Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 9, 1997, 201–219

REMOVABLE SINGULARITIES FOR NONLINEAR ELLIPTIC EQUATIONS

Haïm Brezis — Louis Nirenberg

Dedicated with friendship to O. A. Ladyzhenskaya

1. Introduction

In their celebrated work on nonlinear elliptic equations of the form

$$\partial_{x_i} a_i(x,u,\nabla u) = g(x,u,\nabla u), \tag{1}$$

O. A. Ladyzhenskaya and N. N. Ural'tseva [L-U] proved many basic results including, in particular, regularity for solutions in $L^\infty \cap H^1$. In this paper, under some conditions, we prove a removable singularity result for a subclass of (1),

$$\frac{\partial}{\partial x_i}\left(a_{il}(x,u)\frac{\partial u}{\partial x_l}\right) = g(x,u,\nabla u). \tag{2}$$

The interest in removable singularities arose because of recent work on the following type of problems in a domain Ω in $\mathbb{R}^n$:

$$\begin{aligned} \Delta u - u|\nabla u|^2 &= f(x) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3}$$

The first results treated f in $H^{-1}(\Omega)$, and established the existence of a solution u in $H^1_0(\Omega)$ with $u|\nabla u|$ in $L^2(\Omega)$; see L. Boccardo, F. Murat and J. P. Puel [B-M-P], A. Bensoussan, L. Boccardo and F. Murat [B-B-M], R. Landes [L], T. Del Vecchio [De]—other references may be found in these papers.

1991 *Mathematics Subject Classification.* Primary 35J.

Subsequently, the case where f is in L^1 was considered by L. Boccardo and T. Gallouët [B-G]; they proved the existence of a solution u in H_0^1, with $u|\nabla u|^2$ in L^1. With F. Murat (see [B-G-M]) they then treated the case of $f = f_1 + f_2$, with f_1 in H^{-1} and f_2 in L^1—obtaining a solution in the same class (see also the references therein).

A natural question is whether one might permit f to be a measure—for example, a delta function. If $n = 1$, any measure is in H^{-1}, so a solution exists. In this paper we make the observation that if $n \geq 2$, and f is a Dirac delta function, then no solution exists. This is a consequence of our removable singularity theorem for (2) in a domain Ω.

We now state our conditions.

We assume uniform ellipticity: for some constants $c_0, C_0 > 0$,

$$c_0|\xi|^2 \leq a_{il}(x,u)\xi_i\xi_l \leq C_0|\xi|^2 \quad \forall x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^n, \tag{4}$$

and that the $a_{il}(x,u)$ and $g(x,u,p)$ are smooth. Concerning g we also assume (5)–(9) below.

$$\begin{cases} \text{For every } m \geq 0, \text{ there exists } A_m \text{ such that for } |u| \leq m, \\ |g(x,u,p)| \leq A_m(1+|p|^2) \quad \forall x \in \Omega,\ \forall p \in \mathbb{R}^n. \end{cases} \tag{5}$$

There exist positive numbers α, M such that for all $x \in \Omega$ and $p \in \mathbb{R}^n$,

$$(\operatorname{sgn} u)g(x,u,p) \geq \alpha|p|^2 - h(|u|)^2 \quad \text{for } |u| \geq M. \tag{6}$$

Here h is a C^1 function on $[M,\infty)$ satisfying:

$$h(s) \geq \varepsilon_0 > 0 \quad \forall s \geq M, \tag{7}$$

$$\int_M^\infty \frac{ds}{h(s)} = \infty \tag{8}$$

and

$$\limsup_{s\to\infty} \frac{h'(s)}{h(s)} < \frac{\alpha}{2C_0}. \tag{9}$$

Our first result is

Theorem 1. *Let K be a compact set in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with* $\operatorname{cap} K = 0$ (*here* cap *means Newtonian capacity*). *Let u be a smooth function in $\Omega \setminus K$ satisfying* (2) *in $\Omega \setminus K$. Assume the conditions* (4)–(9). *Then u is smooth in Ω.*

Note that *no a priori assumptions* are made about the behavior of u near K.

For example, the equations

$$-\Delta u + u|\nabla u|^2 = f(x) \tag{10}$$

and

$$-\Delta u + \frac{u}{(1+u^2)^{1/2}}|\nabla u|^2 + c(x)u = f(x) + \gamma u^2, \quad \gamma \in \mathbb{R}, \tag{11}$$

with $f(x)$ and $c(x)$ smooth, fit our framework.

There is a more general form of Theorem 1, which however we derive from it, where, instead of (6), we assume, for all $x \in \Omega$ and $p \in \mathbb{R}^n$,

$$(\operatorname{sgn} u)g(x,u,p) \geq |u|^a(\alpha|p|^2 - k(|u|)^2) \quad \text{for } |u| \geq M, \tag{6'}$$

with $\alpha > 0$, $M > 0$ and $a > -1$. Here k is a C^1 function on $[M, \infty)$ satisfying

$$s^a k(s) \geq \varepsilon_0 > 0 \quad \forall s \geq M, \tag{7'}$$

$$\int_M^\infty \frac{ds}{k(s)} = \infty \tag{8'}$$

and

$$\limsup_{s\to\infty} \frac{k'(s)}{s^a k(s)} < \frac{\alpha}{2C_0}. \tag{9'}$$

Corollary 1. *Let K and u be as in Theorem* 1. *Assume* (4), (5), (6′), (7′), (8′) *and* (9′). *Then u is smooth in Ω.*

Remark 1. Condition (8) on h (or (8′) on k) is rather sharp; see the examples in Section 5 and Theorem 2. For any $\varepsilon > 0$, if we take $h(s) = s^{1+\varepsilon}$ or even $s \log^{1+\varepsilon} s$ the conclusion need not hold.

Remark 2. A closed set K of measure zero with positive capacity need not be a removable set. If K is a smooth hypersurface it need not be removable; for example, if $K = \partial B_{1/2}(0)$, the function $u = 0$ for $|x| < 1/2, u = 1$ for $|x| > 1/2$ satisfies (10) with $f = 0$ in $B_1 \setminus K$.

Corollary 2, which corresponds to $a = -1$ in Corollary 1, is different—this is a borderline case. The conditions we impose on g, in addition to (5), are

$$(\operatorname{sgn} u)g(x,u,p) \geq \frac{1}{|u|}(\alpha|p|^2 - k(|u|)^2) \quad \text{for } |u| \geq M \tag{12}$$

with $M > 0$,

$$\alpha > C_0, \tag{13}$$

$$\frac{k(s)}{s} \geq \varepsilon_0 > 0 \quad \forall s \geq M, \tag{14}$$

$$\int_M^\infty \frac{ds}{k(s)} = \infty \tag{15}$$

and

$$\limsup_{s\to\infty} \frac{sk'(s)}{k(s)} - 1 < \frac{\alpha - C_0}{2C_0}. \tag{16}$$

Corollary 2. *Let K be a compact set in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with* $\operatorname{cap} K = 0$. *Let u be a smooth function in $\Omega \setminus K$ satisfying* (2) *in $\Omega \setminus K$. Assume the conditions* (4), (5) *and* (12)–(16). *Then u is smooth in Ω.*

For example, the equation

$$-\Delta u + \alpha \frac{u}{1+u^2} |\nabla u|^2 + c(x)u = f(x) + \gamma u \log^2(1+u^2) \tag{17}$$

with $f(x)$ and $c(x)$ smooth, $\alpha > 1$ and $\gamma \in \mathbb{R}$ satisfies the conditions of Corollary 2.

Remark 3. When $a = -1$ the additional condition (13), $\alpha > C_0$, is needed; see the counterexample in Section 5 with $\alpha = C_0$. When ($6'$) holds with $a < -1$, even with large α, removable singularity fails; see Section 5.

For linear elliptic operators L, there are classical results stating that if u is a solution of $Lu = 0$ in the punctured ball $B(0) \setminus \{0\}$ then u is a solution in the entire ball provided $|u|$ satisfies a suitable growth condition near the origin. J. Serrin [Se1], [Se2] has proved similar results for a class of nonlinear equations; see the book of L. Véron [Ve2] and also the recent work for degenerate elliptic equations by L. Capogna, D. Danielli and N. Garofalo [C-D-G]. For some very special nonlinear elliptic operators, however, the same conclusion holds *without any restriction near the origin.* The first such example was given by L. Bers [B]; he proved that if u satisfies the minimal surface equation in a punctured disc in $\mathbb{R}^2$ then it may be extended as a smooth solution to the whole disc. E. De Giorgi and G. Stampacchia [D-S] have generalized this result to higher dimensions and J. Serrin [Se3] has similar results for more general equations. Since then, a similar result was established for the equation

$$\Delta u - |u|^{p-1}u = 0 \quad \text{for } p \geq n/(n-2)$$

in $B \setminus \{0\}$, when $n \geq 3$ (see H. Brezis and L. Véron [B-V]); the case $p = (n+2)/(n-2)$ is treated by C. Loewner and L. Nirenberg [L-N]. Study of removable sets has also been made in L. Véron [Ve1] and P. Baras and M. Pierre [B-P].

In proving Theorem 1 we rely on some of the deep regularity results for $H^1 \cap L^\infty$ distribution solutions of equations like (1), due to O. A. Ladyzhenskaya and N. N. Ural'tseva [L-A] (see also M. Giaquinta [G]). In particular, according to Theorem 1.2[1] in Chapter 7 of [G], any $L^\infty \cap H^1$ weak solution of (1) in Ω belongs to $C^{1,\alpha}_{\mathrm{loc}}$ for some α in $(0,1)$. Standard elliptic regularity theory then yields that u is smooth in Ω—even analytic if a_{il} and g are analytic.

To prove Theorem 1, we need thus only establish the following facts under the conditions of Theorem 1:

[1]The condition there that the a_α are Hölder continuous in (x,u) uniformly in p is meant for the $\partial a_\alpha / \partial p_\beta$.

PROPERTY 1. *$u \in L^\infty_{\mathrm{loc}}(\Omega)$.*

PROPERTY 2. *$u \in H^1_{\mathrm{loc}}(\Omega)$.*

PROPERTY 3. *u is a weak (distribution) solution of (2) in all of Ω.*

As we shall see in Section 4, Properties 2 and 3 follow easily from Property 1. The main ingredient for the proof of Property 1 is the following basic lemma in which

$$L = \frac{\partial}{\partial x_i}\left(\alpha_{il}(x)\frac{\partial}{\partial x_l}\right)$$

is an operator with bounded measurable coefficients $\alpha_{il}(x)$ which is elliptic (possibly degenerate):

$$0 \le \alpha_{il}(x)\xi_i\xi_l \le C_0|\xi|^2, \quad C_0 > 0, \quad \forall \xi \in \mathbb{R}^n. \tag{18}$$

LEMMA 1. *Let K be a compact set in a domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, with $\operatorname{cap} K = 0$. Let v be a $C^{0,1}_{\mathrm{loc}}$ function in $\Omega \setminus K$, $v \ge M > 0$, satisfying (in the weak sense)*

$$-Lv + \alpha|\nabla v|^2 \le h(v)^2 \quad \text{in } \Omega \setminus K, \tag{19}$$

where $\alpha > 0$ and h is a C^1 function on $[M, \infty)$ such that (7)–(9) hold. Then $v \in L^\infty_{\mathrm{loc}}(\Omega) \cap H^1_{\mathrm{loc}}(\Omega)$.

Lemma 1 is proved in Section 3.

2. Proofs of Corollaries 1 and 2 using Theorem 1

PROOF OF COROLLARY 1. Let $\varrho(t)$ be a smooth function on $\mathbb{R}$ with $\varrho(0) = 0$, $\varrho' > 0$, satisfying

$$\varrho(t) = (\operatorname{sgn} t)\frac{|t|^{1+a}}{1+a} \quad \text{for } |t| \ge M' > M,$$

with M' to be chosen. Set

$$z = \varrho(u), \quad \text{so} \quad \nabla z = \varrho'(u)\nabla u. \tag{20}$$

Now

$$\begin{aligned} Lz &= \varrho'(u)Lu + \varrho''(u)a_{il}(x,u)u_{x_i}u_{x_l} \\ &= \varrho'(u)g + \frac{\varrho''(u)}{(\varrho'(u))^2}a_{il}(x,u)z_{x_i}z_{x_l} =: \widetilde{g}(x,z,\nabla z), \end{aligned} \tag{21}$$

with $\widetilde{g}$ smooth. Clearly $\widetilde{g}$ satisfies (5), with different constants A_m, while for $|z| \ge (M')^{1+a}/(1+a)$ we have

$$(\operatorname{sgn} z)\widetilde{g}(x,z,p) \ge \alpha|p|^2 - |u|^{2a}k(|u|)^2 - \frac{|a|C_0|p|^2}{(M')^{1+a}}.$$

Let α' be less than α and such that (9′) holds with α' in place of α. Now fix $M' > M$ so that

$$\frac{|a|C_0}{(M')^{1+a}} \le \alpha - \alpha'.$$

Then

$$(\operatorname{sgn} z)\widetilde{g}(x,z,p) \ge \alpha'|p|^2 - h(|z|)^2$$

where

$$h(s) = t^a k(t) \quad \text{with } s = \frac{t^{a+1}}{a+1}.$$

We have to check that h satisfies (7)–(9) with α' in place of α. By (7′), $k(s) \ge \varepsilon_0$ for $s \ge (M')^{1+a}/(1+a)$. From (8′),

$$\int^\infty \frac{ds}{h(s)} = \int^\infty \frac{t^a\,dt}{t^a k(t)} = \infty.$$

Moreover, for $s \ge (M')^{1+a}/(1+a)$,

$$h'(s) = \frac{dh}{ds} = \frac{dh}{dt}\cdot\frac{dt}{ds} = (at^{a-1}k(t) + t^a k'(t))t^{-a} = a\frac{k(t)}{t} + k'(t).$$

Since $1 + a > 0$ we find

$$\limsup_{s\to\infty} \frac{h'(s)}{h(s)} = \limsup_{t\to\infty} \frac{k'(t)}{t^a k(t)} < \frac{\alpha'}{2C_0}.$$

It follows that $\widetilde{g}$ satisfies conditions (5) and (6) and h satisfies (7)–(9) with α' in place of α.

Applying Theorem 1 we see that z is smooth in Ω; consequently, so is u. □

Proof of Corollary 2. We may assume $M > 1$. The proof is similar to the preceding. Let ϱ be a smooth function on $\mathbb{R}$ with $\varrho(0) = 0$, $\varrho' > 0$, satisfying

$$\varrho(t) = (\operatorname{sgn} t)\log|t| \quad \text{for } |t| \ge M.$$

Set $z = \varrho(u)$, so $\nabla z = \varrho'(u)\nabla u$.

As above, (21) holds, with this ϱ, and $\widetilde{g}$ satisfies (5), with different constants A_m. For $|z| \ge \log M$ we have

$$(\operatorname{sgn} z)\widetilde{g}(x,z,p) \ge \frac{1}{|u|^2}(\alpha|p|^2|u|^2 - k(|u|)^2) - C_0|p|^2 = (\alpha - C_0)|p|^2 - \frac{k(|u|)^2}{u^2}.$$

Setting $\alpha - C_0 = \widetilde{\alpha} > 0$, and

$$h(s) = k(t)/t \quad \text{with } s = \log t,$$

we see that

$$(\operatorname{sgn} z)\widetilde{g}(x,z,p) \ge \widetilde{\alpha}|p|^2 - h(|z|)^2.$$

Now

$$\int^\infty \frac{ds}{h(s)} = \int^\infty \frac{dt}{k(t)} = \infty,$$

while

$$\frac{h'(s)}{h(s)} = \frac{t}{k(t)}\left(\frac{k'(t)}{t} - \frac{k(t)}{t^2}\right)t$$

since $dt/ds = t$. Thus

$$\frac{h'(s)}{h(s)} = \frac{tk'(t)}{k(t)} - 1.$$

By (16) we find

$$\limsup_{s\to\infty} \frac{h'(s)}{h(s)} = \limsup_{t\to\infty} \frac{tk'(t)}{k(t)} - 1 < \frac{\alpha - C_0}{2C_0} = \frac{\widetilde{\alpha}}{2C_0}.$$

So $\widetilde{g}$ satisfies the conditions of Theorem 1 with $\widetilde{\alpha}$ in place of α. By the theorem, z is smooth in Ω, and hence so is u. □

3. Proof of Lemma 1

Since $\operatorname{cap} K = 0$, there is a sequence $\zeta_j \in C_0^\infty(\Omega)$, $0 \le \zeta_j \le 1$, such that each $\zeta_j \equiv 1$ near K, with

$$\int |\nabla \zeta_j|^2 \to 0 \quad \text{as } j \to \infty \tag{22}$$

(see [D-S] and [Se1]). Thus $\|\zeta_j\|_{L^2} \to 0$ as $j \to \infty$. Set $\eta_j = 1 - \zeta_j$. By restricting Ω we may always assume that $v \in C^{0,1}$ near and up to $\partial\Omega$. Set

$$\sigma(s) = \int_M^s \frac{dt}{h(t)} \quad \text{for } s \ge M. \tag{23}$$

For any $\varepsilon > 0$ let χ_ε be a smooth nondecreasing function on $\mathbb{R}$, $0 \le \chi_\varepsilon \le 1$ with $\chi_\varepsilon(s) = 0$ for $s \le 0$, $\chi_\varepsilon(s) = 1$ for $s \ge \varepsilon$.

For $t \ge t_0 = \max_{\partial\Omega} v$, multiply (19) by $\eta_j^2 \chi_\varepsilon(v-t)/h(v)^2$ and integrate. Using Green's theorem we find that

$$\alpha J = \alpha \int \eta_j^2 \chi_\varepsilon(v-t)|\nabla\sigma(v)|^2 \le \int \eta_j^2 \chi_\varepsilon(v-t) - \int \alpha_{il} v_{x_l}\left[\eta_j^2\, \frac{\chi_\varepsilon(v-t)}{h(v)^2}\right]_{x_i}.$$

Setting

$$\mu(t) = \operatorname{meas}\{x \in \Omega \setminus K : v(x) > t\},$$

we see that, since $\chi'_\varepsilon \ge 0$,

$$\begin{aligned} \alpha J \le \mu(t) &+ 2C_0 \int \eta_j |\nabla \eta_j| \frac{\chi_\varepsilon(v-t)}{h(v)^2} |\nabla v| \\ &+ 2C_0 \int \eta_j^2 |\nabla v|^2 \frac{\chi_\varepsilon(v-t)}{h(v)^2} \cdot \frac{h'(v)^+}{h(v)}. \end{aligned}$$

In view of (9) we may choose t_1 so large that

$$2C_0 \frac{h'(s)}{h(s)} \le \alpha' < \alpha \quad \text{for } s \ge t_1.$$

We take $t \geq t_1$. Then the last integral above may be absorbed in αJ and we find, using (7),

$$(\alpha - \alpha')J \leq \mu(t) + \frac{\alpha - \alpha'}{2} \int \eta_j^2 \chi_\varepsilon(v - t) \frac{|\nabla v|^2}{h(v)^2} + C \int |\nabla \eta_j|^2$$

with C independent of j and ε. Thus

$$\frac{1}{2}(\alpha - \alpha')J \leq \mu(t) + C \int |\nabla \zeta_j|^2.$$

Using (22) let $j \to \infty$; we obtain

$$\int_{\Omega \setminus K} \chi_\varepsilon(v - t)|\nabla \sigma(v)|^2 \leq C\mu(t)$$
$$= C \text{ meas}\{x \in \Omega \setminus K : \sigma(v(x)) > \sigma(t)\} = C\nu(\sigma(t))$$

where

$$\nu(s) = \text{ meas } \{x \in \Omega \setminus K : \sigma(v(x)) > s\}.$$

Setting $\sigma(t) = s$ we find, on letting $\varepsilon \to 0$, that

(24) $$\int_{\Omega \setminus K,\, \sigma(v) > s} |\nabla \sigma(v)|^2 \leq C\nu(s).$$

This is true for $s \geq s_1 = \sigma(t_1)$, and we rewrite it as

(25) $$\int_{\Omega \setminus K} |\nabla(\sigma(v) - s)^+|^2 \leq C\nu(s) \quad \text{for } s \geq s_1.$$

We pause for a moment to present a simple lemma which will be used several times.

Lemma 2. *Let u be a function in $H^1_{\text{loc}}(\Omega \setminus K)$, $\operatorname{cap} K = 0$, with*

(26) $$\int_{\Omega \setminus K} |\nabla u|^2 < \infty.$$

Then $u \in H^1_{\text{loc}}(\Omega)$.

The lemma seems funny but it requires a proof; if $n = 1$ and K is a point the conclusion is wrong!

Proof of Lemma 2. Let G be open with $K \subset G$ and $\overline{G} \subset \Omega$. Let $\psi \in C_0^\infty(\Omega)$, $0 \leq \psi \leq 1$, with $\psi \equiv 1$ near G. Let

$$\tau_j = \psi(1 - \zeta_j), \quad \zeta_j \text{ as above.}$$

For $k > 0$ we consider the truncation

$$u_k = \begin{cases} k & \text{where } u > k, \\ u & \text{where } -k \leq u \leq k, \\ -k & \text{where } u < -k. \end{cases}$$

The function $\tau_j u_k$ belongs to $H_0^1(\Omega)$ and

$$\|\nabla(\tau_j u_k)\|_{L^2(\Omega)} \le \|\tau_j \nabla u_k\|_{L^2(\Omega)} + \|u_k \nabla\psi\|_{L^2(\Omega)} + \|u_k \nabla\zeta_j\|_{L^2(\Omega)}.$$

But

$$\|\tau_j \nabla u_k\|^2_{L^2(\Omega)} \le \int_{\Omega\setminus K} |\nabla u|^2 < \infty \quad \text{by (26)}$$

and $\|u_k\nabla\psi\|_{L^2(\Omega)} \le \|u\nabla\psi\|_{L^2(\Omega)} < \infty$ since $u \in H^1_{\text{loc}}(\Omega\setminus K)$ and $\operatorname{supp}|\nabla\psi| \subset \Omega\setminus K$. Therefore

$$\|\nabla(\tau_j u_k)\|_{L^2(\Omega)} \le C + k\|\nabla\zeta_j\|_{L^2(\Omega)}$$

where C is independent of j and k. For fixed k, let $j \to \infty$. We infer that $\psi u_k \in H_0^1(\Omega)$ and $\|\nabla(\psi u_k)\|_{L^2(\Omega)} \le C$ independent of k. Letting $k\to\infty$ we conclude that $\psi u \in H_0^1(\Omega)$ and in particular $u \in H^1(G)$. □

We now return to the proof of Lemma 1. In view of Lemma 2, $(\sigma(v)-s)^+ \in H_0^1(\Omega)$ for $s \ge s_1$. Next we rely on a result which is implicitly contained in P. Hartman and G. Stampacchia [H-S]:

Lemma 3. *Let $\varrho \in H^1(\Omega)$, $|\varrho| \le C_1$ on $\partial\Omega$, satisfying*

$$\int_{|\varrho|>s} |\nabla\varrho|^2 \le C\nu^a(s) \quad \textit{for all } s \ge s_1 \ge C_1. \tag{27}$$

where

$$\nu(s) = \operatorname{meas}\{x\in\Omega : |\varrho(x)| > s\} \quad \textit{and} \quad a > \frac{n-2}{n}.$$

Then $\varrho \in L^\infty(\Omega)$.

Proof. Replacing ϱ by $|\varrho|$ we may always assume that $\varrho \ge 0$. Using Hölder's and Sobolev inequalities we find, for all $s > C_1$,

$$\|(\varrho - s)^+\|_{L^1} \le S\|\nabla(\varrho-s)^+\|_{L^2}\nu(s)^{(n+2)/(2n)},$$

where S depends only on n. Combining this with (27) yields, for $s \ge s_1$,

$$\int_s^\infty \nu(\sigma)\,d\sigma = \|(\varrho-s)^+\|_{L^1} \le C\nu(s)^p$$

with $p = (n+2)/(2n) + a/2 > 1$.

The function $f(s) = \int_s^\infty \nu(\sigma)\,d\sigma$ satisfies

$$f'(s) \le -Cf(s)^{1/p} \quad \text{for } s \ge s_1.$$

Integrating this differential inequality we see that $f(s) = 0$ for s sufficiently large. □

Completion of Proof of Lemma 1. Since

$$\int_{\Omega,\,\sigma(v)>s} |\nabla\sigma(v)|^2 \le C\nu(s)$$

we find by Lemma 3 (with $a = 1$) that $\sigma(v)$ is bounded. Now we use the assumption (21), which implies that $\sigma(s) \nearrow \infty$ as $s \to \infty$. Consequently, v is bounded.

Finally, we prove that $v \in H^1_{\mathrm{loc}}(\Omega)$. With τ_j as in the proof of Lemma 2, multiply (19) by τ_j^2 and integrate. We find, since v is bounded,

$$\delta \int \tau_j^2 |\nabla v|^2 \leq C + 2C_0 \int \tau_j |\nabla \tau_j| \cdot |\nabla v|,$$

from which it follows as before that $\int \tau_j^2 |\nabla v|^2 \leq C$ independent of j. Letting $j \to \infty$ and applying Lemma 2 once more, in a smaller set, we find that $v \in H^1_{\mathrm{loc}}(\Omega)$. □

4. Proof of Theorem 1

Recall that to prove the theorem we need only establish

Property 1. $u \in L^\infty_{\mathrm{loc}}(\Omega)$.

Property 2. $u \in H^1_{\mathrm{loc}}(\Omega)$.

Property 3. *u is a weak (distribution) solution of (2) in all of Ω.*

We set

$$\alpha_{il}(x) = a_{il}(x, u(x)). \tag{28}$$

Then $\alpha_{il}(x)$ are smooth in $\Omega \setminus K$, bounded measurable on Ω and satisfy the uniform ellipticity condition: for some $c_0, C_0 > 0$,

$$c_0|\xi|^2 \leq \alpha_{il}(x)\xi_i\xi_l \leq C_0|\xi|^2 \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n.$$

Let

$$L = \frac{\partial}{\partial x_i}\left(\alpha_{il}(x)\frac{\partial}{\partial x_l}\right).$$

Then for $v(x) = f(u(x))$, where f is a C^2 function,

$$Lv = f'(u)Lu + f''(u)\alpha_{il}u_{x_i}u_{x_l} = f'(u)g + f''(u)\alpha_{il}u_{x_i}u_{x_l}. \tag{29}$$

Thus, if f is C^2 and convex then $Lf(u) \geq f'(u)Lu$. By approximation we find Kato's inequality [K]

$$Lw^+ \geq (\mathrm{sign}^+ w)Lw, \quad \text{in the sense of distributions,} \tag{30}$$

for any smooth function w.

Proof of Theorem 1. We divide the proof in 3 steps.

Proof of Property 1. Set

$$v = M + (u - M)^+.$$

We will prove that v satisfies the conditions of Lemma 1. This will imply that $v \in L^\infty_{\text{loc}}(\Omega)$ and therefore $u^+ \in L^\infty_{\text{loc}}(\Omega)$; similarly $u^- \in L^\infty_{\text{loc}}(\Omega)$.

Using (30) we see that, in the weak sense in $\Omega \setminus K$,

$$Lv \geq \operatorname{sign}^+(u-M)Lu = \operatorname{sign}^+(u-M)g =: H.$$

On the set where $u > M$ we have $v = u$ and, by (6),

$$H = g \geq \alpha|\nabla u|^2 - h(u)^2.$$

Therefore

$$H \geq \alpha|\nabla v|^2 - h(v)^2. \tag{31}$$

While on the set where $u \leq M$ we have $H = 0$, $v = M$ and $\nabla v = 0$ a.e. (see e.g. [St] or [G-T]), so that (31) also holds there.

Hence we find that, in the weak sense, $Lv \geq \alpha|\nabla v|^2 - h(v)^2$ in $\Omega \setminus K$. By Lemma 1, $v \in L^\infty_{\text{loc}}(\Omega)$ and thus $u^+ \in L^\infty_{\text{loc}}(\Omega)$. □

Proof of Property 2. Let τ_j be as in the proof of Lemma 2. With λ to be chosen, multiply equation (2) by $\sinh(\lambda u)\tau_j^2$ and integrate. Using Green's theorem we find

$$\lambda c_0 \int \cosh(\lambda u)|\nabla u|^2\tau_j^2 \leq A' \int (1+|\nabla u|^2)|\sinh \lambda u|\tau_j^2 + 2C_0 \int |\sinh \lambda u| \cdot |\nabla u|\tau_j|\nabla \tau_j|$$

where $A' = A_m$ is taken from assumption (5) with $m = \|u\|_{L^\infty(\operatorname{supp}\psi)}$. If we choose $\lambda > (A' + C_0)/c_0$ we obtain

$$\int |\nabla u|^2\tau_j^2 \leq C$$

with C independent of j. Passing to the limit as $j \to \infty$ we conclude that $\int_{\Omega\setminus K} |\nabla u|^2\psi^2 < \infty$. Applying Lemma 2 once more we conclude that $u \in H^1_{\text{loc}}(\Omega)$.

Proof of Property 3. We have to show that, for any function $\varphi \in C_0^\infty(\Omega)$,

$$\int a_{il}\frac{\partial u}{\partial x_l} \cdot \frac{\partial \varphi}{\partial x_i} + \int g\varphi = 0. \tag{32}$$

As before, we multiply the equation (2) by $\varphi(1-\zeta_j)$ and integrate. We find

$$\int \left[a_{il}\frac{\partial u}{\partial x_l} \cdot \frac{\partial \varphi}{\partial x_i} + g\varphi\right](1-\zeta_j) = \int \varphi a_{il}\frac{\partial u}{\partial x_l} \cdot \frac{\partial \zeta_j}{\partial x_i} \to 0.$$

Letting $j \to \infty$, the left hand side tends to the left hand side of (32). □

5. Examples, counterexamples and connection with the strong maximum principle

As we have already mentioned in Remarks 1 and 3 the assumptions in the theorem and corollaries are rather sharp. We present simple examples where some of the assumptions fail and point singularities are not removable if $n \geq 2$.

Example 1. For any $\varepsilon > 0$, the function $u(x) = r^{-1/\varepsilon}$, $r = |x|$, satisfies

$$-\Delta u + |\nabla u|^2 = \frac{1}{\varepsilon^2} u^{2+2\varepsilon} - C u^{1+2\varepsilon} \quad \text{in } B \setminus \{0\}$$

where

$$B = \{x \in \mathbb{R}^n : |x| < 1\} \quad \text{and} \quad C = \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon} + 2 - n\right).$$

Here, (6) holds with $h(s) \simeq \frac{1}{\varepsilon} s^{1+\varepsilon}$ as $s \to \infty$ and thus $\int^\infty ds/h(s) < \infty$.

Example 2. For any $\varepsilon > 0$ the function $u(x) = e^{r^{-1/\varepsilon}}$ satisfies

$$-\Delta u + |\nabla u|^2 = h(u)^2 \quad \text{in } B \setminus \{0\}$$

with $h(s) \simeq \frac{1}{\varepsilon} s \log^{1+\varepsilon} s$ as $s \to \infty$ and thus $\int^\infty ds/h(s) < \infty$.

Example 3. For any positive constant C let

$$G(x) = \begin{cases} C/|x|^{n-2} & \text{if } n \geq 3, \\ -C \log |x| & \text{if } n = 2. \end{cases}$$

The function $u(x) = e^{G(x)}$ satisfies

$$-\Delta u + \frac{1}{u} |\nabla u|^2 = 0 \quad \text{in } B \setminus \{0\}.$$

Here, condition (12) holds with $\alpha = C_0 = 1$ and thus assumption (13) is not satisfied.

Example 4. Given any $\varepsilon > 0$ and $\alpha > 0$ there is a smooth positive function u on $B \setminus \{0\}$ satisfying

$$-\Delta u + \frac{\alpha}{u^{1+\varepsilon}} |\nabla u|^2 = 0 \quad \text{in } B \setminus \{0\} \tag{33}$$

and

$$\lim_{x \to 0} u(x) = \infty.$$

To construct u consider a function of the form $u(x) = \Phi(G(x))$ where G is as in Example 3 and $\Phi : \mathbb{R} \to (0, \infty)$ is a smooth function such that

$$\Phi''(t) = \frac{\alpha [\Phi'(t)]^2}{\Phi(t)^{1+\varepsilon}} \quad \forall t \in \mathbb{R} \tag{34}$$

and

$$\lim_{t \to \infty} \Phi(t) = \infty.$$

Clearly, u satisfies (33) whenever (34) holds. The differential equation (34) has a simple solution. Namely, set

$$H(x) = \int_1^s e^{(\alpha/\varepsilon)\sigma^{-\varepsilon}} d\sigma, \quad s \in (0,\infty).$$

Note that H is increasing on $(0,\infty)$ and

$$\lim_{s\to 0} H(s) = -\infty, \quad \lim_{s\to\infty} H(s) = \infty.$$

Thus the inverse function $\Phi = H^{-1} : \mathbb{R} \to (0,\infty)$ is well defined and we have

$$H'(\Phi(t))\Phi'(t) = 1 \quad \forall t \in \mathbb{R},$$

so that

$$\Phi'(t) = e^{-(\alpha/\varepsilon)\Phi(t)^{-\varepsilon}}$$

and then (34) holds by differentiating this relation.

Connection with the strong maximum principle. Consider a smooth positive function u in $\Omega \setminus K$ ($\operatorname{cap} K = 0$) satisfying

$$-\Delta u + |\nabla u|^2 = f(x) \quad \text{in } \Omega \setminus K, \tag{35}$$

where $f(x)$ is smooth in Ω. By Theorem 1 we know that u is smooth in Ω. We present a different proof of this fact. It relies on removable singularities for bounded solutions of linear elliptic equations and uses also the strong maximum principle.

Set

$$v = e^{-u}. \tag{36}$$

Then v is smooth in $\Omega \setminus K, 0 < v < 1$ in $\Omega \setminus K$ and it satisfies, in $\Omega \setminus K$,

$$-\Delta v + f(x)v = 0. \tag{37}$$

Multiplying (37) by $v\tau_j^2$ (τ_j has been defined in the proof of Lemma 2) we find easily that $v \in H^1_{\text{loc}}(\Omega)$. As in the proof of Property 3 we see that equation (37) holds in the weak sense in all of Ω. Standard regularity theory implies that v is smooth in Ω. The strong maximum principle yields that $v > 0$ in Ω (we cannot have $v \equiv 0$ in Ω since $v > 0$ in $\Omega \setminus K$). Thus $u = -\log v$ is also smooth in Ω.

Instead of (35) consider now the more general equation

$$-\Delta u + |\nabla u|^2 + c(x)u = f(x) \quad \text{in } \Omega \setminus K, \tag{38}$$

where u is positive and smooth in $\Omega \setminus K$, $c(x)$ and $f(x)$ are smooth in Ω. Theorem 1 applies and so u is smooth in Ω. If we try the same method as above we see that $v = e^{-u}$ satisfies the nonlinear equation in $\Omega \setminus K$

$$-\Delta v + f(x)v = -c(x)v\log v. \tag{39}$$

As above we find easily that $v \in H^1_{\mathrm{loc}}(\Omega)$ and that (39) holds in the weak sense in all of Ω (note that $t \log t$ remains bounded as $t \to 0$). Standard regularity theory implies that $v \in C^{2,\alpha}(\Omega)$ for all $\alpha < 1$. However, we cannot invoke the classical strong maximum principle since the function $t \mapsto t \log t$ is not Lipschitz near $t = 0$. But the form due to J. L. Vázquez [Va] applies, since

$$\int_0^{1/2} \frac{ds}{s|\log s|^{1/2}} = \infty.$$

Therefore $v > 0$ in Ω and $u = \log v$ belongs to $C^{2,\alpha}(\Omega)$ for all $\alpha < 1$. Going back to (38) we conclude that u is smooth in Ω.

Similarly, if we start with a positive smooth solution u of

$$-\Delta u + |\nabla u|^2 = h(u)^2 \quad \text{in } \Omega \setminus K$$

the change of unknown $v = e^{-u}$ yields

$$-\Delta v + v[h(-\log v)]^2 = 0 \quad \text{in } \Omega \setminus K,$$

which we write as

$$-\Delta v + \beta(v) = 0 \quad \text{with} \quad \beta(t) = t[h(-\log t)]^2.$$

We assume that β is continuous nondecreasing near 0, $\beta(0) = 0$ and[2]

$$\int_0^{1/2} \frac{ds}{(s\beta(s))^{1/2}} = \infty.$$

We may then invoke [Va] to conclude as above that $v \in C^{1,\alpha}(\Omega)$ for all $\alpha < 1$ and $v > 0$ in Ω. In terms of h the conditions on β mean that

$$\frac{h'(s)}{h(s)} \le \frac{1}{2} \quad \text{for } s \ge s_1 \quad \text{and} \quad \int^\infty \frac{ds}{h(s)} = \infty;$$

these are essentially the assumptions of Theorem 1.

Finally, we point out that assumption (8) plays an essential role in Theorem 1. More precisely, let h be any C^1 function on $[M, \infty]$ satisfying

(40) $$h(s) \ge \varepsilon_0 > 0 \quad \forall s \ge M,$$

(41) $$\frac{h'(s)}{h(s)} \le \delta_0 < \frac{1}{2} \quad \forall s \ge M,$$

(42) $$\int_M^\infty \frac{ds}{h(s)} < \infty$$

for some positive constants M, ε_0 and δ_0.

[2]This is an analogue for second order equations of the classical Osgood condition for uniqueness in first order ordinary differential equations.

THEOREM 2. *Under the assumptions* (40)–(42) *there exists* $R > 0$ *and a* C^2 *radial function* u *on* $B_R \setminus \{0\}$ *such that*

$$u \geq M \quad \text{in } B_R \setminus \{0\}, \tag{43}$$

$$-\Delta u + |\nabla u|^2 = h(u)^2 \quad \text{in } B_R \setminus \{0\}, \tag{44}$$

$$\lim_{x \to 0} u(x) = \infty. \tag{45}$$

As above we will seek u of the form $u = -\log v$; v would satisfy $\Delta v = \beta(v)$ with

$$\beta(t) = t[h(-\log t)]^2 \quad \text{for } 0 < t \leq t_0 = e^{-M}.$$

From (41) we see that β is increasing on $(0, t_0]$ and $\beta(t) \leq Ct^{1-2\delta_0}$, so that $\lim_{t\to 0} \beta(t) = 0$. It is convenient to extend β by $\beta(t_0)$ for $t > t_0$ and by 0 for $t \leq 0$.

We shall construct a radial function $v \in C^{1,\alpha}(\overline{B}_1)$, for all $\alpha \in (0, 1)$, satisfying

$$-\Delta v + \beta(v) = 0 \quad \text{in } B_1, \tag{46}$$

$$v > 0 \quad \text{in } B_1 \setminus \{0\}, \tag{47}$$

$$v(0) = 0. \tag{48}$$

By restricting v to B_R with R sufficiently small we have $v < t_0$ on B_R and then $u = -\log v$ satisfies (43)–(45).

REMARK 4. The existence of such a function v is an example of the "failure" of the strong maximum principle when β is not Lipschitz. It is closely related to the results of J. L. Vázquez [Va], except that he constructs a solution $v \geq 0$ of (46) in an annulus $\{r_1 < |x| < r_2\}$ with $v(x) > 0$ when $|x|$ is near r_1 and $v(x) = 0$ when $|x|$ is near r_2.

It is easy to see that given any positive constant c there is a unique (radial) solution $v = v_c$ of (46) with

$$v = c \quad \text{on } \partial B_1. \tag{49}$$

The maximum principle implies that $v \geq 0$ in B_1, $v(r)$ is nondecreasing on $[0, 1]$ and furthermore

$$0 \leq v_{c_1} - v_{c_2} \leq c_1 - c_2 \quad \text{if } c_2 \leq c_1. \tag{50}$$

In fact, if w and w' are sub- and supersolutions, i.e.,

$$\Delta w' - \beta(w') \leq 0 \leq \Delta w - \beta(w),$$

and if $w \leq w'$ on ∂B_1, then

$$w \leq w' \quad \text{in } B_1.$$

To see this, suppose $\omega := w - w'$ is positive somewhere. Let D be a component of the region where it is positive. Since β is nondecreasing, $\Delta\omega \geq 0$ in D, while $\omega \leq 0$ on ∂D. By the maximum principle, $\omega \leq 0$ in D; contradiction.

Our goal is to prove that for some $c > 0$, v_c vanishes only at the origin. We need some lemmas.

Lemma 4. *There is a constant $c_1 > 0$ such that*

$$v_c(0) > 0 \quad \forall c \geq c_1.$$

Proof. The function $w(x) = a|x|^2 + b$, $a > 0$, $b > 0$, is a subsolution for (46) provided

$$\beta(a + b) \leq 2na$$

and this holds, for example, when $a \geq \frac{1}{2n}\beta(t_0)$. If $c \geq a + b$ we have

$$v_c(0) \geq w(0) = b > 0.$$

Our next lemma is a special case of a result of I. Diaz (see Theorems 1.5 and 1.9 in [Di]). For the convenience of the reader we present the proof.

Lemma 5. *There is a constant $c_2 > 0$ such that*

$$v_c(0) = 0 \quad \forall c \leq c_2.$$

Proof. It suffices to construct a radial supersolution z for (46) such that $z(0) = 0$ and $z(1) > 0$. Following an idea of [B-B-C], we set

$$\varphi(s) = \int_0^s \beta(\sigma)\, d\sigma, \quad s \geq 0,$$

and

$$\gamma(t) = \int_0^t \frac{ds}{(2\varphi(s))^{1/2}}, \quad t \geq 0.$$

Note that, by (42),

$$\int_0^{t_0} \frac{ds}{(s\beta(s))^{1/2}} < \infty$$

and, since

$$\frac{s}{2}\beta\left(\frac{s}{2}\right) \leq \varphi(s) \leq s\beta(s), \tag{51}$$

we see that $\gamma(t) < \infty$. The function $t \mapsto \gamma(t)$ is increasing and $\lim_{t\to\infty} \gamma(t) = \infty$, since $\beta(t) = \beta(t_0)$ for $t \geq t_0$. Therefore the inverse function $h = \gamma^{-1}$ is well defined. We have $\gamma(h(r)) = r$ for all $r > 0$ and differentiation yields

$$h'(r) = (2\varphi(h(r)))^{1/2}, \tag{52}$$

$$h''(r) = \beta(h(r)). \tag{53}$$

In view of the fact that β is nondecreasing, we find that h' is convex and thus

$$h'(r)/r \le h''(r) = \beta(h(t)). \tag{54}$$

It is easy to see, with the help of (53) and (54), that

$$z(r) = h(r/n^{1/2}) \tag{55}$$

is a desired supersolution, i.e., $-\Delta z + \beta(z) \ge 0$.

Proof of Theorem 2. Let $P = \{c > 0 : v_c(0) > 0\}$. Applying (50), Lemmas 4 and 5 we find that P is an open interval of the form $P = (c^\star, \infty)$ with $c^\star > 0$.

Claim. $v^\star = v_{c^\star}$ *has the required properties.*

Since $v^\star(0) = 0$, it suffices to check that

$$v^\star(r) > 0 \quad \forall r \in (0, 1].$$

We argue by contradiction and assume that, for some $0 < r_0 < 1$,

$$v^\star(r) = 0 \quad \forall r \in [0, r_0].$$

With the help of $v^\star$ we shall now construct a radial supersolution y of (46) such that

$$y(0) = 0, \tag{56}$$

$$y(1) > v^\star(1) = c^\star. \tag{57}$$

This will imply that $v_c \le y$ for all $c \le y(1)$. In particular, $v_c(0) \le y(0) = 0$ for all $c \le y(1)$ and thus $c^\star \ge y(1)$—a contradiction with (57).

We first construct a radial solution w of

$$-\Delta w + \beta(w) = 0 \quad \text{in } B_{r_0}$$

with $w(0) = 0$ and $w(r_0) > 0$. This is possible by Lemma 5 (applied in B_{r_0} instead of B_1). Extend the function w to B_1 by choosing

$$\widetilde{w}(r) = \begin{cases} w(r) & \text{for } 0 < r \le r_0, \\ w(r_0) & \text{for } r > r_0. \end{cases}$$

Note that, in the weak sense on B_1, $\Delta\widetilde{w} \le H$ where

$$H = \begin{cases} \beta(w) = \beta(\widetilde{w}) & \text{for } 0 < r \le r_0, \\ 0 & \text{for } r > r_0. \end{cases}$$

The function $y = v^\star + \widetilde{w}$ has the desired properties since

$$-\Delta y + \beta(y) \ge -\beta(v^\star) - H + \beta(v^\star + \widetilde{w}) \ge 0. \qquad \square$$

Acknowledgments. We thank V. Maz'ya for useful discussions concerning the connection with removable singularities in linear elliptic equation (see Section 5). The second author was partly supported by grant ARO-DAAL-03-92-G-0143 and NSF grant DMS-9400912. He wishes also to express his thanks to the Forschungsinstitut ETH for its hospitality.

References

[B-P] P. Baras et M. Pierre, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 1, 185–206.

[B-B-C] P. Bénilan, H. Brezis and M. G. Crandall, *A semilinear equation in $L^1(\mathbb{R}^N)$*, Ann. Scuola Norm. Sup. Pisa (4) **2** (1975), 523–555.

[B-B-M] A. Bensoussan, L. Boccardo and F. Murat, *On a nonlinear partial differential equation having natural growth terms and unbounded solution*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), 347–364.

[B] L. Bers, *Isolated singularities of minimal surfaces*, Ann. of Math. (2) **53** (1951), 364–386.

[B-G] L. Boccardo and T. Gallouët, *Strongly nonlinear elliptic equations having natural growth terms and L^1 data*, Nonlinear Anal. **19** (1992), 573–579.

[B-G-M] L. Boccardo, T. Gallouët and F. Murat, *A unified presentation of two existence results for problems with natural growth*, Progress in Partial Differential Equations: the Metz Surveys 2 (M. Chipot, ed.), Longman, 1993, pp. 127–137.

[B-M-P] L. Boccardo, F. Murat et J. P. Puel, *Existence de solutions non bornées pour certaines équations quasi-linéaires*, Portugal. Math. **41** (1982), 507–534.

[B-V] H. Brezis and L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980), 1–6.

[C-D-G] L. Capogna, D. Danielli and N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. Math. **118** (1996), 1153–1196.

[D-S] E. De Giorgi and G. Stampacchia, *Sulle singolarità eliminabile delle ipersuperficie minimali*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **38** (1965), 352–357.

[De] T. Del Vecchio, *Strongly nonlinear problems with Hamiltonian having natural growth*, Houston J. Math. **16** (1990), 7–24.

[Di] J. I. Diaz, *Nonlinear Partial Differential Equations and Free Boundaries*, vol. I, Pitman, 1985.

[G] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Ann. of Math. Stud., vol. 105, Princeton Univ. Press, 1983.

[G-T] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[H-S] P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential-functional equations*, Acta Math. **115** (1966), 271–310.

[K] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148.

[L-U] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Nauka, Moscow, 1964; Academic Press, New York, 1968; 2nd Russian ed., Nauka, 1973.

[L] R. LANDES, *Solvability of perturbed elliptic equations with critical growth exponent for the gradient*, J. Math. Anal. Appl. **139** (1989), 63–77.

[L-N] C. LOEWNER AND L. NIRENBERG, *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis, Academic Press, 1974, pp. 245–272.

[Se1] J. SERRIN, *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.

[Se2] ———, *Isolated singularities of solutions of quasilinear equations*, Acta Math. **113** (1965), 219–240.

[Se3] ———, *Removable singularities of solutions for elliptic equations II*, Arch. Rational Mech. Anal. **20** (1965), 163–169.

[St] G. STAMPACCHIA, *Equations elliptiques du second ordre à coefficients discontinus*, Les Presses de l'Univ. de Montréal, 1966.

[Va] J. L. VÁZQUEZ, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

[Ve1] L. VÉRON, *Singularités éliminables d'équations elliptiques non linéaires*, J. Differential Equations **41** (1981), 87–95.

[Ve2] ———, *Singularities of Solutions of Second Order Quasilinear Equations*, Longman, 1996.

Manuscript received January 14, 1997

HAÏM BREZIS
Analyse Numérique
Université P. et M. Curie
4 pl. Jussieu
75252 Paris Cedex 05, FRANCE
and
Mathematics Department
Rutgers University
Hill Center, Busch Campus
New Brunswick, NJ 08903, USA

E-mail address: brezis@ann.jussieu.fr; brezis@math.rutgers.edu

LOUIS NIRENBERG
Courant Institute
New York University
251 Mercer Street
New York, NY 10012, USA

E-mail address: nirenl@cims.nyu.edu