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ELLIPTIC VARIATIONAL PROBLEMS WITH INDEFINITE NONLINEARITIES

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Dedicated to Olga Ladyzhenskaya

1. Introduction

The purpose of this paper is to study the semilinear elliptic problem

$$(1_\lambda) \qquad \begin{cases} -\Delta u + u = \lambda |u|^{q-2}u - h(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{on } \mathbb{R}^N, \end{cases}$$

where $h > 0$ is a positive continuous function on $\mathbb{R}^N$ satisfying some integrability condition, $\lambda > 0$ is a positive parameter and $2 < q < p < 2^* = 2N/(N-2)$, $N \geq 3$. We establish the existence of at least one solution (see Sections 2 and 3). In the final Section 4 we study the equation in (1_λ) with the nonlinearity replaced by $k(x)|u|^{q-2}u - \mu|u|^{p-2}u$, with $1 < q < 2 < p < 2^*$, where k is a positive function satisfying an appropriate integrability condition and $\mu > 0$ is a parameter. In this case we prove the existence of infinitely many solutions.

Some existence results for elliptic problems on unbounded domains with indefinite nonlinearities were obtained in [11] and [12]. In [12] a nonlinearity f has the form $f(x,u) = Q_1(x)|u|^{p-2}u - Q_2(x)|u|^{q-2}u$ with $2 < q < p < 2^*$, where Q_i are continuous positive bounded functions satisfying $Q_1(x) \geq \lim_{|x|\to\infty} Q_1(x) > 0$ and $Q_2(x) \leq \lim_{|x|\to\infty} Q_2(x) > 0$ on $\mathbb{R}^N$. Under these assumptions the corresponding elliptic problem has a variational structure with a mountain pass level satisfying the Palais–Smale condition. We point out here that the nonlinearity f in [12] has a different order of terms $|u|^{p-2}u$ and $|u|^{q-2}u$ than in equation

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(1_λ), which means that problem (1_λ) does not have a mountain pass structure. The paper [11] deals with a nonlinearity f involving concave and convex terms

$$f(x,u) = \frac{\lambda}{(1+|x|)^a}|u|^{q-2}u + \frac{\mu}{(1+|x|)^b}|u|^{p-2}u,$$

with $1 < q < 2 < p < 2^*$. If $a > 0$ and $b > 0$ are sufficiently large then a nonlinear functional generated by f is completely continuous on $H^1(\mathbb{R}^N)$. The existence of infinitely many solutions was obtained using the Bartsch–Willem fountain theorem [5] (see also [6]). In this paper motivated by [1] and [3] we study in both cases problem (1_λ) under different assumptions. In Section 3 we obtain the existence of a solution as a minimizer of a variational functional for problem (1_λ). In the case of a nonlinearity combining convex and concave terms we prove the existence of infinitely many solutions also using the Bartsch–Willem fountain theorem. For related problems for Dirichlet problem on bounded domains we refer to [1] and [2].

In this paper we use standard notation and terminology. We denote by $H^1(\mathbb{R}^N)$ the Sobolev space equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx.$$

By $L^p_r(\mathbb{R}^N)$, $1 \le p < \infty$, we denote the weighted Lebesgue space

$$L^p_r(\mathbb{R}^N) = \left\{u : \int_{\mathbb{R}^N} |u(x)|^p r(x)\,dx < \infty\right\},$$

where r is a positive continuous function on $\mathbb{R}^N$, equipped with the norm

$$\|u\|^p_{r,p} = \int_{\mathbb{R}^N} |u(x)|^p r(x)\,dx.$$

If $r \equiv 1$ on $\mathbb{R}^N$, the norm is denoted by $\|\cdot\|_p$.

In this work we always denote weak convergence in a given Banach space by "$\rightharpoonup$" and strong convergence by "$\to$". The duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle$.

2. Preliminaries

In this and the next section we consider problem (1_λ) under the assumption $2 < q < p < 2^*$. We assume that h is a positive and continuous function on $\mathbb{R}^N$ satisfying

$$(H) \qquad \int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}} < \infty.$$

By E we denote the subspace of $H^1(\mathbb{R}^N)$ defined by

$$E = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} h(x)|u|^p\,dx < \infty\right\}$$

and equipped with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx + \bigg(\int_{\mathbb{R}^N} h(x)|u|^p\,dx\bigg)^{2/p}.$$

It is clear that E is a Banach space. Solutions to problem (1_λ) will be found as critical points of the functional $\Phi : E \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\,dx + \frac{1}{p}\int_{\mathbb{R}^N} h(x)|u|^p\,dx.$$

We commence by showing that there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ the problem does not admit a solution.

PROPOSITION 1. *There exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ problem (1_λ) does not have a solution.*

PROOF. Suppose that $u > 0$ is a solution in E of (1_λ). Then u satisfies

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx + \int_{\mathbb{R}^N} h|u|^p\,dx = \lambda\int_{\mathbb{R}^N} |u|^q\,dx. \tag{2}$$

It follows from the Young inequality that

$$\lambda\int_{\mathbb{R}^N} |u|^q\,dx \le \lambda^{p/(p-q)}\frac{p-q}{p}\int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}} + \frac{q}{p}\int_{\mathbb{R}^N} h|u|^p\,dx.$$

This combined with (2) gives

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx \le \lambda^{p/(p-q)}\frac{p-q}{p}\int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}}. \tag{3}$$

By (2) and the Sobolev embedding theorem we have

$$C_q\bigg(\int_{\mathbb{R}^N} |u|^q\,dx\bigg)^{2/q} \le \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx \le \lambda\int_{\mathbb{R}^N} |u|^q\,dx \tag{4}$$

for some constant $C_q > 0$. We deduce from this inequality the estimate

$$(C_q\lambda^{-1})^{q/(q-2)} \le \int_{\mathbb{R}^N} |u|^q\,dx,$$

which combined with (4) leads to the inequality

$$C_q(C_q\lambda^{-1})^{2/(q-2)} \le \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx.$$

Combining this and (3) we obtain

$$C_q(C_q\lambda^{-1})^{2/(q-2)} \le \lambda^{p/(p-q)}\frac{p-q}{p}\int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}}. \tag{5}$$

If we take

$$\lambda^* = \bigg[C_q^{q/(q-2)}\frac{p}{p-q}\bigg(\int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}}\bigg)^{-1}\bigg]^{(p-q)(q-2)/(q(p-2))},$$

the result follows.

To proceed further we need the following inequality: for every $h > 0$, $k > 0$ and $0 < s < r$ we have

$$k|u|^s - h|u|^r \leq C_{rs} k \left(\frac{k}{h}\right)^{s/(r-s)} \tag{6}$$

for all $u \in \mathbb{R}$, where $C_{rs} > 0$ is a constant depending on s and r (see [1], p. 166).

LEMMA 1. *The functional Φ is coercive.*

PROOF. By virtue of (6) we write the following estimate

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\frac{\lambda}{q}|u|^q - \frac{h}{2p}|u|^p\right) dx &\leq C_{pq} \int_{\mathbb{R}^N} \lambda \left(\frac{\lambda}{h}\right)^{q/(p-q)} dx \\ &= C_{pq} \lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}} = C_1.\end{aligned}$$

It therefore follows that

$$\Phi(u) \geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx + \frac{1}{2p} \int_{\mathbb{R}^N} h|u|^p\, dx - C_1$$

and the coercivity follows.

LEMMA 2. *Let $\{u_m\}$ be a sequence in E such that $\Phi(u_m)$ is bounded. Then there exists a subsequence of $\{u_m\}$, relabelled again by $\{u_m\}$, such that $u_m \rightharpoonup u_0$ in E and*

$$\Phi(u_0) \leq \liminf_{m\to\infty} \Phi(u_m).$$

PROOF. Since Φ is coercive in E we see that $\|u_m\|$ and $\int_{\mathbb{R}^N} h|u_m|^p\, dx$ are bounded. We may also assume that $u_m \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, $u_m \rightharpoonup u_0$ in $L^p_h(\mathbb{R}^N)$ and $u_m \to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$. Writing

$$F(x,u) = \frac{\lambda}{q}|u|^q - h(x)\frac{|u|^p}{p} \quad \text{and} \quad f(x,u) = F_u(x,u),$$

we see that

$$f_u(x,u) = (q-1)\lambda|u|^{q-2} - (p-1)h|u|^{p-2} \leq C_{pq}\lambda\left(\frac{\lambda}{h}\right)^{(q-2)/(p-q)}, \tag{7}$$

where the last inequality follows from (6) and $C_{pq} > 0$ is a constant depending only on p and q.

We now use (7) to derive the following estimate for $\Phi(u_0)-\Phi(u_m)$:

$$\begin{aligned}\Phi(u_0)-\Phi(u_m)&=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx-\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_m|^2+u_m^2)\,dx\\&\quad+\int_{\mathbb{R}^N}(F(x,u_m)-F(x,u_0))\,dx\\&=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx-\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_m|^2+u_m^2)\,dx\\&\quad+\int_{\mathbb{R}^N}\bigg(\int_0^1\int_0^s f_u(x,u_0+t(u_m-u_0))\,dt\,ds\bigg)(u_m-u_0)^2\,dx\\&\le\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx-\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_m|^2+u_m^2)\,dx\\&\quad+C_2\int_{\mathbb{R}^N}\frac{(u_m-u_0)^2}{h^{(q-2)/(p-q)}}\,dx,\end{aligned}$$

where $C_2=C_{pq}\lambda^{(p-2)/(p-q)}$. It remains to show that the last integral tends to 0 as $m\to\infty$. Towards this end we use the following estimate for $R>0$:

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{(u_m-u_0)^2}{h^{(q-2)/(p-2)}}\,dx\\&\quad\le\bigg(\int_{|x|\le R}\frac{dx}{h^{q/(p-q)}}\bigg)^{(q-2)/q}\bigg(\int_{|x|\le R}|u_m-u_0|^q\,dx\bigg)^{2/q}\\&\qquad+\bigg(\int_{|x|\ge R}\frac{dx}{h^{q/(p-q)}}\bigg)^{(q-2)/q}\bigg(\int_{|x|\ge R}|u_m-u_0|^q\,dx\bigg)^{2/q}.\end{aligned}\tag{8}$$

Taking $R>0$ sufficiently large and using the fact that $\{u_m\}$ is bounded in $L^q(\mathbb{R}^N)$ and $(u_m-u_0)^2\to 0$ in $L^{q/2}_{\mathrm{loc}}(\mathbb{R}^N)$ we see that

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\frac{(u_m-u_0)^2}{h^{(q-2)/(p-q)}}\,dx=0.\tag{9}$$

Since the norm in $H^1(\mathbb{R}^N)$ is lower semicontinuous with respect to weak convergence we easily derive from (8) and (9) that

$$\Phi(u_0)\le\liminf_{m\to\infty}\Phi(u_m).$$

LEMMA 3. *If u is a solution of problem (1_λ), then*

$$\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx+\frac{p-q}{p}\int_{\mathbb{R}^N}h|u|^p\,dx\le\lambda\frac{p-q}{p}\int_{\mathbb{R}^N}\frac{dx}{h^{q/(p-q)}}$$

and

$$\|u\|\ge\lambda^{1/(q-2)}K,$$

where $K>0$ is a constant independent of u.

PROOF. If u is a solution of (1_λ), then

$$\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx+\int_{\mathbb{R}^N}h|u|^p\,dx=\lambda\int_{\mathbb{R}^N}|u|^q\,dx \leq \lambda\frac{p-q}{p}\int_{\mathbb{R}^N}\frac{dx}{h^{q/(p-q)}}+\frac{q}{p}\int_{\mathbb{R}^N}h|u|^p\,dx$$

and the first part of the assertion follows. To show the second part we use the Sobolev inequality to get

$$\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx\leq\lambda C_q\|u\|^q,$$

where $C_q>0$ and the result readily follows.

3. Existence result

According to Lemmas 1 and 2, Φ is coercive and lower semicontinuous. Therefore there exists $\overline{u}\in E$ such that $\Phi(\overline{u})=\inf_E\Phi(u)$. To ensure that $\overline{u}\not\equiv 0$ we shall show that $\inf_E\Phi<0$. This can be achieved by taking the parameter $\lambda>0$ sufficiently large.

THEOREM 1. *There exists $\lambda_0>0$ such that for $\lambda\geq\lambda_0$ problem (1_λ) admits a solution in E. If $0<\lambda<\lambda_0$, then a solution does not exist.*

PROOF. We set

$$\widetilde{\lambda}=\inf\left\{\frac{q}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx+\frac{q}{p}\int_{\mathbb{R}^N}h(x)|u|^p\,dx : u\in E,\ \int_{\mathbb{R}^N}|u|^q\,dx=1\right\}.$$

First we check that $\widetilde{\lambda}>0$. To show this we consider the constrained minimization problem

$$M=\inf\left\{\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx : u\in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^q\,dx=1\right\}.$$

It is well known [9] that $M>0$ and there exists a radially symmetric function $v\in H^1(\mathbb{R}^1)$ such that

$$M=\int_{\mathbb{R}^N}(|\nabla v|^2+v^2)\,dx\quad\text{and}\quad\int_{\mathbb{R}^N}|v|^q\,dx=1.$$

Since $E\subset H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx\geq M \tag{10}$$

for all $u\in E$ with $\int_{\mathbb{R}^N}|u|^q\,dx=1$. On the other hand, applying the Hölder inequality we get

$$1=\int_{\mathbb{R}^N}|u|^q\,dx\leq\left(\int_{\mathbb{R}^N}\frac{dx}{h^{q/(p-q)}}\right)^{(p-q)/p}\left(\int_{\mathbb{R}^N}h|u|^p\,dx\right)^{q/p}. \tag{11}$$

It then follows that

$$\widetilde{\lambda} \geq \frac{q}{2}M + \frac{q}{2}\left(\int_{\mathbb{R}^N} \frac{dx}{h^{q/(p-q)}}\right)^{-(p-q)/q}$$

and our claim follows.

Let $\lambda > \widetilde{\lambda}$. Then there exists a function $u \in E$ with $\int_{\mathbb{R}^N} |u|^q\,dx = 1$ such that

$$\lambda > \frac{q}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx + \frac{q}{2}\int_{\mathbb{R}^N} h|u|^p\,dx.$$

This can be rewritten as

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\,dx + \frac{1}{p}\int_{\mathbb{R}^N} h(x)|u|^p\,dx < 0$$

and consequently $\inf_{u\in E}\Phi(u) < 0$. By Lemmas 1 and 2, problem (1_λ) has a solution. We now set

$$\lambda_0 = \inf\{\lambda > 0 : (1_\lambda) \text{ admits a solution}\}.$$

According to Lemma 1, $\lambda_0 > 0$.

We now show that for each $\lambda > \lambda_0$ problem (1_λ) admits a solution. Indeed, given $\lambda > \lambda_0$ there exists $\mu \in (\lambda_0, \lambda)$ such that problem (1_μ) has a solution u_μ which is a subsolution of problem (1_λ). We now consider the variational problem

$$\inf\{\Phi(u) : u \in E \text{ and } u \geq u_\mu\}.$$

By Lemmas 1 and 2 this problem admits a solution $\overline{u}$ (see Theorem 1.2 in [10]). Since u_μ is a subsolution of (1_λ) a minimizer $\overline{u}$ is a solution of problem (1_λ). Since $\Phi(\overline{u}) = \Phi(|\overline{u}|)$ we may assume that $u \geq 0$ on $\mathbb{R}^N$. By Theorem 14.1 of [8] (p. 234), u is continuous on $\mathbb{R}^N$. Therefore applying the Harnack inequality (see Theorem 8.18 of [7], p. 194) we deduce that $u > 0$ on $\mathbb{R}^N$. It remains to show that problem (1_{λ_0}) has also a solution. Let $\lambda_m \to \lambda_0$ and $\lambda_m > \lambda_0$ for each m. By the preceding part of the proof problem (1_{λ_m}) has a solution u_m for each m. By Lemma 3 the sequence $\{u_m\}$ is bounded in E. Therefore we may assume that $u_m \rightharpoonup u_0$ in E, $u_m \rightharpoonup u_0$ in $L^p_h(\mathbb{R}^N)$ and $u_m \to u_0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$. Obviously u_0 is a solution of (1_{λ_0}). Since u_m and u_0 are solutions of (1_{λ_m}) and (1_{λ_0}), respectively, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla(u_m - u_0)|^2\,dx &+ \int_{\mathbb{R}^N} h(|u_m|^{p-2}u_m - |u_0|^{p-2}u_0)(u_m - u_0)\,dx \\ &= \lambda_m \int_{\mathbb{R}^N} (|u_m|^{q-2}u_m - |u_0|^{q-2}u_0)(u_m - u_0)\,dx \\ &\quad + (\lambda_m - \lambda_0)\int_{\mathbb{R}^N} |u_0|^{q-2}u_0(u_m - u_0)\,dx = J_{1,m} + J_{2,m}.\end{aligned}$$

We now observe that $\{u_m\}$ is bounded in $L^q(\mathbb{R}^N)$ and consequently $J_{2,m} \to 0$ as $m \to \infty$. It follows from the Hölder inequality that

$$\begin{aligned}|J_{1,m}| \le \sup_{m\ge1}\lambda_m\bigg[&\bigg(\int_{|x|\le R}|u_m|^q\,dx\bigg)^{(q-1)/q}\bigg(\int_{|x|\le R}|u_m-u_0|^q\,dx\bigg)^{1/q}\\&+\bigg(\int_{|x|\le R}|u_0|^q\,dx\bigg)^{(q-1)/q}\bigg(\int_{|x|\le R}|u_m-u_0|^q\,dx\bigg)^{1/q}\\&+\bigg(\int_{|x|>R}h^{-p/(p-q)}\,dx\bigg)^{(p-q)/p}\bigg(\int_{|x|>R}h|u_m|^p\,dx\bigg)^{q/p}\\&+\bigg(\int_{|x|>R}|u_m|^q\,dx\bigg)^{1/q}\bigg(\int_{|x|>R}|u_0|^q\,dx\bigg)^{(1-q)/q}\\&+\bigg(\int_{|x|>R}|u_0|^q\,dx\bigg)^{(q-1)/q}\bigg(\int_{|x|>R}|u_m-u_0|^q\,dx\bigg)^{1/q}\bigg].\end{aligned}$$

For a given $\varepsilon > 0$ we choose $R_\varepsilon > 0$ such that

$$\int_{|x|>R}\frac{dx}{h^{q/(p-q)}} < \varepsilon \quad\text{and}\quad \int_{|x|>R}|u_0|^q\,dx < \varepsilon.$$

Then letting $m \to \infty$ we see that $\limsup_{m\to\infty} J_{1,m} \le C\varepsilon$ for some constant $C > 0$ independent of m and ε. Since $\varepsilon > 0$ is arbitrary, $\lim_{m\to\infty} J_{1,m} = 0$. Hence $u_m \to u$ in $H^1(\mathbb{R}^N)$ and by Lemma 3, $u_0 \not\equiv 0$. By the Harnack inequality, $u_0 > 0$ on $\mathbb{R}^N$ and this completes the proof.

4. Convex and concave nonlinearities

In the case where the right-hand side of the equation in (1_λ) involves convex and concave nonlinearities we establish the existence of infinitely many solutions. Our approach is based on the Bartsch–Willem fountain theorem [5]. We consider the equation

$$-\Delta u + u = k(x)|u|^{q-2}u - \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{12}$$

where $1 < q < 2 < 2^* = 2N/(N-2)$, $N \ge 3$, and $\mu > 0$ is a parameter. Throughout this section it is assumed that k is a positive continuous function on $\mathbb{R}^N$ such that

$$k \in L^s(\mathbb{R}^N) \quad \text{with} \quad s = \frac{2N}{2N - qN + 2q}. \tag{K}$$

It follows from the Hölder and Sobolev inequalities that

$$\int_{\mathbb{R}^N} k(x)|u|^q\,dx \le \bigg(\int_{\mathbb{R}^N}|u|^{2^*}\,dx\bigg)^{q/2^*}\bigg(\int_{\mathbb{R}^N}|k|^s\,dx\bigg)^{1/s} \le S^{-q/2^*}\|u\|^q\|k\|_s, \tag{$*$}$$

where S is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. It is easy to check that the functional $u \to \int_{\mathbb{R}^N} k(x)|u|^q\,dx$ (from $H^1(\mathbb{R}^N)$ into $\mathbb{R}$)

is completely continuous. By $\Psi : H^1(\mathbb{R}^N) \to \mathbb{R}$ we denote the variational functional for (12) defined by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx - \frac{1}{2}\int_{\mathbb{R}^N} k(x)|u|^q\,dx + \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p\,dx.$$

Let $\{e_k\}$, $k = 1, 2, \ldots$, be an orthonormal basis for $H^1(\mathbb{R}^N)$. We set

$$X(j) = \operatorname{span}(e_1, \ldots, e_j),$$
$$X_k = \bigoplus_{j \ge k} X(j) \quad \text{and} \quad X^k = \bigoplus_{j \le k} X(j).$$

Theorem 2 (Bartsch–Willem [5]). *Let $F : H^1(\mathbb{R}^N) \to \mathbb{R}$ be a C^1-functional satisfying the following conditions:*

- (A_1) *There exists an integer k_0 such that for every $k \ge k_0$ there exists $R_k > 0$ such that $F(u) \ge 0$ for every $u \in X_k$ with $\|u\| = R_k$.*
- (A_2) *$b_k = \inf_{B_k} F(u) \to 0$ as $k \to \infty$, where $B_k = \{u \in X_k : \|u\| \le R_k\}$.*
- (A_3) *For every $k \ge 1$ there exist $r_k \in (0, R_k)$ and $d_k < 0$ such that $F(u) \le d_k$ for every $u \in X^k$ with $\|u\| = r_k$.*
- (A_4) *Every sequence $u_n \in X^n$ with $F(u_n) < 0$ and $F'|_{X^n}(u_n) \to 0$ as $n \to \infty$ has a subsequence which converges to a critical point of F.*

Then for each $k \ge k_0$, F has a critical value $c_k \in [d_k, b_k]$.

Theorem 3. *Equation* (12) *admits infinitely many solutions in $H^1(\mathbb{R}^N)$.*

Proof. It suffices to check that the functional Ψ satisfies the assumptions of Theorem 2. Let

$$\lambda_k = \sup_{u \in X_k - \{0\}} \frac{\|u\|_{k,q}}{\|u\|}.$$

It is clear that $\{\lambda_k\}$ is a decreasing sequence. Since $u \to \int_{\mathbb{R}^N} k(x)|u|^q\,dx$ is a completely continuous functional on $H^1(\mathbb{R}^N)$, we can show as in [11] that $\lambda_k \to 0$ as $k \to \infty$. By C_p we denote the best Sobolev constant for the embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, $2 \le p \le 2^*$, that is,

$$C_p = \inf\biggl\{\|u\|^2 : u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^p\,dx = 1\biggr\}.$$

If $p = 2^*$, then $C_p = S$. Let $u \in X_k$. Then

$$\Psi(u) \ge \frac{1}{2}\|u\|^2 - \frac{\lambda_k^q}{q}\|u\|^q + \frac{\mu}{p}C_p\|u\|^p \ge \frac{1}{2}\|u\|^2 - \frac{\lambda_k^q}{q}\|u\|^q.$$

Letting $R_k = (2\lambda_k^q/q)^{1/(2-q)}$, we see that $\frac{1}{2}R_k^2 = (\lambda_k^q/q)R_k^q$. It is clear that $R_k \to 0$ as $k \to \infty$ and $\Psi(u) \ge 0$ for $\|u\| = R_k$, $u \in X_k$, $k \ge k_0$. This proves

(A_1) and since $R_k \to 0$, (A_2) also holds. To check (A_3) we observe that on the finite-dimensional space X^k all norms are equivalent. Hence

$$\Psi(u) \le \frac{1}{2}\|u\|^2 - A\|u\|^q + B\mu\|u\|^p$$

for some constants $A > 0$ and $B > 0$. Since $q < 2 < p$, taking r_k sufficiently small, we satisfy (A_3).

It remains to check the Palais–Smale condition (A_4). First, for n sufficiently large we have

$$\begin{aligned} 1 + \|u_n\| &\ge \Psi(u_n) - \frac{1}{p}\langle \Psi'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{p}\right)\|u_n\|_{k,s} \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \left|\frac{1}{p} - \frac{1}{q}\right| S^{-2^*/q}\|k\|_s\|u_n\|^q. \end{aligned}$$

by $(*)$. This inequality shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Therefore we may assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 < p < 2^*$ and $u_n \to u$ in $L^q_k(\mathbb{R}^N)$. This obviously implies that

$$\langle \Psi(u_n) - \Psi(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Finally, we observe that

$$\begin{aligned} &\|u_n - u\|^2 \\ &\quad \le \mu \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}u_n - |u|^{p-2})(u_n - u)\,dx + \|u_n - u\|^2 \\ &\quad = \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle + \int_{\mathbb{R}^N} k(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\,dx \to 0 \end{aligned}$$

and this completes the proof.

We close with the following remark. Assumption (K) used in the proof of Theorem 3 guarantees the complete continuity of the functional $u \to \int_{\mathbb{R}^N} k(x)|u|^q\,dx$ on $H^1(\mathbb{R}^N)$. This assumption can be replaced by a more general condition which also ensures the complete continuity of this functional. Namely, let $Q(x, l)$ be a cube of the form

$$Q(x, l) = \{y \in \mathbb{R}^N : |y_j - x_j| < l/2,\ j = 1, \ldots, N\}, \quad l > 0.$$

It can be shown that if $k \in L^1(\mathbb{R}^N) \cap L^{(q+\varepsilon)/\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $2 < r < r + \varepsilon < 2^*$ and

$$\lim_{|x|\to\infty} \int_{Q(x,l)} k^{(r+\varepsilon)/\varepsilon}(y)\,dy = 0$$

for some $l > 0$, then the functional $u \to \int_{\mathbb{R}^N} k|u|^q\,dx$, with $1 < q < 2$ is completely continuous on $H^1(\mathbb{R}^N)$ (for details we refer to [6], Proposition A.3.1 in the Appendix, p. 256).

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