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EXISTENCE OF SOLUTIONS FOR THE DISCRETE COAGULATION-FRAGMENTATION MODEL WITH DIFFUSION

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Dedicated to Olga Ladyzhenskaya

1. Introduction

We consider the following infinite system of reaction-diffusion equations:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= d_1\Delta u_1 - u_1\sum_{j=1}^{\infty} a_{1j}u_j + \sum_{j=1}^{\infty} b_{1j}u_{1+j},\\ \frac{\partial u_i}{\partial t} &= d_i\Delta u_i + \frac{1}{2}\sum_{j=1}^{i-1}(a_{i-j,j}u_{i-j}u_j - b_{i-j,j}u_i)\\ &\quad - u_i\sum_{j=1}^{\infty} a_{ij}u_j + \sum_{j=1}^{\infty} b_{ij}u_{i+j}, \quad i = 2,3,\ldots,\end{aligned} \tag{1.1}$$

on $\Omega_T = \Omega \times (0,T)$, subject to the initial condition

$$u_i(0,x) = U_i(x) \quad \text{for } x \in \Omega,\ i = 1,2,\ldots \tag{1.2}$$

and Neumann boundary condition

$$\frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T),\ i = 1,2,\ldots, \tag{1.3}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, ν is an outward normal vector of Ω, and $U_i \in L^\infty(\Omega)$, $i = 1, 2, \ldots$, are given nonnegative functions. We will refer to problem (1.1)–(1.3) as problem (P).

The system (1.1) is a generalization of the discrete coagulation-fragmentation model which describes the dynamics of cluster growth. Such models arise in polymer science [9, 10, 16, 17], atmospheric physics [8] and colloidal chemistry [15], to give a few examples. In models analyzed in this paper the clusters are assumed to be discrete, i.e. they consist of a finite number of smaller particles (resp. polymers and monomers in polymer physics).

The variable u_i represents the concentration of *i-clusters*, that is, clusters consisting of i identical elementary particles. The *coagulation rates* a_{ij} and *fragmentation rates* b_{ij} are nonnegative constants such that $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, and d_i are the *diffusion coefficients*, $d_i > 0$, $i = 1, 2, \ldots$ The coefficient a_{ij} represents reaction in which an $(i + j)$-cluster is formed from an i-cluster and a j-cluster, whereas b_{ij} is due to break-up of an $(i+j)$-cluster into i- and j-clusters. The first two terms in the ith equation describe the rate of change of i-clusters caused by coagulation of smaller clusters and by fragmentation. The last two terms represent the interaction of i-clusters with themselves and all others and break-up of larger clusters into i-clusters. Since the size of clusters is not limited *a priori*, an infinite number of variables is introduced.

A model taking into account the diffusion of clusters (in absence of fragmentation) was given in [7]. Some properties of such a model are shown in [14]. We generalize some results from the previous paper [4], where existence and uniqueness of solutions and mass conservation are studied under the hypothesis that $a_{ij} = r_i r_j$, $b_{ij} = 0$, $r_i = O(i)$, $i, j \geq 1$.

After the manuscript of this article was accomplished the author learned about results obtained by Collet and Poupaud in [5]. They prove the existence of global-in-time solutions only in the case of equal diffusion coefficients in each equation and for a fairly restricted range of coagulation and fragmentation coefficients.

For a physical interpretation of the coefficients in the model we refer the reader to [2, 7, 8, 9, 16]. A survey of different variants of the model and numerical methods for the case without diffusion can be found in [6].

We assume two physically meaningful growth conditions of the coagulation coefficients, namely

(H1) $a_{ij} \leq Aij$ for $i, j \geq 1$, $A > 0$, and
(H2) $a_{ij} \leq A(i + j)$ for $i, j \geq 1$, $A > 0$.

Many properties of the coagulation-fragmentation model without diffusion are proved in [2] under the hypothesis (H2).

The hypothesis (H2) is considered separately because in this case solutions to problem (P) preserve for all $t \geq 0$ the total mass M_0 contained in the initial distribution, i.e.

$$\sum_{i=1}^{\infty} i \int_{\Omega} u_i(x,t)\, dx = \sum_{i=1}^{\infty} i \int_{\Omega} U_i(x)\, dx = M_0 \quad \text{for } t \geq 0. \tag{1.4}$$

From now on we assume that $M_0 < \infty$.

It is known that in the case

$$a_{ij} = r_i r_j, \quad r_i = i^{\alpha}, \quad \alpha > 1/2, \quad b_{ij} = 0, \quad d_i = d > 0, \quad i,j \geq 1,$$

(1.4) holds only locally in time. This is related to the *sol-gel transition* (see [9, 10, 16, 17]) for more details.

Similarly to the previous paper [4] solutions of problem (P) are constructed as limits of solutions of finite systems.

The paper is organized as follows. In Section 2 we prove some uniform bounds for solutions of finite systems approximating the original one. In Section 3 problem (P) is studied under the assumption that the diffusion coefficients d_i are arbitrary positive constants. In comparison with the model without diffusion [2], in order to show existence of global-in-time solutions we need the following additional restriction on the fragmentation coefficients:

(H3) For each $i \geq 1$ there exists $\gamma_i > 0$ such that

$$b_{i,j-i} \leq \gamma_i a_{ij} \quad \text{for } j \geq i+1.$$

In this case we are able to prove the existence of global-in-time solutions satisfying

$$\sum_{i=1}^{\infty} i \int_{\Omega} u_i(x,t)\, dx \leq M_0.$$

In Section 4 we assume that:

(H4) there exist $M \geq 1$ and $d > 0$ such that $d_i = d$ for $i \geq M$.

This enables us to prove the existence of solutions under more general assumptions on the reaction rate coefficients and also to show (1.4). If the diffusion coefficients are the same in each equation the existence of solutions is proved without any growth condition on b_{ij} (see Theorem 4.2).

We shall denote by L the L^2-realization of the Laplace operator subject to homogeneous Neumann boundary condition:

$$Lu = \Delta u \quad \text{with} \quad D(L) = \{u : u \in H^2(\Omega),\ \partial u/\partial \nu = 0\},$$

and Ω_t stands for $\Omega \times (0,t)$.

2. Approximation of solutions

Solutions of the infinite system are constructed in the process of approximation by the following systems (P^N) of $2N$ equations defined for any integer $N \geq 2$:

$$\frac{\partial u_1^N}{\partial t} = d_1 \Delta u_1^N - u_1^N \sum_{i=1}^{N} a_{1j} u_j^N + \sum_{j=1}^{N-1} b_{1j} u_{1+j}^N,$$

$$\vdots$$

$$(2.1)\qquad \begin{aligned} \frac{\partial u_i^N}{\partial t} &= d_i \Delta u_i^N + \frac{1}{2} \sum_{j=1}^{i-1} (a_{i-j,j} u_{i-j}^N u_j^N - b_{i-j,j} u_i^N) \\ &\quad - \sum_{j=1}^{N} a_{ij} u_i^N u_j^N + \sum_{j=1}^{N-i} b_{ij} u_{i+j}^N \quad \text{for } i = 2, \ldots, N, \\ \frac{\partial u_i^N}{\partial t} &= d_i \Delta u_i^N + \frac{1}{2} \sum_{j=i-N}^{N} a_{j,i-j} u_j^N u_{i-j}^N \quad \text{for } N+1 \leq i \leq 2N, \end{aligned}$$

subject to boundary and initial conditions as in (1.2), (1.3).

Notice that this system corresponds to the first $2N$ equations of the system (1.1) where

$$a_{ij} = 0 \quad \text{for } i > N \text{ or } j > N \quad \text{and} \quad b_{ij} = 0 \quad \text{for } i + j > N.$$

In this section we prove some properties of (2.1) which will be useful for further analysis.

PROPOSITION 2.1. *For any $N \geq 2$ the system* (P^N) *has a unique nonnegative solution $\{u_k^N\}_{k=1}^{2N}$ defined on $\Omega \times (0, T_{\max})$: for $k = 1, \ldots, 2N$,*

$$\begin{aligned} &u_k^N \in C([0, T_{\max}[; L^2(\Omega)) \quad \textit{and satisfies } (1.2), \\ &u_k^N \in L^\infty_{\mathrm{loc}}([0, T_{\max}[; L^\infty(\Omega)), \\ &u_k^N \in W^{1,\infty}_{\mathrm{loc}}(]0, T_{\max}[; L^2(\Omega)), \\ &u_k^N \in L^\infty_{\mathrm{loc}}(]0, T_{\max}[; D(L)), \end{aligned}$$

and the equations of (P^N) *are satisfied a.e. in $\Omega \times (0, T_{\max})$.*

PROOF. The system (P^N) can be rewritten in the form

$$\frac{\partial u_i^N}{\partial t} = d_i L u_i^N + f_i(u_1^N, \ldots, u_{2N}^N) + u_i^N g_i(u_1^N, \ldots, u_{2N}^N), \quad i = 1, \ldots, 2N,$$

where L_i is the infinitesimal generator of an analytic semigroup of bounded linear operators on $L^2(\Omega)$, and f_i, g_i are locally Lipschitz continuous functions on $\mathbb{R}^{2N}$ with $f_i \geq 0$ on $\mathbb{R}_+^{2N}$. By the general theory of reaction-diffusion equations (see e.g. [11], [13] or [12]) there exists a unique nonnegative, local-in-time solution of (P^N). □

Multiplying the ith equation in (2.1) by a number g_i and then summing them up we obtain

$$\begin{aligned}(2.2)\qquad \sum_{i=1}^{2N} g_i \frac{\partial u_i^N}{\partial t} - \sum_{i=1}^{2N} g_i d_i \Delta u_i^N \\ = \frac{1}{2} \sum_{1\le i,j\le N} (g_{i+j} - g_i - g_j) a_{ij} u_i^N u_j^N \\ - \frac{1}{2} \sum_{1<i+j\le N} (g_{i+j} - g_i - g_j) b_{ij} u_{i+j}^N \quad \text{on } \Omega\times(0,T_{\max}).\end{aligned}$$

Setting $g_i = i$ for $i = 1,\ldots,2N$ and integrating over Ω_t, we obtain for $0 \le t < T_{\max}$ the mass conservation formula

$$(2.3)\qquad \sum_{i=1}^{2N} \int_\Omega i u_i^N(x,t)\,dx = \sum_{i=1}^{2N} \int_\Omega i U_i(x)\,dx \le M_0$$

and taking $g_i = 1$ for $i = 1,\ldots,2N$ we have

$$\begin{aligned}(2.4)\qquad \sum_{i=1}^{2N} \int_\Omega u_i^N(x,t)\,dx + \frac{1}{2} \int\int_{\Omega_t} \sum_{1\le i,j\le N} a_{ij} u_i^N u_j^N \\ = \frac{1}{2} \int\int_{\Omega_t} \sum_{1\le i+j\le N} b_{ij} u_{i+j}^N + \sum_{i=1}^{2N} \int_\Omega U_i(x)\,dx.\end{aligned}$$

The following lemma shows that under some restrictions on the reaction coefficients the solutions to (P^N) are global in time.

Lemma 2.2. *Under the hypothesis* (H3) *the solutions of* (P^N) *are global in time and for* $i \le N$,

$$(2.5)\qquad \|u_i^N\|_{L^\infty(\Omega\times(0,\infty))} \le K_i,$$

uniformly with respect to N, *where*

$$(2.6)\qquad K_i = k_i + 1 + \frac{\sum_{j=1}^{i-1} a_{i-j,j} K_{i-j} K_j}{\sum_{j=1}^{i-1} b_{i-j,j} + a_{ii}}, \quad k_i = \max\{\gamma_i, \|U_i\|_\infty\}$$

for $i \ge 2$ *and* $K_1 = k_1$. *For* $N < i \le 2N$,

$$\|u_i^N\|_{L^\infty(\Omega_t)} \le \|U_i\|_{L^\infty(\Omega)} + t\Big(\sum_{j=i-N}^{N} a_{i-j,j} K_{i-j} K_j\Big) \quad \textit{for } t > 0.$$

Proof. In the first equation of (P^N) take the test function

$$\phi_1 = (u_1 - k_1)_+|_{[0,t]},$$

where $k_1 = \max\{\gamma_1, \|U_1\|_\infty\}$ and $0 \le t < T_m$ where T_m is the maximal existence time. Integrating yields

$$\frac{1}{2}\|(u_1^N(\cdot,t) - k_1)_+\|_2^2 + d_1 \int\int_{\Omega_t} |\nabla(u_1^N - k_1)_+|^2$$
$$= -a_{11} \int\int_{\Omega_t} (u_1^N - k_1)_+^2 - \int\int_{\Omega_t} \sum_{j=2}^{N} (a_{1j}u_1^N - b_{1j-1})u_j^N(u_1^N - k_1)_+ \le 0$$

where the last inequality follows from (H3) and from the nonnegativity of solutions. Thus, we have

$$\|u_1^N(\cdot,t)\|_{L^\infty(\Omega)} \le k_1 \quad \text{for } 0 \le t < T_m,$$

uniformly with respect to N. Let $\|u_j^N\|_{L^\infty(\Omega_t)} \le K_j$ for $1 \le j \le i-1$. We shall show that

$$\|u_i^N\|_{L^\infty(\Omega_t)} \le K_i, \tag{2.7}$$

where K_i is given by (2.6). To this end let us test the ith equation with the function

$$\phi_i = p(u_i^N - k_i)_+^{p-1}|_{[0,t]},$$

where $p \ge 2$ and $t < T_m$. We then obtain, taking $k_i \ge \|U_i\|_{L^\infty(\Omega)}$,

$$\|(u_i^N(t) - k_i)_+\|_p^p + d_i p(p-1) \int\int_{\Omega_t} |\nabla(u_i^N - k_i)_+|^2 (u_i^N - k_i)_+^{p-2} \tag{2.8}$$
$$\le \int\int_{\Omega_t} \Big(\sum_{j=1}^{i-1} a_{i-j,j}K_{i-j}K_j - \sum_{j=1}^{i-1} b_{i-j,j}(u_i^N - k_i)_+ - a_{ii}(u_i^N - k_i)_+^2\Big)$$
$$\times p(u_i^N - k_i)_+^{p-1}\,dx\,dt$$
$$- \int\int_{\Omega_t} \Big(\sum_{j=1}^{i-1} a_{ij}u_j^N u_i^N + \sum_{j=i+1}^{N} (a_{ij}u_i^N - b_{i,j-i})u_j^N\Big)$$
$$\times p(u_i^N - k_i)_+^{p-1}\,dx\,dt =: I_1 - I_2.$$

By (H3), $I_2 \ge 0$ if $k_i \ge \gamma_i$. Using the Young inequality "with ε" we find

$$I_1 \le p^{p-1}\Big(\frac{p}{p-1}\Big)^{1-p}\Big(\sum_{j=1}^{i-1} a_{ij}K_iK_j\Big)^p \varepsilon^{1-p}|\Omega|t + \varepsilon \int\int_{\Omega_t} (u_i^N - k_i)_+^p$$
$$- p\Big(\sum_{j=1}^{i-1} b_{i-j,j}\Big)\int\int_{\Omega_t} (u_i^N - k_i)_+^p + |\Omega|t - pa_{ii}\int\int_{\Omega_t} (u_i^N - k_i)_+^p.$$

Taking $\varepsilon = p(\sum_{j=1}^{i-1} b_{i-j,j} + a_{ii})$ yields

$$I_1 \le \Big(\Big(\sum_{j=1}^{i-1} a_{ij}K_iK_j\Big)^p \Big(\sum_{j=1}^{i-1} b_{i-j,j} + a_{ii}\Big)^{1-p} + 1\Big)|\Omega|T_m \tag{2.9}$$

and finally from (2.8) and (2.9) it follows that

$$\sup_{t\in[0,T_m]} \limsup_{p\geq 2} \left(\int_\Omega (u_i^N(x,t)-k_i)^p dx\right)^{1/p} \leq \left(\sum_{j=1}^{i-1} a_{ij}K_iK_j\right)\left(\sum_{j=1}^{i-1} b_{i-j,j}+a_{ii}\right)^{-1}+1, \tag{2.10}$$

which yields (2.7). Therefore, solutions can be prolonged for all $t>0$, which implies (2.5) and (2.6). The last statement of the theorem follows immediately from the maximum principle. □

In Lemma 2.3, below, which will be used in Section 4, we shall use the following formula: for $M<N$,

$$\begin{aligned}\sum_{i=M}^{2N} g_i \frac{\partial u_i^N}{\partial t} - \sum_{i=M}^{2N} g_i d_i \Delta u_i^N &\\ = \frac{1}{2}\sum_{M\leq i,j\leq N} (g_{i+j}-g_i-g_j)a_{ij}u_i^N u_j^N &\\ + \sum_{1\leq i<M\leq j\leq N} (g_{i+j}-g_j)a_{ij}u_i^N u_j^N + \frac{1}{2}\sum_{1\leq i,j\leq M\leq i+j} g_{i+j}a_{ij}u_i^N u_j^N &\\ - \frac{1}{2}\sum_{T_1}(g_{i+j}-g_i-g_j)b_{ij}u_{i+j}^N - \sum_{T_2}(g_{i+j}-g_j)b_{ij}u_{i+j}^N &\\ - \frac{1}{2}\sum_{T_3} g_{i+j}b_{ij}u_{i+j}^N. &\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}T_1 &= \{(i,j): i+j\leq N,\ i,j\geq M\},\\ T_2 &= \{(i,j): i+j\leq N,\ j\geq M,\ 1\leq i\leq M-1\},\\ T_3 &= \{(i,j): i,j\leq M\leq i+j\leq N\},\end{aligned}$$

which follows from (2.2) by setting $g_i=0$ for $1\leq i<M<N$.

Lemma 2.3. *Suppose* (H4) *and*

(i) *for* $1\leq i<M$ *there exists a positive constant* γ_i *such that*

$$b_{i,j-i}\leq \gamma_i a_{ij} \quad \textit{for } j\geq i+1,$$

(ii) $\sum_{i=1}^\infty i^2\|U_i\|_\infty \leq \text{const}.$

Then under the hypothesis (H1) *there exists* $T_m > 0$ *and a constant* $C_1(T)$ *such that*

$$\sup_{t\in[0,T]} \Big\| \sum_{i=1}^{2N} i^2 u_i^N(\cdot,t) \Big\|_{L^\infty(\Omega)} \le C_1(T) \quad \textit{for } T \in [0, T_m[, \tag{2.12}$$

(2.13) *under the hypothesis* (H2), (2.12) *is true for* $T_m = \infty$.

Proof. To show (2.12) we proceed as in [4]. Let $\{u_i^N\}_{i=1}^{2N}$ be the solution to (P^N). Taking in (2.11)

$$g_i = \begin{cases} i^2 & \text{for } i \ge M, \\ 0 & \text{for } 1 \le i < M, \end{cases} \tag{2.14}$$

and using (H1) and Lemma 2.2 we obtain for $\varrho^N = \sum_{i=M}^{2N} i^2 u_i^N$,

$$\begin{aligned} \frac{\partial \varrho^N}{\partial t} - d\Delta \varrho^N &\le A \sum_{M\le i,j\le N} i^2 j^2 u_i^N u_j^N + A \sum_{1\le i<M\le j\le N} i^2 j(i+2j) u_i^N u_j^N \\ &\quad + \frac{A}{2} \sum_{1\le i,j\le M\le i+j} ij(i+j)^2 u_i^N u_j^N \\ &\le A(\varrho^N)^2 + F_1 \varrho^N + F_2, \end{aligned}$$

where

$$F_1 = 3A \sum_{i=1}^{M-1} i^2 K_i, \quad F_2 = \frac{A}{2} \sum_{1\le i,j\le M\le i+j} ij(i+j)^2 K_i K_j,$$

with K_i, $i = 1, \ldots, M$, given by (2.6). Applying the parabolic comparison principle, we obtain

$$\varrho^N(x,t) \le w(t) \quad \text{on } \Omega \times [0, T_m[,$$

where $w : [0, T_m[\to \mathbb{R}$ is the maximal solution of the o.d.e.

$$\frac{dw}{dt} = Aw^2 + F_1 w + F_2, \quad w(0) = \sum_{i=M}^{\infty} i^2 \|U_i\|_{L^\infty(\Omega)}.$$

Hence, for $T < T_m$,

$$\sum_{i=1}^{2N} i^2 u_i^N(x,t) \le C_1(T) \quad \text{for } (x,t) \in \Omega_T,$$

where

$$C_1(T) = \sup_{t\in[0,T]} \Big\{ \sum_{i=1}^{M-1} i^2 K_i + w(t) \Big\}.$$

To show (2.13) we shall first prove that there exists a constant $C_2(T)$ such that

$$\sup_{t\in[0,T]} \Big\| \sum_{i=1}^{2N} i u_i^N(\cdot,t) \Big\|_{L^\infty(\Omega)} \le C_2(T) \quad \text{for all } T \ge 0. \tag{2.15}$$

To this end, take in (2.11)

$$g_i = \begin{cases} 0 & \text{for } 1 \le i < M, \\ i & \text{for } i > M. \end{cases} \tag{2.16}$$

Then $\eta^N = \sum_{i=M}^{2N} iu_i^N$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t}\eta^N - d\Delta\eta^N &\le A \sum_{1\le i<M\le j\le N} i(i+j)u_i^N u_j^N + \frac{A}{2} \sum_{1\le i,j<M\le i+j} (i+j)^2 u_i^N u_j^N \\ &\le G_1\eta^N + G_2, \end{aligned}$$

where

$$G_1 = 2A\sum_{i=1}^{M-1} iK_i, \quad G_2 = \frac{A}{2}\sum_{1\le i,j<M\le i+j} (i+j)^2 K_iK_j.$$

By the parabolic comparison principle, we obtain

$$\eta^N(x,t) \le z(t) \quad \text{on } \Omega\times(0,T),$$

where $z : [0,T] \to \mathbb{R}$ is the solution of the linear o.d.e.

$$\frac{dz}{dt} = G_1 z + G_2, \quad z(0) = \sum_{i=M}^{\infty} i\|U_i\|_{L^\infty(\Omega)}.$$

Hence, the solution to (P^N) can be prolonged for all $T > 0$ and (2.15) follows with

$$C_2(T) = \sup_{t\in[0,T]} \bigg\{ \sum_{i=1}^{M-1} iK_i + z(t) \bigg\}.$$

Now, taking in (2.11) g_i as in (2.14) and using (2.15) we have for $\varrho^N = \sum_{i=M}^{2N} i^2u_i^N$,

$$\begin{aligned} \frac{\partial \varrho^N}{\partial t} - d\Delta\varrho^N &\le 2A\bigg(\sum_{j=M}^{N} ju_j^N\bigg)\sum_{i=M}^{2N} i^2u_i^N \\ &\quad + A\sum_{1\le i<M\le j\le N} (i^2+2ij)(i+j)u_i^Nu_j^N \\ &\quad + \frac{A}{2}\sum_{1\le i,j<M\le i+j} (i+j)^3u_i^Nu_j^N \\ &\le H_1\varrho^N + H_2, \end{aligned}$$

where

$$H_1 = 2AC_2(T) + 6A\sum_{i=1}^{M-1} iK_i, \quad H_2 = \sum_{1\le i,j<M\le i+j} (i+j)^3K_iK_j.$$

Using again the comparison principle we obtain

$$\varrho^N(x,t) \le y(t) \quad \text{for } (x,t) \in \Omega\times(0,T),$$

where $y : [0,T) \to \mathbb{R}$ is the solution of the linear o.d.e. defined for all $T > 0$,

$$\frac{dy}{dt} = H_1 y + H_2, \quad y(0) = \sum_{i=M}^{\infty} i^2 \|U_i\|_{L^\infty(\Omega)}.$$

Thus, (2.13) follows with

$$C_1(T) = \sup_{t\in[0,T]} \Big\{ \sum_{i=1}^{M-1} i^2 K_i + y(t) \Big\}.$$

This completes the proof of Lemma 2.3. □

3. Existence of solutions in the case of arbitrary positive diffusion coefficients

Theorem 3.1. *Under the assumption* (H3), *if*

$$a_{ij} = o(j) \quad \text{for each } i \ge 1 \tag{3.1}$$

then for $T > 0$ there exists a mild solution $\{u_i\}_{i=1}^{\infty}$ to (P) *defined on $\Omega \times (0,\infty)$ such that for $i = 1,2,\ldots$,*

$$\|u_i\|_{L^\infty(\Omega\times(0,\infty))} \le K_i \quad \text{where } K_i \text{ is defined in } (2.6), \tag{3.2}$$

$$u_i \in C([0,T]; L^1(\Omega)), \tag{3.3}$$

$$\sum_{j=1}^{\infty} a_{ij} u_j,\ \sum_{j=i+1}^{\infty} b_{i,j-i} u_j \in L^\infty(0,T; L^1(\Omega)) \quad \text{for any } T > 0 \tag{3.4}$$

and

$$\int_\Omega \sum_{i=1}^{\infty} i u_i(x,t)\,dx \le M_0 \quad \text{for } t \in [0,T]. \tag{3.5}$$

Proof. Let $\{u_i^N\}_{i=1}^{2N}$ be the solution to (P^N) defined on $\Omega \times (0,\infty)$. From (3.1) it follows that for each i, there exists a constant $\widetilde{c}_i$ such that

$$a_{ij} \le \widetilde{c}_i j \quad \text{for } j \ge 1.$$

Hence, by (2.3) we obtain, for each i and $N > i$,

$$\sup_{t\in[0,T]} \int_\Omega \sum_{j=1}^{N} a_{ij} u_j^N(x,t)\,dx \le \widetilde{c}_i M_0 \quad \text{for any } T > 0. \tag{3.6}$$

Similarly, by (H3) and (2.3) we have

$$\sup_{t\in[0,T]} \int_\Omega \sum_{j=i+1}^{N} b_{i,j-i} u_j^N(x,t)\,dx \le \widetilde{c}_i \gamma_i M_0 \quad \text{for any } T > 0. \tag{3.7}$$

Let us denote the reaction terms in the ith equation of (P^N) by

$$F_i^N = f_i^N - u_i^N g_i^N + h_i^N,$$

where

$$f_i^N = \frac{1}{2}\Big(\sum_{j=1}^{i-1} a_{i-j,j} u_{j-i}^N u_j^N - b_{i-j,j} u_i^N\Big),$$

$$g_i^N = \sum_{j=1}^{N} a_{ij} u_j^N, \quad h_i^N = \sum_{j=1}^{N-i} b_{ij} u_{i+j}^N = \sum_{j=i+1}^{N} b_{i,j-i} u_j^N$$

if $i \le N$, and

$$f_i^N = \frac{1}{2} \sum_{j=i-N}^{N} a_{j,i-j} u_j^N u_{i-j}^N, \quad g_i^N = 0, \quad h_i^N = 0$$

for $N < i \le 2N$.

By Lemma 2.2 and (3.6), (3.7) it follows that for $N \ge i$,

$$\|f_i^N - u_i^N g_i^N + h_i^N\|_{L^\infty(0,T;L^1(\Omega))} \le D_i, \tag{3.8}$$

where D_i, $i = 1, 2, \ldots$, are positive constants independent of N.

The operator L is closable in $L^1(\Omega)$ and accretive in this space. Its closure L_1 generates a compact, positive, analytic semigroup in $L^1(\Omega)$ (see [1]). Using (3.8) and applying to each equation the compactness result from [3, Theorem 1(i)] yields, for $i = 1, 2, \ldots$, relative compactness of $\{u_i^N\}_{N=i}^\infty$ in the space $C([0,T]; L^1(\Omega))$. Let $\{N_l\}_{l=1}^\infty$ be the diagonal increasing sequence such that

$$u_i^{N_l} \to u_i \quad \text{in } C([0,T]; L^1(\Omega)), \qquad u_i^{N_l} \to u_i \quad \text{a.e. in } \Omega_T \tag{3.9}$$

as $l \to \infty$, for all $i = 1, 2, \ldots$ For fixed l and $i < N_l$ the function $u_i^{N_l}$ is the mild solution given by the Duhamel formula

$$u_i^{N_l}(t) = e^{td_iL} U_i + \int_0^t e^{(t-s)d_iL} (f_i^{N_l}(s) - u_i^{N_l}(s) g_i^{N_l}(s) + h_i^{N_l}(s))\, ds. \tag{3.10}$$

Notice that thanks to (3.6) and (3.7), for fixed i, (3.4) follows. By (3.9) and Lemma 2.2 it follows that

$$f_i^{N_l} \to \frac{1}{2}\Big(\sum_{j=1}^{i-1} a_{i-j,j} u_{i-j} u_j - b_{i-j,j} u_i\Big) \quad \text{in } C([0,T]; L^1(\Omega))$$

as $l \to \infty$. In order to pass to the limit in (3.10) we shall show that for each $i = 1, 2, \ldots,$

$$g_i^{N_l} \to \sum_{j=1}^{\infty} a_{ij} u_j \qquad \text{in } L^1(\Omega_T), \tag{3.11}$$

$$h_i^{N_l} \to \sum_{j=i+1}^{\infty} b_{i,j-i} u_j \quad \text{in } L^1(\Omega_T) \tag{3.12}$$

as $l \to \infty$. To this end notice that by (3.1) for fixed i and arbitrary $\varepsilon > 0$ there exists l_0 such that for any $l > l_0$,

$$\int\int_{\Omega_T} \sum_{j=N_{l_0}}^{N_l} a_{ij} u_j^{N_l} < \frac{\varepsilon}{3} \quad \text{and} \quad \int\int_{\Omega_T} \sum_{j=N_{l_0}}^{\infty} a_{ij} u_j < \frac{\varepsilon}{3},$$

where ε is such that

$$3M_0 T a_{ij}/j \le \varepsilon \quad \text{for } j > N_{l_0}.$$

Thanks to (3.9) there exists $l_1 \ge l_0$ such that for any $l > l_1$,

$$\int\int_{\Omega_T} \sum_{j=1}^{N_{l_0}-1} a_{ij} |u_j^{N_l} - u_j| \le \frac{\varepsilon}{3}.$$

It follows that for $l > l_1$,

$$\int\int_{\Omega_T} \Big| \sum_{j=1}^{N_l} a_{ij} u_j^{N_l} - \sum_{j=1}^{\infty} a_{ij} u_j \Big| < \varepsilon.$$

Hence (3.11) follows. The proof of (3.12) is similar, so we skip it. By Lemma 2.2, (3.9) and (3.11) it follows that

$$u_i^{N_l} g_i^{N_l} \to u_i \sum_{j=1}^{\infty} a_{ij} u_j \quad \text{in } L^1(\Omega_T) \text{ as } l \to \infty,$$

which enables us to pass to the limit in (3.10).

Using the above reasoning one can construct a solution defined on $\Omega \times (0,\infty)$ which satisfies (3.2), in the following way. Let $\{T_n\}_{n=1}^{\infty}$ be any increasing sequence of positive numbers such that $T_n \to \infty$. Then using a compactness argument there exists a sequence $\{N_l^1\}_{l=1}^{\infty}$ such that for each $i = 1, 2, \ldots$ a solution u_i^1 to (P) on the interval $[0, T_1]$ is defined as the limit of $\{u_i^{N_l^1}|_{[0,T_1]}\}_{l=1}^{\infty}$. Let N_l^2 be a subsequence of $\{N_l^1\}_{l=1}^{\infty}$ such that $\{u_i^{N_l^2}|_{[0,T_2]}\}_{l=1}^{\infty}$ tends to the solution u_i^2 defined on $[0, T_2]$. Of course, the solutions u_i^1 and u_i^2 coincide on $[0, T_1]$. Step by step we define in this way the sequence $\{N_l^n\}_{l=1}^{\infty}$ for any $n \ge 1$. Hence, the solution u_i to (P) on $\Omega \times (0,\infty)$ is defined as the limit of $\{u_i^{N_l^l}\}_{l=1}^{\infty}$, where $\{N_l^l\}_{l=1}^{\infty}$ is the diagonal subsequence.

Passing to the limit we obtain (3.4) and (3.5) from (3.8) and (2.3) respectively. □

REMARK 1. Since the reaction terms in each equation are in the space $L^\infty(0,T;L^1(\Omega))$, we have $u_i \in C^\alpha([\delta,T];L^1(\Omega))$ for any $0 \le \alpha < 1$ and $\delta > 0$, which follows from the regularity result for mild solutions (see e.g. [12, pp. 110–111]) applied separately to each equation.

Assuming some structural assumptions on the coefficients a_{ij} we are able to prove the existence of strong solutions in $L^2(\Omega_T)$.

Theorem 3.2. *Under the assumption* (H3), *suppose that there exists* $\{r_i\}_{i=1}^{\infty}$ *such that*

$$a_{ij} = r_i r_j, \tag{3.13}$$

$$r_i = o(i) \quad \textit{for } i \geq 1, \ \textit{and} \tag{3.14}$$

$$\sum_{i+j=k} b_{ij} \leq Bk \quad \textit{where} \quad B > 0. \tag{3.15}$$

Then there exists a nonnegative solution $\{u_i\}_{i=1}^{\infty}$ *to* (P) *defined on* $\Omega \times (0,\infty)$ *such that* (3.2) *and* (3.5) *hold and for* $i = 1, 2, \dots$,

$$\begin{aligned} &u_i \in C([0,T]; L^2(\Omega)) \quad \textit{for any } T \textit{ and satisfies } (1.2),\\ &u_i \in W^{1,2}_{\mathrm{loc}}(]0,\infty[; L^2(\Omega)) \cap L^2_{\mathrm{loc}}(]0,\infty[; D(L)),\\ &\sum_{i=1}^{\infty} r_i u_i \in L^2(\Omega_T) \quad \textit{for any } T > 0, \end{aligned} \tag{3.16}$$

and the equations (1.1) *are satisfied a.e. on* $\Omega \times (0,\infty)$.

Proof. Let $\{u_i^N\}_{i=1}^{2N}$ be the solution of (P^N). Taking $g_i = 1$ for $i \geq 1$ in (2.2) and integrating on Ω_t we obtain, using (3.13),

$$\sum_{i=1}^{2N} \int_\Omega u_i^N(x,t)\,dx - \sum_{i=1}^{2N} \int_\Omega U_i(x)\,dx + \frac{1}{2} \int\int_{\Omega_t} \bigg(\sum_{i=1}^{N} r_i u_i^N\bigg)^2 = \frac{1}{2} \int\int_{\Omega_T} \sum_{k=2}^{N} \sum_{i+j=k} b_{ij} u_k^N. \tag{3.17}$$

Hence, using (3.15) for any $T > 0$,

$$\bigg\| \sum_{i=1}^{N} r_i u_i^N \bigg\|^2_{L^2(\Omega_T)} \leq BTM_0. \tag{3.18}$$

By (H3) we have for $i = 1, 2, \dots$,

$$\int\int_{\Omega_T} \bigg(\sum_{j=i+1}^{N} b_{i,j-i} u_j^N\bigg)^2 \leq \gamma_i r_i \int\int_{\Omega_T} \bigg(\sum_{j=i+1}^{N} r_j u_j^N\bigg)^2. \tag{3.19}$$

Therefore,

$$\|f_i^N - u_i^N g_i^N + h_i^N\|_{L^2(\Omega_T)} \leq D_i, \tag{3.20}$$

where D_i, $i = 1, 2, \dots$, are positive constants independent of N.

Now, to end the proof, it is sufficient to use the existence result of Theorem 3.1. Since now the reaction terms are in the space $L^2(\Omega)$ the regularity of solutions in (3.16) follows from the standard theory of parabolic equations (see e.g. [11, 13, 12]) applied separately to each equation in (P).

4. Existence of mass preserving solutions

In this section we exploit the hypothesis (H4) which enables us to find better estimates of the terms

$$\sum_{i=1}^{\infty} iu_i, \quad \sum_{j=1}^{\infty} a_{ij}u_j, \quad \sum_{j=i+1}^{\infty} b_{i,j-i}u_j.$$

Then the hypothesis (H3) can be modified in the following way:

(H3)$_M$ For each $i \geq 1$, if $1 \leq i < M$ then there exists $\gamma_i > 0$ such that

$$\begin{aligned} &b_{i,j-i} \leq \gamma_i a_{ij} &&\text{for } j \geq i+1, \text{ and} \\ &b_{ij} = o(j^2) &&\text{for each } i \geq M. \end{aligned} \tag{4.1}$$

THEOREM 4.1. *Under the assumptions* (H4), (H3)$_M$ *and* (H1) *or* (H2), *if*

$$\sum_{i=1}^{\infty} i^2 \|U_i\|_{L^\infty(\Omega)} \leq \text{const}$$

then there exists $T > 0$ such that (P) *has a nonnegative solution $\{u_i\}_{i=1}^{\infty}$ on Ω_T such that for $i = 1, 2, \ldots,$*

$$\begin{aligned} &u_i \in C([0,T]; L^2(\Omega)) \quad \textit{and satisfies } (1.2), \\ &u_i \in L^\infty(\Omega_T) \cap W^{1,2}_{\text{loc}}(]0,T]; L^2(\Omega)), \\ &u_i \in L^2_{\text{loc}}(]0,T]; D(L)), \end{aligned} \tag{4.2}$$

$$\sum_{j=1}^{\infty} a_{ij}u_j, \ \sum_{j=i+1}^{\infty} b_{i,j-i}u_j \in L^\infty(\Omega_T). \tag{4.3}$$

The equations of (1.1) *are satisfied a.e. on Ω_T and*

$$\sum_{k=1}^{\infty} \int_\Omega k u_k(x,t)\, dx = M_0 \quad \textit{for } t \in [0,T]. \tag{4.4}$$

Under the hypothesis (H2) *the solution is defined on $\Omega \times (0,\infty)$ and the above statement is true for any $T > 0$.*

PROOF. We proceed in much the same way as in the proof of Theorem 3.1. By Lemma 2.3 there exists $T_m > 0$ such that for any $N > 1$ and $0 < T < T_m$,

$$\Big\| \sum_{j=1}^{N} a_{ij}u_j^N \Big\|_{L^\infty(\Omega_T)} \leq \Big\| Ai \sum_{j=1}^{N} ju_j^N \Big\|_{L^\infty(\Omega_T)} \leq AiC_1(T) \tag{4.5}$$

and by (4.1),

$$\Big\| \sum_{j=i+1}^{N} b_{i,j-i}u_j^N \Big\|_{L^\infty(\Omega_T)} \leq \widetilde{c}_i C_1(T), \tag{4.6}$$

where $\widetilde{c_i}$ is a positive constant. Under the assumption (H2) the estimates above are valid for any $T > 0$.

It follows that (3.20) still holds true. Using again the compactness argument we can choose a subsequence $\{N_l\}_{l=1}^{\infty}$ such that

$$u_i^{N_l} \to u_i \quad \text{in } C([0,T];L^2(\Omega)), \qquad u_i^{N_l} \to u_i \quad \text{a.e. in } \Omega_T$$

as $l \to \infty$. By Lemma 2.3 it follows that for any $k \le N_l$,

$$\Big\| \sum_{j=k}^{N_l} a_{ij} u_j^{N_l} \Big\|_{L^\infty(\Omega_T)} + \Big\| \sum_{j=k}^{2N_l} j u_j^{N_l}(\cdot,t) \Big\|_{L^\infty(\Omega)} \le \frac{iA+1}{k} C_1(T) \tag{4.7}$$

for $0 \le t \le T$. By Lemma 2.3 and (4.1) it follows that for each $i \ge 1$ and for $\varepsilon > 0$ there exists $k_0 > i$ such that

$$\Big\| \sum_{j=k_0}^{N_l} b_{i,j-i} u_j^{N_l} \Big\|_{L^\infty(\Omega_T)} < \varepsilon,$$

where

$$C_1(T) \frac{b_{i,j-i}}{j^2} < \varepsilon \quad \text{for } j \ge k_0.$$

Proceeding as in the last part of the proof of Theorem 3.1 one shows (3.11) and (3.12) with $L^1(\Omega_T)$ replaced by $L^2(\Omega_T)$. Similarly, by (4.7) for $0 \le t \le T$,

$$\sum_{j=1}^{2N_l} j u_j^{N_l}(\cdot,t) \to \sum_{j=1}^{\infty} j u_j(\cdot,t) \quad \text{in } L^1(\Omega) \quad \text{as } l \to \infty.$$

Hence, using (2.3) we obtain (4.4). Finally, by (4.5) and (4.6) we arrive at (4.3) by passing to the limit. □

The following theorem shows the existence of solutions under the assumption (H2) for arbitrarily fast fragmentation. However, to prove it we need all diffusion coefficients to be the same in each equation.

Theorem 4.2. *In the case* (H2), *if $d_i = d > 0$ for $i \ge 1$ and*

$$\sum_{i=1}^{\infty} i^2 \|U_i\|_{L^\infty} \le \text{const}, \tag{4.8}$$

then for arbitrary $b_{ij} \ge 0$, $i,j \ge 1$, there exists a mild solution $\{u_i\}_{i=1}^{\infty}$ to (P) *defined on $\Omega \times (0,\infty)$ such that for any $T > 0$ and each $i = 1,2\dots,$*

$$\begin{aligned} &u_i \in C([0,T];L^1(\Omega)) \quad \textit{and satisfies } (1.2),\\ &\sum_{j=1}^{\infty} a_{ij} u_j \in L^\infty(\Omega_T), \quad \sum_{j=1}^{\infty} b_{i,j-i} u_j \in L^1(\Omega_T),\\ &\int_\Omega \sum_{i=1}^{\infty} i u_i(x,t)\,dx = M_0 \quad \textit{for} \;\; t \in [0,T]. \end{aligned}$$

PROOF. For any $N > 2$ let $\{u_i^N\}_{i=1}^{2N}$ be the solution of (P^N). Setting $M = 1$ in Lemma 2.3 we have for $T > 0$,

$$\Big\| \sum_{j=1}^{2N} j^2 u_j^N \Big\|_{L^\infty(\Omega_T)} \le C_1(T) \tag{4.9}$$

without any growth condition on b_{ij}. In particular, it follows that the solution is global in time. Now, taking in (2.2) $g_i = i^2$ for $i = 1, \ldots, 2N$, integrating over Ω_t and using (H2) we obtain for $0 < t \le T$,

$$\begin{aligned}\int_\Omega \sum_{i=1}^{2N} i^2 u_i^N(x,t)\,dx + \int\!\!\int_{\Omega_t} \sum_{1<i+j\le N} ijb_{ij}u_{i+j}^N \\ \le |\Omega| TA(C_1(T))^2 + \sum_{i=1}^{\infty} \int_\Omega i^2 U_i(x)\,dx.\end{aligned} \tag{4.10}$$

Hence, for each i and $N > i$ there exists a constant $\widehat{c_i}$ independent of N such that for $i = 1, 2, \ldots,$

$$\int\!\!\int_{\Omega_T} \sum_{j=1}^{N-i} b_{ij} u_{i+j}^N \le \widehat{c_i}(T). \tag{4.11}$$

For each i inequalities (4.10) and (4.11) imply the uniform bound

$$\|f_i^N - u_i^N g_i^N + h_i^N\|_{L^1(0,T;L^1(\Omega))} \le \text{const}.$$

Using the compactness argument from [3, Theorem 1(ii)] one can choose a subsequence $\{N_l\}_{l=1}^\infty$ such that

$$\begin{aligned} &u_i^{N_l} \to u_i \quad \text{in } L^q(0,T;L^1(\Omega)) \text{ for any } 1 \le q < \infty, \\ &u_i^{N_l} \to u_i \quad \text{a.e. in } \Omega_T \end{aligned}$$

as $l \to \infty$, for all $i = 1, 2, \ldots$ In the same way as in the proof of Theorem 3.1 one shows (3.11) using (4.9). To show (3.12) notice that due to (4.10) for each $i \ge 1$ and $k < N_l - i$,

$$\int\!\!\int_{\Omega_T} \sum_{j=k}^{N_l - i} b_{ij} u_{i+j}^{N_l} \le \frac{\widehat{c_i}(T)}{k}.$$

Now, passing to the limit in the Duhamel formula as $l \to \infty$ we obtain

$$u_i^{N_l} \to u_i \quad \text{in } C([0,T]; L^1(\Omega)).$$

By (4.10) for $0 < t \le T$ we have

$$\int_\Omega \sum_{i=k}^{2N} i u_i^N(x,t) dx \le \frac{\text{const}}{k},$$

hence (1.4) follows. $\square$

Remark 2. In [2] the existence of solutions for the model without diffusion is proved under the same assumptions on a_{ij} and b_{ij} as in the above theorem. It is worth pointing out that the result is shown there under the only assumption that the total mass contained in the initial data is finite, $M_0 < \infty$. We assume more, (4.8), but our proof seems to be shorter than that in [2]. It is not known to the author how to transfer the result from [2] to our case.

Remark 3. In general there is no uniqueness of solutions for the system (1.1) even for the case without diffusion (see [2]). In the particular case of $a_{ij} = r_i r_j$ and $r_i = Ai + B$ where A, B are constants $(A > 0, B \geq 0), b_{ij} = 0$ for $i, j \geq 1$, and (H4) holds, uniqueness of solutions is proved in [4].

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