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TOPOLOGY OF AN ATTRACTION DOMAIN, DYNAMICAL ZETA FUNCTIONS AND REIDEMEISTER TORSION

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We dedicate this paper to O. A. Ladyzhenskaya

0. Introduction

In Section 1 of this paper we consider a flow with a circular chain-recurrent set and describe, with the help of the Reidemeister torsion, the connection between the topology of the attraction domain of an attractor and the dynamics of the flow on the attractor. We show in Theorem 3 that the Reidemeister torsion of a level surface of a Lyapunov function and of the attraction domain of an attractor is calculated as a special value of the twisted Lefschetz zeta function build via closed orbits in the attractor. In Section 2 we continue the study of analytical properties of the Nielsen zeta function. The Nielsen zeta function $N_f(z)$ has a positive radius of convergence which has a sharp estimate in terms of the topological entropy of the map f [15]. In Theorem 5 of Section 2 we propose another proof of the positivity of the radius and give an exact algebraic lower estimate for the radius using the Reidemeister trace formula for the generalized Lefschetz numbers.

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1. Topology of an attraction domain, twisted Lefschetz zeta function and Reidemeister torsion

1.1. Preliminaries. Assume that on a smooth compact manifold M of dimension n there is given a tangent vector field X of class C^1, and consider the corresponding system of differential equations

$$\frac{dx}{dt} = X(x). \tag{1}$$

Let $\phi(t,x)$ be the trajectory of (1) passing through the point x for $t = 0$. We shall say that a set I is an *attractor* or an *asymptotically stable compact invariant set* for the system (1) if for any neighbourhood U of I there is a neighbourhood W with $I \subset W \subset U$ such that

1) for any $x \in W$, $\phi(t,x) \in U$ for $t \in [0,\infty)$,
2) for any $x \in W$, $\phi(t,x) \to I$ as $t \to \infty$.

By a *Lyapunov function* $V(x)$ for an attractor I we mean a function that satisfies the following conditions:

1) $V \in C^1(U - I)$, $V \in C(U)$,
2) $V(x) > 0$, $x \in U - I$; $V(x) = 0$, $x \in I$,
3) the derivative along the system (1) satisfies $dV(x)/dt < 0$ in $U - I$.

Such a Lyapunov function for I always exists [36]. Suppose that S is a level surface of a Lyapunov function V in U. The condition 3) and the Implicit Function Theorem imply that S is a compact smooth $(n-1)$-dimensional manifold transverse to the trajectories of (1), and these trajectories intersect S on the descending side of the Lyapunov function V. Any two level surfaces of V are diffeomorphic. Note that the manifold S is determined up to diffeomorphism by the behaviour of the trajectories of (1) in $U - I$ and does not depend on the choice of the Lyapunov function V and its level. Let $N \supset I, \dim N = n$, be a compact smooth manifold with boundary $\partial N = S$.

In this article we will study the dependence of the topology of the *attraction domain*

$$D = \{x \in M - I : \phi(t,x) \to I \text{ as } t \to \infty\}$$

of the attractor I and of the level surface S of the Lyapunov function V on the dynamical properties of the system (1) on the attractor. The investigation of the topological structure of the level surfaces of Lyapunov functions was initiated by Wilson [36]. Note that the attraction domain D is diffeomorphic to $S \times \mathbb{R}^1$, since each trajectory of (1) in D intersects the $(n-1)$-dimensional manifold S exactly once. Hence the homology groups of D and S are isomorphic.

If (1) has a finite number of closed orbits and is a Morse–Smale system then in [11] (see also Subsection 1.2 below) we have described the topology of the attraction domain with the help of an analog of Morse inequalities.

In Subsection 1.3 we consider the flow (1) with a circular chain-recurrent set $R \subset I$, i.e. there is a smooth map $\theta : U \to \mathbb{R}^1/\mathbb{Z}$, U a neighbourhood of R in N, on which $\frac{d}{dt}(\theta \circ \phi(t,x)) > 0$. In other words, there is a cross-section of the flow (1) on R, namely, a level set of θ on $\mathrm{int}(U)$. For instance, if R is finite, i.e. consists of finitely many closed orbits, then R is circular. More generally, if ϕ on R has no stationary points and the topological dimension of R is 1, then R is circular. For example, if ϕ is a nonsingular Smale flow, so that R is hyperbolic and 1-dimensional, then R is circular.

If the flow (1) has a circular chain-recurrent set R then we describe in Theorem 3 the connection between the topology of the attraction domain and the dynamics of the flow on the attractor with the help of the twisted Lefschetz zeta function and Reidemeister torsion.

1.2. Morse–Smale systems. We assume in this subsection that the system (1) is given in $\mathbb{R}^n$ and is a Morse–Smale system on the manifold N, i.e. the following conditions are satisfied:

1) the set Ω of nonwandering trajectories of (1) is the union of a finite number of hyperbolic stationary points and hyperbolic closed trajectories,
2) the stable and unstable manifolds of stationary points and closed trajectories intersect transversely.

The stable and unstable manifolds of a stationary point or a closed trajectory p are denoted by $W^{\mathrm{s}}(p)$ and $W^{\mathrm{u}}(p)$. Let a_k be the number of stationary points p of (1) in I such that $\dim W^{\mathrm{u}}(p) = k$, and b_k the number of closed orbits q of (1) in I such that $\dim W^{\mathrm{u}}(q) = k$. Set $M_k = a_k + b_k + b_{k+1}$, $B_k = \dim H_k(D;Q) = \dim H_k(S;Q)$; finally, $\chi(D) = \chi(S)$ is the Euler characteristic of D and S.

Theorem 1 ([11]). *The numbers B_k and M_k satisfy the following inequalities:*

$$
\begin{aligned}
&B_0 \le M_0 + M_{n-1} - M_n,\\
&B_1 - B_0 \le M_1 - M_0 + M_{n-2} - M_{n-1} + M_n,\\
&B_2 - B_1 + B_0 \le M_2 - M_1 + M_0 + M_{n-3} - M_{n-2} + M_{n-1} - M_n,\\
&\vdots\\
&\sum_{i=0}^{n-1}(-1)^i B_i = \chi(S) = \chi(D) = (1 + (-1)^{n-1}) \sum_{i=0}^{n}(-1)^i M_i.
\end{aligned}
\tag{2}
$$

The last identity in Theorem 1 is also true in a more general situation. Namely, assume that (1) is an autonomous system of differential equations having a finite number of stationary points in the attractor I. Denote by $\mathrm{Index}(p)$ the index of the vector field X at the stationary point p.

Theorem 2 ([11]).

$$\chi(D) = \chi(S) = ((-1)^n - 1)\sum_{p\in I}\mathrm{Index}(p). \tag{3}$$

Proof. The vector field X is directed on ∂N into N. Therefore from the Poincaré–Hopf theorem, by replacing X by $-X$ we obtain

$$\chi(N) = (-1)^n\sum_{p\in I}\mathrm{Index}(p).$$

It is known that $\chi(\partial N) = (1+(-1)^{n-1})\chi(N)$. Hence

$$\chi(D) = \chi(S) = ((-1)^n - 1)\sum_{p\in I}\mathrm{Index}(p).$$

Since for a hyperbolic stationary point p with $\dim W^{\mathrm{u}}(p) = k$ the index of the vector field at p is $(-1)^k$, we obtain

Corollary 1. *Suppose the stationary points on I are hyperbolic, and a_k is the number of stationary points of I with* $\dim W^{\mathrm{u}}(p) = k$. *Then*

$$\chi(D) = \chi(S) = ((-1)^n - 1)\sum_{k=0}^{n}(-1)^k a_k. \tag{4}$$

For $n = 3$, S is a union of a finite number of spheres with handles. Suppose m is the number of connected components, and h is the total number of handles of the manifold S. Then $\chi(S) = 2m - 2h$. Hence we obtain

Corollary 2. *We have*

$$m - h = -\sum_{p\in I}\mathrm{Index}(p). \tag{5}$$

If the stationary points are hyperbolic then

$$m - h = a_0 - a_1 + a_2 - a_3.$$

1.3. Reidemeister torsion. In 1936 Reidemeister [31] classified up to PL equivalence the lens spaces S^3/Γ where Γ is a finite cyclic group of fixed point free orthogonal transformations. He used a certain new invariant which was quickly extended by Franz [17], who used it to classify the generalized lens spaces S^{2n+1}/Γ. This invariant is a ratio of determinants concocted from a Γ-equivariant chain complex of S^{2n+1} and a nontrivial character $\varrho : \Gamma \to U(1)$ of Γ. Such a ϱ determines a flat bundle E over S^{2n+1}/Γ such that E has holonomy ϱ. The new invariant is now called the *Reidemeister torsion* or *R-torsion* of E. The Reidemeister torsion is closely related to the K_1 groups of algebraic K-theory. The results of Reidemeister and Franz were extended by de Rham [32] to spaces of constant curvature $+1$.

Later Milnor [27] identified the Reidemeister torsion with the Alexander polynomial, which plays a fundamental role in the theory of knots and links. Then Cheeger [5] and Müller [28] proved that the Reidemeister torsion coincides with the analytical torsion of Ray and Singer [30].

Recently a connection between the Lefschetz type dynamical zeta functions and the Reidemeister torsion was established by D. Fried [19, 20]. The work of Milnor [26] was the first indication that such a connection exists.

In [13] a connection was established between the Reidemeister torsion and Reidemeister zeta function. We obtained an expression for the Reidemeister torsion of the mapping torus of the dual map of a group endomorphism in terms of the Reidemeister zeta function of the endomorphism. The result is obtained by expressing the Reidemeister zeta function in terms of the Lefschetz zeta function of the dual map, and then applying the theorem of D. Fried. This means that the Reidemeister torsion counts the fixed point classes of all iterates of f, i.e. periodic point classes of f.

Like the Euler characteristic, the Reidemeister torsion is algebraically defined. Roughly speaking, the Euler characteristic is a graded version of the dimension, extending the dimension from a single vector space to a complex of vector spaces. In a similar way, the Reidemeister torsion is a graded version of the absolute value of the determinant of an isomorphism of vector spaces. Let $d^i : C^i \to C^{i+1}$ be a cochain complex C^* of finite-dimensional vector spaces over $\mathbb{C}$ with $C^i = 0$ for $i < 0$ and for large i. If the cohomology $H^i = 0$ for all i we say that C^* is *acyclic*. If one is given positive densities Δ_i on C^i then the *Reidemeister torsion* $\tau(C^*, \Delta_i) \in (0, \infty)$ for acyclic C^* is defined as follows:

Definition 1. Consider a chain contraction $\delta^i : C^i \to C^{i-1}$, i.e. a linear map such that $d \circ \delta + \delta \circ d = \mathrm{id}$. Then $d + \delta$ determines a map $(d+\delta)_+ : C^+ := \bigoplus C^{2i} \to C^- := \bigoplus C^{2i+1}$ and a map $(d+\delta)_- : C^- \to C^+$. Since the map $(d+\delta)^2 = \mathrm{id} + \delta^2$ is unipotent, $(d+\delta)_+$ must be an isomorphism. One defines $\tau(C^*, \Delta_i) := |\det(d+\delta)_+|$ (see [20]).

Reidemeister torsion is defined in the following geometric setting. Suppose K is a finite complex and E is a flat, finite-dimensional, complex vector bundle with base K. We recall that a flat vector bundle over K is essentially the same thing as a representation of $\pi_1(K)$ when K is connected. If $p \in K$ is a base point then one may move the fibre at p in a locally constant way around a loop in K. This defines an action of $\pi_1(K)$ on the fibre E_p of E above p. We call this action the *holonomy representation* $\varrho : \pi \to \mathrm{GL}(E_p)$. Conversely, given a representation $\varrho : \pi \to \mathrm{GL}(V)$ of π on a finite-dimensional complex vector space V, one may define a bundle $E = E_\varrho = (\widetilde{K} \times V)/\pi$. Here $\widetilde{K}$ is the universal cover of K, and π acts on $\widetilde{K}$ by covering transformations and on V by ϱ. The holonomy of E_ϱ is ϱ, so the two constructions give the equivalence of flat bundles and representations of π.

If K is not connected then it is simpler to work with flat bundles. One then defines the holonomy as a representation of the direct sum of π_1 of the components of K. In this way, the equivalence of flat bundles and representations is recovered.

Suppose now that one has on each fibre of E a positive density which is locally constant on K. In terms of ϱ_E this assumption just means $|\det \varrho_E| = 1$. Let V denote the fibre of E.

Then the cochain complex $C^i(K;E)$ with coefficients in E can be identified with the direct sum of copies of V associated with each i-cell σ of K. The identification is achieved by choosing a base point in each component of K and a base point from each i-cell. By choosing a flat density on E we obtain a preferred density Δ_i on $C^i(K;E)$. One defines the *R-torsion* of (K,E) to be $\tau(K;E) = \tau(C^*(K;E), \Delta_i) \in (0,\infty)$.

The Reidemeister torsion of an acyclic bundle E on K has many nice properties. Suppose that A and B are subcomplexes of K. Then we have a multiplicative law:

$$\tau(A \cup B; E) \cdot \tau(A \cap B; E) = \tau(A;E) \cdot \tau(B;E), \tag{6}$$

which is interpreted as follows. If three of the bundles $E|_{A\cup B}$, $E|_{A\cap B}$, $E|_A$, $E|_B$ are acyclic then so is the fourth and the equality (6) holds.

Another property is the simple homotopy invariance of the Reidemeister torsion. Suppose K' is a subcomplex of K obtained by an elementary collapse of an n-cell σ in K. This means that $K = K' \cup \sigma \cup \sigma'$ where σ' is an $(n-1)$-cell of K such that $\partial\sigma' = \sigma' \cap K'$ and $\sigma' \subset \partial\sigma$, i.e. σ' is a free face of σ. So one can push σ' through σ into K' giving a homotopy equivalence. Then $H^*(K;E) = H^*(K';E)$ and

$$\tau(K;E) = \tau(K';E). \tag{7}$$

By iterating elementary collapses and their inverses, one obtains a homotopy equivalence of complexes that is called *simple*. Plainly, by iterating (7), the Reidemeister torsion is a simple homotopy invariant. In particular, τ is invariant under subdivision. This implies that for a smooth manifold, one can unambiguously define $\tau(K;E)$ to be the torsion of any smooth triangulation of K.

In the case $K = S^1$ is a circle, let A be the holonomy of a generator of the fundamental group $\pi_1(S^1)$. The bundle E is acyclic iff $I - A$ is invertible and then

$$\tau(S^1;E) = |\det(I - A)|. \tag{8}$$

Note that the choice of a generator is irrelevant as $I - A^{-1} = (-A^{-1})(I - A)$ and $|\det(-A^{-1})| = 1$.

These three properties of the Reidemeister torsion are the analogues of the properties of Euler characteristic (cardinality law, homotopy invariance and normalization on a point), but there are differences. Since a point has no acyclic representations ($H^0 \neq 0$) one cannot normalize τ on a point as we do for the Euler characteristic, and so one must use S^1 instead. The multiplicative cardinality law for the Reidemeister torsion can be made additive just by using $\log \tau$, so the difference here is inessential. More important for some purposes is that the Reidemeister torsion is not an invariant under a general homotopy equivalence: as mentioned earlier this is in fact why it was invented.

It might be expected that the Reidemeister torsion counts something geometric (like the Euler characteristic). D. Fried showed that it counts the periodic orbits of a flow and the periodic points of a map.

1.4. The Reidemeister torsion of the level surface of a Lyapunov function and of the attraction domain of the attractor. In this subsection we consider the flow (1) with a circular chain-recurrent set $R \subset I$. The Reidemeister torsion of the attraction domain D and of the level surface S is the relevant topological invariant of D and S which is calculated in Theorem 3 and Corollary 3 via closed orbits of the flow (1) in the attractor I.

The point $x \in M$ is called *chain-recurrent* for the flow (1) if for any $\varepsilon > 0$ there exist points $x_1 = x, x_2, \ldots, x_n = x$ and real numbers $t(i) \geq 1$ such that $\varrho(\phi(t(i), x_i), x_{i+1}) < \varepsilon$ for $1 \leq i < n$. Let $R \subset I$ be a set of chain-recurrent points of (1) on the manifold N defined above. We assume in this section that R is *circular*, i.e. there is a smooth map $\theta : U \to \mathbb{R}^1/\mathbb{Z}$, U a neighbourhood of R in N, on which $\frac{d}{dt}(\theta \circ \phi(t, x)) > 0$. In other words, there is a cross-section of the flow (1) on R, namely, a level set of θ on $\operatorname{int}(U)$. For instance, if R is finite, i.e. consists of finitely many closed orbits, then R is circular. More generally, if ϕ on R has no stationary points and the topological dimension of R is 1, then R is circular. For example, if ϕ is a nonsingular Smale flow, so that R is hyperbolic and 1-dimensional, then R is circular.

If $U \subset N$ is such that $\bigcap_{t\in\mathbb{R}^1} \phi_t(U) = J$ is compact and $J \subset \operatorname{int}(U)$, then we say that U is an *isolating neighbourhood* of the isolated invariant set J. According to Conley [6], there is a continuous function $G : N \to \mathbb{R}^1$ such that G is decreasing on $N - R$ and $G(R)$ is nowhere dense in $\mathbb{R}^1$. Taking an open neighbourhood W of $G(R)$ and $U = G^{-1}(W)$, we see that U is an isolating neighbourhood for some isolating invariant set J and that $J \to R$ as $W \to G(R)$ [18]. This proves that the chain-recurrent set R can be approximated by the isolated invariant set J. In particular, we can make J circular. Further, there are finitely many points $x_i < x_{i+1}$ in $\mathbb{R}^1 - G(R)$ such that $G^{-1}[x_i, x_{i+1}]$ isolates an invariant set J_i so that $J = \bigcup J_i$ is as close as we like to R. In particular, we can make J circular. In the sequel we need isolating blocks [6]. A compact isolating neighbourhood B_i of J_i is said to be an *isolating block* if:

1) B_i is a smooth manifold with corners,
2) $\partial B_i = b_i^+ \cup b_i^- \cup b_i^0$ where each term is a compact manifold with boundary,
3) the trajectories $\phi(t,x)$ are tangent to b_i^0 and $\partial b_i^0 = (b_i^+ \cup b_i^-) \cap b_i^0$,
4) the trajectories $\phi(t,x)$ are transverse to b_i^+ and b_i^-, enter B_i through b_i^+ and exit through b_i^-.

Since J_i is circular there is a smooth map $\theta_i : B_i \to \mathbb{R}^1/\mathbb{Z}$ such that $\frac{d}{dt}(\theta_i \circ \phi(t,x)) > 0$ on B_i. By perturbing θ_i, we can make θ_i transverse to $0 \in \mathbb{R}^1/\mathbb{Z}$ on $B_i, b_i^+, b_i^-, b_i^0, \partial b_i^+, \partial b_i^-, \partial b_i^0$. Now let $Y_i = \theta_i^{-1}(0) \cup b_i^-$, and $Z_i = b_i^-$. Then (Y_i, Z_i) is a simplicial pair. We define a continuous map $r_i : Y_i \to Y_i$ as follows. If $y_i \notin Z_i$ then $r_i(y) = \phi(\tau, y)$ where $\tau = \tau(y) > 0$ is the smallest positive time t for which $\phi(t,y) \in Y_i$. Since $\phi(t,x)$ exits transversely through $b_i^- = Z_i$, we see that $\tau(y)$ is near 0 for y near Z_i. Thus τ extends continuously to Z_i if we set $\tau|_{Z_i} = 0$. Now let E be a flat complex vector bundle of finite dimension on B_i. There is a bundle map $\alpha_i : r_i^*(E|_{Y_i}) \to E|_{Y_i}$ defined by pulling back along the trajectory from y to $r_i(y)$, using the flat connection on E. This determines an endomorphism

$$(\alpha_i)_* : H^*(Y_i, Z_i; r_i^* E) \to H^*(Y_i, Z_i; E). \tag{9}$$

Since there is a natural induced map $r_i^* : H^*(Y_i, Z_i; E) \to H^*(Y_i, Z_i; r_i^* E)$ we obtain the endomorphism

$$\beta_i = (\alpha_i)_* r_i^* : H^*(Y_i, Z_i; E) \to H^*(Y_i, Z_i; E). \tag{10}$$

So the *relative Lefschetz number*

$$L(\beta_i) = \sum_{k=0}^{n-1} (-1)^k \operatorname{Tr}(\beta_i)_k \tag{11}$$

is defined. According to Atiyah and Bott [2] the numbers $L(\beta_i)$ can be computed from the fixed point set of r_i in $Y_i - Z_i$. If $\operatorname{Fix}(r_i) - Z_i$ is a finite set of points p with the Lefschetz index $\operatorname{Index}_L(r_i, p)$ and $(\alpha_i)_p : E_p \to E_p$ is the endomorphism of the fibre at p, then one has the relative Lefschetz formula

$$L(\beta_i) = \sum_p \operatorname{Index}_L(r_i, p) \cdot \operatorname{Tr}(\alpha_i)_p. \tag{12}$$

We see that $L(\beta_i^n), n \geq 1$, counts the periodic points of period n for β_i which are not in Z_i, i.e. the closed orbits of system (1) that wrap n times around $\mathbb{R}^1/\mathbb{Z}$

under θ_i, with a weight coming from the holonomy of E around these closed orbits. Now, consider the *twisted Lefschetz zeta function* [18] for E and (B_i, b_i^-):

$$L_i(z) \equiv L_{\beta_i}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(\beta_i^n)}{n} z^n\right). \tag{13}$$

We now turn to the R-torsion of pairs. Suppose that L is a CW-subcomplex of K and consider the relative cochain complex

$$C^*(K, L; E) = \ker(C^*(K; E) \to C^*(L; E|_L)).$$

Then one has a natural isomorphism $|C^*(K, L; E)| \cong \bigotimes_j |V|$, where j runs over the i-cells in $K - L$. So our flat density on E gives a density Δ_i on the relative i-cochains in E. Thus we again have an R-torsion denoted by $\tau(K, L; E, D_i)$ for some choice of positive densities D_i on $H^i(K, L; E)$. If $H^i(K, L; E) = 0$, we say that E is *acyclic for* (K, L), and we simply write $\tau(K, L; E)$ when the D_i are chosen standard. Let $\varrho_E : \pi_1(N, p) \to \mathrm{GL}(E_p)$ be the holonomy representation for an acyclic bundle E on an orientable manifold $N, \dim N = n$, let ϱ_E^* be the contragredient representation of ϱ_E and E^* the flat complex vector bundle with holonomy ϱ_E^*. We suppose that $\det \varrho_E = 1$. Let $L_i^*(z)$ be the twisted Lefschetz zeta function for E^* and (B_i, b_i^-), and

$$L^*(z) = \prod_i L_i^*(z), \quad L(z) = \prod_i L_i(z). \tag{14}$$

Theorem 3. *We have*

$$\tau(D; E) = \tau(S; E) = |L(1)|^{-1} \cdot |L^*(1)|^{\varepsilon(n)}, \tag{15}$$

where $\varepsilon(n) = (-1)^n$.

Proof. Consider the function $G : N \to \mathbb{R}^1$ defined at the beginning of this subsection. Smoothing the level set $G^{-1}(x_i)$ by sliding it along the flow, one obtains a smooth region $N_i \subset N$ with

$$G^{-1}((-\infty, x_i - \varepsilon)) \subset N_i \subset G^{-1}((-\infty, x_i + \varepsilon))$$

such that the trajectories $\phi(t, x)$ are transverse to ∂N_i, for large i we have $N_i = N$ and $\partial N^- = \emptyset$. If ε is small then $N_{i+1} - N_i$ isolates J_i. Thus by the properties of the Reidemeister torsion [18] one finds

$$\tau(N; E) = \prod_i \tau(N_{i+1}, N_i; E) = \prod_i \tau(B_i, b_i; E). \tag{16}$$

D. Fried proved [18] that E is acyclic for (B_i, b_i^-) iff $I - \beta_i$ is invertible and then

$$\tau(B_i, b_i; E) = |L_i(z)|^{-1}|_{z=1}. \tag{17}$$

So we have

$$\tau(N;E)=\prod_i|L_i(1)|^{-1}=|L(1)|^{-1}. \tag{18}$$

Since $\partial N=S$, from the multiplicative law (6) for the Reidemeister torsion it follows that

$$\tau(N;E)=\tau(N,S;E)\cdot\tau(S;E). \tag{19}$$

Using Milnor's duality theorem for the Reidemeister torsion [27] we have

$$\tau(N,S;E)=\tau(N;E^*)^{(-1)^n}. \tag{20}$$

Hence

$$\tau(N;E^*)=\prod_i|L_i^*(1)|^{-1}=|L^*(1)|^{-1}. \tag{21}$$

Since the attraction domain D is diffeomorphic to $S\times\mathbb{R}^1$, we get $\tau(D;E)=\tau(S;E)$ by the simple homotopy invariance of the Reidemeister torsion. Now from (18)–(21) we have

$$\begin{aligned}\tau(D;E)=\tau(S;E)&=\tau(N;E)\cdot\tau^{-1}(N,S;E)\\&=\tau(N;E)\cdot\tau(N;E^*)^{(-1)^{n+1}}=|L(1)|^{-1}\cdot|L^*(1)|^{(-1)^n}.\end{aligned}$$

Suppose now that the system (1) on the manifold N is a nonsingular almost Morse–Smale system. This means that (1) has finitely many hyperbolic prime periodic orbits γ and no other chain-recurrent points. Over the orbit γ lies a strongly unstable bundle $E^{\mathrm{u}}(\gamma)$ of some dimension $u(\gamma)$. Let $\delta(\gamma)$ be $+1$ if E^{u} is orientable and -1 if it is not. Let $\varepsilon(\gamma)=(-1)^{u(\gamma)}$.

Corollary 3. *We have*

$$\begin{aligned}&\tau(D;E)=\tau(S;E)\\&\quad=\prod_\gamma|\det(I-\delta(\gamma)\varrho_E(\gamma))|^{\varepsilon(\gamma)}\times\bigg(\prod_\gamma|\det(I-\delta(\gamma)\varrho_E^*(\gamma))|^{\varepsilon(\gamma)}\bigg)^{(-1)^{n+1}}.\end{aligned}$$

Proof. According to D. Fried [18], if J_i is a prime hyperbolic closed orbit γ then

$$|L_i(1)|^{-1}=|\det(I-\delta(\gamma)\varrho_E(\gamma))|^{\varepsilon(\gamma)}.$$

Now, the statement follows from Theorem 3.

2. Nielsen zeta function

2.1. Preliminaries. We assume everywhere in this section X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Taking a dynamical point of view, we consider the iterates of f. In the theory of discrete dynamical systems the following zeta functions are known: the *Artin–Mazur zeta function* [1]

$$\zeta_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n\right),$$

where $F(f^n)$ is the number of isolated fixed points of f^n; the *Ruelle zeta function* [33]

$$\zeta_f^g(z) := \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}(f^n)} \prod_{k=0}^{n-1} g(f^k(x))\right),$$

where $g : X \to \mathbb{C}$ is a weight function (if $g = 1$ we recover $\zeta_f(z)$); the *Lefschetz zeta function*

$$L_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right),$$

where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{Tr}[f_{*k}^n : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})]$$

are the Lefschetz numbers of the iterates of f; reduced mod 2 Artin–Mazur and Lefschetz zeta functions [16]; twisted Artin–Mazur and Lefschetz zeta functions [18], which have coefficients in the group ring $\mathbb{Z}H$ of an abelian group H.

The above zeta functions are directly analogous to the Hasse–Weil zeta function of an algebraic manifold over a finite field [35]. The Lefschetz zeta function is always a rational function of z and is given by the formula

$$L_f(z) = \prod_{k=0}^{\dim X} \det(I - f_{*k} \cdot z)^{(-1)^{k+1}}.$$

This immediately follows from the trace formula for the Lefschetz numbers of the iterates of f. The Artin–Mazur zeta function has a positive radius of convergence for a dense set in the space of smooth self-maps of a compact smooth manifold [1]. Manning proved the rationality of the Artin–Mazur zeta function for diffeomorphisms of a compact smooth manifold satisfying Smale's Axiom A [25]. The knowledge that a zeta function is rational is important because it shows that the infinite sequence of coefficients is closely interconnected, and is given by the finite set of zeros and poles of the zeta function.

The Artin–Mazur zeta function and its modification count periodic points of a map geometrically, while the Lefschetz type zeta functions do this algebraically (with a weight given by index theory). Another way to count the periodic points is given by Nielsen theory. Let $p : \widetilde{X} \to X$ be the universal covering of X and $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ a lifting of f, i.e. $p \circ \widetilde{f} = f \circ p$. Two liftings $\widetilde{f}$ and $\widetilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\widetilde{f}' = \gamma \circ \widetilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\widetilde{f})) \subset \mathrm{Fix}(f)$ is called the *fixed point class of f determined by the lifting class* $[\widetilde{f}]$. A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted by $R(f)$ (see [31]). This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$. The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra the Nielsen number of a map is equal to the least number of fixed points of maps with the same homotopy type as f (see [34]).

If we consider the iterates of f, we may define several zeta functions connected with Nielsen fixed point theory (see [9, 10, 15]). We assume throughout this article that $R(f^n) < \infty$ for all $n > 0$. The *Reidemeister zeta function* of f and the *Nielsen zeta function* of f are defined as power series:

$$R_f(z) := \exp\bigg(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\bigg), \quad N_f(z) := \exp\bigg(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\bigg).$$

$R_f(z)$ and $N_f(z)$ are homotopy invariants.

2.2. Radius of convergence of the Nielsen zeta function. In this section we find a sharp estimate for the radius of convergence of the Nielsen zeta function in terms of the topological entropy of the map. It follows from this estimate that the Nielsen zeta function has a positive radius of convergence.

2.2.1. Radius and topological entropy. The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition. Let $f : X \to X$ be a self-map of a compact metric space. For given $\varepsilon > 0$ and $n \in \mathbb{N}$, a subset $E \subset X$ is said to be (n, ε)-*separated* under f if for each pair $x \neq y$ in E there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \varepsilon$. Let $s_n(\varepsilon, f)$ denote the largest cardinality of any (n, ε)-separated subset E under f. Thus $s_n(\varepsilon, f)$ is the greatest number of orbit segments $x, f(x), \ldots, f^{n-1}(x)$ of length n that can be distinguished from one another provided we can only distinguish between points of X that are at least ε apart. Now let

$$h(f, \varepsilon) := \limsup_n \frac{1}{n} \log s_n(\varepsilon, f), \quad h(f) := \limsup_{\varepsilon \to 0} h(f, \varepsilon).$$

The number $0 \leq h(f) \leq \infty$, which is shown to be independent of the metric d used, is called the *topological entropy* of f. If $h(f, \varepsilon) > 0$ then, up to resolution $\varepsilon > 0$, the number $s_n(\varepsilon, f)$ of distinguishable orbit segments of length n grows exponentially with n. So $h(f)$ measures the growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution. A basic relation between Nielsen numbers and topological entropy was found by N. Ivanov [22] and independently by Aronson and Grines. We present here a very short proof of Jiang [24] of Ivanov's inequality.

LEMMA 1 ([22]). *We have*

$$h(f) \geq \limsup_n \frac{1}{n} \log N(f^n).$$

PROOF. Let δ be such that every loop in X of diameter $< 2\delta$ is contractible. Let $\varepsilon > 0$ be a small number such that $d(f(x), f(y)) < \delta$ whenever $d(x, y) < 2\varepsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of f^n. Thus $|E_n| = N(f^n)$. By the definition of $h(f)$, it suffices to show that E_n is (n, ε)-separated. Suppose it is not so. Then there are two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \varepsilon$ for $0 \leq i < n$, hence for all $i \geq 0$. Pick a path c_i from $f^i(x)$ to $f^i(y)$ of diameter $< 2\varepsilon$ for $0 \leq i < n$ and let $c_n = c_0$. By the choice of δ and ε, $f \circ c_i \simeq c_{i+1}$ for all i, so $f^n \circ c_0 \simeq c_n = c_0$. This means x, y are in the same fixed point class of f^n, contradicting the construction of E_n.

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

We denote by R the radius of convergence of the Nielsen zeta function $N_f(z)$. Let $h = \inf h(g)$ over all maps g of the same homotopy type as f.

THEOREM 4 ([15]). *For a continuous map of a compact polyhedron X into itself,*

$$R \geq \exp(-h) > 0. \tag{22}$$

PROOF. The inequality $R \geq \exp(-h)$ follows from the previous lemma, the Cauchy–Hadamard formula, and the homotopy invariance of the radius R of the Nielsen zeta function $N_f(z)$. Consider a smooth compact manifold M which is a regular neighbourhood of X, and a smooth map $g : M \to M$ of the same homotopy type as f. It is known [29] that the entropy $h(g)$ is finite. Thus $\exp(-h) \geq \exp(-h(g)) > 0$.

2.2.2. Algebraic lower estimate for the radius of convergence. In this subsection we propose another proof of positivity of the radius R and give an exact algebraic lower estimate for R using the Reidemeister trace formula for generalized Lefschetz numbers. Pick a base point $x_0 \in X$ and a path w from x_0 to $f(x_0)$. Let $\widetilde{f}_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ be the composition

$$\pi = \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(X, f(x_0)) \xrightarrow{w_*} \pi_1(X, x_0).$$

The fundamental group $\pi = \pi_1(X, x_0)$ splits into $\widetilde{f}_*$-conjugacy classes. Two elements $\alpha, \beta \in \pi_1(X, x_0)$ are said to be $\widetilde{f}_*$-*conjugate* if there is a $\gamma \in \pi_1(X, x_0)$ such that $\beta = \widetilde{f}_*(\gamma)\alpha\gamma^{-1}$. Let π_f denote the set of $\widetilde{f}_*$-conjugacy classes, and $\mathbb{Z}\pi_f$ the abelian group freely generated by π_f. We will use the bracket notation $a \to [a]$ for both projections $\pi \to \pi_f$ and $\mathbb{Z}\pi \to \mathbb{Z}\pi_f$. Let x be a fixed point of f. Take a path c from x_0 to x. The $\widetilde{f}_*$-conjugacy class in π of the loop $c \cdot (f \circ c)^{-1}$, which is evidently independent of the choice of c, is called the *coordinate* of x. Two fixed points are in the same fixed point class F iff they have the same coordinates. This $\widetilde{f}_*$-conjugacy class is thus called the *coordinate* of the fixed point class F and denoted by $\mathrm{cd}_\pi(F, f)$ (compare with description in Section 1).

The *generalized Lefschetz number* or *Reidemeister trace* [31] is defined as

$$L_\pi(f) := \sum_F \mathrm{Index}(F, f) \cdot \mathrm{cd}_\pi(F, f) \in \mathbb{Z}\pi_f, \tag{23}$$

the summation being over all essential fixed point classes F of f. The *Nielsen number* $N(f)$ is the number of nonzero terms in $L_\pi(f)$, and the indices of the essential fixed point classes appear as the coefficients in $L_\pi(f)$. This invariant used to be called the Reidemeister trace because it can be computed as an alternating sum of traces on the chain level as follows [31, 34]. Assume that X is a finite cell complex and $f : X \to X$ is a cellular map. A cellular decomposition e_j^d of X lifts to a π-invariant cellular structure on the universal covering $\widetilde{X}$. Choose an arbitrary lift $\widetilde{e}_j^d$ for each e_j^d. They constitute a free $\mathbb{Z}\pi$-basis for the cellular chain complex of $\widetilde{X}$. The lift $\widetilde{f}$ of f is also a cellular map. In every dimension d, the cellular chain map $\widetilde{f}$ gives rise to a $\mathbb{Z}\pi$-matrix $\widetilde{F}_d$ with respect to the above basis, i.e. $\widetilde{F}_d = (a_{ij})$ if $\widetilde{f}(\widetilde{e}_i^d) = \sum_j a_{ij}\widetilde{e}_j^d$, where $a_{ij} \in \mathbb{Z}\pi$. Then we have the *Reidemeister trace formula*

$$L_\pi(f) = \sum_d (-1)^d [\mathrm{Tr}\, \widetilde{F}_d] \in \mathbb{Z}\pi_f. \tag{24}$$

Now we describe an alternative approach to the Reidemeister trace formula proposed recently by Jiang [24]. This approach is useful when we study the periodic points of f, i.e. the fixed points of the iterates of f.

The *mapping torus* T_f of $f : X \to X$ is the space obtained from $X \times [0, \infty)$ by identifying $(x, s+1)$ with $(f(x), s)$ for all $x \in X, s \in [0, \infty)$. On T_f there is a natural semiflow $\phi : T_f \times [0, \infty) \to T_f, \phi_t(x, s) = (x, s+t)$ for all $t \geq 0$. Then the map $f : X \to X$ is the return map of the semiflow ϕ. A point $x \in X$ and a positive number $\tau > 0$ determine the *orbit curve* $\phi_{(x,\tau)} := \phi_t(x)_{0 \leq t \leq \tau}$ in T_f.

Take the base point x_0 of X as the base point of T_f. It is known that the fundamental group $H := \pi_1(T_f, x_0)$ is obtained from π by adding a new generator z and adding the relations $z^{-1}gz = \widetilde{f}_*(g)$ for all $g \in \pi = \pi_1(X, x_0)$. Let H_c denote the set of conjugacy classes in H. Let $\mathbb{Z}H$ be the integral group ring of H, and let $\mathbb{Z}H_{\rm c}$ be the free abelian group with basis $H_{\rm c}$. We again use the bracket notation $a \to [a]$ for both projections $H \to H_{\rm c}$ and $\mathbb{Z}H \to \mathbb{Z}H_{\rm c}$. If F^n is a fixed point class of f^n, then $f(F^n)$ is also a fixed point class of f^n and $\operatorname{Index}(f(F^n), f^n) = \operatorname{Index}(F^n, f^n)$. Thus f acts as an index-preserving permutation on the fixed point classes of f^n. By definition, an *n-orbit class* O^n of f is the union of elements of an orbit of this action. In other words, two points $x, x' \in \operatorname{Fix}(f^n)$ are in the same n-orbit class of f if and only if some $f^i(x)$ and some $f^j(x')$ are in the same fixed point class of f^n. The set $\operatorname{Fix}(f^n)$ splits into a disjoint union of n-orbit classes. A point x is a fixed point of f^n or a periodic point of period n if and only if the orbit curve $\phi_{(x,n)}$ is a closed curve. The free homotopy class of the closed curve $\phi_{(x,n)}$ will be called the *H-coordinate* of the point x, written $\operatorname{cd}_H(x,n) = [\phi_{(x,n)}] \in H_{\rm c}$. It follows that periodic points x of period n and x' of period n' have the same H-coordinate if and only if $n = n'$ and x, x' belong to the same n-orbit class of f. Thus it is possible, equivalently, to define $x, x' \in \operatorname{Fix} f^n$ to be in the same n-orbit class if and only if they have the same H-coordinate.

Recently, Jiang [24] has considered the *generalized Lefschetz number with respect to H*,

$$L_H(f^n) := \sum_{O^n} \operatorname{Index}(O^n, f^n) \cdot \operatorname{cd}_H(O^n) \in \mathbb{Z}H_{\rm c}, \tag{25}$$

and proved the following trace formula:

$$L_H(f^n) = \sum_d (-1)^d [\operatorname{Tr}(z\widetilde{F}_d)^n] \in \mathbb{Z}H_{\rm c}, \tag{26}$$

where $\widetilde{F}_d$ are the $\mathbb{Z}\pi$-matrices defined above and $z\widetilde{F}_d$ is regarded as a $\mathbb{Z}H$-matrix.

For any set S let $\mathbb{Z}S$ denote the free abelian group with the specified basis S. The norm in $\mathbb{Z}S$ is defined by

$$\Big\| \sum_i k_i s_i \Big\| := \sum_i |k_i| \in \mathbb{Z}, \tag{27}$$

when the s_i in S are all different.

For a $\mathbb{Z}H$-matrix $A = (a_{ij})$, define its norm by $\|A\| := \sum_{i,j} \|a_{ij}\|$. Then we have the inequalities $\|AB\| \le \|A\| \cdot \|B\|$ when A, B can be multiplied, and $\|\operatorname{Tr} A\| \le \|A\|$ when A is a square matrix. For a matrix $A = (a_{ij})$ in $\mathbb{Z}S$, its *matrix of norms* is defined to be the matrix $A^{\rm norm} := (\|a_{ij}\|)$, which is a matrix of nonnegative integers. In what follows, the set S will be π, H or $H_{\rm c}$. We denote by $s(A)$ the *spectral radius* of A, $s(A) = \lim_n \|A^n\|^{1/n}$, which coincides with the largest modulus of an eigenvalue of A.

Theorem 5. *For any continuous map f of any compact polyhedron X into itself the Nielsen zeta function has a positive radius of convergence R, which admits the following estimates:*

$$R \geq \frac{1}{\max_d \|z\widetilde{F}_d\|} > 0 \tag{28}$$

and

$$R \geq \frac{1}{\max_d s(\widetilde{F}_d^{\mathrm{norm}})} > 0. \tag{29}$$

Proof. By homotopy invariance we can suppose that f is a cell map of a finite cell complex. By the definition, the Nielsen number $N(f^n)$ is the number of nonzero terms in $L_\pi(f^n)$. The norm $\|L_H(f^n)\|$ is the sum of the absolute values of the indices of all the n-orbit classes O^n. It equals $\|L_\pi(f^n)\|$, the sum of the absolute values of the indices of all the fixed point classes of f^n, because any two fixed point classes of f^n contained in the same n-orbit class O^n must have the same index. From this we have

$$\begin{aligned} N(f^n) &\leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \left\| \sum_d (-1)^d [\mathrm{Tr}\,(z\widetilde{F}_d)^n] \right\| \\ &\leq \sum_d \|[\mathrm{Tr}\,(z\widetilde{F}_d)^n]\| \leq \sum_d \|\,\mathrm{Tr}\,(z\widetilde{F}_d)^n\| \leq \sum_d \|(z\widetilde{F}_d)^n\| \leq \sum_d \|z\widetilde{F}_d\|^n \end{aligned}$$

(see [24]). The radius of convergence R is given by the Cauchy–Hadamard formula:

$$1/R = \limsup_n \,(N(f^n)/n)^{1/n} = \limsup_n \,(N(f^n))^{1/n}.$$

Therefore we have

$$R = \frac{1}{\limsup_n (N(f^n))^{1/n}} \geq \frac{1}{\max_d \|z\widetilde{F}_d\|} > 0.$$

The inequalities

$$\begin{aligned} N(f^n) &\leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \left\| \sum_d (-1)^d [\mathrm{Tr}\,(z\widetilde{F}_d)^n] \right\| \leq \sum_d \|[\mathrm{Tr}\,(z\widetilde{F}_d)^n]\| \\ &\leq \sum_d \|\,\mathrm{Tr}\,(z\widetilde{F}_d)^n\| \leq \sum_d \mathrm{Tr}\,((z\widetilde{F}_d)^n)^{\mathrm{norm}} \leq \sum_d \mathrm{Tr}\,((z\widetilde{F}_d)^{\mathrm{norm}})^n \\ &\leq \sum_d \mathrm{Tr}\,((\widetilde{F}_d)^{\mathrm{norm}})^n \end{aligned}$$

and the definition of the spectral radius give the estimate

$$R = \frac{1}{\limsup_n (N(f^n))^{1/n}} \geq \frac{1}{\max_d s(\widetilde{F}_d^{\mathrm{norm}})} > 0.$$

EXAMPLE 1. Let X be a surface with boundary, and $f : X \to X$ be a map. Fadell and Husseini [7] devised a method of computing the matrices of the lifted chain map for surface maps. Suppose $\{a_1, \ldots, a_r\}$ is a free basis for $\pi_1(X)$. Then X has the homotopy type of a bouquet B of r circles which can be decomposed into one 0-cell and r 1-cells corresponding to the a_i, and f has the homotopy type of a cellular map $g : B \to B$. By homotopy invariance, we can replace f with g in computations. The homomorphism $\widetilde{f}_* : \pi_1(X) \to \pi_1(X)$ induced by f and g is determined by the images $b_i = \widetilde{f}_*(a_i), i = 1, \ldots, r$. The fundamental group $\pi_1(T_f)$ has a presentation $\pi_1(T_f) = \langle a_1, \ldots, a_r, z \mid a_i z = z b_i, i = 1, \ldots, r\rangle$. Let

$$D = (\partial b_i / \partial a_j)$$

be the Jacobian in Fox calculus (see [4]). Then, as pointed out in [7], the matrices of the lifted chain map $\widetilde{g}$ are

$$\widetilde{F}_0 = (1), \quad \widetilde{F}_1 = D = (\partial b_i / \partial a_j).$$

Now, we can find estimates for the radius R from (28) and (29).

2.3. Polyhedra with finite fundamental group and the Nielsen zeta function. Let W be the complex vector space of complex-valued class functions on the finite fundamental group $\pi_1(X)$. A *class function* is a function which takes the same value on every element of a usual congruence class. The map $\widetilde{f}_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ induces a linear map $B : W \to W$ defined by

$$B(g) := g \circ \widetilde{f}_*.$$

REMARK 1. The characteristic functions of the congruence classes in $\pi_1(X, x_0)$ form a basis of W, and are mapped to one another by B (the map need not be a bijection). Therefore the trace of B is the number of elements of this basis which are fixed by B. By Theorem 5 of [13], this is equal to the Reidemeister number of f.

THEOREM 6. *Let X be a connected, compact polyhedron with finite fundamental group π. Suppose that the action of π on the rational homology of the universal cover $\widetilde{X}$ is trivial, i.e. for every covering translation $\alpha \in \pi$, $\alpha_* = \mathrm{id} : H_*(\widetilde{X}; \mathbb{Q}) \to H_*(\widetilde{X}; \mathbb{Q})$. If $L(f^n) \neq 0$ for every $n > 0$, then*

$$N_f(z) = R_f(z) = \frac{1}{\det(1 - Bz)}. \tag{30}$$

If $L(f^n) = 0$ only for a finite number of n, then

$$N_f(z) = \exp(P(z)) \cdot R_f(z) = \exp(P(z)) \cdot \frac{1}{\det(1 - Bz)}, \tag{31}$$

where $P(z)$ is a polynomial.

Proof. If $L(f^n) \neq 0$ for every $n > 0$, then $N(f^n) = R(f^n) = \operatorname{Tr} B^n$ for every $n > 0$ (see Remark 1 above and Theorems 5 and 9 of [13]). We now calculate directly

$$\begin{aligned} R_f(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{Tr} B^n}{n} z^n\right) = \exp\left(\operatorname{Tr} \sum_{n=1}^{\infty} \frac{B^n}{n} z^n\right) \\ &= \exp(\operatorname{Tr}(-\log(1 - Bz))) = \frac{1}{\det(1 - Bz)}. \end{aligned}$$

If $L(f^n) = 0$, then $N(f^n) = 0$. So $N_f(z)/R_f(z) = \exp(P(z))$, where $P(z)$ is a polynomial whose degree is the maximal n such that $L(f^n) = 0$.

Corollary 4. *Let $\widetilde{X}$ be a compact 1-connected polyhedron which is a rational homology n-sphere, where n is odd. Let π be a finite group acting freely on $\widetilde{X}$ and let $X = \widetilde{X}/\pi$. Then Theorem 6 applies.*

Corollary 5. *If X is a closed 3-manifold with finite π, then Theorem 6 applies.*

Example 2 ([3]). Let $f : S^2 \vee S^4 \to S^2 \vee S^4$ be a continuous map of the bouquet of spheres such that $f|_{S^4} = \mathrm{id}_{S^4}$ and the degree of $f|_{S^2} : S^2 \to S^2$ is -2. Then $L(f) = 0$, hence $N(f) = 0$ since $S^2 \vee S^4$ is simply connected. For $k > 1$ we have $L(f^k) = 2 + (-2)^k \neq 0$, therefore $N(f^k) = 1$. From this we obtain by direct calculation

$$N_f(z) = \exp(-z) \cdot \frac{1}{1 - z}. \tag{32}$$

2.4. Concluding remarks and open questions. We would like to mention that in all known cases the Nielsen zeta function is a nice function. By this we mean that it is a product of the exponential of a polynomial with a function some power of which is rational. Maybe this is a general pattern.

For the case of an almost nilpotent fundamental group (i.e. group with polynomial growth, in view of Gromov's theorem [21]) we believe that some power of the Reidemeister zeta function is a rational function. We intend to prove this conjecture by identifying the Reidemeister number on the nilpotent part of the fundamental group with the number of fixed points in the direct sum of the duals of the quotients of successive terms in the central series. We then hope to show that the Reidemeister number of the whole induced endomorphism on the fundamental group is the sum of the numbers of orbits of such fixed points under the action of the finite quotient group (i.e. the quotient of the whole group by the nilpotent part). The situation for fundamental groups with exponential growth is very different. There one can expect the Reidemeister number to be infinite as long as the induced endomorphism is injective.

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