in \mathbb{R}^N \text{ and } |z| \ge 1;$$

(H$_7$) there are $a_4 > 0$ and $\nu > \max\left\{0, \frac{\alpha-2+N}{2-\alpha+N}\right\}$ such that

$$|\overline{H}_z(x,z)| \le a_4 |z|^\nu, \quad \forall x \in \mathbb{R}^N \text{ and } |z| \le 1;$$

(H$_8$) $|\overline{H}_z| \in L^\infty(\mathbb{R}^N \times B_R)$ for any $R > 0$, where $B_R = \{z \in \mathbb{R}^2 : |z| \le R\}$, and

$$|z|^{-1} |\overline{H}_z(x,z)| \to 0 \quad \text{as } |z| \to \infty \text{ uniformly in } x \in \mathbb{R}^N.$$

We remark that, under the above assumptions, (ES) is a nonlinear Schrödinger equation group with the Schrödinger operator $A = -\Delta + q(x)$. Conditions like (Q) arise in mathematical physics, e.g., when one deals with the systems associated with the generalized harmonic oscillator $A = -\Delta + (q^+(x) - q^-(x))$ where $0 \le q^+(x) \to \infty$ as $|x| \to \infty$ and $q^-(x)$ is bounded, or particularly, the anharmonic oscillator $A = -\Delta + q(x)$ in which $q(x)$ is a polynomial of degree $2m$ with the property that the coefficient of the leading term is positive (see [9], [10]).

The purpose of this paper is to show that (ES) has infinitely many solutions if $\overline{H}(x,z)$ is even in z and satisfies the above assumptions. Precisely, we have

Theorem 1.1. *Let H be of the form* (1.1) *with q satisfying* (Q) *and $\overline{H}$ satisfying* (H$_1$)–(H$_4$). *Suppose, in addition, that $\overline{H}(x,z)$ is even with respect to $z \in \mathbb{R}^2$. Then* (ES) *has infinitely many solutions.*

Theorem 1.2. *Let H be of the form* (1.1) *with q satisfying* (Q$_\alpha$) *and $\overline{H}$ satisfying* (H$_5$)–(H$_8$). *Suppose, in addition, that $\overline{H}(x,z)$ is even with respect to $z \in \mathbb{R}^2$. Then* (ES) *has infinitely many solutions.*

Remark 1.3. The existence of at least one solution (u,v) to the elliptic systems like (ES) on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $u|_{\partial\Omega} = 0 =$

$v|_{\partial\Omega}$ has been studied by Benci–Rabinowitz [3], Clément–de Figueiredo–Mitidieri [4], de Figueiredo–Felmer [7] and Szulkin [11] using a variational approach.

2. Two theoretical propositions

The following two abstract propositions will be used for proving the previous results.

Let E be a real Hilbert space with norm $\|\cdot\|$. Suppose that E has an orthogonal decomposition $E = E_1 \oplus E_2$ with both E_1 and E_2 being infinite-dimensional. Let $\{v_n\}$ (resp. $\{w_n\}$) be an orthogonal basis for E_1 (resp. E_2), and set

$$X_n = \operatorname{span}\{v_1, \dots, v_n\} \oplus E_2, \quad X^m = E_1 \oplus \operatorname{span}\{w_1, \dots, w_m\}.$$

For a functional $I \in C^1(E, \mathbb{R})$ we set $I_n = I|_{X_n}$. Recall that we say that *I satisfies the* (PS)* *condition* if any sequence $\{u_n\}$ with $u_n \in X_n$ for which $0 \le I(u_n) \le \text{const}$ and $I_n'(u_n) \equiv \nabla I_n(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence. We also say that *I satisfies the* (PS)** *condition* if for each $n \in \mathbb{N}$, I_n satisfies the Palais–Smale condition, i.e., any sequence $\{u_k\} \subset X_n$ for which $I(u_k)$ is bounded and $I_n'(u_k) \to 0$ as $k \to \infty$ has a convergent subsequence.

Proposition 2.1. *Let E be as above and let $I \in C^1(E, \mathbb{R})$ be even, satisfy* (PS)* *and* (PS)**, *and $I(0) = 0$. Suppose, moreover, that I satisfies, for each $m \in \mathbb{N}$,*

(I$_1$) *there is $R_m > 0$ such that*

$$I(u) \le 0, \quad \forall u \in X^m \text{ with } \|u\| \ge R_m;$$

(I$_2$) *there are $r_m > 0$ and $a_m > 0$ with $a_m \to \infty$ as $m \to \infty$ such that*

$$I(u) \ge a_m, \quad \forall u \in (X^{m-1})^\perp \text{ with } \|u\| = r_m;$$

(I$_3$) *I is bounded from above on bounded sets of X^m.*

Then I has a sequence $\{c_k\}$ of critical values with $c_k \to \infty$ as $k \to \infty$.

This proposition is a version of the symmetric Mountain Pass Theorem of Ambrosetti–Rabinowitz. The main difference between them is that in the former case E_1 is infinite-dimensional, while in the latter case E_1 is finite-dimensional (see [1] or [8, Theorem 9.12]). Such a result is also a special case of Bartsch–Willem [2, Theorem 3.1], and so its proof is omitted.

Now we turn to another result which seems to us to be new even though its proof is simpler.

Proposition 2.2. *Let E be as above and let $I \in C^1(E,\mathbb{R})$ be even, satisfy* (PS)* *and* (PS)***, and $I(0)=0$. Suppose, moreover, that I satisfies, for each $m \in \mathbb{N}$,*

(I$_4$) *there are $r_m > 0$ and $a_m > 0$ such that*

$$a_m \le I(u), \quad \forall u \in X^m \text{ with } \|u\| = r_m;$$

(I$_5$) *there is $b_m > 0$ with $b_m \to 0$ as $m \to \infty$ such that*

$$I(u) \le b_m, \quad \forall u \in (X^{m-1})^\perp.$$

Then I has a sequence $\{c_k\}$ of critical values with $0 < c_k \to 0$ as $k \to \infty$.

Proof. Let Σ denote the family of closed (in E) subsets of $E\backslash\{0\}$ symmetric with respect to the origin, and $\gamma : \Sigma \to \mathbb{N} \cup \{0, \infty\}$ be the genus map [8]. Set

$$\Sigma_n^m = \{A \in \Sigma : A \subset X_n \text{ and } \gamma(A) \ge n+m\}, \quad c_n^m = \sup_{A \in \Sigma_n^m} \inf_{u \in A} I(u).$$

Since for each $A \in \Sigma_n^m$, $A \subset X_n$ and $\gamma(A) \ge n+m$, it is known that $A \cap (X^{m-1})^\perp \ne \emptyset$. Thus by (I$_5$) we have

$$\inf_{u \in A} I(u) \le \sup_{u \in (X^{m-1})^\perp} I(u) \le b_m. \tag{2.1}$$

Since $\gamma(\partial B_{r_m} \cap X_n^m) = n+m$ where $B_{r_m} = \{u \in E : \|u\| \le r_m\}$ and $X_n^m = X_n \cap X^m = \operatorname{span}\{v_1, \dots, v_n, w_1, \dots, w_m\}$, one sees that $\partial B_{r_m} \cap X_n^m \in \Sigma_n^m$ and so by (I$_4$),

$$\inf_{\partial B_{r_m} \cap X_n^m} I(u) \ge a_m. \tag{2.2}$$

Combining (2.1) and (2.2) shows

$$a_m \le c_n^m \le b_m. \tag{2.3}$$

Since I satisfies (PS)**, using the genus theory and a positive rather than a negative gradient flow (see [8, Appendix A, Remark A.17–(iii)]), a standard argument [1, 8] shows that c_n^m is a critical value of I_n. By (2.3), noting that a_m and b_m are independent of n, we see that $c_n^m \to c^m$ as $n \to \infty$ and

$$a_m \le c^m \le b_m. \tag{2.4}$$

Finally, taking into account that I satisfies the (PS)* condition, we conclude that c^m is a critical value of I, and so by (I$_5$) and (2.4), $0 < c^m \le b_m \to 0$ as $m \to \infty$. The proof is complete.

3. Spaces associated with the Schrödinger operator

In this section we recall some embedding properties of the Hilbert space on which we will work. We refer to [6, Section 2] or [5].

Suppose q satisfies (Q) and let A denote the selfadjoint extension of $-\Delta+q(x)$ acting in $L^2 \equiv L^2(\mathbb{R}^N)$, defined as a sum of quadratic forms. Let $|A|$ be the absolute value of A, $|A|^{1/2}$ the square root of $|A|$, $\{E(\nu) : -\infty < \nu < \infty\}$ the resolution of A, and $U = I - E(0) - E(-0)$. Set $W = \mathcal{D}(|A|^{1/2})$. Then W is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_0 = (|A|^{1/2}u, |A|^{1/2}v)_{L^2} + (u, v)_{L^2}$$

and norm $\|u\|_0^2 = \langle u, u\rangle_0$, where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. Clearly W is continuously embedded in $W^{1,2}(\mathbb{R}^N)$ (see [6]). Moreover, we have

Lemma 3.1. *If q satisfies* (Q) *then W is compactly embedded in L^p for $p \in [2, \overline{N})$ where $\overline{N} = \frac{2N}{N-2}$ if $N \geq 3$, $\overline{N} = \infty$ if $N = 2$, and $p \in [2, \infty]$ if $N = 1$.*

Proof. See [6, Lemma 2.1].

Lemma 3.2. *If q satisfies* (Q_α) *then W is compactly embedded in L^p for all $1 \leq p \in \big(\frac{2N}{2-\alpha+N}, \overline{N}\big)$.*

Proof. See [6, Lemma 2.2]. We only mention that (Q_α) implies (Q), and since $\alpha < 2$, one has $\frac{2N}{2-\alpha+N} < 2$, and if further $\alpha < 2 - N$ then $\frac{2N}{2-\alpha+N} < 1$.

Now by Lemma 3.1, A has a compact resolution, and so $\sigma(A)$, the spectrum of A, consists of eigenvalues (counted with multiplicities)

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \to \infty$$

with a corresponding system of eigenfunctions $\{h_n\}$, $Ah_n = \lambda_n h_n$, which forms an orthogonal basis in L^2. Let n^- (resp. n^0) denote the number of negative (resp. null) eigenvalues, and $\overline{n} = n^- + n^0$. Set

$$W^- = \operatorname{span}\{h_1, \ldots, h_{n^-}\}, \quad W^0 = \operatorname{span}\{h_{n^-+1}, \ldots, h_{\overline{n}}\}, \quad W^+ = (W^- \oplus W^0)^\perp.$$

Then $W = W^- \oplus W^0 \oplus W^+$ is a natural orthogonal decomposition. Based on this decomposition we introduce the following inner product in W:

$$\langle u, v\rangle_1 = (|A|^{1/2}u, |A|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}$$

and norm $\|u\|_1 = \langle u, u\rangle_1^{1/2}$ for all $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in W = W^- \oplus W^0 \oplus W^+$. It is easy to see that $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent norms on W. We note that W^-, W^0 and W^+ are orthogonal to each other with respect to both $\langle\cdot,\cdot\rangle_1$ and $(\cdot,\cdot)_{L^2}$.

Let

$$a(u, v) = (|A|^{1/2}Uu, |A|^{1/2}v)_{L^2}$$

be the quadratic form associated with A. Then for $u \in \mathcal{D}(A)$ and $v \in W$,

$$a(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + q(x)uv) \tag{3.1}$$

and so by continuity, (3.1) holds for all $u, v \in W$. Clearly, W^-, W^0 and W^+ are orthogonal to each other with respect to $a(\cdot,\cdot)$, and moreover

$$a(u,v) = \langle (p^+ - p^-)u, v\rangle_1, \tag{3.2}$$

$$a(u,u) = \|u^+\|_1^2 - \|u^-\|_1^2, \tag{3.3}$$

where $p^\pm : W \to W^\pm$ are the orthogonal projectors.

Now we turn to the product space $E = W \times W$ with the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle (u,v),(\varphi,\psi)\rangle = \langle u,\varphi\rangle_1 + \langle v,\psi\rangle_1$$

and norm $\|(u,v)\|^2 = \|u\|_1^2 + \|v\|_1^2$. Define

$$\begin{aligned} E^0 &= W^0 \times W^0, \\ E^- &= \{(u^- + u^+, u^- - u^+) : u^- + u^+ \in W^- \oplus W^+\}, \\ E^+ &= \{(u^- + u^+, -u^- + u^+) : u^- + u^+ \in W^- \oplus W^+\}. \end{aligned}$$

Then $E = E^- \oplus E^0 \oplus E^+$ is an orthogonal decomposition of E. For any $z = (u,v) \in E$ we have the unique representation $z = z^- + z^0 + z^+$, where

$$\begin{aligned} z^- &= \tfrac{1}{2}(u^- + v^- + u^+ - v^-, u^- + v^- - u^+ + v^+) \in E^-, \\ z^0 &= (u^0, v^0) \in E^0, \\ z^+ &= \tfrac{1}{2}(u^- - v^- + u^+ + v^+, -u^- + v^- + u^+ + v^+) \in E^+. \end{aligned}$$

Consider the quadratic form defined on E by

$$Q((u,v),(\varphi,\psi)) = a(u,\psi) + a(v,\varphi).$$

Then by (3.1),

$$Q((u,v),(\varphi,\psi)) = \int_{\mathbb{R}^N} [\nabla u \nabla \psi + q(x)u\psi + \nabla v \nabla \varphi + q(x)v\varphi], \tag{3.4}$$

and by (3.2) and (3.3),

$$Q(z) \equiv Q((u,v),(u,v)) = \|z^+\|^2 - \|z^-\|^2 \tag{3.5}$$

for all $z = (u,v) \in E$.

Finally, in virtue of Lemmas 3.1 and 3.2, we have

Lemma 3.3. *E is compactly embedded in $(L^p(\mathbb{R}^N))^2$ for all $p \in [2, \overline{N})$ if q satisfies* (Q), *and for all $1 \le p \in \left(\frac{2N}{2-\alpha+N}, \overline{N}\right)$ if q satisfies* (Q_α).

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let the assumptions of Theorem 1.1 be satisfied and let E be the product space defined in the previous section. By (H_4), we have

$$|\overline{H}_z(x,z)| \le C_1 + C_2|z|^{\overline{\gamma}-1}, \quad \forall (x,z); \tag{4.1}$$

here (and in the sequel) C_i (or C) stands for generic positive constants. This, together with (H_3), shows that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|\overline{H}_z(x,z)| \le \varepsilon|z| + C_\varepsilon|z|^{\overline{\gamma}-1}, \quad \forall (x,z), \tag{4.2}$$

and

$$\overline{H}(x,z) \le C_3|z|^2 + C_4|z|^{\overline{\gamma}}, \quad \forall (x,z). \tag{4.3}$$

Let

$$J(z) = \int_{\mathbb{R}^N} \overline{H}(x,z)\,dx, \quad \forall z \in E.$$

By (4.1)–(4.3) and Lemma 3.3, a standard argument shows that $J \in C^1(E,\mathbb{R})$ with

$$J'(z)y = \int_{\mathbb{R}^n} \overline{H}_z(x,z)y\,dx, \quad \forall z, y \in E,$$

where $J' \equiv \nabla J$ represents the gradient of J, and J' is a compact operator (see [6]). Define

$$I(z) = \tfrac{1}{2}Q(z) - J(z) = \tfrac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \textstyle\int_{\mathbb{R}^N} \overline{H}(x,z)\,dx$$

for all $z = (u,v) \in E$. Then $I \in C^1(E,\mathbb{R})$ and for $z = (u,v)$ and $y = (\varphi,\psi) \in E$, by (3.4),

$$\begin{aligned} I'(z)y = &\int_{\mathbb{R}^N} (\nabla u \nabla \psi + q(x)u\psi + \nabla v \nabla \varphi + q(x)v\varphi) \\ &- \int_{\mathbb{R}^N} \left(\frac{\partial \overline{H}}{\partial u}(x,u,v)\varphi + \frac{\partial \overline{H}}{\partial v}(x,u,v)\psi \right). \end{aligned}$$

Hence, any critical point of I corresponds to a $W^{1,2}(\mathbb{R}^N,\mathbb{R}^2)$ solution of (ES). We will use Proposition 2.1 to look for critical points of I.

Let $e_1, e_2, \dots$ be an orthonormal basis for E^+, and $g_1, g_2, \dots$ be an orthonormal basis for $E^- \oplus E^0$. Set $E_1 = E^- \oplus E^0, E_2 = E^+, X_n = \mathrm{span}\{g_1,\dots,g_n\} \oplus E^+, X^m = E^- \oplus E^0 \oplus \mathrm{span}\{e_1,\dots,e_m\}$, and $I_n = I|_{X_n}$.

LEMMA 4.1. *I satisfies* $(\mathrm{PS})^*$ *and* $(\mathrm{PS})^{**}$.

PROOF. See [6, Lemma 3.2] where $(\mathrm{PS})^*$ was verified. However, the verification of $(\mathrm{PS})^{**}$ can be checked along the same lines and so it is omitted here.

Lemma 4.2. *I satisfies* (I_1).

Proof. By (H_1) and (H_2) one has

$$\overline{H}(x,z) \ge \underline{b}|z|^\mu, \quad \forall x \in \mathbb{R}^N \text{ and } |z| \ge 1,$$

which, together with the fact that $|z|^\mu \le |z|^2$ for $|z| \le 1$, yields, for any $0 < \varepsilon \le \underline{b}$,

$$\overline{H}(x,z) \ge \varepsilon(|z|^\mu - |z|^2), \quad \forall (x,z). \tag{4.4}$$

In virtue of Lemma 3.3, there is $d > 0$ such that $\|z\|_{L^2}^2 \le d\|z\|^2$ for all $z \in E$. Taking $\varepsilon = \min\{\frac{1}{4d}, \underline{b}\}$, we have by (4.4), for $z = z^- + z^0 + z^+ \in X^m$,

$$\begin{aligned} I(z) &= \tfrac{1}{2}\|z^+\|^2 - \tfrac{1}{2}\|z^-\|^2 - \int_{\mathbb{R}^N} \overline{H}(x,z)\,dx \\ &\le \tfrac{1}{2}\|z^+\|^2 - \tfrac{1}{2}\|z^-\|^2 + \varepsilon\|z\|_{L^2}^2 - \varepsilon\|z\|_{L^\mu}^\mu \\ &\le \|z^+\|^2 - \tfrac{1}{4}\|z^-\|^2 + \tfrac{1}{4}\|z^0\|^2 - \varepsilon\|z\|_{L^\mu}^\mu. \end{aligned} \tag{4.5}$$

Using L^2 orthogonality, the Hölder inequality $(1/\mu + 1/\mu' = 1)$ and $\dim(E^0 \oplus \mathrm{span}\{e_1, \ldots, e_m\}) < \infty$, we have

$$\|z^0 + z^+\|_{L^2}^2 = (z^0 + z^+, z)_{L^2} \le \|z^0 + z^+\|_{L^{\mu'}}\|z\|_{L^\mu} \le C(m)\|z^0 + z^+\|_{L^2}\|z\|_{L^\mu},$$

and so

$$C'(m)\|z^0 + z^+\|^\mu \le \|z\|_{L^\mu}^\mu \tag{4.6}$$

where $C'(m) > 0$ depends on m but not on $z \in X^m$. (4.5) and (4.6) imply

$$I(z) \le \|z^0 + z^+\|^2 - \tfrac{1}{4}\|z^-\|^2 - \varepsilon C'(m)\|z^0 + z^+\|^\mu \tag{4.7}$$

for all $z \in X^m$. Since $\mu > 2$, (4.7) implies that there is $R_m > 0$ such that $I(z) \le 0$ for all $z \in X^m$ with $\|z\| \ge R_m$, proving (I_1).

Lemma 4.3. *I satisfies* (I_2).

Proof. Set

$$\eta_m = \sup_{z \in (X^m)^\perp \setminus \{0\}} \|z\|_{L^{\overline{\gamma}}}/\|z\|.$$

Clearly, $\eta_m \ge \eta_{m+1} > 0$. Moreover, one has

$$\eta_m \to 0 \quad \text{as } m \to \infty. \tag{4.8}$$

Indeed, if not, then $\eta_m \to \eta > 0$. Consequently, there is a sequence $z_m \in (X^m)^\perp$ with $\|z_m\| = 1$ and $\|z_m\|_{L^{\overline{\gamma}}} \ge \eta/2$. Since $\langle z_m, e_k\rangle \to 0$ as $m \to \infty$ for each k, one sees $z_m \to 0$ weakly in E, and so by Lemma 3.3, $\|z_m\|_{L^{\overline{\gamma}}} \to 0$, yielding a contradiction. Therefore (4.8) must be true.

By (4.2) with $\varepsilon = 1/(4d)$ and $C = C_\varepsilon$, one has for $z \in (X^{m-1})^\perp$,

$$I(z) = \tfrac{1}{2}\|z\|^2 - \int_{\mathbb{R}^N} \overline{H}(x,z) \ge \tfrac{1}{4}\|z\|^2 - C\|z\|_{L^{\overline{\gamma}}}^{\overline{\gamma}} \ge \tfrac{1}{4}\|z\|^2 - C\eta_{m-1}^{\overline{\gamma}}\|z\|^{\overline{\gamma}}.$$

Consequently, taking $r_m = (2\overline{\gamma}C\eta_{m-1}^{\overline{\gamma}})^{-1/(\overline{\gamma}-1)}$ and $a_m = \big(\frac{1}{4} - \frac{1}{2\overline{\gamma}}\big)r_m^2$, one obtains $I(z) \geq a_m$ for all $z \in (X^{m-1})^{\perp}$ with $\|z\| = r_m$. Since $\overline{\gamma} > 2$, (4.8) shows that $a_m \to \infty$ as $m \to \infty$. (I_2) follows.

Lemma 4.4. *I satisfies* (I_3).

Proof. (I_3) follows directly from (4.7).

Now we give the following

Proof of Theorem 1.1. Clearly $I(0) = 0$ and I is even since $\overline{H}(x,z)$ is even in $z \in \mathbb{R}^2$. Lemmas 4.1–4.4 show that I satisfies all the assumptions of Proposition 2.1. Hence I has a positive critical value sequence $c_k \to \infty$ as $k \to \infty$. Let $z_k = (u_k, v_k)$ be the critical point of I such that $I(z_k) = c_k$. Then (u_k, v_k) are entire solutions of (ES). The proof is complete.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 will rely on an application of Proposition 2.2. Let the assumptions of Theorem 1.2 be satisfied. Below, all the symbols E, E_1, E_2, X_n, X^m and so on still have the same meaning as in Section 4.

By (H_5) and (H_7) one sees that

$$\overline{H}(x,z)\begin{cases} \geq (\min_{x\in\mathbb{R}^N,|\xi|=1} \overline{H}(x,\xi))|z|^{\beta} & \text{if } |z| \leq 1,\\ \leq (\max_{x\in\mathbb{R}^N,|\xi|=1} \overline{H}(x,\xi))|z|^{\beta} & \text{if } |z| \geq 1,\end{cases}$$
$$\overline{H}(x,z) \leq a_4|z|^{1+\nu}, \quad \forall x \in \mathbb{R}^N \text{ and } |z| \leq 1.$$

These, jointly with (H_6), show that $1 + \nu \leq \beta$ and

$$a_3|z|^{\beta} \leq \overline{H}(x,z) \leq a_4|z|^{\beta}, \quad \forall (x,z). \tag{5.1}$$

Note also that by (H_7),

$$1 + \nu > \frac{2N}{2 - \alpha + N},$$

and by (H_7) and (H_8),

$$|\overline{H}_z(x,z)| \leq a_5(|z|^{\nu} + |z|), \quad \forall (x,z).$$

Consider again the functional J defined on E by

$$J(z) = \int_{\mathbb{R}^N} \overline{H}(x,z)\,dx.$$

The above argument, together with Lemma 3.3, shows that J is well defined, $J \in C^1(E,\mathbb{R})$ with

$$J'(z)y = \int_{\mathbb{R}^N} \overline{H}_z(x,z)y\,dx, \quad \forall z, y \in E, \tag{5.2}$$

and J' is compact (see [6]).

Now define the functional I on E by

$$I(z) = J(z) - \tfrac{1}{2}Q(z) = J(z) - \tfrac{1}{2}\|z^+\|^2 + \tfrac{1}{2}\|z^-\|^2.$$

Then $I \in C^1(E,\mathbb{R})$, and by (3.4) and (5.2), critical points of I give rise to solutions of $(\mathrm{ES})_1$. We will verify that I satisfies the assumptions of Proposition 2.2.

Lemma 5.1. *I satisfies* $(\mathrm{PS})^*$ *and* $(\mathrm{PS})^{**}$.

Proof. See [6, Section 4, Step 3].

Lemma 5.2. *I satisfies* (I_4).

Proof. For any $z \in X^m$, we have by (5.1),

$$I(z) \geq a_3\|z\|_{L^\beta}^\beta - \tfrac{1}{2}\|z^+\|^2 + \tfrac{1}{2}\|z^-\|^2. \tag{5.3}$$

Since $\dim(E^0 \oplus \operatorname{span}\{e_1,\dots,e_m\}) < \infty$, one has $(\beta' = \beta/(\beta-1) > 2)$

$$\|z^0+z^+\|_{L^2}^2 = (z^0+z^+,z)_{L^2} \leq \|z^0+z^+\|_{L^{\beta'}}\|z\|_{L^\beta} \leq C(m)\|z^0+z^+\|_{L^2}\|z\|_{L^\beta}$$

and so by Lemma 3.3,

$$C'(m)\|z^0+z^+\|^\beta \leq a_3\|z\|_{L^\beta}^\beta,$$

which, together with (5.3), yields

$$I(z) \geq C'(m)\|z^0+z^+\|^\beta - \tfrac{1}{2}\|z^0+z^+\|^2 + \tfrac{1}{2}\|z^-\|^2$$

for all $z = z^- + z^0 + z^+ \in X^m$, where $C'(m)$ is a constant depending only on m. Therefore, since $\beta < 2$, there are $r_m > 0$ and $a_m > 0$ such that $I(z) \geq a_m$ for all $z \in X^m$ with $\|z\| = r_m$, i.e., I satisfies (I_4).

Lemma 5.3. *I satisfies* (I_5).

Proof. Let $z \in (X^{m-1})^\perp$. By (5.1) we have

$$I(z) = \int_{\mathbb{R}^N} \overline{H}(x,z) - \tfrac{1}{2}\|z\|^2 \leq a_4\|z\|_{L^\beta}^\beta - \tfrac{1}{2}\|z\|^2. \tag{5.4}$$

Let ξ_m be defined by

$$\xi_m = \sup_{z\in(X^m)^\perp\setminus\{0\}} \|z\|_{L^\beta}/\|z\|.$$

Similarly to the proof of Lemma 4.3, one obtains

$$0 < \xi_m \to 0 \quad \text{as } m \to \infty. \tag{5.5}$$

Now by (5.4), for $z \in (X^{m-1})^\perp$, we have

$$I(z) \leq a_4\xi_{m-1}^\beta\|z\|^\beta - \tfrac{1}{2}\|z\|^2. \tag{5.6}$$

Let

$$b_m = (1-\beta/2)a_4\xi_{m-1}^\beta(a_4\beta\xi_{m-1}^\beta)^{\beta/(2-\beta)}.$$

Then by (5.5) and since $\beta < 2$, $b_m \to 0$ as $m \to \infty$, and by (5.6),

$$I(z) \le b_m, \quad \forall z \in (X^{m-1})^\perp,$$

i.e., I satisfies (I_5).

Now we turn to

PROOF OF THEOREM 1.2. Clearly by (H_5), $I(0) = 0$, and since $\overline{H}(x, z)$ is even with respect to $z \in \mathbb{R}^2$, I is even. Lemmas 5.1–5.3 show that I satisfies all the assumptions of Proposition 2.2. Therefore I has a sequence of positive critical values, $\{c_k\}$, satisfying $c_k \to 0$ as $k \to \infty$. Let $z_k = (u_k, v_k)$ be the critical points of I corresponding to c_k, i.e., $I'(z_k) = 0$ and $I(z_k) = c_k$. Then (u_k, v_k) are entire solutions of (ES). The proof is thereby complete.

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