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EXISTENCE AND MULTIPLICITY RESULTS FOR WAVE EQUATIONS WITH TIME-INDEPENDENT NONLINEARITY

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Abstract. We shall study the existence of time-periodic solutions for a semilinear wave equation with a given time-independent nonlinear perturbation and small forcing. Since the distribution of eigenvalues of the linear part varies with the period, the solvability of the problem depends essentially on the frequency. The main idea of this paper is to consider the situation where the period is not prescribed and hence treated as a parameter. The description of the distribution of eigenvalues as a function of the period enables us to show that under certain conditions the interaction between the nonlinearity and the spectrum of the wave operator induces multiple solutions. Our basic new result states that the autonomous equation admits at least two nontrivial solutions (free vibrations) for a restricted (but infinite) set of periods such that the nonlinearity interacts with one simple eigenvalue. As a corollary we prove that the semilinear wave equation with time-independent nonlinearity and small forcing admits an infinite sequence of pairs of periodic solutions with corresponding period tending to zero. The results are obtained via generalized topological degree theory.

1. Introduction

We shall consider semilinear wave equation of the form

$$\text{(1.1)}\qquad \begin{cases} \partial_t^2 v - \alpha_0^2 \partial_x^2 v - g(x,v) = h(x,t), \\ v(0,t) = v(L,t) = 0 \qquad\qquad (x \in \,]0,L[,\ t \in \mathbb{R}). \end{cases}$$

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We assume that the forcing term h in (1.1) is τ-periodic in t, τ being a parameter, and look for τ-periodic solutions for equation (1.1).

The linear wave equation models a string with lenght L with constant density and fixed endpoints. The unknown function v is the displacement of the string from equilibrium, x is the distance along string's equilibrium position, and t is time. The constant $\alpha_0^2 > 0$ is related to the tension of the string.

After rescaling we obtain the equation

$$\begin{cases} \omega^2\partial_t^2 u - \widehat{\alpha}_0^2\partial_x^2 u - \widehat{g}(x,u) = h_\omega(x,t), \\ u(0,t) = u(\pi,t) = 0 & (x \in \,]0,\pi[,\ t \in \mathbb{R}), \\ u(x,t) = u(x,t+2\pi), \end{cases} \tag{1.2}$$

where $\omega = 2\pi/\tau$, $u(x,t) = v((L/\pi)x, \omega^{-1}t)$, $h_\omega(x,t) = h((L\pi)x, \omega^{-1}t)$, $\widehat{\alpha}_0 = \alpha_0\pi/L$ and $\widehat{g}(x,u) = g((L/\pi)x,u)$. The "normalized frequency" $\omega = 2\pi/\tau$, has a major role in our considerations. Throughout this paper we assume that $\omega \in \alpha_0\mathbb{Q}_+$. Without loss of generality we may assume that $L = \pi$ and hence we can simplify the notations by denoting $\widehat{g}$ again by g, $\widehat{\alpha}_0$ by α_0 and $h_\omega(x,t) = h(x,\omega^{-1}t)$.

We shall study the existence of *weak solutions* of (1.2), i.e. solutions of the operator equation

$$L_\omega u - N(u) = h_\omega, \quad u \in D(L_\omega) \tag{1.3}$$

in $H = L_2(\Omega;\mathbb{R})$ with $\Omega = \,]0,\pi[\, \times \,]0,2\pi[$, where N is the Nemytskiĭ operator generated by g and $L_\omega\colon D(L_\omega) \subset H \to H$ is the abstract realization of the wave operator $\omega^2\partial_t^2 - \alpha_0^2\partial_x^2$. The operator L_ω is self adjoint and its kernel is infinite dimensional. Hence the Leray–Schauder degree is not applicaple and we shall employ the extension introduced in [7].

Another method used in this paper is the reduction to suitable invariant subspaces. Indeed, if the wave operator L_ω is reduced by a closed linear subspace V and $N(V) \subset V$, any solution of the reduced equation

$$L_\omega|_V u - N|_V(u) = h_\omega, \quad u \in D(L_\omega) \cap V, \quad h_\omega \in V \tag{1.4}$$

is also a solution for the original operator equation (1.3). If the reduction yields $\dim \operatorname{Ker} L_\omega|_V < \infty$, then the use of Leray–Schauder degree is possible. The method of reductions to suitable subspaces was already used by O. Vejvoda (see [22]) and J. M. Coron ([13], see also [8], [11]). The same idea was employed earlier in the study of periodic solutions for ordinary differential equations (see for instance [20]).

In order to study of the solvability of (1.3) we have to deal with the question how the nonlinearity interacts with the spectrum of the linear part. The distribution of eigenvalues of L_ω changes dramatically by considering different periods and also by reduction to suitable invariant subspaces. It is relevant to consider the problem with different frequencies, because there is no a-priori reason for any prescribed period for the external forces. In a special case the forcing term does not depend on time (for instance $h = 0$). Then $h_\omega = h$ is independent of $\omega = 2\pi/\tau$ and the frequency ω is a free parameter. For any $h = h(x)$ the solution may be time-independent and we possibly get a solution for the ODE

$$\begin{cases} -\alpha_0^2 u'' - g(x,u) = h(x), \\ u(0) = u(\pi) = 0 \qquad (x \in \,]0,\pi[). \end{cases} \tag{1.5}$$

Semilinear wave equations with prescribed period τ and $\alpha_0 = 1$ are widely studied by different methods, e.g. degree theory, variational methods and saddle point reduction. We mention here the papers [1]–[3], [6], [12], [14], [16], [17], [19] and the references therein. The existence of multiple periodic solutions for (1.1) with special nonlinearity is examined both numerically and theoretically in connection of modeling suspension bridges, see [18], [21], for instance.

In this paper the nonlinearity g is a Caratheodory function satisfying the condition

$$a \le \frac{g(x,s)}{s} \le b \quad \text{for all } s \ne 0. \tag{1.6}$$

The classical condition for the nonlinearity $g = g(s)$ assumes the boundedness of the derivative $g'(s)$, that is, $a \le g'(s) \le b$ for all $s \in \mathbb{R}$ (see [12]). The relaxation of the regularity of g leads to the condition, where the difference quotient of g remains between a and b (see [15]). Our condition (1.6) can be viewed as a further generalization. Indeed, if g is differentiable and (1.6) holds, then the derivative g' may be unbounded.

We say that the nonlinearity g *interacts* with the spectrum of L_ω, if $[a,b] \cap \sigma(L_\omega) \ne \emptyset$. The interaction, which clearly depens on the frequency, is crucial for the existence of multiple solutions. We recall the following two standard results (cf. [7]).

Theorem 1.1. *Assume that* (1.6) *holds,* $\omega \in \alpha_0\mathbb{Q}_+$ *and* $a < b$ *are such that* $[a,b] \cap \sigma(L_\omega) = \emptyset$. *Then the equation*

$$\begin{cases} \omega^2 \partial_t^2 u - \alpha_0^2 \partial_x^2 u - g(x,u) = 0, \\ u(0,t) = u(\pi,t) = 0 \qquad (x \in \,]0,\pi[,\ t \in \mathbb{R}), \\ u(x,t) = u(x,t+2\pi), \end{cases} \tag{1.7}$$

has only the trivial solution $u = 0$.

Theorem 1.2. *Assume that* $0 < a < b$, $g(x,\cdot)$ *is nondecreasing and the condition* (1.6) *holds. If* $\omega \in \alpha_0\mathbb{Q}_+$ *and* $[a,b]$ *are such that* $[a,b] \cap \sigma(L_\omega) = \emptyset$, *then the equation*

$$
\begin{cases}
\omega^2\partial_t^2 u - \alpha_0^2\partial_x^2 u - g(x,u) = f, \\
u(0,t) = u(\pi,t) = 0 & (x \in \,]0,\pi[, \ t \in \mathbb{R}), \\
u(x,t) = u(x,t+2\pi),
\end{cases}
\tag{1.8}
$$

admits at least one weak solution for any $f \in L_2(\Omega;\mathbb{R})$.

We shall prove the existence of multiple solutions for (1.8) whenever certain interaction between the nonlinearity and the spectrum occurs. We prove that the homogeneous equation (1.7) admits at least two nontrivial solutions for a certain infinite set of periods such that the nonlinearity interacts with one simple eigenvalue. We show that the conclusion remains valid for any sufficiently small forcing term h which is even in t. As a corollary we obtain the result that the equation (1.1) with small forcing admits an infinite sequence of pairs of periodic solutions with corresponding period tending to zero. The results are obtained using a generalization of the Leray–Schauder topological degree and reduction to invariant subspaces.

The paper is organized as follows. In Section 2 we recall the formulation of problem (1.2) in Hilbert space $L_2(\Omega;\mathbb{R})$ and give the properties of topological degree needed in the sequel. Section 3 is devoted to the study of the distribution of eigenvalues as a function of the frequency. We obtain a characterization for the spectrum of the wave operator on any given compact interval as a function of ω. In Section 4 we prove a priori estimates for the homotopy equation. These estimates improve the previous results and we show that they hold uniformly in ω on a certain set of frequencies. Section 5 contains new existence and multiplicity results. In all theorems the solvability is obtained for an infinite set of frequencies.

2. Prerequisites

We recall first the basic properties of the linear wave operator. Denote $H = L_2(\Omega;\mathbb{R})$, where $\Omega = \,]0,\pi[\times]0,2\pi[$. We shall use the notations $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\|\cdot\|$ for the inner product and norm in any real Hilbert space. A possible subscript V indicates that we are dealing with a subspace V of H. Define

$$
\phi_{j,k}(x,t) = \begin{cases}
\dfrac{\sqrt{2}}{\pi}\sin(jx)\sin(kt) & j \in \mathbb{Z}_+, \ k \in \mathbb{Z}_+, \\
\dfrac{1}{\pi}\sin(jx) & j \in \mathbb{Z}_+, \ k = 0, \\
\dfrac{\sqrt{2}}{\pi}\sin(jx)\cos(kt) & j \in \mathbb{Z}_+, \ -k \in \mathbb{Z}_+.
\end{cases}
$$

Then the set $\{\phi_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z}\}$ forms an orthonormal basis in H and each $u \in H$ has a representation

$$u = \sum_{j\in\mathbb{Z}_+,k\in\mathbb{Z}} \langle u, \phi_{j,k}\rangle \phi_{j,k}.$$

The wave operator $\omega^2\partial_t^2 - \alpha_0^2\partial_x^2$ with periodic Dirichlet boundary conditions has in H the abstract realization

$$L_\omega u = \sum_{j,k} \lambda_{j,k}^\omega \langle u, \phi_{j,k}\rangle \phi_{j,k}$$

with $\lambda_{j,k}^\omega = \alpha_0^2 j^2 - \omega^2 k^2$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, and with the domain

$$D(L_\omega) = \bigg\{ u \in L_2(\Omega) \ \bigg|\ \sum_{j,k} |\alpha_0^2 j^2 - \omega^2 k^2|^2 |\langle u, \phi_{j,k}\rangle|^2 < \infty \bigg\}.$$

Thus the othonormal basis $\{\phi_{j,k}\}$ consists of the eigenvectors of the operator L_ω. We shall always assume that $\omega \in \alpha_0\mathbb{Q}_+$, i.e. α_0/ω is rational. Otherwise we encounter the hard problem of "small divisors", see [6], [11], for instance. For any $\omega \in \alpha_0\mathbb{Q}_+$ the operator L_ω is self adjoint and has a compact partial inverse K_ω on $\operatorname{Im} L_\omega$. For more details on related abstract operators and their properties we refer to [5].

Let the function $(x, s) \to g(x, s)$ from $[0, \pi] \times \mathbb{R}$ to $\mathbb{R}$ be measurable in $x \in [0, \pi]$ for each $s \in \mathbb{R}$ and continuous in s for almost all $x \in [0, \pi]$. Moreover, we assume that g satisfies the linear growth condition

$$|g(x, s)| \le c_0|s| + k_0(x), \tag{2.1}$$

where $c_0 \ge 0$ and $k_0 \in L_2(]0, \pi[)$. Denote by $N\colon H \to H$ the bounded continuous Nemytskiĭ operator generated by g, that is,

$$N(u)(x, t) = g(x, u(x, t)) \quad \text{for all } u \in H,\ (x, t) \in \Omega.$$

For any $f \in H$ a function $u \in H$ is called a weak solution of the problem

$$\begin{cases} \omega^2\partial_t^2 u - \alpha^2\partial_x^2 u - g(x, u) = f(x, t), \\ u(0, t) = u(\pi, t) = 0 & (x \in]0, \pi[,\ t \in \mathbb{R}), \\ u(x, t) = u(x, t + 2\pi), \end{cases} \tag{2.2}$$

if and only if

$$\int_\Omega u(\omega^2\psi_{tt} - \alpha^2\psi_{xx})\, dx\, dt = \int_\Omega (g(x, u) + f)\psi\, dx\, dt \tag{2.3}$$

for all $\psi \in C^2$, where C^2 stands for twice continuously differentiable functions $\psi\colon [0, \pi] \times \mathbb{R} \to \mathbb{R}$ such that $\psi(0, t) = \psi(\pi, t) = 0$ for all $t \in \mathbb{R}$ and $\psi(x, \cdot\,)$ is 2π-periodic. It is easy to see that $u \in H$ satisfies (2.2) if and only if

$$u \in D(L_\omega) \quad \text{and} \quad L_\omega u - N(u) = f.$$

Let us consider more closely the spectrum $\sigma(L_\omega)$ of the operator L_ω. For any $\omega \in \alpha_0\mathbb{Q}_+$ it is easy to see that $\mathrm{Ker} L_\omega$ is infinite dimensional. Indeed, if $\omega = \alpha_0 p/q \in \alpha_0\mathbb{Q}_+$, where p and q are relative primes, then $\lambda_{j,k} = 0$ if and only if $j = lq$ and $|k| = lp$, $l \in \mathbb{Z}_+$. It is easy to see that L_ω has a real pure point spectrum $\sigma(L_\omega) = \{\lambda^\omega_{jk} \mid \lambda^\omega_{j,k} = \alpha_0^2 j^2 - \omega^2 k^2,\ j \in \mathbb{Z}_+,\ k \in \mathbb{Z}\}$. The eigenvalues are isolated and all nonzero eigenvalues have finite geometric multiplicities. Note that the spectrum is unbounded from below and from above. Clearly the eigenvalues $\lambda = 0$ and $\lambda = \lambda^\omega_{j,0}$, $j \in \mathbb{Z}_+$, being constant in ω, play a special role. In order to deal with these exceptional "constant eigenvalues" it is possible to look for solutions of (1.3) in suitable invariant subspaces. Indeed, if the operator L_ω is reduced by a closed subspace $V \subset H$, then $\sigma(L_\omega) = \sigma(L_\omega|_V) \cup \sigma(L_\omega|_{V^\perp})$, implying that the spectrum of L_ω in V is thinner than the spectrum of L_ω in H. The main problem is to find natural conditions ensuring $N(V) \subset V$.

Let $\omega \in \alpha_0\mathbb{Q}_+$ be fixed and denote by P and $Q = I - P$ the orthogonal projections to $\mathrm{Ker}\, L_\omega$ and $\mathrm{Im}\, L_\omega = (\mathrm{Ker}\, L_\omega)^\perp$, respectively. The equation

$$L_\omega u - N(u) = f, \quad u \in D(L_\omega),$$

can be written equivalently as

$$Q(u + K_\omega Q N(u)) + PN(u) = (K_\omega Q - P)f, \quad u \in H. \tag{2.4}$$

Since $\dim \mathrm{Ker}\, L_\omega = \infty$ the Leray–Schauder degree cannot be applied to tackle the solvability of equation (2.4). This difficulty can be dealt with if N is monotone or of class (S_+) or more generally pseudomonotone by using the extension of degree constructed in [7]. Indeed, recall first some basic definitions of mappings of monotone type. In all definitions we assume that N is bounded in the sense that it takes any bounded set into a bounded set, and demicontinuous, i.e. $u_k \to u$ (norm convergence) implies $N(u_k) \rightharpoonup N(u)$ (weak convergence). The mapping N is *monotone* if

$$\langle N(u) - N(v), u - v\rangle \geq 0 \quad \text{for all } u, v \in H.$$

Any monotone map is a special case of *pseudomonotone* maps, that is, the conditions $u_k \rightharpoonup u$ and $\limsup\langle N(u_k), u_k - u\rangle \leq 0$ imply that $N(u_k) \rightharpoonup N(u)$ and $\langle N(u_k), u_k\rangle \to \langle N(u), u\rangle$. If there exists a constant $\nu > 0$ such that for all $u, v \in H$

$$\langle N(u) - N(v), u - v\rangle \geq \nu\|u - v\|^2,$$

N is *strongly monotone*. Such mappings belong to the wider *class* (S_+) defined by the property: $u_k \rightharpoonup u$ and $\limsup\langle N(u_k), u_k - u\rangle \leq 0$ imply that $u_k \to u$. Any map of the form $I - C\colon H \to H$, where C is compact is said to be of *Leray–Schauder type*. Note that in Hilbert space any map of Leray–Schauder type belongs to the class (S_+). Moreover, the class (S_+) is contained in the class of pseudomonotone maps. The class (S_+) is stable under a large class of perturbations, for instance, compact or pseudomonotone perturbations are allowed. If N is of class (S_+), then the topological degree constructed in [7] is, as a special case, well-defined for mappings of the type

$$Q(I + K_\omega QN) + PN: H \to H.$$

This degree theory is a unique extension of the classical Leray–Schauder degree. It is single-valued and has the usual properties of degree, such as additivity of domains and invariance under homotopies. Let the corresponding degree function be d_H and simplify our notations by setting

$$\deg_H(L_\omega - N, G, f) \equiv d_H(Q(I - K_\omega QN) + PN, G, (K_\omega Q - P)f)$$

for any open bounded set $G \subset H$ such that $f \notin (L_\omega - N)(\partial G \cap D(L_\omega))$. Similar, degree theory exists for mappings of the type $L_\omega + N$, where $N: H \to H$ is bounded, demicontinuous and of class (S_+) (see [7] and also the remarks in Section 6). By a *reference map* we refer to any linear injection $L_\omega - N_0$ for which the degree is well-defined. For any reference map we have (see [7])

$$\deg_H(L_\omega - N_0, G, f) \neq 0 \quad \text{for any } f \in (L_\omega - N_0)(D(L) \cap G).$$

Typical reference maps are

$$L_\omega - P, \quad L_\omega - \lambda I, \quad \text{where } \lambda \notin \sigma(L_\omega),$$

and

$$L_\omega - \lambda I + P_\lambda, \quad \text{where } \lambda \in \sigma(L_\omega),\ \lambda \neq 0,$$

and P_λ is the orthogonal projection onto $\mathrm{Ker}(L_\omega - \lambda I)$. Only the last one is actually used in this paper. For different types of reference maps, their application and the calculation of the value of degree we refer to [4], [7] and [10]. The following continuation theorem is a direct consequence of the basic properties of the degree.

Theorem 2.1. *Assume that $L_\omega - N_0$ is a reference map, $N: H \to H$ is bounded and demicontinuous, $G \subset H$ is open and bounded, $f_0 \in (L_\omega - N_0)(G \cap D(L_\omega))$ and $f \in H$. Assume that*

$$L_\omega u - \mu N(u) - (1-\mu)N_0(u) \neq \mu f + (1-\mu) f_0 \tag{2.5}$$

for all $u \in \partial G \cap D(L_\omega)$ and $0 \leq \mu \leq 1$. If N is of class (S_+), then

$$\deg_H(L_\omega - N, G, f) = \deg_H(L_\omega - N_0, G, f_0) \neq 0$$

and the equation

$$L_\omega u - N(u) = f, \quad u \in D(L_\omega)$$

admits at least one solution $u \in G \cap D(L_\omega)$. *A solution exists also when* N *is pseudomonotone,* G *is convex and* (2.5) *holds.*

3. On the distribution of eigenvalues

Assume that the nonlinearity g is time-independent and satisfies, for almost all $x \in [0,\pi]$, the condition

$$a \le \frac{g(x,s)}{s} \le b \quad \text{for all } s \ne 0.$$

Essential to our considerations is the set of interacting eigenvalues $[a,b] \cap \sigma(L_\omega)$. By Theorem 1.1 the existence of nontrivial solution for homogeneous equation (1.7) requires some interaction between the nonlinearity and the spectrum of the linear part.

In this section we prove some some new results on the spectrum of L_ω in a compact interval $[a,b]$ as a function of the frequency ω. Note that there are two different types of eigenvalues. Indeed, the set

$$\sigma_{\text{const}} := \{0\} \cup \{\alpha_0^2 j^2 \mid j \in \mathbb{Z}_+\} \subset \sigma(L_\omega)$$

remains constant in $\omega \in \alpha_0\mathbb{Q}_+$. Moreover, the eigenvalue $\lambda = 0$ is special in the sense that the corresponding eigenspace $\operatorname{Ker}(L_\omega)$ is always infinite dimensional but depends on ω. Denote $\sigma_\omega = \sigma(L_\omega) \setminus \sigma_{\text{const}}$, i.e.

$$\sigma_\omega = \{\lambda_{j,k}^\omega = \alpha_0^2 j^2 - \omega^2 k^2 \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z},\ k \ne 0,\ \lambda_{j,k}^\omega \ne 0\}.$$

All the results of this section characterize the set $\sigma_\omega \cap [a,b]$, where $a < b$. We show first that the set of frequencies such that $\sigma_\omega \cap [a,b] \ne \emptyset$ is dense in $\alpha_0\mathbb{Q}_+$ and for all sufficienly small frequencies $\sigma_\omega \cap [a,b] \ne \emptyset$. On the other hand, in Lemma 3.2 we prove that there exists an infinite number of frequencies such that $\sigma_\omega \cap [a,b] = \emptyset$. The third result (Lemma 3.3) is quite technical, it states, roughly speaking, that for a given interval $[0,R]$ and given integer $n \ge 0$ we can find frequencies such that $\sigma_\omega \cap [0,R]$ contains exactly n eigenvalues. As a corollary we obtain the result used later on that for a certain set of frequencies the set $\sigma_\omega \cap [a,b]$, where $0 \notin [a,b]$, contains exactly one eigenvalue with multiplicity two.

In our basic existence result (Theorem 5.1) the nonlinearity interacts with one simple eigenvalue. Due to the asymmetry of the spectrum $\sigma(L_\omega)$ we face two cases. First, if $0 < a < b$ the set $\sigma_{\text{const}} \cap [a,b]$ may be empty or nonempty. In both situations we obtain existence results. For instance, if $\sigma_{\text{const}} \cap [a,b] = \emptyset$, we shall prove (using Corollary 3.4 and reduction to suitable subspace V) that there exist infinitely many frequencies $\omega \in \alpha_0\mathbb{Q}_+$ such that $[a,b] \cap \sigma(L_\omega|_V)$ contains exactly one simple eigenvalue. Secondly, if $a < b < 0$, then always $\sigma_{\text{const}} \cap [a,b] = \emptyset$ and by similar argument we conclude that there exist infinitely many frequencies

$\omega \in \alpha_0\mathbb{Q}_+$ such that $[a,b]$ contains exactly one simple eigenvalue of the reduced operator $L_\omega|_V$.

Our first result shows that for almost all frequencies there is some interaction between σ_ω and g.

Lemma 3.1. *Let $[a,b]$ be a given compact interval in $\mathbb{R}$. Then we have:*

(a) *The set $\{\omega \in \alpha_0\mathbb{Q}_+ | [a,b] \cap \sigma_\omega \neq \emptyset\}$ is dense in $\mathbb{R}_+$.*
(b) *There exists a limit value $\widetilde{\omega} > 0$ such that $[a,b] \cap \sigma_\omega \neq \emptyset$ for all $\omega \in \alpha_0\mathbb{Q}_+$ with $\omega \leq \widetilde{\omega}$.*

Proof. (a) Let $r > 0$ be a given real number. Then there exist a sequence $(r_l)_{l=1}^\infty$ of rational numbers such that

$$r_l = \frac{p_l}{q_l} \neq r, \quad p_l \to \infty, \quad q_l \to \infty \quad \text{and} \quad r_l \to r.$$

Take

$$s_l \in \left[\sqrt{\frac{p_l^2}{q_l^2} - \frac{b}{\alpha_0^2 q_l^2}}, \sqrt{\frac{p_l^2}{q_l^2} - \frac{a}{\alpha_0^2 q_l^2}}\right] \cap \mathbb{Q}_+$$

such that $s_l \neq p_l/q_l$ and denote $\omega_l = \alpha_0 s_l$. Then it is easy to see that $\omega_l/\alpha_0 \to r$ and $\lambda^{\omega_l}_{p_l,q_l} \in [a,b] \setminus \{0\}$ for all $l \in \mathbb{Z}_+$. Thus r is in the closure of the set $\{\omega/\alpha_0 \in \mathbb{Q}_+ \mid [a,b] \cap \sigma_\omega \neq \emptyset\}$ completing the proof.

(b) Denote $j_0 = \min\{j \in \mathbb{Z}_+ \mid j^2 > b/\alpha_0^2\}$ and $c_0^2 = j_0^2 - b/\alpha_0^2$ and $d_0^2 = j_0^2 - a/\alpha_0^2$. If $\widetilde{\omega} \leq \alpha_0(d_0 - c_0)$, then it is easy to see that for any $\omega \leq \widetilde{\omega}$, $k_0\omega/\alpha_0 \in [c_0, d_0]$ for some $k_0 \in \mathbb{Z}_+$. But then $\lambda^\omega_{j_0,k_0} \in [a,b]$ and the proof is complete. $\square$

It is easy to see that the set $N = \{\omega \in \alpha_0\mathbb{Q}_+ \mid [a,b] \cap \sigma_\omega \neq \emptyset\}$ is a union of intervals (in $\alpha_0\mathbb{Q}_+$). Note also that in case $a, b \notin \alpha_0^2\mathbb{Q}$ the set N is relatively open in $\alpha_0\mathbb{Q}_+$. Indeed, if $\omega \in N$ and $a, b \notin \alpha_0^2\mathbb{Q}$ then there exist $j, k \in \mathbb{Z}_+$ such that $a < \lambda^\omega_{j,k} < b$ and clearly $a < \lambda^{\omega'}_{j,k} < b$ for all ω' close enough to ω.

On the other hand the interaction does not occur for certain sufficiently large values of ω. This is verified in our next result.

Lemma 3.2. *Let $[a,b]$ be a given compact interval in $\mathbb{R}$, and $\omega_1 = \alpha_0 r$, where $r = p/q \in \mathbb{Q}_+$. Denote $\omega_n = \alpha_0 r n$, where $n \in \mathbb{Z}_+$. If $\alpha_0\omega_n \geq q\max\{|a|,|b|\}$ then $[a,b] \cap \sigma_{\omega_n} = \emptyset$.*

Proof. For any $\lambda^{\omega_n}_{jk} \in \sigma_{\omega_n}$ we have

$$|\lambda^{\omega_n}_{jk}| = \alpha_0^2|j - nrk|(j + nrk) = \alpha_0^2\frac{|jq - pnk|}{q}(j + nrk) \geq \alpha_0^2\frac{(1 + nr)}{q}.$$

Hence, if $\alpha_0\omega_n \geq q\max\{|a|,|b|\}$, then $|\lambda^{\omega_n}_{jk}| > \max\{|a|,|b|\}$ and the conclusion follows. $\square$

The existence of multiple solutions requires some interaction between the nonlinearity and the spectrum of the linear part. The following result is crucial.

Lemma 3.3. *Let $r_0 \in \mathbb{R}$, $r_0 > 0$ be given. Take any rational sequence $(s_i/q_i)_{i=1}^{\infty}$ such that $s_i/q_i \neq r_0$, $s_i/q_i \to r_0$, $s_i \to \infty$ and $q_i \to \infty$. Denote $p_i = s_i q_i - 1$ and $\omega_i = \alpha_0 p_i/q_i$. Then $\omega_i \to \infty$ and, for all $l \in \mathbb{Z}_+$,*

$$\lambda^{\omega_i}_{ls_i,l} \to 2r_0\alpha_0^2 l^2 \quad \text{as } i \to \infty. \tag{3.1}$$

Moreover, for any $n \in \mathbb{Z}_+$, there exists $i_n \in \mathbb{Z}_+$ such that for all $i \geq i_n$ the first n positive eigenvalues in σ_{ω_i} are

$$0 < \lambda^{\omega_i}_{s_i,\pm 1} < \lambda^{\omega_i}_{2s_i,\pm 2} < \ldots < \lambda^{\omega_i}_{ns_i,\pm n}. \tag{3.2}$$

Proof. By the fact that $p_i/q_i = s_i - 1/q_i$ we get

$$\begin{aligned}\lambda^{\omega_i}_{ls_i,l} &= \alpha_0^2 l^2\left(s_i - \frac{p_i}{q_i}\right)\left(s_i + \frac{p_i}{q_i}\right)\\ &= \alpha_0^2 l^2\left(s_i - s_i + \frac{1}{q_i}\right)\left(s_i + s_i + \frac{1}{q_i}\right) = \alpha_0^2 l^2\left(2\frac{s_i}{q_i} + \frac{1}{q_i^2}\right) \to 2\alpha_0^2 r_0 l^2\end{aligned}$$

and consequently (3.1) holds.

The main part of the proof is to show (3.2). Indeed, let $n \in \mathbb{Z}_+$ be given and denote $c_n = 2r_0\alpha_0^2(n+1/2)^2$. It is sufficient to show that for all sufficiently large values of i

$$[0, c_n] \cap \sigma_{\omega_i} = \{\lambda^{\omega_i}_{s_i,\pm 1}, \lambda^{\omega_i}_{2s_i,\pm 2}, \ldots, \lambda^{\omega_i}_{ns_i,\pm n}\}. \tag{3.3}$$

Indeed, assume the contrary. Then we can find a subsequence of $(i)_{i=1}^{\infty}$ (denoted for simplicity again by $(i)_{i=1}^{\infty}$) such that

$$0 < \lambda^{\omega_i}_{j_i,k_i} \leq c_n, \quad j_i, k_i \in \mathbb{Z}_+,\ (j_i, k_i) \neq (ls_i, l),\ l = 1, \ldots, n.$$

Since $p_i/q_i^2 = s_i/q_i - 1/q_i^2 \to r_0$ and $j_i q_i - p_i k_i \geq 1$ we have

$$\lambda^{\omega_i}_{j_i,k_i} = \alpha_0^2 \frac{1}{q_i^2}(j_i q_i - p_i k_i)(j_i q_i + p_i k_i) \geq \alpha_0^2 \frac{p_i}{q_i^2} k_i$$

and thus (k_i) is bounded. Taking a subsequence if necessary we can assume that $k_i \equiv k_0 \in \mathbb{Z}_+$. Hence

$$\lambda^{\omega_i}_{j_i,k_i} = \lambda^{\omega_i}_{j_i,k_0} = \alpha_0^2\left(j_i - \frac{p_i}{q_i}k_0\right)\left(j_i + \frac{p_i}{q_i}k_0\right),$$

where $j_i + (p_i/q_i)k_0 \to \infty$ and therefore

$$j_i - \frac{p_i}{q_i}k_0 = j_i - s_i k_0 + \frac{k_0}{q_i} \to 0$$

implying $j_i = k_0 s_i$ for all sufficiently large values of i. Hence we get

$$\lim_{i\to\infty} \lambda^{\omega_i}_{j_i,k_i} = \lim_{i\to\infty} \lambda^{\omega_i}_{k_0 s_i,k_0} = \lim_{i\to\infty} \alpha_0^2 \frac{k_0^2}{q_i}\left(2s_i - \frac{1}{q_i}\right) = 2r_0\alpha_0^2 k_0^2.$$

Consequently $2\alpha_0^2 r_0 k_0^2 \le c_n = 2\alpha_0^2 r_0 (n+1/2)^2$ implying $k_0 \le n$, a contradiction completing the proof. $\square$

Note that analogous result as above holds if $r_0 < 0$. In fact the result is better, because all negative eigenvalues of L_ω belong to σ_ω. Take any rational sequence $(s_i/q_i)_{i=1}^\infty$ such that $s_i/q_i \ne -r_0$, $s_i/q_i \to -r_0$, $s_i \to \infty$ and $q_i \to \infty$. Denote $p_i = s_i q_i + 1$ and $\omega_i = \alpha_0 p_i/q_i$. Then $\omega_i \to \infty$ and

$$\lambda^{\omega_i}_{ls_i,l} \to 2r_0\alpha_0^2 l^2, \quad l = 1, 2, \ldots, \text{ as } i \to \infty.$$

Moreover, for any $n \in \mathbb{Z}_+$ there exists $i_n \in \mathbb{Z}_+$ such that for all $i \ge i_n$ the n greatest negative eigenvalues of L_{ω_i} are

$$\lambda^{\omega_i}_{ns_i,\pm n} < \lambda^{\omega_i}_{(n-1)s_i,\pm(n-1)} < \ldots < \lambda^{\omega_i}_{s_i,\pm 1} < 0.$$

As a direct consequence of the above lemma we obtain the following.

Corollary 3.4. *Let $[a,b]$ be any compact interval such that $0 \notin [a,b]$. Then there exist infinitely many frequencies $\omega \in \alpha_0\mathbb{Q}_+$ such that $[a,b] \cap \sigma_\omega$ contains exactly one eigenvalue having multiplicity two.*

Proof. Assume $0 < a < b$ and choose $c \in \mathbb{R}$ such that $a < c < b$ and $4c > b$. Denote $r_0 = c(2\alpha_0^2)^{-1}$ and apply Lemma 3.3. Consequently there exist sequences $(\lambda^{\omega_i}_{s_i,\pm1})_{i=1}^\infty$ and $(\lambda^{\omega_i}_{2s_i,\pm2})_{i=1}^\infty$ such that, for all sufficiently large values of i, $0 < a < \lambda^{\omega_i}_{s_i,\pm1} < b < \lambda^{\omega_i}_{2s_i,\pm2}$ and $[a,b] \cap \sigma_{\omega_i} = \{\lambda^{\omega_i}_{s_i,\pm1}\}$. If $a < b < 0$ we can apply the corresponding result on negative side. $\square$

4. A priori estimates

In this section we shall prove a-priori estimates for a solution set of a certain homotopy equation which will be used in Section 5. The idea of the proofs is adopted from [9] and [15]. However, we shall sharpen and improve the estimates and show explicitly their dependence on the frequency $\omega = 2\pi/\tau$.

Let $g\colon [0,\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the Caratheodory conditions. Moreover, we impose on g the following conditions: for almost all $x \in [0,\pi]$

$$a \le \frac{g(x,s)}{s} \le b \quad \text{for all } s \ne 0, \tag{4.1}$$

where $b > a > 0$ are constants. We assume that there exist further constants $\overline{a}$, $\overline{b}$ and $d > c > 0$ such that for almost all $x \in [0,\pi]$

$$\begin{cases} a \le \dfrac{g(x,s)}{s} \le \overline{a} & \text{for all } |s| > d, \\[2mm] \overline{b} \le \dfrac{g(x,s)}{s} \le b & \text{for all } 0 < |s| \le c, \end{cases} \tag{4.2}$$

and

$$a < \overline{a} < \overline{b} < b \quad \text{with } 4\overline{b} > b. \tag{4.3}$$

In view of applications we notice that if $g = g(u)$ satifies the condition (4.1) and

$$\liminf_{s\to 0} \frac{g(s)}{s} = b, \quad \limsup_{|s|\to\infty} \frac{g(s)}{s} = a,$$

then conditions (4.2) and (4.3) trivially hold. As earlier, we denote $\sigma_{\mathrm{const}} = \{0\} \cup \{\alpha_0^2 j^2 \mid j \in \mathbb{Z}_+\}$ and $\sigma_\omega = \sigma(L_\omega) \setminus \sigma_{\mathrm{const}}$. Assume that $[a,b] \cap \sigma_{\mathrm{const}} = \emptyset$, that is, the set of interacting eigenvalues in H is

$$\sigma(L_\omega) \cap [a,b] = \sigma_\omega \cap [a,b]. \tag{4.4}$$

By Corollary 3.4 there exist infinitely many frequencies $\omega \in \alpha_0\mathbb{Q}_+$ such that

$$\sigma_\omega \cap [a,b] = \{\lambda^\omega_{s,1}, \lambda^\omega_{s,-1}\} \subset [a_1, b_1], \tag{4.5}$$

where a_1, b_1 are some fixed constants such that $\overline{a} < a_1 < b_1 < \overline{b}$ and $4b_1 > b$ (the notation $\lambda^\omega_{s,\pm 1}$ refers to Lemma 3.3). Moreover, by Lemma 3.3 we can take ω in such a way that there exist real constants $\underline{\lambda}$ and $\overline{\lambda}$ such that

$$\max\{\lambda^\omega_{j,k} \mid \lambda^\omega_{j,k} < \lambda^\omega_{s,1}\} \le \underline{\lambda} < a, \quad \min\{\lambda^\omega_{j,k} \mid \lambda^\omega_{j,k} > \lambda^\omega_{s,1}\} \ge \overline{\lambda} > b. \tag{4.6}$$

From now on we assume that $\omega \in \alpha_0\mathbb{Q}_+$ is fixed such that the conditions (4.4)–(4.6) hold. Denote $\lambda(\omega) = \lambda^\omega_{s,1} = \lambda^\omega_{s,-1}$. Crucial for the estimates is the decomposition $H = H^\omega_1 \oplus H^\omega_2 \oplus H^\omega_3$ defined by the projections

$$\begin{aligned}
P^\omega_1 u &= \sum_{\lambda^\omega_{jk} < \lambda(\omega)} \langle u, \phi_{j,k}\rangle \phi_{j,k}, \\
P^\omega_2 u &= \sum_{\lambda^\omega_{jk} = \lambda(\omega)} \langle u, \phi_{j,k}\rangle \phi_{j,k}, \\
P^\omega_3 u &= \sum_{\lambda^\omega_{jk} > \lambda(\omega)} \langle u, \phi_{j,k}\rangle \phi_{j,k},
\end{aligned}$$

where $H^\omega_i = \operatorname{Im} P^\omega_i$, $i = 1,2,3$. In the sequel we shall use the following immediate consequences of the decomposition:

$$\begin{aligned}
\langle L_\omega u - \lambda(\omega)u, P^\omega_1 u\rangle &\le (\underline{\lambda} - \lambda(\omega))\|P^\omega_1 u\|^2, \\
\langle L_\omega u - \lambda(\omega)u, P^\omega_3 u\rangle &\ge (\overline{\lambda} - \lambda(\omega))\|P^\omega_3 u\|^2,
\end{aligned}$$

which are valid for any $u \in D(L_\omega)$. Let $y_\omega \in H^\omega_2 = \operatorname{Ker}(L_\omega - \lambda(\omega)I)$ be a fixed such that $\|y_\omega\| = 1$. Denote

$$\begin{aligned}
F^\omega_\mu(u) &= L_\omega u - \lambda(\omega)u - \mu(N(u) - \lambda(\omega)u) + (1-\mu)P^\omega_2 u, \quad 0 \le \mu \le 1, \\
K_\omega &= \{u \in D(L_\omega) \mid F^\omega_\mu(u) = (1-\mu)y_\omega \text{ for some } \mu \in [0,1]\}, \\
D^+_\omega &= \{u \mid \langle P^\omega_2 u - y_\omega, P^\omega_2 u\rangle \ge 0\}, \\
D^-_\omega &= \{u \mid \langle P^\omega_2 u - y_\omega, P^\omega_2 u\rangle < 0\}
\end{aligned}$$

and

$$\widetilde{u} = P_1^\omega u + P_2^\omega u - P_3^\omega u, \quad \widehat{u} = P_1^\omega u - P_2^\omega u - P_3^\omega u.$$

Note that the homotopy equation $F_\mu^\omega(u) = (1-\mu)y_\omega$ with $\mu = 0$ takes the form

$$L_\omega u - \lambda(\omega)u + P_2^\omega u = y_\omega,$$

which has a unique solution $u = y_\omega$. We shall prove the following two estimates:

Estimate 1. Assume that (4.1)–(4.3) are satisfied. Moreover, assume that (4.4) holds and $\omega \in \alpha_0\mathbb{Q}_+$ satisfies the conditions (4.5), (4.6). Then any $u \in K_\omega \cap D_\omega^-$ satisfies the inequality

$$0 > (\overline{b} - \underline{\lambda})\|P_1^\omega u\|^2 + (\overline{\lambda} - b)\|P_3^\omega u\|^2 - (\overline{b} - a)\int_{\Omega_1} |P_1^\omega u + P_2^\omega u|^2,$$

where $\Omega_1 = \{(x,t) \in \Omega \mid |u| > c,\ u\widetilde{u} > 0\}$.

Estimate 2. Assume that (4.1)–(4.3) are satisfied. Moreover, assume that (4.4) holds and $\omega \in \alpha_0\mathbb{Q}_+$ satisfies the conditions (4.5), (4.6). Then any $u \in K_\omega \cap D_\omega^+$ satisfies the inequality

$$0 \ge (a - \underline{\lambda})\|P_1^\omega u\|^2 + (\overline{\lambda} - \overline{a})\|P_3^\omega u\|^2 + (a_1 - \overline{a})\|P_2^\omega u\|^2 - c_1\|P_1^\omega u\| - c_2,$$

where c_1, c_2 are positive constants independent of ω.

Proof of Estimate 1. Assume that $u \in K_\omega \cap D_\omega^-$. Then $0 < \|P_2^\omega u\| < 1$. We first conclude from the equality $\langle F_\mu^\omega(u) - (1-\mu)y_\omega, \widetilde{u}\rangle = 0$ that

$$\mu\langle N(u) - \lambda(\omega)u, \widetilde{u}\rangle \le -(\lambda(\omega) - \underline{\lambda})\|P_1^\omega u\|^2 - (\overline{\lambda} - \lambda(\omega))\|P_3^\omega u\|^2 \le 0$$

and hence

$$\langle N(u) - \lambda(\omega)u, \widetilde{u}\rangle \le \mu\langle N(u) - \lambda(\omega)u, \widetilde{u}\rangle.$$

Combining the results we obtain the basic estimate

$$\underline{\lambda}\|P_1^\omega u\|^2 + \lambda(\omega)\|P_2^\omega u\|^2 - \overline{\lambda}\|P_3^\omega u\|^2 \ge \int_\Omega g(x,u)\widetilde{u}. \tag{4.7}$$

By conditions (4.1)–(4.3) together with assumptions on ω we get

$$\begin{aligned}
\int_\Omega g(u)\widetilde{u} \ge{}& \overline{b}\int_{|u|\le c, u\widetilde{u}>0} u\widetilde{u} + a\int_{\Omega_1} u\widetilde{u} + b\int_{|u|\le d, u\widetilde{u}<0} u\widetilde{u} + \overline{a}\int_{|u|>d, u\widetilde{u}<0} u\widetilde{u} \\
={}& \overline{b}\int_\Omega u\widetilde{u} - (\overline{b} - a)\int_{\Omega_1} u\widetilde{u} \\
&+ (b - \overline{b})\int_{|u|\le d, u\widetilde{u}<0} u\widetilde{u} - (\overline{b} - \overline{a})\int_{|u|>d, u\widetilde{u}<0} u\widetilde{u} \\
\ge{}& \overline{b}\|P_1^\omega u\|^2 + \overline{b}\|P_2^\omega u\|^2 - \overline{b}\|P_3^\omega u\|^2 \\
&- (\overline{b} - a)\int_{\Omega_1} |P_1^\omega u + P_2^\omega u|^2 - (b - \overline{b})\|P_3^\omega u\|^2,
\end{aligned}$$

where in the last estimate the nonnegative term

$$-(\overline{b}-\overline{a})\int_{|u|>d,u\widetilde{u}<0} u\widetilde{u}$$

is dropped. Thus by the basic estimate (4.7) and by neglecting the strictly positive term $(\overline{b}-\lambda(\omega))\|P_2^\omega u\|^2$ we finally obtain

$$0>(\overline{b}-\underline{\lambda})\|P_1^\omega u\|^2+(\overline{\lambda}-b)\|P_3^\omega u\|^2-(\overline{b}-a)\int_{\Omega_1}|P_1^\omega u+P_2^\omega u|^2. \qquad \square$$

Proof of Estimate 2. Assume that $u\in K_\omega\cap D_\omega^+$. From the equality

$$\langle F_\mu^\omega(u)-(1-\mu)y_\omega,\widehat{u}\rangle=0$$

we obtain in analogy with (4.7) the estimate

$$\underline{\lambda}\|P_1^\omega u\|^2-\lambda(\omega)\|P_2^\omega u\|^2-\overline{\lambda}\|P_3^\omega u\|^2\geq\int_\Omega g(x,u)\widehat{u}. \tag{4.8}$$

Denote $\Omega_2=\{(x,t)\in\Omega\,|\,|u|\leq d\}$ and notice that $|\widehat{u}|\leq d+2|P_1^\omega u|$ on Ω_2. By the assumptions on g and ω we get

$$\begin{aligned}\int_\Omega g(u)\widehat{u}\geq a&\int_{|u|>d,u\widehat{u}>0}u\widehat{u}+\overline{a}\int_{|u|>d,u\widehat{u}<0}u\widehat{u}\\
&+a\int_{|u|\leq d,u\widehat{u}>0}u\widehat{u}+b\int_{|u|\leq d,u\widehat{u}<0}u\widehat{u}\\
=&\,\overline{a}\int_\Omega u\widehat{u}-(\overline{a}-a)\int_{|u|>d,u\widehat{u}>0}u\widehat{u}\\
&-(\overline{a}-a)\int_{|u|\leq d,u\widehat{u}>0}u\widehat{u}+(b-\overline{a})\int_{|u|\leq d,u\widehat{u}<0}u\widehat{u}\\
\geq&\,\overline{a}\int_\Omega u\widehat{u}-(\overline{a}-a)\int_{|u|>d,u\widehat{u}>0}|P_1^\omega u|^2\\
&-(\overline{a}-a)\int_{\Omega_2}|u\widehat{u}|-(b-\overline{a})\int_{\Omega_2}|u\widehat{u}|\\
\geq&\,\overline{a}\|P_1^\omega u\|^2-\overline{a}\|P_2^\omega u\|^2\\
&-\overline{a}\|P_3^\omega u\|^2-(\overline{a}-a)\|P_1^\omega u\|^2-(b-a)\int_{\Omega_2}|u\widehat{u}|,\end{aligned}$$

where

$$(b-a)\int_{\Omega_2}|u\widehat{u}|\leq(b-a)d\int_\Omega(d+2|P_1^\omega u|)\leq c_2+c_1\|P_1^\omega u\|$$

with constants c_1 and c_2 independent of ω. Hence it follows from the estimate (4.8) that

$$0\geq(a-\underline{\lambda})\|P_1^\omega u\|^2+(\overline{\lambda}-\overline{a})\|P_3^\omega u\|^2+(\lambda(\omega)-\overline{a})\|P_2^\omega u\|^2-c_1\|P_1^\omega u\|-c_2,$$

and since $\lambda(\omega)\geq a_1$ the desired Estimate 2 follows. $\square$

Lemma 4.1. *Assume that* (4.1)–(4.6) *hold and let* $y_\omega \in H_2^\omega$, $\|y_\omega\| = 1$, *be given. Then there exist constants* $R > 1$ *and* $\rho \in]0,1[$ (*independent of* ω *and* y_ω) *such that*

$$L_\omega u - \mu N(u) - (1-\mu)(\lambda(\omega)u - P_2^\omega u) \neq (1-\mu)y_\omega$$

for all $0 \le \mu \le 1$, $u \in D_\omega^- \cap D(L_\omega)$ *with* $0 < \|P_2 u\| \le \rho$ *and for all* $u \in D(L_\omega)$ *with* $\|u\| \ge R$.

Proof. Assume first that $u \in K_\omega \cap D_\omega^-$. Now $\|P_2^\omega u\| < 1$ and by Estimate 1 we get

$$0 > (\bar{b} - \underline{\lambda})\|P_1^\omega u\|^2 + (\overline{\lambda} - b)\|P_3^\omega u\|^2 - (\bar{b} - a)\|P_1^\omega u\|^2 - (\bar{b} - a).$$

Hence there exists $R_1 > 1$ such that $\|u\| < R_1$. If $u \in K_\omega \cap D_\omega^+$ it is clear by Estimate 2 that there exists $R \ge R_1$ such that $\|u\| < R$.

To prove the second part of the lemma assume again that $u \in K_\omega \cap D_\omega^-$. Recall that $\Omega_1 = \{(x,t) \in \Omega \mid |u| > c,\ u\widetilde{u} > 0\}$ and it depends on ω via the solution u. By dropping the P_3^ω-term from Estimate 1 we get

$$0 > \int_{\Omega_1} [(\bar{b} - \underline{\lambda})|P_1^\omega u|^2 - (\bar{b} - a)|P_1^\omega u + P_2^\omega u|^2].$$

Thus the set Ω_1 has positive measure and denoting $\gamma = \max_\Omega |P_2^\omega u|$ we obtain

$$0 > \int_{\Omega_1} [(a - \underline{\lambda})|P_1^\omega u|^2 - 2(\bar{b} - a)\gamma|P_1^\omega u| - (\bar{b} - a)\gamma^2].$$

Denote $z = |P_1^\omega u|$ and notice that in Ω_1 we have $z + \gamma + |P_3^\omega u| > c$ and $z + \gamma - |P_3^\omega u| > 0$ implying $z > c/2 - \gamma$. Consider now the function

$$E(z) = (a - \underline{\lambda})z^2 - 2(\bar{b} - a)\gamma z - (\bar{b} - a)\gamma^2, \quad z > c/2 - \gamma.$$

It is easy to see that $E(z) = 0$ has only one nonnegative root z_0 which is of the form $z_0 = \text{const} \times \gamma$ and $E(z) < 0$ for all $0 \le z < z_0$ and $E(z) > 0$ for all $z > z_0$. Thus $E(z) > 0$ for all $z > c/2 - \gamma$ provided $z_0 \le c/2 - \gamma$. On the other hand we know that

$$\int_{\Omega_1} E(|P_1^\omega u|) < 0.$$

Hence γ cannot be arbitrarily small and consequently there exists γ_0 such that $\gamma \ge \gamma_0$. Since the space H_2^ω is two-dimensional all its norms are equivalent and we conclude that there exists a further constant $0 < \rho < 1$ such that

$$\|P_2^\omega u\| > \rho \quad \text{for all } u \in D_\omega^- \cap K_\omega,$$

completing the proof. □

5. Existence results

As before, we assume that $g\colon[0,\pi]\times\mathbb{R}\to\mathbb{R}$ is a function satisfying the Caratheodory conditions and

$$a\le\frac{g(x,s)}{s}\le b,\quad\text{for all } s\ne 0,\ \text{for a.a. } x\in[0,\pi],$$

where $b>a>0$. Define

$$V_e=\{u\in H\mid u(x,2\pi-t)=u(x,t)\text{ for a.a. } x\in\,]0,\pi[,\ t\in\,]0,2\pi[\}.$$

It is easy to see that

$$V_e=\overline{\mathrm{sp}}\{\phi_{jk}\mid j\in\mathbb{Z}_+,\ -k\in\mathbb{N}\}.$$

Clearly the operator L_ω, $\omega\in\alpha_0\mathbb{Q}_+$, is completely reduced by V_e, i.e. $P_eD(L_\omega)\subset D(L_\omega)$ and $P_eL_\omega u=L_\omega P_e u$ for all $u\in D(L_\omega)$, where we have denoted by P_e the orthogonal projection onto V_e. Moreover, $N(V_e)\subset V_e$. More generally, for any closed subspace V of H spanned by any given set of eigenvectors we denote

$$\sigma_{\mathrm{const}}(V)=\sigma_{\mathrm{const}}\cap\sigma(L_\omega|_V).$$

The set $[a,b]\cap\sigma_{\mathrm{const}}(V)$ is independent of the frequency. The only way to make $[a,b]\cap\sigma_{\mathrm{const}}(V)$ thinner is to choose the invariant subspace V appropriately. Denote

$$\sigma_\omega(V)=\sigma_\omega\cap\sigma(L_\omega|_V).$$

By Lemma 3.1 we know that for almost all $\omega\in\alpha_0\mathbb{Q}_+$ the set $[a,b]\cap\sigma_\omega$ is nonempty. However, by Lemma 3.2 there exist infinitely many frequencies ω such that $[a,b]\cap\sigma_\omega=\emptyset$ and by Corollary 3.4 infinitely many frequencies ω such that $[a,b]\cap\sigma_\omega$ contains exactly one eigenvalue with multiplicity two. We refer to the set

$$\sigma(L_\omega|_V)\cap[a,b]$$

as the set of interacting eigenvalues in V. In order to obtain multiple solutions it is necessary that the interval $[a,b]$ contains some eigenvalues of $L_\omega|_V$. The application of topological degree requires the nonlinearity to be of class (S_+). To this end we assume that

$$(5.1)\qquad (g(x,s)-g(x,\widetilde{s}))(s-\widetilde{s})\ge\nu|s-\widetilde{s}|^2\quad\text{for all } s,\widetilde{s}\in\mathbb{R},\ \text{a.a. } x\in[0,\pi],$$

where $\nu>0$ is constant. By condition (5.1) the Nemytskiĭ operator N generated by g is strongly monotone and hence of class (S_+). Before stating the main result we recall that for any τ-periodic function $h(x,t)$, $x\in[0,L]$, $t\in\mathbb{R}$, we denote $h_\omega(x,t)=h(x,\omega^{-1}t)$ with the assumption that $L=\pi$.

Theorem 5.1. *Assume that* (4.1)–(4.6) *and* (5.1) *hold. Then there exists an* $\varepsilon = \varepsilon(\omega) > 0$ *such that the equation*

$$\begin{cases} \partial_t^2 u - \alpha_0^2 \partial_x^2 u - g(x,u) = h(x,t), & \\ u(0,t) = u(\pi,t) = 0 & (x \in \,]0,\pi[,\ t \in \mathbb{R}), \\ u(x,\,\cdot\,) & \text{is } 2\pi/\omega\text{-periodic}, \end{cases}$$

admits at least two weak solutions for any $2\pi/\omega$*-periodic function* h *such that* $h_\omega \in V_e$ *and* $\|h_\omega\| < \varepsilon(\omega)$. *If* $h = 0$, *both solutions are nontrivial and time-dependent.*

Proof. Assume first that $h = 0$. We already know that the equation has a trivial solution $u = 0$. Choose $y \in V_e \cap H_2^\omega$, $\|y\| = 1$ and define

$$G_+ = \{u \in V_e \mid \|u\| < R,\ P_2^\omega u = \theta y,\ \rho < \theta < R\}.$$

Now $V_e \cap H_2^\omega$ is one-dimensional, the set $G_+ \subset V_e$ is open and bounded in V_e and $0 \notin G_+$. By Lemma 4.1

$$L_\omega u - \mu N(u) - (1-\mu)(\lambda(\omega)u - P_2^\omega u) \neq (1-\mu)y$$

for all $0 \leq \mu \leq 1$, $u \in \partial G_+ \cap D(L_\omega)$. It is easy to see that

$$F_0^\omega(u) = L_\omega u - \lambda(\omega)u + P_2^\omega u = y, \quad u \in D(L_\omega)$$

if and only if $u = y \in G_+ \cap D(L_\omega)$. Hence by Theorem 2.1

$$\deg_V(F_1, G_+, 0) = \deg_V(F_0, G_+, y) \neq 0$$

and consequently $0 \in F_1(G_+ \cap D(L_\omega))$.

Replacing y by $-y$ and G_+ by the open bounded set

$$G_- = \{u \in V \mid \|u\| < R,\ P_2^\omega u = -\theta y,\ \rho < \theta < R\}$$

yields another nontrivial solution for the equation

$$L_\omega u - N(u) = 0.$$

Denote these solutions by u_1 and u_2. Since $\langle u_1, y\rangle > 0$ and $\langle u_2, y\rangle < 0$, the solutions do not coincide and they are time-dependent. Choose G to be G_+ and G_-, respectively. The set $F_1(\partial G \cap D(L_\omega))$ is closed and we denote by Δ_0 the open component of $H \setminus F_1(\partial G \cap D(L_\omega))$ containing the origin. Hence there exists $\varepsilon(\omega) > 0$ such that $B(0, \varepsilon(\omega)) \subset \Delta_0$. For any h such that $h_\omega \in V$ and $\|h_\omega\| < \varepsilon(\omega)$ we have

$$F_1(u) \neq \mu h_\omega \quad \text{for all } u \in \partial G \cap D(L_\omega),\ 0 \leq \mu \leq 1.$$

Hence by the homotopy invariance of the degree we obtain

$$\deg_V(F_1, G, h_\omega) = \deg_V(F_1, G, 0) \neq 0$$

implying $h_\omega \in F_1(G \cap D(L_\omega))$. □

By the previous theorem we can construct a sequence $(\omega_i)_{i=0}^\infty$, $\omega_i \to \infty$, such that for all i the corresponding homogenous equation admits at least two nontrivial solutions. On the other hand, using Theorem 1.1 and Lemma 3.2 it is possible to find a another sequence of frequencies $(\overline{\omega}_i)_{i=0}^\infty$ such that $\omega_i - \overline{\omega}_i \to 0$ but the corresponding equation has only the trivial solution. Indeed, we get the following result.

Corollary 5.2. *Assume that* (4.1)–(4.4) *and* (5.1) *hold. Then there exists a sequence* $(\tau_i)_{i=1}^\infty$ *such that* $\tau_i \to 0+$ *and the equation*

$$\begin{cases} \partial_t^2 u - \alpha_0^2 \partial_x^2 u - g(x,u) = 0, & \\ u(0,t) = u(\pi,t) = 0 & (x \in \,]0,\pi[,\ t \in \mathbb{R}), \\ u(x,\cdot) & \textit{is } \tau_i\textit{-periodic}, \end{cases}$$

admits at least two nontrivial and time-dependent weak solutions for all $i \in \mathbb{Z}_+$. *Moreover, there exists another sequence* $(\overline{\tau}_i)_{i=1}^\infty$ *such that* $\overline{\tau}_i \to 0+$, $|1 - \tau_i/\overline{\tau}_i| = o(\tau_i)$ *and the equation has only a trivial* $\overline{\tau}_i$*-periodic solution for all* i *sufficiently large.*

Proof. We can choose a sequence $(\omega_i)_{i=1}^\infty$ such that $\omega_i \to \infty$ and (4.5), (4.6) hold. Then $\tau_i = 2\pi/\omega_i \to 0$ and the first conclusion follows from Theorem 5.1. By Lemma 3.2 it is easy to construct another sequence $(\overline{\omega}_i)_{i=1}^\infty$ such that $\overline{\omega}_i \to \infty$, $|\overline{\omega}_i - \omega_i| \to 0$ and $[a,b] \cap \sigma(L_{\overline{\omega}_i}) = \emptyset$ for all sufficiently large i. Consequently, by Theorem 1.1 there exists only trivial $\overline{\tau}_i$-periodic solution for $\overline{\tau}_i = 2\pi/\overline{\omega}_i$. Moreover,

$$\frac{1}{\tau_i}\left(1 - \frac{\tau_i}{\overline{\tau}_i}\right) = \frac{\omega_i - \overline{\omega}_i}{2\pi} \to 0$$

completing the proof. □

In the preceeding two results we have assumed that $[a,b] \cap \sigma_{\mathrm{const}} = \emptyset$ (condition (4.4)). Assume now that $[a,b] \cap \sigma_{\mathrm{const}} \neq \emptyset$. To be more precise, assume that

$$\alpha_0^2 p^2 < a < \alpha_0^2 (p+1)^2 < \ldots < \alpha_0^2 (p+n)^2 < b < \alpha_0^2 (p+n+1)^2 \tag{5.2}$$

for some $p \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. In the case $n = 1$ there is exactly one eigenvalue in $[a,b] \cap \sigma_{\mathrm{const}}$ and we obtain the following variant of Theorem 5.1.

Theorem 5.3. *Assume that* (4.1), (4.2) *and* (5.1) *hold. Moreover, assume that* (5.2) *holds with* $n = 1$. *Then for infinitely many frequencies* $\omega \in \alpha_0 \mathbb{Q}_+$ *there exists an* $\varepsilon = \varepsilon(\omega) > 0$ *such that the equation*

$$\begin{cases} \partial_t^2 u - \alpha_0^2 \partial_x^2 u - g(x,u) = h(x,t), & \\ u(0,t) = u(\pi,t) = 0 & (x \in \,]0,\pi[,\ t \in \mathbb{R}), \\ u(x,\cdot) & \textit{is } 2\pi/\omega\textit{-periodic}, \end{cases}$$

admits at least two weak solutions for any $2\pi/\omega$-periodic forcing term h such that $h_\omega \in H$ and $\|h_\omega\| < \varepsilon(\omega)$. If $h = 0$, both solutions are nontrivial.

Proof. Denote $\underline{\lambda} = \alpha_0^2 p^2$ and $\overline{\lambda} = \alpha_0^2(p+2)^2$. Then $\underline{\lambda} < a < b < \overline{\lambda}$ and by Lemma 3.1 there exists infinitely many frequencies $\omega \in \alpha_0\mathbb{Q}_+$ such that

$$[\underline{\lambda}, \overline{\lambda}] \cap \sigma_\omega = \emptyset.$$

Hence for all such frequencies we have

$$\max\{\lambda_{j,k}^\omega \mid \lambda_{j,k}^\omega < \alpha_0^2(p+1)^2\} \le \underline{\lambda} < a, \quad \min\{\lambda_{j,k}^\omega \mid \lambda_{j,k}^\omega > \alpha_0^2(p+1)^2\} \ge \overline{\lambda} > b.$$

Now we can proceed as in the proof of Theorem 5.1. Indeed, the Estimates 1 and 2 of Section 4 hold with $a_1 = b_1 = \alpha_0^2(p+1)^2$, $\lambda(\omega)$ replaced by $\alpha_0^2(p+1)^2$ and P_2^ω replaced by orthogonal projection P_2 onto the subspace H_2 spanned by the eigenvector $\phi_{p+1,0}$. Note that the space H_2 is now one dimensional and hence Lemma 4.1 (with y_ω replaced by $y = \pm\phi_{p+1,0}$) suffices to determine disjoint open bounded sets

$$G_\pm = \{u \in H \mid \|u\| < R,\ P_2 u = \pm\theta y,\ \rho < \theta < R\}$$

such that for both $G = G_+$, $y = \phi_{p+1,0}$ and $G = G_-$, $y = -\phi_{p+1,0}$ we have

$$L_\omega u - \mu N(u) - (1-\mu)(\alpha_0^2(p+1)^2 u - P_2 u) \ne (1-\mu) y$$

for all $0 \le \mu \le 1$, $u \in \partial G \cap D(L_\omega)$. Hence the conclusion follows as in the proof of Theorem 5.1. $\square$

In the case where $[a,b] \cap \sigma_{\mathrm{const}}$ contains more than one eigenvalue, our method requires further conditions on g and on the forcing term h. For any $j_0 \in \mathbb{Z}_+$, $j_0 \ge 2$ we define

$$\begin{aligned} V_{j_0} = &\{u \in H \mid u(x + 2\pi/j_0, t) = u(x,t) \text{ for a.a. } x \in \,]0, \pi - 2\pi/j_0[,\ t \in \,]0, 2\pi[\\ &\quad \text{and } u(2\pi/j_0 - x, t) = -u(x,t) \text{ for a.a. } x \in \,]0, 2\pi/j_0[,\ t \in \,]0, 2\pi[\} \\ = &\,\overline{\mathrm{sp}}\{\phi_{j,k} \mid j \in j_0\mathbb{Z}_+,\ k \in \mathbb{Z}\}. \end{aligned}$$

We obtain the following result.

Theorem 5.4. *Assume that* (4.1), (4.2) *and* (5.1) *hold. Moreover, assume that $g = g(u)$ is odd and* (5.2) *holds with $n \ge 2$. Then for infinitely many frequencies $\omega \in \alpha_0\mathbb{Q}_+$ there exists an $\varepsilon = \varepsilon(\omega) > 0$ such that the equation*

$$\begin{cases} \partial_t^2 u - \alpha_0^2 \partial_x^2 u - g(u) = h(x,t), & \\ u(0,t) = u(\pi,t) = 0 & (x \in \,]0,\pi[,\ t \in \mathbb{R}), \\ u(x, \cdot) & \textit{is } 2\pi/\omega\textit{-periodic}, \end{cases}$$

admits at least two weak solutions for any $2\pi/\omega$-periodic forcing term h such that $h_\omega \in V_{j_0}$, $j_0 = p+n$, and $\|h_\omega\| < \varepsilon(\omega)$.

Proof. Only a slight modification of the proof of Theorem 5.3 is needed. We first observe that $N(V_{j_0}) \subset V_{j_0}$ and

$$\sigma_{\mathrm{const}}(V_{j_0}) \cap [a,b] = \{\alpha_0^2(p+n)^2\}.$$

Thus we face the same situation as in Theorem 5.3; the space H is only replaced by V_{j_0}. Thus the conclusion follows. $\square$

We close this section by a result, where we show how the reduction to an invariant subspace can be used to remove the assumption on the monotonocity of g (cf. [13]). To this end we define

$$\begin{aligned} V = Z &:= \{u \in H \mid u(\pi - x, t+\pi) = u(x,t) \text{ for a.a. } x \in \,]0,\pi[,\ t \in \,]0,\pi[\} \\ &= \overline{\mathrm{sp}}\{\phi_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z} \text{ with } j+k \text{ is odd}\}. \end{aligned}$$

We consider the solvability with prescribed period. In fact we shall seek solutions with frequency an integer multiple of the prescribed one. Note that if the precribed period is τ_1 and we find free vibrations with period τ_1/n, where n is a positive integer, then these solutions are also τ_1-periodic thus being solutions of the original, τ_1-periodic homoneneous equation. As an example of many possible variants we prove the following result.

Theorem 5.5. *Let $\tau_1 = 2\pi/\omega_1$, where $\omega_1 = \alpha_0 p/q \in \alpha_0\mathbb{Q}_+$ is given such that p, q are odd. Assume that $0 < a < \alpha_0^2 < b < 4\alpha_0^2$. Suppose $g = g(u)$ satisfies the conditions (4.1) and (4.2). Denote $\omega_n = n\omega_1$. Then there exists $\varepsilon_n > 0$ such that the equation*

$$\begin{cases} \partial_t^2 u - \alpha_0^2\partial_x^2 u - g(u) = h(x,t), & \\ u(0,t) = u(\pi,t) = 0 & (x \in]0,\pi[,\ t \in \mathbb{R}), \\ u(x,\cdot) & \textit{is } \tau_1/n\textit{-periodic}, \end{cases}$$

admits at least two weak solutions for any τ_1/n-periodic forcing term h such that $h_{\omega_n} \in Z$ and $\|h_{\omega_n}\| < \varepsilon_n$, provided that n is odd and $n \geq q^2 b(\alpha_0^2 p)^{-1}$.

Proof. By Lemma 3.2 $[a,b] \cap \sigma_{\omega_n} = \emptyset$ for all $n \geq q^2 b(\alpha_0^2 p)^{-1}$. Moreover, if $\lambda_{jk}^{\omega_n} = 0$ is an eigenvalue of the restriction $L_{\omega_n}|_Z$, then necessarily $jq = np|k|$. By the definition of the subspace Z this is not possible whenever n is odd. Thus $\operatorname{Ker} L_{\omega_n} = \{0\}$ for all $n \geq q^2 b(\alpha_0^2 p)^{-1}$, n odd. Consequently, for any τ_1/n-periodic forcing term h such that $h_{\omega_n} \in Z$ the operator equation

$$L_{\omega_n} u - N(u) = h_{\omega_n}, \quad u \in D(L_{\omega_n}|_Z)$$

can be written in the form

$$u - (L_{\omega_n}|_Z)^{-1} N(u) = (L_{\omega_n}|_Z)^{-1} h_{\omega_n}.$$

Since the inverse of $L_{\omega_n}|_Z$ is compact, we can use the Leray–Schauder degree theory and no monotonicity for g is needed. Moreover, the estimates given in Section 3 are valid in an obviously modified form (see the proof of Theorem 5.3). Hence the proof can be completed exactly as the proof of Theorem 5.3. □

6. Concluding remarks

In many applications it is natural to consider equations of the form

$$\begin{cases} \omega^2\partial_t^2 u - \alpha_0^2\partial_x^2 u + g(x,u) = h_\omega(x,t), \\ u(0,t) = u(\pi,t) = 0 & (x \in\,]0,\pi[,\ t \in \mathbb{R}), \\ u(x,t) = u(x,t+2\pi). \end{cases} \tag{6.1}$$

From purely abstract point of view we may replace L_ω by $-L_\omega$ in all our considerations. Hence our results have counterparts concerning the equation

$$L_\omega u + N(u) = h_\omega, \qquad u \in D(L_\omega). \tag{6.2}$$

Note however that due to the asymmetry of the spectrum of the wave operator the conditions are different for the solvability of (1.2) and (6.1), respectively. The main difference is caused by the fact that all the "constant eigenvalues" in σ_{const} are nonnegative. As pointed out in Section 3 the distribution of eigenvalues in σ_ω as a function of ω can be treated in a similar way on negative and positive side. Assume that L_ω is replaced by $-L_\omega$. Then Theorem 5.1 and Corollary 5.2 still hold, but now without assumption (4.4) which is trivially satisfied. Note that if we replace L_ω by $-L_\omega$, then Theorems 5.2–5.3 have no analogies due to the asymmetry of the spectrum.

Another variant of our results is obtained, if the nonlinearity instead the condition (4.2) satisfies the opposite one

$$\begin{cases} a \le \dfrac{g(x,s)}{s} \le \overline{a} & \text{for all } 0 < |s| \le c, \\ \overline{b} \le \frac{g(x,s)}{s} \le b & \text{for all } |s| > d. \end{cases} \tag{6.3}$$

We indicate briefly how this situation can be handled. Indeed, assume that (4.1), (6.3), (4.3)–(4.6) hold and denote $\widehat{L}_\omega = -L_\omega + bI$, $\widehat{g}(x,s) = -g(x,s) + bs$, $\widehat{N} = -N + bI$, $\widehat{a} = 0$, $\widehat{b} = b - a$ and $\widehat{\lambda}(\omega) = -\lambda(\omega) + b$. Then

$$\widehat{a} \le \frac{\widehat{g}(x,s)}{s} \le \widehat{b} \quad \text{for all } s \ne 0$$

and condition (4.2) holds for $\widehat{g}$ with $\overline{a}$, $\overline{b}$ replaced by $b-\overline{b}$ and $b-\overline{a}$, respectively. Thus we can proceed as before. The homotopy equation used in this case is

$$\widehat{L}_\omega u - \widehat{\lambda}(\omega)u - \mu(\widehat{N}(u) - \widehat{\lambda}(\omega)u) + (1-\mu)P_2^\omega u = (1-\mu)y_\omega,$$

which is equivalent to

$$L_\omega u - \lambda(\omega)u - \mu(N(u) - \lambda(\omega)u) - (1-\mu)P_2^\omega u = -(1-\mu)y_\omega.$$

It is easy to see that the estimates of Section 5 are valid with P_1^ω replaced by P_3^ω and vice versa. Hence Theorem 5.1, Corollary 5.2 and Theorems 5.3–5.5 remain true whenever (4.2) replaced by (6.3).

Clearly by the remarks above we can also deal with the case, where g is decreasing and $a < b < 0$.

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