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LERAY–SCHAUDER CONTINUATION THEOREMS IN THE ABSENCE OF A PRIORI BOUNDS

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Dedicated to O. A. Ladyzhenskaya

1. Introduction

In 1934, Leray and Schauder have published their fundamental paper *Topologie et équations fonctionnelles* [37], which is the founding father of algebraic topology in infinite-dimensional spaces and a milestone in nonlinear functional analysis and nonlinear differential equations. The style of this paper is still amazingly modern and its influence in contemporary mathematics considerable. This paper was among the thirty-seven most quoted mathematical papers for the period 1950–1965 and its influence still increased in the early seventies, with the development of bifurcation theory, global analysis and the use of topological techniques in critical point theory. The reader can consult the references [53, 32, 40] to get a first idea of the tremendous bibliography related to the consequences and extensions of [37], and the celebrated books of Ladyzhenskaya *et al.* for striking applications to Navier–Stokes equations [33] and to nonlinear elliptic [34] or parabolic partial differential equations [35].

The central topic of this paper is the study of *continuation theorems* for proving the existence of a solution to some equations. Let X and Y be topological spaces, $A \subset X$, and $f : X \to Y$, $g : X \to Y$ two continuous mappings. The fundamental idea of the *continuation method* to solve the equation

$$f(x) = g(x) \tag{1}$$

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in A consists in embedding (1) in a one-parameter family of equations

$$F(x,\lambda) = G(x,\lambda), \tag{2}$$

where the continuous functions $F : X \times [0,1] \to Y$, $G : X \times [0,1] \to Y$ are chosen in such a way that:

1. $F(\cdot,1) = f$, $G(\cdot,1) = g$.
2. *The equation* $F(x,0) = G(x,0)$ *has a nonempty set of solutions in* A.
3. *One of those solutions at least can be continued into a solution in* A *of* (2) *for each* $\lambda \in [0,1]$, *giving in particular a solution of* (1) *in* A *when* $\lambda = 1$.

A situation in which Assertion 3 could be violated is when all solutions of (2) leave A after some $\lambda^* \in]0,1[$. This is the case for example for the family of equations in a real variable

$$(1-\lambda)x + \lambda(x^2+1) = 0,$$

with $A = [-1,1]$ and $\lambda^* = 1/3$. A way to avoid such a situation consists in "closing the boundary", i.e. in introducing the *boundary condition*:

$$F(x,\lambda) \neq G(x,\lambda) \quad \textit{for each } (x,\lambda) \in \partial A \times [0,1].$$

When the boundary condition is satisfied, Assertion 3 can still be violated, however, as shown by the family of equations in a real variable

$$x^2 + 2\lambda - 1 = 0,$$

in which the two solutions when λ is small have disappeared after coalescing at $\lambda = 1/2$. Such a situation can be eliminated by reinforcing Assumption 2 into

$2'$. *Equation* $F(x,0) = G(x,0)$ *has a* "robust" *nonempty set of solutions in* A.

A precise way of expressing Assumption $2'$ can be made, for some classes of spaces X, Y and mappings F, G, through the introduction of an "algebraic" count of the number of solutions of $F(x,0) = G(x,0)$ in A. This is called the *topological degree* or the *fixed point index*, and was already developed by Kronecker, Poincaré and Brouwer for continuous mappings between oriented manifolds of the same finite dimension. It was the merit of Leray and Schauder to extend this concept to an important class of mappings defined in a (possibly infinite-dimensional) Banach space.

2. The Leray–Schauder continuation theorem

Let X be a Banach space and $I = [0,1]$. If $A \subset X \times I$ and $\lambda \in I$, we shall write $A_\lambda = \{x \in X : (x,\lambda) \in A\}$. For $a \in X$ and $r > 0$, $B(a,r)$ will denote the open ball of center a and radius r. Let $\Omega \subset X \times I$ be a bounded open set with closure $\overline{\Omega}$ and boundary $\partial\Omega$, and let $F : \overline{\Omega} \to X$ be a mapping. We denote by Σ the (possibly empty) set defined by

$$\Sigma = \{(x, \lambda) \in \overline{\Omega} : x = F(x, \lambda)\}.$$

The following assumptions were introduced by Leray and Schauder in [37].

(H_0) $F : \overline{\Omega} \to X$ *is completely continuous.*

Recall that a *completely continuous* mapping is a continuous mapping which takes bounded subsets into relatively compact ones.

(H_1) $\Sigma \cap \partial\Omega = \emptyset$ (A priori estimate).

(H_2) Σ_0 *is a finite nonempty set* $\{a_1, \ldots, a_\mu\}$ *and the corresponding topological degree* $\deg[I - F(\cdot, 0), \Omega_0, 0]$ *is different from zero* (Degree condition).

As mentioned above, this topological degree is some "algebraic count" of the number of elements of Σ_0, such that, in particular, $\Sigma_0 \neq \emptyset$ when the degree is not zero. In the special case where F is of class C^1,

$$\deg[I - F(\cdot, 0), \Omega_0, 0] = \sum_{j=1}^{\mu} (-1)^{\sigma_j},$$

where σ_j is the sum of the algebraic multiplicities of the eigenvalues of $F'_x(a_j, 0)$ contained in $]1, \infty[$.

The *Leray–Schauder continuation theorem* goes as follows.

Theorem 1. *If conditions* (H_0), (H_1) *and* (H_2) *hold, then* Σ *contains a continuum* $\mathcal{C}$ *along which* λ *takes all values in* I.

In other words, under the above assumptions, Σ contains a compact connected subset $\mathcal{C}$ connecting Σ_0 to Ω_1. In particular, the equation $x = F(x, 1)$ has a solution in Ω_1.

Leray and Schauder observed furthermore that, by refining Assumption (H_2), one could obtain a more precise conclusion, which is reminiscent of more recent results in bifurcation theory [50]:

If the local index *of* a_1, *defined by* $\deg[I - F(\cdot, 0), B(a_1, r), 0]$ *with* $r > 0$ *small, is different from zero, then* $(a_1, 0)$ *belongs to a continuum in* Σ *which contains one of the other points* $(a_2, 0), \ldots, (a_\mu, 0)$, *or to a continuum in* Σ *along which* λ *takes all the values in* I.

Notice that the conclusion of Theorem 1 still holds if the finiteness of the set Σ_0 is dropped from Assumption (H_2). Hence, *from now on, we shall refer to Assumption* (H_2) *as being the condition*

(H_2) $\deg[I - F(\cdot, 0), \Omega_0, 0] \neq 0$ (Degree condition).

Conditions (H_0) and (H_2) are in general the easiest ones to check. In particular, as already noticed by Leray and Schauder, (H_2) holds if $0 \in \Omega_0$ and

$F(\cdot,0)=0$, or if Σ_0 is a finite nonempty set $\{a_1,\dots,a_\mu\}$ such that μ is odd and $I-F(\cdot,0)$ is one-to-one on a neighbourhood of each a_j.

Condition (H_1) requires the a priori knowledge of some properties of the solution set Σ and is in general very difficult to check. An important situation in which it holds had already been emphasized by Leray and Schauder and corresponds to the case where $\Omega = X$ and the set of possible solutions of the deformation is *a priori bounded.* In their own words [37]: *Soit une famille d'équations … qui dépendent continûment du paramètre* k $(k_1 \le k \le k_2)$

$$x-\mathcal{F}(x,k)=0.$$

L'une des conséquences de notre théorie est la suivante: il suffit de savoir majorer a priori toutes les solutions que possèdent ces équations et de vérifier, pour une valeur particulière k_0 de k, *une certaine condition d'unicité pour avoir le droit d'affirmer que l'équation* $x-\mathcal{F}(x,k)=0$ *possède au moins une solution quel que soit* k.

In the notations of the present paper, this important special case can be stated as follows. Introduce the condition

(H_1') Σ *is bounded* (A priori bound).

Corollary 1. *Assume that conditions* (H_0), (H_1') *and* (H_2) *hold. Then the conclusion of Theorem* 1 *is valid.*

Indeed, by assumption, there is some $R>0$ such that $\Sigma \subset B(0,R)\times I$ and, if we take $\Omega = B(0,R)\times I$, then $\Sigma \cap \partial\Omega = \emptyset$.

Most of the applications of the Leray–Schauder continuation theorem to differential or integral equations are devoted to situations where the set of possible solutions of the deformation is a priori bounded.

When this is not the case, one can try to determine an open set Ω for which the assumptions (H_1) and (H_2) hold. To obtain it explicitly seems to be in many cases an almost hopeless task, and we shall now describe some recent results where this explicit determination is replaced by another condition. To motivate it, we first rephrase the conclusion of Corollary 1 in the following form, where the a priori bound condition on Σ is weakened into an a priori bound condition on Σ_0 only, and the conclusion takes the form of an alternative (see e.g. [50] and [20]). We introduce the condition

(H_1'') Σ_0 *is bounded.*

Corollary 2. *Assume that conditions* (H_0), (H_1'') *and* (H_2) *(with* Ω_0 *an open bounded neighbourhood of* Σ_0*) hold. Then there exists a continuum* $\mathcal{C}\subset\Sigma$ *meeting* Σ_0 *which either meets* Σ_1 *or is unbounded.*

The idea will then be to introduce a functional over $X\times[0,1]$ whose restriction to Σ is proper and avoids at least two values. Recall that a mapping $g: X\to Y$

between topological spaces is called *proper* if $g^{-1}(K)$ is compact for each compact set $K \subset Y$. Using the connectedness of the continuum $\mathcal{C}$ given in Corollary 2, it will then be possible to prove that $\mathcal{C}$ has to meet Σ_1.

3. The use of a continuous functional in global continuation theorems

The following continuation theorem was first stated and proved in [6]. We keep the notations and terminology of Section 1.

Theorem 2. *Assume that conditions (H_0), (H_1'') and (H_2) hold, with Ω_0 an open bounded neighbourhood of Σ_0. Assume moreover that there exists a continuous function $\varphi : X \times I \to \mathbb{R}$ and two real numbers c_-, c_+ having the following properties:*

1. *φ is proper on Σ.*
2. *$c_- < \min_{x\in\Sigma_0} \varphi(x,0) \le \max_{x\in\Sigma_0} \varphi(x,0) < c_+$.*
3. *$\varphi(\Sigma) \cap \{c_-, c_+\} = \emptyset$.*

Then Σ contains a continuum $\mathcal{C}$ along which λ takes all values in I.

Proof. If the conclusion is not true, then, by Corollary 2, Σ contains an unbounded continuum meeting Σ_0. Notice that, by the complete continuity of F, Σ is locally compact. As φ is proper, $\varphi(\mathcal{C})$ is unbounded and connected, and hence is an unbounded interval which, by the first part of assumption 2, contains $\min_{x\in\Sigma_0} \varphi(x,0)$ and $\max_{x\in\Sigma_0} \varphi(x,0)$. Consequently, it contains c_- or c_+, a contradiction with the second part of assumption 2. $\square$

A rather direct consequence of Theorem 2 which is easier to apply is the following.

Corollary 3. *Assume that conditions (H_0), (H_1'') and (H_2) hold, with Ω_0 an open bounded neighbourhood of Σ_0. Assume moreover that there exist a continuous mapping $\varphi : X \times I \to \mathbb{R}_+$ and an unbounded increasing sequence $(c_k)_{k\in\mathbb{N}}$ satisfying the following conditions.*

(h_1) *There exists $R > 0$ such that $\varphi(u,\lambda) \ne c_k$ for all $k \in \mathbb{N}$ and $(u,\lambda) \in \Sigma$ with $\|u\| \ge R$.*

(h_2) *$\varphi^{-1}([0, c_n[) \cap \Sigma$ is bounded for each $n \in \mathbb{N}$.*

Then there exists a continuum $\mathcal{C} \subset \Sigma$ along which λ takes all values in I.

This corollary can be applied [5, 42] to the study of *T-periodic solutions of second order superlinear differential equations* of the form

$$u'' + g(u) = p(t,u,u'), \tag{3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following *superlinearity condition*:

$$\lim_{|x|\to\infty} g(x)/x = \infty, \tag{4}$$

and where $p : [0,T] \times \mathbb{R} \times \mathbb{R}$ is continuous and satisfies the following *linear growth condition*:

$$|p(t,x,y)| \le a + b|x| + c|y|, \tag{5}$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ and some nonnegative a, b, c. We embed this problem in the one-parameter family

$$u'' + g(u) = -(1-\lambda)\frac{u'}{1+|u'|} + \lambda p(t,u,u') := q(t,u,u',\lambda) \quad (\lambda \in I). \tag{6}$$

By standard techniques which we shall not describe here (see e.g. [40]), the T-periodic solutions of this equation can be obtained as the solutions of an equation of the form

$$u = F(u,\lambda)$$

in the space $X = \{u \in C^1([0,T]) : u(0) - u(T) = u'(0) - u'(T) = 0\}$, with F completely continuous on X. It follows from the choice of the deformation and the superlinearity condition on g that Σ_0 is bounded and it follows from a general result in [5] that $|\deg[I - F(\cdot,0),\Omega_0,0]| = 1$. Thus conditions (H_1'') and (H_2) hold. Now we take $\delta(x,y) = \min\{0, 1/(x^2+y^2)\}$ and

$$\varphi(u,\lambda) = \left|\frac{1}{2\pi}\int_0^T [u'^2(t) - u(t)q(t,u(t),u'(t),\lambda)]\delta(u(t),u'(t))\,dt\right|,$$

so that $\varphi(u,\lambda)$ is the absolute value of the *winding number* or *Poincaré's index* with respect to 0 (and hence a nonnegative integer) of the closed curve $(u(t),u'(t))$, when (u,λ) is a solution of (6) such that $u^2(t) + u'^2(t) \ge 1$ for all $t \in [0,T]$. A differential inequality type argument using the superlinearity of g and the linear growth condition on p shows the existence of some $R > 0$ such that this last inequality holds when $(u,\lambda) \in \Sigma$ and $\|u\| > R$. Now, the superlinearity of g implies that, for the autonomous equation

$$u'' + g(u) = 0,$$

the orbits with sufficiently large maximal amplitude A are closed and their period $\tau(A)$ is such that $\lim_{A\to\infty} \tau(A) = 0$. A delicate perturbation argument based upon the linear growth condition on p then implies that, along Σ, $\varphi(u,\lambda) \to \infty$ when $\|u\| \to \infty$. Thus we can take $c_k = k + 1/2$ in Corollary 1. We refer to [6] for the details. Thus, we have the following

Theorem 3. *If g satisfies* (4) *and p satisfies* (5), *then equation* (3) *has at least one T-periodic solution.*

As the equation

$$u'' + u' + u^3 = 0$$

only has the trivial T-periodic solution, the conclusion above is optimal for the class of equations considered.

The interest of this approach, as compared to other techniques like shooting methods and critical point theory, is that it easily extends to the case of some *functional differential equations,* as shown in [7]. Let $r > 0$ and, for $x : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, denote as usual by x_t the function

$$x_t : [-r, 0] \to \mathbb{R}, \quad \theta \mapsto x(t + \theta).$$

Consider the second order functional differential equation

$$u''(t) + g(u(t)) = p(t, u(t), u'(t), u_t, u'_t), \tag{7}$$

where $g : \mathbb{R} \to \mathbb{R}$ and $p : \mathbb{R} \times \mathbb{R}^2 \times \mathcal{C} \to \mathbb{R}$ are continuous and p is T-periodic in t. The following existence result is proved in [7], using Theorem 2.

Theorem 4. *If* (4) *holds and if there exist $K, L \geq 0$ such that*

$$|p(t, x, y, \eta, \psi)| \leq K(|x| + |y|) + L$$

for all $(t, x, y, \eta, \psi) \in \mathbb{R} \times \mathbb{R}^2 \times C([-r, 0))$, then equation (7) *has at least one T-periodic solution.*

The case of functional differential equations with other boundary conditions has been considered by M. Henrard [29].

Notice also that existence results have been obtained as well in [6] and [7] for periodic solutions of *planar ordinary differential systems* of the form

$$z'(t) = -J[H'(z(t)) + p(t, z(t))],$$

or *planar functional differential systems* of the form

$$z'(t) = J[H'(z(t)) + p(t, z(t), z_t)],$$

where J denotes the symplectic matrix, $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is such that $H' = H'_z$ satisfies a suitable superlinear condition and p satisfies some growth restrictions with respect to H. We refer to [6] and [7] for a precise statement of the existence theorems. See also a recent result of Precup [48] on a version based upon Granas' transversality technique [17] and applications to periodic boundary value problems with impulses.

The above techniques also allow proving the existence of T-periodic solutions for some *fourth order superlinear differential equations*. Consider the periodic boundary value problem

$$
\begin{aligned}
& u'''' = g(u) + p(t, u, u', u'', u'''), \\
& u(T) - u(0) = u'(T) - u'(0) = u''(T) - u''(0) = u'''(T) - u'''(0) = 0,
\end{aligned} \tag{8}
$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (4) and $p : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is a Carathéodory function. The following existence result is proved in [44], using Theorem 2.

Theorem 5. *If g satisfies* (4) *and*

$$|p(t, x, y, z, w)| \le q(t)$$

for almost every $t \in [0,T]$, all $(x,y,z,w) \in \mathbb{R}^4$, and some $q \in L^1([0,T], \mathbb{R}^+)$, then problem (8) *has at least one solution.*

The proof of this result requires a very detailed study of the oscillatory properties of the solutions of the associated family of equations

$$
\begin{aligned}
& u'''' = g(u) + q(t, u, u', u'', u''', \lambda), \quad \lambda \in [0,1], \\
& u(T) - u(0) = u'(T) - u'(0) = u''(T) - u''(0) = u'''(T) - u'''(0) = 0,
\end{aligned}
$$

where

$$q(t, x, y, z, w, \lambda) := -(1-\lambda)\frac{y}{1+|y|} + \lambda p(t, x, y, z, w) \quad \text{for } \lambda \in [0,1].$$

4. The use of a continuous functional in localized continuation theorems

More precise results have been obtained in [4] for the *superlinear Sturm–Liouville problem*

$$u''(t) + g(u(t)) = p(t, u(t), u'(t)), \tag{9}$$

$$ax(0) + bx'(0) = 0, \quad cx(T) + dx'(T) = 0, \tag{10}$$

when g is continuous and satisfies (4), and $p : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies (5). In this situation one can prove the existence of infinitely many solutions with arbitrarily large norms.

This result requires an extension of Theorem 2 which goes as follows (see [4]). We consider again $F : X \times I \to X$ completely continuous and keep the notations of Theorem 2. If ω is a (possibly unbounded) open subset of X such that $S = \Sigma_\lambda \cap \overline{\omega}$ is compact and $S \subset \omega$ (i.e. there is no solution of $x = F(x, \lambda)$ on $\partial\omega$), then there exists an open bounded set $\mathcal{U}$ such that $S \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \omega$. For all such $\mathcal{U}$, the Leray–Schauder degree $\deg[I - F(\cdot, \lambda), \mathcal{U}, 0]$ is the same, by the excision property. We will denote it by $\deg[I - F(\cdot, \lambda), \omega, 0]$.

Let $\mathcal{O} \subset X \times I$ be open in $X \times I$. Denote by Σ^* the (possibly empty) set

$$\Sigma^* = \{(x, \lambda) \in \overline{\mathcal{O}} : x = F(x, \lambda)\} = \overline{\mathcal{O}} \cap \Sigma.$$

We introduce the conditions:

(H_1^*) Σ_0^* *is bounded in* X *and* $\Sigma_0^* \subset \mathcal{O}_0$.
(H_2^*) $\deg[I - F(\cdot, 0), \mathcal{O}_0, 0] \neq 0$,

so that $\Sigma_0^* \neq \emptyset$.

THEOREM 6. *Assume that conditions* (H_0), (H_1^*) *and* (H_2^*) *hold and that there exist a continuous functional* $\varphi : X \times I \to \mathbb{R}$ *and constants* c_-, c_+ *with the following properties:*

1. φ *is proper on* Σ^*.
2. $c_- < \min_{u \in \Sigma_0^*} \varphi(u, 0) \leq \max_{u \in \Sigma_0^*} \varphi(u, 0) < c_+$.
3. $\varphi(\Sigma^* \cap \mathcal{O}) \cap \{c_-, c_+\} = \emptyset$ *and* $\varphi(\Sigma^* \cap \partial\mathcal{O}) \cap [c_-, c_+] = \emptyset$.

Then Σ^* *contains a continuum along which* λ *takes all values in* I.

The proof of this result follows the same lines as that of Theorem 2.

Let us now consider a consequence of Theorem 6 which is useful for the application we have in mind. Assume that $\varphi : X \times I \to \mathbb{R}_+$ is continuous and $(c_k)_{k \in \mathbb{N}}$ is an unbounded increasing sequence that satisfies the following conditions:

(h_1^*) *There exists* $R > 0$ *such that* $\varphi(u, \lambda) \neq c_k$ *for all* $k \in \mathbb{N}$ *and* $(u, \lambda) \in \Sigma^*$ *with* $\|u\| \geq R$.
(h_2^*) $\varphi^{-1}([0, c_n[) \cap \Sigma^*$ *is bounded for each* $n \in \mathbb{N}$.

Let k_0 be an integer such that $c_{k_0} > \sup\{\varphi(u, \lambda) : (u, \lambda) \in \Sigma^*, \|u\| \leq R\}$.

COROLLARY 4. *Assume that conditions* (H_0), (h_1^*) *and* (h_2^*) *hold and that*

$$\deg[I - F(\cdot, 0), (\mathcal{O}^k)_0, 0] \neq 0$$

for each integer $k > k_0$, *where* $\mathcal{O}^k = \varphi^{-1}(]c_k, c_{k+1}[)$. *Then, for each of those integers, equation*

$$u = F(u, 1) \tag{11}$$

has at least one solution u_k *such that* $\varphi(u_k, 1) \in \,]c_k, c_{k+1}[$. *Moreover,*

$$\lim_{j \to \infty} \|u_j\| = \infty.$$

PROOF. Let, for $k \geq k_0$, $\Sigma^k = \overline{\mathcal{O}^k} \cap \Sigma^*$. By ($h_2^*$), $(\Sigma^k)_0$ is bounded. But, by (h_1^*), $\varphi(x, \lambda) \neq c_k$ and $\varphi(x, \lambda) \neq c_{k+1}$ for all $(x, \lambda) \in \Sigma^k$, so $(\Sigma^k)_0 \subset (\mathcal{O}^k)_0$. We now prove that φ is proper on Σ^k. Let K be a compact subset of $\mathbb{R}$. Then $\varphi^{-1}(K) \cap \Sigma^k$ is closed and included in Σ^k, which is compact, so it is also compact.

Thus, all conditions of Theorem 6 with $\Sigma = \Sigma^k$, $\mathcal{O} = \mathcal{O}^k$ are satisfied and equation (11) will have at least one solution $u \in \bigcap (\overline{\mathcal{O}^k})_1$. If the last conclusion of Corollary 4 is not true, we can find a bounded subsequence (u_{k_j}) of solutions of (11) with $\varphi(u_{k_j}) \in]c_{k_j}, c_{k_j+1}[$. So $\varphi(u_{k_j}) \to \infty$ as $j \to \infty$. Thus we get a contradiction, as the sequence (u_{k_j}) is precompact. $\square$

Now, we will use Corollary 4 to prove the existence of solutions for the problem (9)–(10) when g is continuous and satisfies (4) and $p : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies (5). To avoid some technical problems, we suppose that $|g(x)| \geq |x|$. Thanks to the superlinearity of g, this condition is satisfied for $|x|$ sufficiently large. If this is not the case for all x, take $E = \operatorname{conv}\{x \in \mathbb{R} : |g(x)| < |x|\}$. Then the function

$$\widetilde{g}(x) = \begin{cases} x & \text{if } x \in E, \\ g(x) & \text{otherwise,} \end{cases}$$

has this property and the growth condition (5) is still valid for the function $\widetilde{p}(t,x,y) = p(t,x,y) + \widetilde{g}(x) - g(x)$. Let $f(x) = x + x^3$. This (odd) function satisfies the conditions imposed in Section 3 for the computation of the degree.

We consider the homotopy

$$u''(t) + f(u(t)) = \lambda q(t, u(t), u'(t)) \tag{12}$$

where $q(t,x,y) = p(t,x,y) - g(x) + f(x)$. For $\lambda = 1$, this is equation (9).

Suppose for the moment that $ad - bc \neq 0$; this means that the two lines $D = \{(x,y) \in \mathbb{R}^2 : ax + by = 0\}$ and $A = \{(x,y) \in \mathbb{R}^2 : cx + dy = 0\}$ are distinct.

Let

$$r_1(u) = \frac{au + bu'}{|ad - bc|^{1/2}}, \quad r_2(u) = \frac{cu + du'}{|ad - bc|^{1/2}}$$

and $\delta : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\delta(x,y) = \min\{1, 1/(x^2 + y^2)\}$. We now define on $C^1([0,T], \mathbb{R}) \times I$ the continuous functional φ by

$$\begin{aligned}\varphi(u,\lambda) \\ = \frac{2}{\pi} \bigg| \int_0^T [u'(t)^2 + u(t)(f(u(t)) - \lambda q(t, u(t), u'(t)))] \delta(r_1(u)(t), r_2(u)(t))\, dt \bigg|.\end{aligned}$$

If (u, λ) is a solution of (12) such that $[r_1(u)(t)]^2 + [r_2(u)(t)]^2 \geq 1$ for all $t \in [0,T]$, we get

$$\varphi(u,\lambda) = \frac{2}{\pi} \bigg| \int_0^T \frac{u'(t)^2 - u(t)u''(t)}{[r_1(u)(t)]^2 + [r_2(u)(t)]^2}\, dt \bigg| = \frac{2}{\pi} \bigg| \int_0^T \frac{d}{dt} \arctan\left(\frac{r_2(u)(t)}{r_1(u)(t)}\right) dt \bigg|.$$

Thus, for a solution of (12) satisfying the boundary condition (10), it turns out that $\varphi(u,\lambda)$ counts the number of quarters of lap if this is understood as the passage from one of the lines to the other.

When we are in the "pathological" case where $ad - bc = 0$, we define φ by

$$\varphi(u,\lambda) = \frac{2}{\pi}\left|\int_0^T [u'(t)^2 + u(t)(f(u(t)) - \lambda q(t,u(t),u'(t)))]\delta(u(t),u'(t))\,dt\right| + 1.$$

If (u,λ) is a solution of (12)–(10) with $u(t)^2 + u'(t)^2 \geq 1$ for all $t \in I$, then

$$\varphi(u,\lambda) = 2/\pi\left|\int_0^T \frac{d}{dt}\arctan\left(\frac{u'(t)}{u(t)}\right)dt\right| + 1$$

is just the number of quarters of lap plus one.

Consistently with the notations of Section 2, we set

$$\Sigma = \{(u,\lambda) \in C^1([0,T]) \times I : (u,\lambda) \text{ is a solution of (12)–(10)}\}.$$

It is now possible, although not trivial, to show that the functional φ satisfies all the conditions of Corollary 4, and to prove the following existence result for second order nonlinear Sturm–Liouville problems (see [9] and [4] for details, and [8] for the special case of Neumann conditions).

Theorem 7. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous, satisfies* (4) *and that $p : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies* (5). *Then there exists $k_0 \in \mathbb{N}$ such that, for each $j > k_0$, the problem* (9)–(10) *has at least one solution u_j such that $\varphi(u_j,1) \in\,]2j, 2(j+1)[$. Moreover, $\|u_j\| \to \infty$ as $j \to \infty$.*

Theorem 6 has also been applied by M. Henrard [28] to prove the existence of infinitely many solutions for *singular superlinear boundary value problems* of the type

$$\begin{aligned}
&u''(t) + \frac{n}{t}u'(t) + g(u(t)) = p(t,u(t),u'(t)),\\
&u'(0) = 0, \quad \alpha u(T) + \beta u'(T) = 0,
\end{aligned}$$

when g is superlinear and p has at most linear growth. His results extend those of Castro and Kurepa [12, 13]. The same author has also considered the extension of the results of Sections 2 and 3 to *general linear boundary conditions* which contain the periodic, the Sturm–Liouville and the three-point boundary conditions as special cases [27, 31, 30]. Garcia-Huidobro, Manásevich and Zanolin [24, 25] have extended the results of Theorem 7 to the *Dirichlet problem for superlinear equations with ϕ-laplacians*

$$(\phi(u'))' + g(u) = p(t,u,u'), \quad u(0) = A,\ u(T) = B,$$

when ϕ belongs to a suitable class of increasing homeomorphisms of $\mathbb{R}$, g grows faster than ϕ at infinity and p is bounded.

Theorem 7 can be refined by the use of a slightly more general form of Corollary 4 [11], and a different choice of the functional φ.

Assume that $\varphi : X \times I \to \mathbb{R}$ is continuous and $(c_k)_{k\in\mathbb{Z}}$ is an increasing double sequence with $c_k < 0$ for $k < 0$, $c_k > 0$ for $k > 0$ and $\lim_{k\to\pm\infty} c_k = \pm\infty$ that satisfies the following conditions:

(H_1^*) *There exists $R > 0$ such that $\varphi(u,\lambda) \neq c_k$ for all $k \in \mathbb{Z}$ and $(u,\lambda) \in \Sigma$ with $\|u\| \geq R$.*

(H_2^*) *$\varphi^{-1}(]c_{-n}, c_n[) \cap \Sigma$ is bounded for each $n \in \mathbb{Z}^+$.*

Let k_0 be a positive integer such that

$$\min\{-c_{-k_0}, c_{k_0}\} > \sup\{|\varphi(u,\lambda)| : (u,\lambda) \in \Sigma, \|u\| \leq R\}. \tag{13}$$

Corollary 5. *Assume that conditions (H_0), (H_1^*) and (H_2^*) hold and that there is $k_0 \in \mathbb{Z}^+$ satisfying* (13) *such that*

$$\deg[(I - F(\cdot,0)), (\mathcal{O}^k)_0, 0] \neq 0$$

for some integer k with $|k| > k_0$ and

$$\mathcal{O}^k = \begin{cases} \varphi^{-1}(]c_k, c_{k+1}[) & \text{if } k > 0, \\ \varphi^{-1}(]c_{k-1}, c_k[) & \text{if } k < 0. \end{cases} \tag{14}$$

Then there is at least one solution $\widetilde{u}$ for (11) *with $\varphi(\widetilde{u},1) \in\,]c_k, c_{k+1}[$ if $k > 0$ and $\varphi(\widetilde{u},1) \in\,]c_{k-1}, c_k[$ if $k < 0$. In particular, if (H_3^*) holds for every $k \in \mathbb{Z}$ with $|k| > k_0$, then, for each $n \in \mathbb{N}$ with $n > k_0$, equation* (11) *has at least two solutions u_n and w_n such that $\varphi(u_n,1) \in\,]c_n, c_{n+1}[$ and $\varphi(w_n,1) \in\,]c_{-(n+1)}, c_{-n}[$. Moreover, $\lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|w_n\| = \infty$.*

Proof. For $k \in \mathbb{Z}$ with $|k| > k_0$, let $\Sigma^k = \overline{\mathcal{O}^k} \cap \Sigma$. By (H_2^*), $(\Sigma^k)_0$ is bounded and hence compact. But, by (H_1^*), $\varphi(x,\lambda) \neq c_k$ and $\varphi(x,\lambda) \neq c_{k+1}$ (if $k > 0$) or $\varphi(x,\lambda) \neq c_{k-1}$ (if $k < 0$), for all $(x,\lambda) \in \Sigma^k$, so that $\Sigma^k \subset \mathcal{O}^k$ and also $(\Sigma^k)_0 \subset (\mathcal{O}^k)_0$. Thus we have proved the condition (H_1). We now prove that φ is proper on Σ^k. Let K be a compact subset of $\mathbb{R}$. Then $\varphi^{-1}(K) \cap \Sigma^k$ is closed and included in Σ^k which is compact, so it is also compact. Thus, all conditions of Theorem 6 with $\Sigma^* = \Sigma^k$, $\mathcal{O} = \mathcal{O}^k$ and $(c_-, c_+) = (c_k, c_{k+1})$ for $k > 0$ or $(c_-, c_+) = (c_{k-1}, c_k)$ for $k < 0$ are satisfied and equation (11) will have at least one solution $u \in (\overline{\mathcal{O}^k})_1$. The rest of the proof is essentially similar to that of Corollary 4. $\square$

Corollary 5 can now be used to prove the following sharp existence theorem for *superlinear second order equations with Dirichlet conditions* (see [11]). We want to apply the abstract theory to the problem

$$u''(t) + g(u(t)) = p(t, u(t), u'(t)), \quad u(0) = u(T) = 0, \tag{15}$$

where g is continuous and satisfies (4), and $p : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies (5). Without loss of generality we can assume that $g(x)x > 0$ for $x \neq 0$, moving if necessary a bounded term from f to p.

Problem (15) will be solved via the continuation principle described in Corollary 5. To this end, we consider the homotopy

$$u''(t) + h(u(t), \lambda) = \lambda p(t, u(t), u'(t)), \quad u(0) = u(T) = 0, \quad \lambda \in I, \tag{16}$$

where

$$h(x, \lambda) = \lambda g(x) + (1 - \lambda) f(x), \quad \lambda \in I, \tag{17}$$

and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, which is also odd and satisfies the following conditions:

$$f(x) > 0 \quad \text{for } x > 0, \qquad \lim_{x\to\infty} \frac{f(x)}{x} = \infty, \qquad \frac{d}{dx}\left(\frac{f(x)}{x}\right) > 0 \quad \text{for } x > 0.$$

An adequate choice for this auxiliary function is given by $f(x) = x^3$. Let

$$q(t, x, y, \lambda) = h(x, \lambda) - \lambda p(t, x, y, \lambda).$$

Clearly, the parametrized problem (16) moves (15) to the autonomous ordinary differential equation with the homogeneous Dirichlet boundary conditions

$$u'' + f(u) = 0, \quad u(0) = 0 = u(T). \tag{18}$$

In what follows, we have to consider the time-map $\tau(s)$, which is the time needed for a solution (u, v) of the planar system $u' = v$, $v' = -f(u)$ to move from the point $(0, s)$ to the point $(0, -s)$ crossing once the half-plane $u > 0$. The maximum value $m(s) > 0$ reached by such a solution u is such that $2F(m(s)) = s^2$ (with $F(x) = \int_0^x f(s)\,ds$), and, by the oddness of f the time $\tau(s)$ is the same which is taken by a solution to move from $(0, -s)$ to $(0, s)$ across the half-plane $u < 0$, with u reaching its minimum value $-m(s)$. From the energy integral associated with (18), we get

$$\tau(s) = 2\int_0^{m(s)} \frac{dx}{\sqrt{s^2 - 2F(x)}} \quad \text{for } s > 0.$$

In [47, Th. 8] (see also [51, Th. 1.3.2]) it is proved that the assumption $f(x)/x$ increasing implies that $s \mapsto \tau(s)$ is decreasing, and using also the superlinear growth condition for f, we see that the continuous map $s \mapsto \tau(s)$ satisfies $\lim_{s\to\infty} \tau(s) = 0$. Hence, an elementary analysis shows that there exists $n_0 \in \mathbb{Z}^+$ such that problem (18) has two sequences of solutions $(\widetilde{u}_n)_n$ and $(-\widetilde{u}_n)_n$ with $n \geq n_0$ with $\widetilde{u}_n$ and $-\widetilde{u}_n$ having both $n - 1$ zeros in $]0, T[$ and such that $\widetilde{u}_n'(0) = \widetilde{s}_n > 0$, with $\widetilde{s}_n \to \infty$ as $n \to \infty$. Actually, to find $\widetilde{u}_n$, we only have to find $\widetilde{s}_n$ such that $\tau(\widetilde{s}_n) = T/n$ and then define $\widetilde{u}_n$ as the solution of $u'' + f(u) = 0$ with $u(0) = 0$ and $u'(0) = \widetilde{s}_n$.

Our goal now is to prove that a similar result can be derived for problem (15). Problem (16) can easily be written as a fixed point problem in the space $X = C^1([0,T])$. Consistently with the notations of Section 2, we denote here by $\Sigma \subset C^1([0,T]) \times I$ the set of solutions (u,λ) of the boundary value problem (16). We introduce the following functional φ which is a slight modification of that considered in [9]. Let $\delta : \mathbb{R}^2 \to \mathbb{R}$ and $\eta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\delta(x,y) = \min\left\{1, \frac{1}{x^2+y^2}\right\}, \quad \eta(x) = \min\{1, \max\{-1, x\}\}.$$

Then we define the continuous functional φ on $C^1([0,T]) \times I$ by

$$\varphi(u,\lambda) = \eta(u'(0))\left|\frac{1}{\pi}\int_0^T [u'(t)^2 + u(t)q(t,u(t),u'(t),\lambda)]\delta(u(t),u'(t))\,dt\right|. \tag{19}$$

To describe the meaning of $\varphi(u,\lambda)$, suppose that (u,λ) is a solution of (16) such that

$$u(t)^2 + u'(t)^2 \geq R^2 \geq 1 \quad \text{for all } t \in [0,T]. \tag{20}$$

In this case, we get, letting $v(t) = u'(t)$,

$$\varphi(u,\lambda) = \operatorname{sign}(u'(0))\left|\frac{1}{\pi}\int_0^T \left(\frac{d}{dt}\arctan\frac{u(t)}{v(t)}\right)dt\right|.$$

By the assumptions on f, g and p, it follows that $y^2 + q(t,x,y,\lambda)x \to \infty$ as $x^2+y^2 \to \infty$, uniformly with respect to $t \in [0,T]$ and $\lambda \in [0,1]$. Therefore, there is $d > 0$ such that $y^2 + q(t,x,y,\lambda)x > 0$ for all (x,y) such that $x^2+y^2 \geq d^2$ and each $t \in [0,T]$ and $\lambda \in [0,1]$. Thus, if $R \geq (1+d^2)^{1/2}$ in (20), we obtain $v(t)u'(t) - u(t)v'(t) > 0$ for all $t \in [0,T]$. Hence assuming (20) to be satisfied for such an R, we see that the above integrands are positive. By evaluating now $|\varphi(u,\lambda)|$ for $(u,\lambda) \in \Sigma$ and (u,u') satisfying (20) with R sufficiently large (see [11] for details), we find that the functional φ defined in (19) satisfies (H_1^*) and (H_2^*) of Corollary 5 with respect to the double sequence $(c_k)_{k\in\mathbb{Z}}$ with $c_0 = 3/8$ and $c_k = (|k| - (1/2))\operatorname{sign}(k)$ for all $k \neq 0$. This leads to the following result.

Theorem 8. *Let f and p satisfy* (4) *and* (5), *respectively. Then there is $k_0 \in \mathbb{Z}^+$ such that for each $n > k_0$, the boundary value problem* (15) *has at least two solutions u_n and w_n with $u_n'(0) > 0$ and $w_n'(0) < 0$ such that*

$$\lim_{n\to\infty}\min_{t\in[0,T]}(|u_n(t)| + |u_n'(t)|) = \lim_{n\to\infty}\min_{t\in[0,T]}(|w_n(t)| + |w_n'(t)|) = \infty.$$

These solutions have the following nodal properties. For n odd, $u_n'(T) < 0$, and for n even, $u_n'(T) > 0$. Moreover, u_n has exactly $n+1$ zeros in $[0,T]$. For n odd, $w_n'(T) > 0$, and for n even, $w_n'(T) < 0$. Moreover, w_n has exactly $n+1$ zeros in $[0,T]$. All the zeros of u_n and w_n are simple and all the local maxima or minima of u_n and w_n are strict. Between any two consecutive zeros of a solution, as well *as between* 0 *and the first zero or between the last zero and* T, *there is only one critical point of the solution.*

One can consult [3] for another proof of this result.

5. The use of two functionals in continuation theorems

We describe in this section some results of [10] showing how continuation theorems using two functionals can provide sharp existence conditions for the periodic solutions of some second order differential equations with *linear growth.* We keep the notations of Section 2.

Theorem 9. *Assume that conditions* (H_0), (H_1''), (H_2) *and* (H_3'') *hold, with* Ω_0 *an open bounded neighbourhood of* Σ_0. *Assume moreover that there exist two continuous functionals* $\psi, \eta : X \times [0,1] \to \mathbb{R}$ *and real numbers* $R > 0$, $d \geq 0$ *such that* ψ *is proper on* Σ, $\psi(u,\lambda) \geq -d$, *and* $\eta(u,\lambda) \in \mathbb{Z}$ *for each* $(u,\lambda) \in \Sigma$ *with* $\|u\| \geq R$. *Suppose moreover that for each* $k \in \mathbb{Z}$ *there is a sequence* $\{c_n^{(k)}\}_n$ *with*

$$\lim_{n\to\infty} c_n^{(k)} = \infty$$

and there is an index n_k^* *such that* $\psi(u,\lambda) \neq c_n^{(k)}$ *for all* $(u,\lambda) \in \Sigma \cap \eta^{-1}(k)$ *and* $n \geq n_k^*$. *Then equation* (11) *has at least one solution.*

Proof. Assume, by contradiction, that (11) has no solution. Then, according to [20], there exists a closed unbounded connected set $\mathcal{C} \subset \Sigma$ such that $\mathcal{C} \cap (\Sigma_0 \times \{0\}) \neq \emptyset$ (see also [6, proof of Lemma 1]). Let $R_0 \geq R$ be a fixed radius such that $B(0,R_0) \supset \Sigma_0$. Consider

$$\mathcal{D}_0 := \Sigma \cap (B[0,R_0] \times [0,1]) \supset \mathcal{C} \cap (B[0,R_0] \times [0,1]),$$

a compact set (by the local compactness of Σ), so that the following constants are defined:

$$a_0 := \max\{\psi(u,\lambda) : (u,\lambda) \in \mathcal{D}_0\}, \quad K := \max\{|\eta(u,\lambda)| : (u,\lambda) \in \mathcal{D}_0\}.$$

Consider now only the sequences $\{c_n^{(k)}\}_n$ with $k \in I_K := \mathbb{Z} \cap [-K,K]$. For any $k \in I_K$, we can find an index $\widetilde{n}_k \geq n_k^*$ such that $a_0 < c_{\widetilde{n}_k}^{(k)}$. In order to simplify the notation, we set $c_k^{\#} := c_{\widetilde{n}_k}^{(k)}$. Choose now a constant $b_0 > \max\{\max\{c_k^{\#} : k \in I_K\}, d\}$. In this manner, we have $a_0 < c_k^{\#} < b_0$ for all $k \in I_K$. Finally, as a last step, we use the properness of $\psi_{|\Sigma}$ and find a radius $R_1 > R_0$ such that $\|u\| < R_1$ for all $(u,\lambda) \in \Sigma \cap \psi^{-1}([-b_0,b_0])$. By the definition of R_1, it follows that $|\psi(u,\lambda)| > b_0$ for all $(u,\lambda) \in \Sigma \setminus (B(0,R_1) \times [0,1])$, which in turn implies that $\psi(u,\lambda) > b_0$ for all $(u,\lambda) \in \Sigma$ with $\|u\| \geq R_1$ (by (h_0') and since $R_1 > R_0 \geq R$). After these preliminary choices of the constants $a_0, c_k^{\#}$ (with $k \in I_K$), b_0 and R_1, we proceed as follows.

By Whyburn's lemma (see [50]), there exists a subcontinuum $\mathcal{C}_1$ of $\mathcal{C}$ joining $\partial B(0,R_0)\times[0,1]$ to $\partial B(0,R_1)\times[0,1]$; more precisely, we have

$$\mathcal{A}:=\mathcal{C}_1\cap(\partial B(0,R_0)\times[0,1])\neq\emptyset,\quad \mathcal{B}:=\mathcal{C}_1\cap(\partial B(0,R_1)\times[0,1])\neq\emptyset,$$
$$R_0\le\|u\|\le R_1,\quad (u,\lambda)\in\mathcal{C}_1.$$

Now, by the property (h_1') for η and the choice $R_0\ge R$, it follows that there exists $\overline{k}\in\mathbb{Z}$ such that $\eta(u,\lambda)=\overline{k}$ for all $(u,\lambda)\in\mathcal{C}_1$. Indeed, η is continuous and takes only discrete values outside $B(0,R_0)\times[0,1]$, so that η is constant outside that set. In particular, $\eta(u,\lambda)=\overline{k}$ for all $(u,\lambda)\in\mathcal{A}$. On the other hand, $\mathcal{A}\subset\mathcal{D}_0$, so that $|\eta(u,\lambda)|\le K$ for all $(u,\lambda)\in\mathcal{A}$. In conclusion, $-K\le\overline{k}\le K$, i.e. $\overline{k}\in I_K$. Consider now the set $\psi(\mathcal{C}_1)$. It is a compact (since $\mathcal{C}_1$ is compact) connected subset of $\mathbb{R}$, i.e. a closed bounded interval. Thus, we set $\psi(\mathcal{C}_1)=[\alpha,\beta]$. We have

$$\alpha=\inf\psi(\mathcal{C}_1)\le\inf\psi(\mathcal{A})\le\sup\psi(\mathcal{A})\le\sup\psi(\mathcal{D}_0)=a_0,$$
$$\beta=\sup\psi(\mathcal{C}_1)\ge\sup\psi(\mathcal{B})\ge\inf\psi(\mathcal{B})>b_0.$$

Hence, $[a_0,b_0]\subset\psi(\mathcal{C}_1)$ and we can conclude that there is $(\overline{u},\overline{\lambda})\in\mathcal{C}_1$ such that $\psi(\overline{u},\overline{\lambda})=c^{\#}_{\overline{k}}$. On the other hand, we also have $\eta(\overline{u},\overline{\lambda})=\overline{k}$, and this contradicts (h_1'). □

We consider some applications of Theorem 9 to the solvability of the *periodic problem for second order equations with linear growth*

$$x''+g(x)=p(t,x,x'),\quad x(0)-x(T)=x'(0)-x'(T)=0, \tag{21}$$

with $g:\mathbb{R}\to\mathbb{R}$ and $p:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ continuous and bounded,

$$\lim_{|x|\to\infty}g(x)\operatorname{sign}(x)=\infty \tag{g_1}$$

and

$$(G_1)\ \ \forall c_1>0,\ \exists c_2>0:\quad AB>0\,\&\,|\sqrt{G(B)}-\sqrt{G(A)}|<c_1\Rightarrow|B-A|<c_2.$$

Following [15] we say that the problem (21) is *asymptotically resonant* if there is $k\in\mathbb{Z}^+$ such that for $\tau_g(c)=2\int_{h(c)}^{c}[2(G(c)-G(s))]^{-1/2}\,ds$,

$$\lim_{c\to\infty}\tau_g(c)=T/k.$$

Let us now define

$$\tau^*:=\limsup_{c\to\infty}\tau_g(c),\quad \tau_*:=\liminf_{c\to\infty}\tau_g(c)$$

and consider the interval $[\tau_*,\tau^*]\subset[0,\infty]$. With the above notations, we have the following existence theorem.

Theorem 10. *Assume* (g_1) *and* (G_1) *and suppose that problem* (21) *is not asymptotically resonant, i.e.*

$$[\tau_*, \tau^*] \neq \{T/k\} \quad (k \in \mathbb{Z}^+). \tag{22}$$

Then problem (21) *has at least one solution.*

Theorem 10 extends Theorem 1 of [15], Theorem 2.2 of [16] and the main result of [49], as far as the existence of at least one solution is concerned. Indeed, condition (22), generalizing the corresponding assumption in [15], has been already assumed in [14] and [49] together with more restrictive conditions on the function g.

The evaluation of τ_*, τ^* is particularly simple if we assume that

(g_2) *g is odd.*

In this case, $\tau_g(c) = 2T_g(c) := 4\int_0^c [2(G(c) - G(s))]^{-1/2}\, ds$, and so $\tau_* = 2T_*$ and $\tau^* = 2T^*$, where

$$T_* = \liminf_{c\to\infty} \sqrt{2}\int_0^c \frac{du}{\sqrt{G(c)-G(u)}}, \quad T^* = \limsup_{c\to\infty} \sqrt{2}\int_0^c \frac{du}{\sqrt{G(c)-G(u)}}.$$

Following [47], [18], [16], it is easy to see that

$$[T_*, T^*] \supset [\pi/\sqrt{G^*}, \pi/\sqrt{G_*}],$$

where

$$G_* := \liminf_{x\to\infty} 2G(x)/x^2, \quad G^* := \limsup_{x\to\infty} 2G(x)/x^2$$

and we use the convention $1/\infty = 0$, $1/0^+ = \infty$. Set also

$$g_* := \liminf_{x\to\infty} g(x)/x, \quad g_* := \limsup_{x\to\infty} g(x)/x$$

and recall that, according to the generalized L'Hospital rule, $g_* \le G_* \le G^* \le g^*$. Finally, denote by $\lambda_k = (2\pi k/T)^2, k \in \mathbb{Z}^+$, the kth positive eigenvalue of the differential operator $x \mapsto -x''$ in the space of T-periodic functions.

Corollary 6. *Assume* (g_1), (g_2) *and* (G_1). *Then problem* (21) *has at least one solution provided that*

$$[G_*, G^*] \neq \{\lambda_k\} \quad (k \in \mathbb{Z}^+). \tag{23}$$

To compare this result with previous theorems, we remark that the condition assumed in [39] was $[g_*, g^*] \cap \{\lambda_k : k \in \mathbb{Z}_+\} = \emptyset$ and observe that $[g_*, g^*] \supset [G_*, G^*]$, so that the result in [39] implies $[G_*, G^*] \cap \{\lambda_k : k \in \mathbb{Z}_+\} = \emptyset$ and thus (23) follows. On the other hand, in [16], one can find conditions like (23), but for g globally lipschitzian in $\mathbb{R}$, an assumption which is not required here. Finally, to apply the theorem of [49], one should replace (G_1) with the more restrictive assumption (G_2). Note also that (23) is always satisfied if $G^* = \infty$, with no condition on G_*.

As a further application of Theorem 10, we have the following

Corollary 7. *Assume* (g_1) *and* (G_1) *and suppose that*

$$\tau^* > T. \tag{24}$$

Then problem (21) *has at least one solution.*

Corollary 7 can be compared with previous results of Opial [46] and [18]. Actually, under condition (g_1) it was proved in [46] that if

$$\liminf_{c\to-\infty} T_g(c) + \liminf_{c\to\infty} T_g(c) > T, \tag{25}$$

then problem (21) is solvable. Subsequently, in [18] Opial's assumption was improved to

$$\liminf_{c\to-\infty} T_g(c) + \limsup_{c\to\infty} T_g(c) > T. \tag{26}$$

Clearly, (24) contains all the above conditions as particular cases. In our situation, however, the extra hypothesis (G_1) has to be required.

A comparison of all these results in the odd case shows that (under (g_2)) condition (25) reads as $T_* > T/2$, while (26) becomes $T^* + T_* > T$, and finally (24) gives $T^* > T/2$. Note that this last condition is satisfied whenever

$$\liminf_{x\to\infty} 2G(x)/x^2 < (2\pi/T)^2.$$

With the techniques of this section, one can also investigate some cases of asymmetric behaviour of g at $-\infty$ and ∞. We refer to [42, 10, 43] for the details.

6. Continuation theorems in metric ANRs

In a survey paper on applications of the topology of function spaces [52], Schauder observed the possibility of extending the Leray–Schauder theory to more general linear and even nonlinear spaces: *Anderseits kann ich eine ähnliche Theorie des Abbildungsgrades auch dann entwickeln, wenn es sich um allgemeinere Räume handelt, etwa um lineare metrische Räume, in welchen es beliebig kleine, konvexe Umgebungen der Null gibt*[1]. *Auch nichtlineare Räumen könnten betrachtet werden*[2].

The announced joint paper was never written because of the known tragic circumstances, but Leray [36], in a paper dedicated "à la mémoire du profond

[1]Eine gemeinsame Arbeit von Herrn Leray und mir über verschiedene topologische Invarianten in möglichst allgemeinen Räumen wird erscheinen.

[2]Etwa metrische Räume, deren Umgebungen, z.B., den Umbegungen in Banachschen Räumen homöomorph sind.

mathématicien polonais Jules Schauder, victime des massacres de 1940", described those extensions. In particular, if F is a continuous mapping of a compact space C into itself, and C is a retract of an open subset G of a locally convex space X, Leray proposed to define the corresponding *fixed point index* through the topological degree of the mapping $\varrho \circ F$, where ϱ is the retraction of G onto C. This approach was independently developed by Granas [26] and by Browder [1] to provide a theory of the index of fixed points for completely continuous mappings defined on the closure of an open subset of an absolute neighbourhood retract (ANR), i.e. a metric space which is homeomorphic to a neighbourhood retract of a Banach space. See [45] for a recent survey.

The continuation theorem of Section 2 has been extended to this setting by A. Capietto [2]. Let X be a metric ANR, $\mathcal{O} \subset X \times I$ an open set and $F : X \times I \to X$ be a completely continuous map, so that

$$\Sigma^* = \{(x, \lambda) \in \mathcal{O} : x = F(x, \lambda)\}$$

is locally compact. When Σ_0^* is bounded (and hence compact) and $\Sigma_0^* \subset \mathcal{O}_0$, the fixed point index $i_X[F(\cdot, 0), \mathcal{O}_0]$ is well defined by

$$i_X[F(\cdot, 0), \mathcal{O}_0] := i_X[F(\cdot, 0), \mathcal{U}],$$

for any open bounded set $\mathcal{U}$ such that $\Sigma_0^* \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{O}_0$. The considerations of Section 3 can be extended as follows.

Theorem 11. *Assume that Σ_0^* is bounded, $\Sigma_0^* \subset \mathcal{O}_0$, and*

$$i_X[F(\cdot, 0), \mathcal{O}_0] \neq 0.$$

Assume moreover that there exists a continuous function $\varphi : X \times I \to \mathbb{R}$ and real numbers c_-, c_+ such that the following conditions hold:

1. *φ is proper on Σ^*.*
2. *$c_- < \inf\{\varphi(x, 0) : x \in \Sigma_0^*\} \le \sup\{\varphi(x, 0) : x \in \Sigma_0^*\} < c_+$.*
3. *$\varphi(x, \lambda) \notin \{c_-, c_+\}$ for each $x \in \mathcal{O}_\lambda \cap \Sigma_\lambda^*$, $\lambda \in]0, 1[$, and*
   $\varphi(x, \lambda) \notin [c_-, c_+]$ for each $x \in (\partial\mathcal{O})_\lambda \cap \Sigma_\lambda^*$, $\lambda \in]0, 1[$.

Then Σ^* contains a continuum $\mathcal{C}$ along which λ takes all values in I.

The proof of this result follows the same lines as that of Theorem 6, with the Leray–Schauder degree replaced by the fixed point index.

One can now state a consequence of Theorem 11 which is a slight variant of Corollary 3. Let $\Sigma = \{(x, \lambda) \in X \times I : x = F(x, \lambda)\}$.

Corollary 8. *Let $A \subset X$. Assume that Σ_0 is bounded and*

$$i_X[F(\cdot, 0), \Sigma_0] \neq 0.$$

Assume moreover that there exists a continuous function $\psi : X \times I \to \mathbb{R}$ *satisfying the following conditions:*

(h_1') *There exists a bounded subset* $C \subset X$ *such that, for each* $\lambda \in [0,1]$, *one has* $\Sigma_\lambda \setminus A \subset C$.

(h_2') *For every* $n \in \mathbb{Z}_+$ *there exists a bounded subset* $C_n \subset X$ *such that for every* $\lambda \in [0,1]$ *one has* $A \cap \Sigma_\lambda \cap \psi^{-1}(\{n\}) \subset C_n$.

Then Σ *contains a continuum* $\mathcal{C}$ *along which* λ *takes all values in* I.

We refer to [2] for the proof of this result and applications to periodic solutions of ordinary differential equations on manifolds. In particular, Corollary 8 can be applied to prove an interesting result of Furi et Pera [21] on the existence of periodic solutions for the spherical pendulum equation. See also [22, 23, 19, 38].

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