LONG TIME BEHAVIOR OF FLOWS
MOVING BY MEAN CURVATURE, II

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Dedicated to Olga A. Ladyzhenskaya

In our papers [7]–[9] we studied the evolution of a nonparametric surface whose boundary is fixed and interior points move with normal speed equal to the mean curvature. In the classical setting the problem is to find a smooth function $u = u(x,t)$, $(x,t) \in \overline{\Omega} \times [0, \infty)$, satisfying the following conditions:

$$(1) \qquad \frac{u_t}{\sqrt{1 + |Du|^2}} = H[u] \quad \text{in } \Omega \times [0, \infty),$$

$$(2) \qquad u = \varphi \qquad \text{on } \partial\Omega \times [0, \infty),$$

$$(3) \qquad u = u_0 \qquad \text{in } \Omega \times \{0\}.$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $Du = (D_1 u, \ldots, D_n u)$ is the gradient of u, $D_i u = \partial u / \partial x_i$, $u_t = \partial u / \partial t$, $H[u] = \operatorname{div}(Du / \sqrt{1 + |Du|^2})$ is the mean curvature of the graph of u, $\varphi = \varphi(x)$ and $u_0 = u_0(x)$ are given smooth functions. It is well known that, in general, the classical solvability of (1)–(3) fails if Ω is not mean convex. The latter means that the mean curvature $H_{\partial\Omega}(x)$ of $\partial\Omega$ with respect to the inward normal at $x \in \partial\Omega$ is nonnegative. Similarly, if Ω is not mean convex then the classical solution may not exist for the corresponding stationary problem

$$(4) \qquad H[\Phi] = 0 \quad \text{in } \Omega,$$

$$(5) \qquad \Phi = \varphi \quad \text{on } \partial\Omega.$$

It is also well known that if the domain Ω is mean convex then, under suitable compatibility conditions, there exists a global in t classical solution of (1)–(3) which is unique and tends to the solution Φ of (4), (5) as $t \to \infty$; see [2].

In this paper, as in [7]–[9], we continue to study the problem (1)–(3) in domains which are not assumed to be mean convex. More specifically, let

$$\partial_- \Omega = \{x \in \partial\Omega : H_{\partial\Omega}(x) < 0\},$$
$$\partial_+ \Omega = \{x \in \partial\Omega : H_{\partial\Omega}(x) > 0\},$$
$$\partial_0 \Omega = \{x \in \partial\Omega : H_{\partial\Omega}(x) = 0\},$$

and assume that $\partial_- \Omega \neq \emptyset$. Note that $\partial_+ \Omega \neq \emptyset$ because Ω is bounded. It was shown in [7]–[9] that while the problem (1)–(3) may not have a classical solution, it always has a generalized solution (under some hypotheses and in a sense to be explained shortly). The behavior of this generalized solution was studied in [7]–[9] under the assumption that the stationary problem (4), (5) has a classical solution. This assumption allowed us, in particular, to establish that the constructed generalized solution to (1)–(3) becomes smooth in $\overline{\Omega}$ for sufficiently large t.

The purpose of this paper is to investigate the behavior of solutions to (1)–(3) without assuming that the stationary problem (4), (5) has a classical solution.

It will be useful to review briefly here some of the relevant results from [7]–[9]. In [7] the problem (1)–(3) was studied under the assumption that $\varphi \equiv 0$. Then $\Phi \equiv 0$ is a classical solution of the stationary problem (4), (5). In [8] and [9] it was assumed that there exists a solution $\Phi \in C^2(\overline{\Omega})$ to the problem (4), (5) and $u_0 - \Phi \in C_0^\infty(\Omega)$. The generalized solution of (1)–(3) was defined as a limit

$$(6) \qquad\qquad u(x, t) = \lim_{\varepsilon \searrow 0} u^\varepsilon(x, t)$$

of solutions to regularized problems

$$(7) \qquad \frac{u_t^\varepsilon}{\sqrt{1 + |Du^\varepsilon|^2}} = H[u^\varepsilon] + \varepsilon \Delta u^\varepsilon \quad \text{in } \Omega \times [0, \infty),$$

$$(8) \qquad\qquad u^\varepsilon = \varphi \qquad\qquad \text{on } \partial\Omega \times [0, \infty),$$

$$(9) \qquad\qquad u^\varepsilon = u_0 \qquad\qquad \text{in } \Omega \times \{0\},$$

where $\Delta = \sum_{i=1}^n D_i D_i$ is the Laplace operator.

For the regularized problems we proved estimates for u^ε uniform with respect to ε and this allowed us to pass to the limit for some sequence $\varepsilon_k \to 0$ and to prove that the limit function u satisfies conditions (1) and (3). However, the boundary condition (2) is guaranteed to be satisfied for all t only on the "good" part of the boundary, namely, on $\partial_+ \Omega$. On the rest of $\partial\Omega$ the boundary condition is satisfied only in some weak sense (to be described later). Simple examples show

that the solutions to (1)–(3) may actually "detach" from the boundary data on the "bad" part of the boundary.

The proof of existence of a generalized solution does not depend on the existence of a classical solution Φ to (4), (5). The only place where we essentially used the existence of $\Phi \in C^2(\overline{\Omega})$ was the proof of the following amazing fact: for $t \geq T$, where T depends on u_0, the solution u becomes "classical", that is, $u \in C^\infty(\overline{\Omega} \times [T, \infty))$ and

(10) $$u = \varphi \quad \text{on } \partial\Omega \times [T, \infty).$$

Furthermore, the availability of Φ enabled us to investigate the asymptotic behavior of u for large t. In particular, we proved that

$$u(\cdot, t) \to \Phi \quad \text{as } t \to \infty$$

uniformly in $\overline{\Omega}$.

However, if the problem (4), (5) has no classical solutions, the question regarding the behavior of generalized solutions to the problem (1)–(3) as $t \to \infty$ remained open.

In this paper we answer this question and prove that the solution u constructed in [7], [8] has a limit as $t \to \infty$ and the limiting function coincides with the generalized solution Φ of (4), (5). The latter means that Φ is a minimizer of the area functional

(11) $$A(v) = \int_\Omega \sqrt{1 + |Dv|^2}\, dx + \int_{\partial\Omega} |v - \varphi|\, d\mathcal{H}_{n-1}$$

in $W^{1,1}(\Omega)$; here $\mathcal{H}_{n-1}$ denotes the Hausdorff measure of dimension $n-1$ on $\partial\Omega$.

Such a solution of (4), (5) always exists and is unique [1]. The function Φ is analytic and satisfies the minimal surface equation (4) in Ω. But it may not satisfy the condition $\Phi = \varphi$ on a part of $\partial_-\Omega$. Geometrically, in this case, the surface $S \subset \mathbb{R}^{n+1}$ minimizing the area functional (11) is the union of graph(u) and a part of the vertical cylinder $\partial\Omega \times \mathbb{R}$ bounded by $\{(x, \varphi(x)) : x \in \partial\Omega\}$ and $\{(x, \Phi(x)) : x \in \partial\Omega\}$. Thus, the boundary of S coincides with the prescribed graph $\{(x, \varphi(x)) : x \in \partial\Omega\}$. The properties of the function Φ near points $x \in \partial_-\Omega$ where $\Phi(x) \neq \varphi(x)$ were studied by L. Simon [10] and F. H. Lin [5]. It was proved that near such points the restriction of Φ to $\partial_-\Omega$ is smooth and Φ is Hölder continuous with exponent $1/2$ in $\Omega \cup \partial_-\Omega$, provided the data are sufficiently smooth. The derivatives of $\Phi(y)$ in tangential directions are bounded while the normal derivative becomes unbounded as $y \to x \in \partial_-\Omega$ if $\varphi(x) \neq \Phi(x)$.

Analogous results were obtained by one of the authors of this paper for solutions of (1)–(3); see [11], [12]. In general, the behavior of solutions, in both elliptic and parabolic cases, in the neighborhood of $\partial_0\Omega$ can be rather

pathological [1]. Because we are not imposing here any special conditions on $H_{\partial\Omega}$ we cannot use the results of [10], [5], [11], [12]. We only note that for the case where $\partial_0\Omega = \emptyset$ one of the present authors [13] studied the behavior of solutions near the so-called contact set

$\{x \in \partial_-\Omega : \Phi(x) = \varphi(x);$ in any neighborhood of x on $\partial_-\Omega$ there exist

both types of points, that is, points where $\Phi \neq \varphi$ and points where $\Phi = \varphi\}$

and proved that the trace of Φ on $\partial_-\Omega$ is C^1 in a neighborhood of the contact set. This result is optimal near the contact set. Furthermore, this result enables us to prove uniqueness of generalized solutions to the problem (1)–(3). However, since the uniqueness does not imply the convergence of $u(\cdot, t)$ as $t \to \infty$, we do not use it here.

As pointed out earlier, we impose no conditions on the mean curvature of $\partial\Omega$. We assume only that all data are C^∞ smooth and the compatibility condition of order zero,

$$(12) \qquad\qquad u_0 = \varphi \quad \text{on } \partial\Omega,$$

holds.

Below, unless stated otherwise, we denote by $M_0, M_1, \ldots$ and c various constants depending on initial and boundary data.

THEOREM 1. *For any $\varepsilon \in (0, 1]$ there exists a unique solution $u^\varepsilon \in C^\infty(\Omega \times [0, \infty)) \cap C(\overline\Omega \times [0, \infty)) \cap C^\infty(\overline\Omega \times (0, \infty))$ to the problem (7)–(9). The following estimates hold uniformly with respect to ε:*

$$(13) \qquad\qquad \sup_{\Omega \times (0,\infty)} |u^\varepsilon| \leq M_0,$$

$$(14) \qquad\qquad \sup_{\Omega \times (0,\infty)} |u_t^\varepsilon| \leq M_1,$$

$$(15) \qquad\qquad \sup_{(0,\infty)} \int_\Omega [\sqrt{1 + |Du^\varepsilon|^2} + \varepsilon|Du^\varepsilon|^2]\, dx \leq M_2,$$

$$(16) \qquad\qquad \sup_{\Omega' \times (0,\infty)} |D^\alpha u^\varepsilon| \leq c(\Omega', \alpha) \quad \forall \Omega' \Subset \Omega,\ \forall \alpha.$$

PROOF. As already mentioned Theorem 1 can be proved in the same way as the corresponding results in [7], [8]. We use this opportunity to make some remarks concerning the proof of (14). Note first that one cannot apply the maximum principle to u_t^ε because the compatibility condition (12) by itself does not guarantee the continuity of u_t^ε at $\partial\Omega \times \{0\}$. The same remark applies as well to the situation considered in [8]. Though we assumed there one more compatibility condition, namely, $H[u_0] = 0$ in the vicinity of $\partial\Omega$, it is not sufficient to deduce continuity of u_t^ε from the ε-problem (7)–(9).

To overcome this noncompatibility let us "improve" the boundary condition (8) by replacing $\varphi(x)$ by

$$\varphi^\delta(x,t) = \varphi(x) + \delta\psi(t/\delta)L^\varepsilon[u_0](x), \quad \delta \in (0,1),$$

where $L^\varepsilon[u_0] = \sqrt{1 + |Du_0|^2}\big(H[u_0] + \varepsilon\Delta u_0\big)$ and ψ is a smooth nonnegative function defined for nonnegative arguments so that

$$\psi(0) = 0, \quad \psi'(0) = 1 = \sup|\psi'|, \quad \operatorname{supp}\psi = [0,2].$$

Let $v^{\varepsilon,\delta}$ be a solution to the problem (7)–(9) with φ replaced by φ^δ. For the corresponding $u^{\varepsilon,\delta}$ the compatibility conditions of orders zero and one are fulfilled. The derivative $w = u_t^{\varepsilon,\delta}$ belongs to $C(\overline{\Omega} \times [0,\infty)) \cap C^\infty(\Omega \times (0,\infty))$ and satisfies the parabolic equation

$$(17) \qquad w_t = a_{ij}(x,t)D_iD_jw + b_i(x,t)D_iw \quad \text{in } \Omega \times [0,\infty)$$

with coefficients a_{ij}, b_i smooth in $\Omega \times [0,\infty)$. Then, by the maximum principle we have

$$(18) \qquad \sup_{\Omega \times (0,\infty)} |u_t^{\varepsilon,\delta}| \le \max\{\sup_{\partial\Omega \times (0,\infty)} |u_t^{\varepsilon,\delta}|; \sup_\Omega |L^\varepsilon u_0|\} = \sup_\Omega |L^\varepsilon u_0| \le M_1.$$

Now we have the desired estimate for $u_t^{\varepsilon,\delta}$. On the other hand, we can apply the maximum principle to the function $w = u^\varepsilon - u^{\varepsilon,\delta}$, which satisfies an equation of the form (17) and it is continuous in $\overline{\Omega} \times [0,\infty)$. This gives us the estimate

$$\sup_{\Omega \times (0,\infty)} |u^\varepsilon - u^{\varepsilon,\delta}| \le \delta \sup_\Omega |L^\varepsilon u_0| \le \delta M_1 \to 0 \quad \text{as } \delta \to 0.$$

This and (18) imply the estimate (14).

REMARK 1. The inequalities (16) actually hold for any $\Omega' \Subset \Omega \cup \partial_+\Omega$. To prove this one can use local barrier functions, for example, as in [4], and obtain gradient estimates in the interior of $\partial_+\Omega$.

We introduce the following notation: for $p \in \mathbb{R}^n$ put

$$(19) \qquad \begin{aligned} F(p) &= \sqrt{1 + |p|^2}, & F^\varepsilon(p) &= F(p) + \frac{\varepsilon}{2}|p|^2, \\ F_i(p) &= \frac{\partial F(p)}{\partial p_i} = \frac{p_i}{F(p)}, & F_i^\varepsilon(p) &= F_i(p) + \varepsilon p_i. \end{aligned}$$

By $\|\cdot\|_{q,\Omega}$ (resp. $(\|\cdot\|_{q,\partial\Omega})$ we denote the $L^q(\Omega)$ (resp. $L^q(\partial\Omega)$) norm.

Below we will use some known properies of the equation

$$(20) \qquad H[v] = f \quad \text{in } \Omega$$

with the mean curvature operator H. In order to make the presentation reasonably self-contained, we recall here some of these properties; for more details see [6], [1], [10].

Let $f \in L^{\infty}(\Omega)$ and let $v \in W^{1,1}(\Omega)$ be a weak solution of (20), that is,

$$\int_{\Omega} [F_i(Dv)\eta_{x_i} + f\eta]\, dx = 0 \quad \forall \eta \in C_0^{\infty}(\Omega).$$

The *conormal derivative* Tv on $\partial\Omega$ of a solution v is defined as

$$(21) \qquad \langle Tv, \eta \rangle := \int_{\partial\Omega} Tv\eta\, d\mathcal{H}_{n-1} = \int_{\Omega} [F_i(Dv)\eta_{x_i} + f\eta]\, dx \quad \forall \eta \in L^1(\partial\Omega).$$

Here, on the right hand side a $W^{1,1}(\Omega)$ extension of η to Ω (denoted again by η) is taken. Clearly,

$$(22) \qquad |\langle Tv, \eta \rangle| \leq \|D\eta\|_{1,\Omega} + \|f\|_{\infty,\Omega} \|\eta\|_{1,\Omega} \leq c\|\eta\|_{1,\partial\Omega}.$$

The $L^{\infty}(\partial\Omega)$ norm of Tv is the infimum of constants c in (22) corresponding to various extensions. It is known [1] that for any $\delta > 0$ there exists an extension for which

$$\|D\eta\|_{1,\Omega} \leq (1+\delta)\|\eta\|_{1,\partial\Omega}, \quad \|\eta\|_{1,\Omega} \leq \delta\|\eta\|_{1,\partial\Omega}.$$

Hence $Tv \in L^{\infty}(\partial\Omega)$ and

$$\|Tv\|_{\infty,\partial\Omega} \leq \inf_{\delta>0}(1 + \varepsilon + \delta\|f\|_{\infty,\Omega}) = 1.$$

DEFINITION 1. A function v is a *generalized solution* to the Dirichlet problem

$$(23) \qquad\qquad H[v] = f \quad \text{in } \Omega; \quad v = \varphi \quad \text{in } \partial\Omega$$

if $v \in W^{1,1}(\Omega)$ is a weak solution of equation (20) and

$$(24) \qquad\qquad -Tv \in \operatorname{supp}(v - \varphi) \quad \text{a.e. on } \partial\Omega.$$

PROPOSITION 1. *For a function $v \in W^{1,1}(\Omega)$ to be a generalized solution of the problem* (23) *is equivalent to the fact that v is a minimizer of the functional*

$$(25) \qquad E(w) = \int_{\Omega} [F(Dw) + fw]\, dx + \int_{\partial\Omega} |\varphi - w|\, d\mathcal{H}_{n-1}$$

on $W^{1,1}(\Omega)$.

PROOF. 1°. Suppose v is a generalized solution to (23). Denote by $I(w)$ and $g(w)$, respectively, the volume and boundary integrals in (25).

It follows from (21) and the convexity of F that for any $w \in W^{1,1}(\Omega)$,

$$(26) \quad I(w) \geq I(v) + \int_{\Omega} [F_i(Dv)(w_{x_i} - v_{x_i}) + f(w - v)] = I(v) + \langle Tv, w - v \rangle.$$

We also have the following identity:

$$
(27) \qquad g(w) - g(v) = \int_{\partial\Omega \cap \{v=\varphi\}} |w - \varphi| \, d\mathcal{H}_{n-1}
$$
$$
+ \int_{\partial\Omega \cap \{v>\varphi\}} [|w - \varphi| - v + \varphi] \, d\mathcal{H}_{n-1}
$$
$$
+ \int_{\partial\Omega \cap \{v<\varphi\}} [|w - \varphi| + v - \varphi] \, d\mathcal{H}_{n-1}.
$$

Furthermore, by Definition 1 we have

$$
\langle Tv, w - v \rangle = \int_{\partial\Omega \cap \{v=\varphi\}} Tv(w - \varphi) \, d\mathcal{H}_{n-1} + \int_{\partial\Omega \cap \{v>\varphi\}} (v - w) \, d\mathcal{H}_{n-1}
$$
$$
+ \int_{\partial\Omega \cap \{v<\varphi\}} (w - v) \, d\mathcal{H}_{n-1}.
$$

From (26) and (27) we get the inequality

$$
(28) \qquad E(w) \geq E(v) + \int_{\partial\Omega \cap \{v=\varphi\}} [|w - \varphi| + Tv(w - \varphi)] \, d\mathcal{H}_{n-1}
$$
$$
+ \int_{\partial\Omega \cap \{v>\varphi\}} [|w - \varphi| + \varphi - w] \, d\mathcal{H}_{n-1}
$$
$$
+ \int_{\partial\Omega \cap \{v<\varphi\}} [|w - \varphi| + w - \varphi] \, d\mathcal{H}_{n-1}.
$$

Because each boundary integral in (28) is nonnegative we conclude that

$$
E(w) \geq E(v) \quad \text{for any } w \in W^{1,1}(\Omega).
$$

2°. Assume now that v is a minimizer of E on the space $W^{1,1}(\Omega)$. Because v is a weak solution of the equation (20) (see [1]), we only have to check (24). This step is standard; see A. Lichnewsky and R. Temam [3]. In the inequality $E(w) \geq E(v)$ take $w = v + \lambda\eta$, with $\lambda > 0$ ($\lambda < 0$) and $\eta \in W^{1,1}(\Omega)$. Passing to the limit as $\lambda \searrow 0$ ($\lambda \nearrow 0$) we obtain

$$
\int_{\Omega} [F_i(Dv)\eta_{x_i} + f\eta] \, dx + \lim_{\lambda \searrow 0} \int_{\partial\Omega} \frac{|v + \lambda\eta - \varphi| - |v - \varphi|}{\lambda} \, d\mathcal{H}_{n-1} = 0.
$$

The first integral here is $\langle Tv, \eta \rangle$, while the second one is equal to $\int_{\partial\Omega} \sigma\eta \, d\mathcal{H}_{n-1}$ with $\sigma \in \text{supp}(v - \varphi)$. Thus we arrive at the equation $Tv + \sigma = 0$ which coincides with (24). □

We will not discuss here the solvability of (23). Various aspects of this problem have been studied in detail by M. Miranda, E. Giusti, N. Trudinger, and others.

PROPOSITION 2. *A generalized solution to* (23) *is unique, possibly, up to an additive constant.*

PROOF. If v and w are two solutions then it follows from (21) and (24) that

$$(29) \quad \int_\Omega [F_i(Dv) - F_i(Dw)](v_{x_i} - w_{x_i})\, dx = \int_{\partial\Omega} [(Tv - Tw)(v - w)]\, d\mathcal{H}_{n-1} \leq 0.$$

On the other hand, by convexity of F the inequality $[F_i(p) - F_i(q)](p_i - q_i) \geq 0$ holds for any $p, q \in \mathbb{R}^n$ with equality for $p = q$ only.

This and (29) imply that

$$[F_i(Dv) - F_i(Dw)](v_{x_i} - w_{x_i}) = 0 \quad \text{a.e. in } \Omega$$

and therefore $Dv = Dw$ a.e. in Ω.

REMARK 2. Later on we will be considering only solutions to (23) for which $v = \varphi$ on $\partial_+\Omega$. By Proposition 2 such a solution is unique.

We now return to solutions u^ε of (7)–(9). It follows from Theorem 1 and Remark 1 that there exists a function

$$u \in C^\infty((\Omega \cup \partial_+\Omega) \times (0, \infty)) \cap L^\infty([0, \infty); W^{1,1}(\Omega))$$

such that for some sequence $\varepsilon_k \searrow 0$ the functions u^{ε_k} and all their derivatives converge uniformly to u and to its respective derivatives on compact subsets of $(\Omega \cup \partial_+\Omega) \times (0, \infty)$. Also, $u^{\varepsilon_k} \to u$ in $L^q(\Omega \times (0, T))$ for all $T > 0$ and $q < \infty$. In addition, u satisfies inequalities analogous to (13)–(16) with the same constants. Moreover, it satisfies (1), (3),

$$(30) \qquad\qquad u = \varphi \quad \text{on } \partial_+\Omega \times (0, \infty),$$

and for any $t \in (0, \infty)$,

$$(31) \qquad\qquad -Tu \in \text{supp}(u - \varphi) \quad \text{a.e. on } \partial\Omega.$$

For the proof of the last statement see [12]. In other words, for each $t \in (0, \infty)$, u is a generalized solution of (23) with

$$f = f^{(t)} = \frac{u_t(\cdot, t)}{\sqrt{1 + |Du(\cdot, t)|^2}}.$$

PROPOSITION 3. *The following estimate holds:*

$$(32) \qquad\qquad \int_0^\infty \int_\Omega \frac{|u_t|^2}{\sqrt{1 + |Du|^2}}\, dx\, dt \leq M_3.$$

PROOF. Multiply the equation (7) by u_t^ε and integrate the result over $\Omega \times [t_1, t_2]$, $0 < t_1 < t_2 < \infty$. Then integrate by parts taking into account vanishing of u_t^ε on $\partial\Omega \times (0, \infty)$. Finally, using the relation

$$\frac{dF^\varepsilon(Du^\varepsilon)}{dt} = F_i^\varepsilon D_i u_t^\varepsilon,$$

we arrive at

$$\int_{t_1}^{t_2} dt \int_\Omega \frac{|u_t^\varepsilon|^2}{\sqrt{1 + |Du^\varepsilon|^2}} \, dx + \int_\Omega F^\varepsilon(Du^\varepsilon) \, dx \Big|_{t_1}^{t_2} = 0.$$

In the first term here write the integral $\int_\Omega$ as the sum of integrals over Ω_δ and $\Omega \setminus \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$ for some small positive δ. Then, using (14), (15), we obtain

$$\int_{t_1}^{t_2} dt \int_{\Omega_\delta} \frac{|u_t^\varepsilon|^2}{\sqrt{1 + |Du^\varepsilon|^2}} \, dx \leq M_3 + O(\delta) \cdot (t_2 - t_1).$$

Passing to the limit, first as $\varepsilon \to 0$ and then as $\delta \to 0$, we get the estimate

$$\int_{t_1}^{t_2} dt \int_\Omega \frac{|u_t|^2}{\sqrt{1 + |Du|^2}} \, dx \leq M_3.$$

Since $0 < t_1 < t_2 < \infty$ are arbitrary, this estimate gives (32).   □

COROLLARY 1. *There exists a sequence $t_k \to \infty$ such that*

(33) $u(\cdot, t_k) \to \overline{u} \quad as \ k \to \infty,$

where the convergence is in $L^q(\Omega)$, $q < \infty$. In addition, the derivatives of u in x of any order tend uniformly on compact subsets of $\Omega \cup \partial_+\Omega$ to the corresponding derivatives of $\overline{u}$ and

(34) $H[\overline{u}] = 0 \quad in \ \Omega, \quad \overline{u} = \varphi \quad on \ \partial_+\Omega.$

THEOREM 2. *$\overline{u}$ is the unique generalized solution of (25) with $f = 0$. The convergence in (33) holds for any sequence $t_k \to \infty$.*

PROOF. To prove the first statement it is sufficient to show that $\overline{u}$ is a minimizer of the area functional (11). We begin with the observation that $u_k := u(\cdot, t_k)$ is a minimizer of E_k, that is,

(35) $E_k(w) \geq E_k(u_k) \quad \forall w \in W^{1,1}(\Omega).$

Here E_k is the functional (25) with $f = f^{(t_k)}$ defined just before the Proposition 3. Because of (32) we can choose t_k in such a way that

$$(36) \qquad r(t_k) := \int_\Omega \frac{|u_t|^2}{\sqrt{1 + |Du|^2}}\, dx \bigg|^{t=t_k} \to 0 \quad \text{as } k \to \infty.$$

Then

$$(37) \qquad \|f^{(t_k)}\|_{1,\Omega} \le (r(t_k)|\Omega|)^{1/2} \to 0 \quad \text{as } k \to \infty$$

and, consequently, by the estimate (13) (applied to u) we obtain

$$(38) \qquad \int_\Omega f^{(t_k)} u_k\, dx \to 0 \quad \text{as } k \to \infty.$$

The functional (11) is lower semicontinuous with respect to convergence in $L^1(\Omega)$. This fact and (33), (38), (35), and (37) imply

$$(39) \qquad A(\overline{u}) \le \liminf_{k \to \infty} A(u_k) = \liminf_{k \to \infty}\left[E_k(u_k) - \int_\Omega f^{(t_k)} u_k\, dx \right]$$
$$= \liminf_{k \to \infty} E_k(u_k) \le \liminf_{k \to \infty} E_k(w) = A(w)$$

for any $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$.

Let now $w \in W^{1,1}(\Omega)$ and $M = \sup_{\partial\Omega} \varphi$. Put

$$v = \begin{cases} w & \text{if } |w| < M, \\ M & \text{if } w \ge M, \\ -M & \text{if } w \le -M. \end{cases}$$

It is easy to see that $A(\overline{u}) \le A(v) \le A(w)$. Thus $\overline{u}$ is a minimizer of A.

To prove the second statement in Theorem 2, consider an arbitrary sequence $t_{k'} \to \infty$. Let $\{k''\}$ be a subsequence of $\{k'\}$ such that $u_{k''} \to v$ where the convergence is the same as described in Corollary 1. We want to prove that $A(v) \le A(w)$ for all $w \in W^{1,1}(\Omega)$. Then the desired result will follow from the uniqueness of the minimizer $v = \overline{u}$ (see Remark 2). It is tempting to try to get (36) for k'. Fix some small $\delta > 0$ and consider

$$r_\delta(t) = \int_{\Omega_\delta} \frac{|u_t|^2}{\sqrt{1 + |Du|^2}}\, dx \bigg|^t = r(t) + O(\delta)$$

with Ω_δ defined in the proof of Proposition 3.

Because of uniform boundedness of $|Du|$ and $|Du_t|$ on Ω_δ the estimate $|dr_\delta(t)/dt| \le c(\delta)$ holds uniformly in t. From this and (32) one immediately concludes that $r_\delta(t) \to 0$ as $t \to \infty$. Indeed, if $r_\delta(t_{k'}) \ge \alpha > 0$ for some $t_{k'} \to \infty$, then $r_\delta(t) \ge \alpha/2$ in the $\alpha/(2c\delta)$-neighborhood of each $t_{k'}$. But this contradicts (32).

Finally, we note that we also have the inequality

$$A(v) \leq A(w) + O(\delta) \quad \forall w \in W^{1,1}(\Omega) \cap L^{\infty}(\Omega).$$

Arguing as above and using the fact that δ is an arbitrary small positive number, we conclude that $A(v) \leq A(w)$ for all $w \in W^{1,1}(\Omega)$ and, consequently, $v = \overline{u}$. $\qquad\square$

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