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STABLE PERIODIC MOTION OF A DELAYED SPRING

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Dedicated to Professor Klaus Kirchgässner

Abstract. It is shown that an autonomous delay differential system for a damped spring with a delayed restoring force has a periodic solution whose orbit is exponentially stable with asymptotic phase.

1. Introduction

The system

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -\mu v(t) + f(x(t-1)) \tag{1}$$

stands for a spring where the action of the position-dependent force $f:\mathbb{R} \to \mathbb{R}$ is delayed by one time unit. The friction coefficient μ is assumed positive, and f is taken from one of the sets $F = F_{\beta\varepsilon}$, $\beta > 0$ and $0 < \varepsilon < a$, formed by all odd, bounded, continuous real functions $\overline{f}$ on $\mathbb{R}$ which satisfy

$$|\overline{f}(\xi)| < a + \varepsilon \quad \text{for all } \xi \in \mathbb{R} \quad \text{and} \quad |\overline{f}(\xi) + a| < \varepsilon \quad \text{for all } \xi \geq \beta.$$

The parameter $a > 0$ is fixed.

Notice that the condition for the force to be restoring with respect to the position $\xi = 0$, namely

$$\xi f(\xi) < 0,$$

is required only for $|\xi| \geq \beta$; it will not be needed for $0 < |\xi| < \beta$ in the sequel.

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The main result is that for μ sufficiently large, β and ε sufficiently small, f Lipschitz continuous, not too steep in the interval $(-\beta,\beta)$ and sufficiently flat outside there exists a periodic solution with strong attraction properties. In case f is continuously differentiable the periodic orbit is stable and hyperbolic.

The proof extends a method introduced in [8] for first order equations

$$\dot{x}(t)=-\mu x(t)+f(x(t-1)).$$

The first step is to find closed sets of initial data to which solutions return. This is achieved in the next section (Corollary 2.1). In addition a sharp lower bound for the return time is derived (Proposition 2.9). Section 3 deals with Lipschitz estimates of the return time and return map. Under the conditions described above the return map becomes a contraction (Theorem 3.1). Its fixed point defines the desired periodic solution. In Section 4 it is shown how to obtain hyperbolic stability of the periodic orbit.

A major difference to the result in [8] is that here a large friction coefficient is essential. In [8] smallness of β and ε together with conditions on f were sufficient to guarantee attracting periodic orbits. The role of friction is not surprising since the analogue of (1) without friction and delay is conservative. A delay should destabilize the conservative system, at least for certain monotone nonlinearities f, and exclude attracting periodic orbits.

Another aspect is that here the method comes considerably closer to its limits. The need for the sharp lower estimate of the return time in Proposition 2.9, which has no counterpart in [8], indicates this, as well as the need to introduce the weight 1/2 in the norm on the state space below.

Other results based on the approach from [8] concern first order equations with analytic and monotone nonlinearities which arise in applications [9], the interaction of instantaneous growth and delayed feedback ([6]), and a system which models automatic position control and involves a state-dependent delay ([10]).

Second order delay differential equations related to (1) were studied by other methods in [1]–[3], [5], [7]. A difference to (1) is that in addition to the delayed feedback instantaneous position-dependent feedback is included and used. In [1], [5], existence of periodic solutions is obtained for a rather large class of nonlinearities. [2], [3], [7] study equations with discontinuous nonlinearities and find, among others, stable periodic orbits. These papers also contain references to a variety of applications.

An open problem is whether the approach developed here can be extended to models for automatic position control similar to the system studied in [10], but based on Newton's law instead of a first order differential equation.

Preliminaries. A solution of (1) on $[0,\infty)$ is defined to be a pair (x,v) of a continuous function $x\colon[-1,\infty)\to\mathbb{R}$ and a differentiable function $v\colon[0,\infty)\to\mathbb{R}$

so that x is differentiable on $(0,\infty)$ and (1) holds for all $t>0$. Analogously one has solutions on $[t_0,\infty)$ for each $t_0\in\mathbb{R}$. Solutions on $\mathbb{R}$ are pairs (x,v) of differentiable real functions defined on $\mathbb{R}$ which satisfy (1) for all $t\in\mathbb{R}$.

Set $C=C([-1,0],\mathbb{R})$. $X=C\times\mathbb{R}$ serves as state space. The norms are given by $\|\phi\|=\max_{t\in[-1,0]}|\phi(t)|$ for all $\phi\in C$ and

$$\|(\phi,u)\|=\frac{1}{2}\|\phi\|+|u|$$

for all $(\phi,u)\in X$.

Each $(\phi,u)\in X$ uniquely determines a solution $(x,v)=(x,v)^{(\phi,u)}$ of (1) on $[0,\infty)$ with $x(t)=\phi(t)$ on $[-1,0]$ and $v(0)=u$. This is most easily shown by the method of steps: For $(\phi,u)\in X$ given and $t\in[0,1]$, insert ϕ into the right hand side of the second equation of (1) and solve the initial value problem given by $x(0)=\phi(0)$, $v(0)=u$ for the resulting ordinary differential system. Repeat on $[1,2],[2,3],\ldots$ Frequently the variation-of-constants formula

$$v(t)=e^{-\mu t}v(t_0)+\int_{t_0}^{t}e^{-\mu(t-s)}f(x(s-1))\,ds$$

for the second component of a solution will be used.

The relations $S_{\mu f}(t,(\phi,u))=(x_t,v(t))$, $(x,v)=(x,v)^{(\phi,u)}$, $x_t(s)=x(t+s)$ define a continuous semiflow $S_{\mu f}\colon[0,\infty)\times X\to X$. In case f is continuously differentiable the restriction of the semiflow to $(1,\infty)\times X$ is continuously differentiable, too. To derive this from the analogous smoothness result in [4, Chapter VII], for the semiflow $\widehat{S}_{\mu f}$ generated by (1) on the space $\widehat{C}=C([-1,0],\mathbb{R}^2)$, use the equation

$$S_{\mu f}=(\mathrm{id}_{\,C}\times(ev_0\circ\mathrm{pr}_2))\circ\widehat{S}_{\mu f}\circ(\mathrm{id}_{\,[0,\infty)}\times(\mathrm{id}_{\,C}\times j)),$$

with the embedding $j\colon\mathbb{R}\ni u\mapsto u_c\in C$, $u_c(t)=u$ for all $t\in[-1,0]$, the projection $\mathrm{pr}_2\colon\widehat{C}\ni(\phi,\psi)\mapsto\psi\in C$, and the evaluation $ev_0\colon C\ni\psi\mapsto\psi(0)\in\mathbb{R}$.

Lipschitz constants of maps $g\colon A\to F$, $A\subset E$, E and F Banach spaces, are defined by

$$\mathrm{Lip}\,(g)=\sup_{x\neq y}\frac{\|g(x)-g(y)\|}{\|x-y\|}\leq\infty.$$

2. Recurrence

In this section sets of initial values are found to which solutions return after an excursion into the ambient space.

Proposition 2.1. *For all $(\phi,u)\in X$ with $|u|\leq(a+\varepsilon)/\mu$ the solution $(x,v)=(x,v)^{(\phi,u)}$ satisfies*

$$|v(t)|<\frac{a+\varepsilon}{\mu}\quad\textit{for all } t>0.$$

PROOF. The variation-of-constants formula

$$v(t) = e^{-\mu t}u + \int_0^t e^{-\mu(t-s)} f(x(s-1))\,ds$$

for all $t > 0$ yields

$$e^{\mu t}v(t) \in u + (-a-\varepsilon, a+\varepsilon)\frac{1}{\mu}(e^{\mu t}-1),$$

hence

$$\begin{aligned} -\frac{a+\varepsilon}{\mu} &= -\frac{a+\varepsilon}{\mu}e^{-\mu t} - \frac{a+\varepsilon}{\mu} + \frac{a+\varepsilon}{\mu}e^{-\mu t} < v(t) \\ &< \frac{a+\varepsilon}{\mu}e^{-\mu t} + \frac{a+\varepsilon}{\mu} - \frac{a+\varepsilon}{\mu}e^{-\mu t} = \frac{a+\varepsilon}{\mu}. \end{aligned}$$ □

For $\beta > 0$, $\varepsilon \in (0, a)$, $\mu > 0$ and for the additional parameter $r \in (0,1)$ let $A = A_{\beta\varepsilon\mu r}$ denote the set of initial data $(\phi, u) \in X$ which satisfy

$$\phi(t) \le -\beta \quad \text{on } [-1,0], \quad \phi(0) = -\beta, \quad u \in \left[r\frac{a-\varepsilon}{\mu}, \frac{a+\varepsilon}{\mu}\right].$$

The aim is to find parameters β, ε, μ, r with β, ε small so that for every $f \in F_{\beta\varepsilon}$ and every $(\phi, u) \in A_{\beta\varepsilon\mu r} = A$ the solution $(x, v) = (x, v)^{(\phi,u)}$ of (1) reaches the set $-A$ at some $B = B(\phi, u, f, \beta, \varepsilon, \mu, r) > 0$ in the sense of $S(B, (\phi, u)) \in -A$, $S = S_{\mu f}$. Fixed points of the return map

$$A \ni (\phi, u) \mapsto -S(B, (\phi, u)) \in A$$

will then define periodic solutions of (1). This follows easily from f being odd which implies that for every solution (x, v) also $(-x, -v)$ is a solution.

It is convenient to introduce the function

$$\Delta\colon \mathbb{R} \times (-\infty, a) \times \mathbb{R} \times (0,1) \ni (\beta, \varepsilon, \mu, r) \mapsto \frac{2\mu}{r(a-\varepsilon)}\beta \in \mathbb{R}.$$

PROPOSITION 2.2. *Let $f \in F_{\beta\varepsilon}$, $(\phi, u) \in A_{\beta\varepsilon\mu r}$, $(x, v) = (x, v)^{(\phi,u)}$, $\Delta = \Delta(\beta, \varepsilon, \mu, r)$.*

(a) *For $0 < t \le 1$,*

$$ue^{-\mu t} + \frac{1}{\mu}(a-\varepsilon)(1-e^{-\mu t}) < v(t) < ue^{-\mu t} + \frac{1}{\mu}(a+\varepsilon)(1-e^{-\mu t})$$

and

$$\begin{aligned} -\beta + \frac{a-\varepsilon}{\mu}t + \left(u - \frac{a-\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu t}) &< x(t) \\ &< -\beta + \frac{a+\varepsilon}{\mu}t + \left(u - \frac{a+\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu t}). \end{aligned}$$

In particular, $\dot{x}(t) > 0$.

(b) *If*

$$2\beta < \frac{a-\varepsilon}{\mu}\left(1+(r-1)\frac{1-e^{-\mu}}{\mu}\right) \tag{2}$$

then $\beta < x(1)$, and there exists a unique $b = b(\phi, u, f, \beta, \varepsilon, \mu, r) \in (0,1)$ so that $x(b) = \beta$. Moreover,

$$b < \Delta \quad \textit{and} \quad x(1+\Delta) < -\beta + (1+\Delta)\frac{a+\varepsilon}{\mu}.$$

Proof. (a) The estimate of $v(t)$ follows as in the proof of Proposition 2.1. Integration and $x(0) = -\beta$ yield the estimate of $x(t)$.

(b) Condition (2) is equivalent to

$$\beta < -\beta + \frac{a-\varepsilon}{\mu} + \left(r\frac{a-\varepsilon}{\mu} - \frac{a-\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu}).$$

The last term is not larger than $x(1)$, by part (a). Existence and uniqueness of b follow by means of $x(0) = -\beta$ and $0 < v(t) = \dot{x}(t)$ for $0 < t \le 1$. Moreover, by part (a),

$$\begin{aligned}\beta = x(b) &> -\beta + \frac{a-\varepsilon}{\mu}b + \left(u - \frac{a-\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu b}) \\ &\ge -\beta + \frac{a-\varepsilon}{\mu}b + (r-1)\frac{a-\varepsilon}{\mu}\frac{1-e^{-\mu b}}{\mu},\end{aligned}$$

hence

$$\frac{a-\varepsilon}{\mu}b < 2\beta + (1-r)\frac{a-\varepsilon}{\mu}b\frac{1-e^{-\mu b}}{\mu b} \le 2\beta + (1-r)\frac{a-\varepsilon}{\mu}b,$$

and thereby

$$r\frac{a-\varepsilon}{\mu}b < 2\beta$$

which gives $b < \Delta$. Finally, by Proposition 2.1,

$$x(1+\Delta) = x(0) + \int_0^{1+\Delta} v(t)\,dt < -\beta + (1+\Delta)\frac{a+\varepsilon}{\mu}. \qquad \square$$

Observe that in case (2) holds, $x(t) > \beta$ for all $t \in (b, 1]$.

Proposition 2.3. *Suppose*

$$-\beta\left(1+\frac{2(a+\varepsilon)}{r(a-\varepsilon)}\right) + \frac{a-\varepsilon}{\mu}\left(1-(1-r)\frac{1-e^{-\mu}}{\mu}\right) > \beta \tag{3}$$

holds. Let $f \in F_{\beta\varepsilon}$, $(\phi,u) \in A_{\beta\varepsilon\mu r}$, $(x,v) = (x,v)^{(\phi,u)}$, $\Delta = \Delta(\beta,\varepsilon,\mu,r)$. Then $x(t) > \beta$ for all $t \in (b, 1+\Delta]$, and there exists a smallest $B = B(\phi,u,f,\beta,\varepsilon,\mu,r)$ in $(1+\Delta,\infty)$ with $x(B) = \beta$.

PROOF. *Step 1.* Observe that (3) implies (2), the hypothesis in part (b) of Proposition 2.2. The estimate of $x(1)$ from below in part (a) of Proposition 2.2 yields

$$\begin{aligned} x(1) &> -\beta + \frac{a-\varepsilon}{\mu} + \left(u - \frac{a-\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu}) \\ &\geq -\beta + \frac{a-\varepsilon}{\mu} + \frac{a-\varepsilon}{\mu}(r-1)\frac{1-e^{-\mu}}{\mu} \\ &= -\beta + \frac{a-\varepsilon}{\mu}\left(1-(1-r)\frac{1-e^{-\mu}}{\mu}\right). \end{aligned}$$

For $1 \leq t \leq 1+\Delta$, by (3),

$$\begin{aligned} x(t) &= x(1) + \int_1^t v(s)ds > x(1) - \Delta\frac{a+\varepsilon}{\mu} \quad \text{(see Proposition 2.1)} \\ &= x(1) - \frac{2(a+\varepsilon)}{r(a-\varepsilon)}\beta \\ &> -\beta\left(1+\frac{2(a+\varepsilon)}{r(a-\varepsilon)}\right) + \frac{a-\varepsilon}{\mu}\left(1-(1-r)\frac{1-e^{-\mu}}{\mu}\right) > \beta. \end{aligned}$$

Step 2. For every $t_e > 1+\Delta$ so that $\beta \leq x(t)$ for all $t \in [1+\Delta, t_e]$ it follows that $x(t-1) \geq \beta$ for all t in the larger interval $[1+\Delta, t_e+1]$, and thereby

$$e^{\mu t}(\dot v(t) + \mu v(t)) = e^{\mu t} f(x(t-1)) \in e^{\mu t}(-a-\varepsilon, -a+\varepsilon).$$

Integration yields

$$e^{\mu t}v(t) - e^{\mu(1+\Delta)}v(1+\Delta) \in \frac{e^{\mu t} - e^{\mu(1+\Delta)}}{\mu}(-a-\varepsilon, -a+\varepsilon),$$

or

$$\begin{aligned} &\left(v(1+\Delta) + \frac{a+\varepsilon}{\mu}\right)e^{-\mu(t-(1+\Delta))} - \frac{a+\varepsilon}{\mu} < v(t) \\ &\quad < \left(v(1+\Delta) + \frac{a-\varepsilon}{\mu}\right)e^{-\mu(t-(1+\Delta))} - \frac{a-\varepsilon}{\mu} \quad \text{for } t \in (1+\Delta, t_e]. \end{aligned}$$

Step 3. Recall $x(1) > \beta$. Suppose $x(t) > \beta$ for all $t > 1$. Then the previous part of the proof yields $\dot x(t) = v(t) \leq -(a-\varepsilon)/2\mu < 0$ for all t sufficiently large, which gives a contradiction to the assumption. □

It is convenient to state separately the result of part 2 of the previous proof, and the integrated version of this inequality.

Proposition 2.4. *Suppose* (3) *holds. Let* $f \in F_{\beta\varepsilon}$, $(\phi, u) \in A_{\beta\varepsilon\mu r}$, $(x, v) = (x, v)^{(\phi,u)}$, $\Delta = \Delta(\beta, \varepsilon, \mu, r)$. *Let* $t_e > 1 + \Delta$ *be given with* $\beta \le x(t)$ *for all* $t \in [1 + \Delta, t_e]$. *Then*

$$\left(v(1+\Delta) + \frac{a+\varepsilon}{\mu}\right)e^{-\mu(t-(1+\Delta))} - \frac{a+\varepsilon}{\mu} < v(t)$$
$$< \left(v(1+\Delta) + \frac{a-\varepsilon}{\mu}\right)e^{-\mu(t-(1+\Delta))} - \frac{a-\varepsilon}{\mu}$$

and

$$-\frac{a+\varepsilon}{\mu}(t-(1+\Delta)) + \left(v(1+\Delta) + \frac{a+\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))}) < x(t) - x(1+\Delta)$$
$$< -\frac{a-\varepsilon}{\mu}(t-(1+\Delta)) + \left(v(1+\Delta) + \frac{a-\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))})$$

for all $t \in (1 + \Delta, t_e + 1]$.

Proposition 2.3 implies that the component $x_B \in C$ of $S(B, (\phi, u))$ satisfies two of the conditions for $S(B, (\phi, u))$ to be in $-A$. We also need

$$-\frac{a+\varepsilon}{\mu} \le v(B) \le -r\frac{a-\varepsilon}{\mu}.$$

Before discussing for which β, ε, μ, r the last inequality holds conditions which imply the inequality $2 + \Delta < B$ are studied. Consider the function

$$p: \mathbb{R} \times (-\infty, a) \times (0, \infty) \times (0, 1) \times \mathbb{R} \to \mathbb{R}$$

given by

$$p(\beta, \varepsilon, \mu, r, s) = -\beta - \frac{3(a+\varepsilon)}{\mu}\Delta + \frac{a-\varepsilon}{\mu}\left(1 - (1-r)\frac{1-e^{-\mu}}{\mu}\right)$$
$$- \frac{a+\varepsilon}{\mu}s + \left((r-1)\frac{a-\varepsilon}{\mu}e^{-\mu} + \frac{2a}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu s})$$

with $\Delta = \Delta(\beta, \varepsilon, \mu, r)$.

Proposition 2.5. *If* $\beta > 0$, $0 < \varepsilon < a$, $\mu > 0$, $0 < r < 1$ *and if* (3),

$$\beta < p(\beta, \varepsilon, \mu, r, 0) \tag{4}$$

and

$$\beta < p(\beta, \varepsilon, \mu, r, 1) \tag{5}$$

hold then

$$2 + \Delta(\beta, \varepsilon, \mu, r) < B(\phi, u, f, \beta, \varepsilon, \mu, r)$$

for all $f \in F_{\beta\varepsilon}$ *and all* $(\phi, u) \in A_{\beta\varepsilon\mu r}$.

Proof. *Step 1.* Let $\Delta = \Delta(\beta, \varepsilon, \mu, r)$ in the sequel.

Claim. For every solution $(x,v)=(x,v)^{(\phi,u)}$ with $(\phi,u)\in A_{\beta\varepsilon\mu r}$ and for every $t_e\ge 1+\Delta$ with $x(t)\ge\beta$ for all $t\in[1+\Delta,t_e]$,

$$x(t)>p(\beta,\varepsilon,\mu,r,t-(1+\Delta))\quad\text{for all } t\in[1+\Delta,t_e].$$

Proof. For $1+\Delta\le t\le t_e$, the last inequality in Proposition 2.4 gives

$$x(t)\ge x(1+\Delta)-\frac{a+\varepsilon}{\mu}(t-(1+\Delta))+\left(v(1+\Delta)+\frac{a+\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))}).$$

As in Step 1 of the proof of Proposition 2.3,

$$x(1+\Delta)>x(1)-\frac{a+\varepsilon}{\mu}\Delta.$$

By Proposition 2.1,

$$\dot v(t)=-\mu v(t)+f(x(t-1))>-\mu\frac{a+\varepsilon}{\mu}-a-\varepsilon\quad\text{for all } t>0,$$

hence

$$v(1+\Delta)>v(1)-2(a+\varepsilon)\Delta.$$

These lower estimates of $x(1+\Delta)$ and $v(1+\Delta)$ yield

$$\begin{aligned}x(t)>{}&x(1)-\frac{a+\varepsilon}{\mu}\Delta-\frac{a+\varepsilon}{\mu}(t-(1+\Delta))\\&+\left(v(1)-2(a+\varepsilon)\Delta+\frac{a+\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))}).\end{aligned}$$

By means of the lower estimates for $x(1)$ and $v(1)$ from Proposition 2.2,

$$\begin{aligned}x(t)>{}&-\beta+\frac{a-\varepsilon}{\mu}+\left(u-\frac{a-\varepsilon}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu})-\frac{a+\varepsilon}{\mu}\Delta-\frac{a+\varepsilon}{\mu}(t-(1+\Delta))\\&+\left(ue^{-\mu}+\frac{1}{\mu}(a-\varepsilon)(1-e^{-\mu})-2(a+\varepsilon)\Delta+\frac{a+\varepsilon}{\mu}\right)\\&\cdot\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))}).\end{aligned}$$

By means of $u\ge r(a-\varepsilon)/\mu$,

$$\begin{aligned}x(t)>{}&-\beta+\frac{a-\varepsilon}{\mu}+(r-1)\frac{a-\varepsilon}{\mu}\frac{1-e^{-\mu}}{\mu}-\frac{a+\varepsilon}{\mu}\Delta-\frac{a+\varepsilon}{\mu}(t-(1+\Delta))\\&+\left(r\frac{a-\varepsilon}{\mu}e^{-\mu}+\frac{1}{\mu}(a-\varepsilon)(1-e^{-\mu})-2(a+\varepsilon)\Delta+\frac{a+\varepsilon}{\mu}\right)\\&\cdot\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))})\\>{}&-\beta-\frac{a+\varepsilon}{\mu}\Delta-\frac{2(a+\varepsilon)}{\mu}\Delta+\frac{a-\varepsilon}{\mu}+(r-1)\frac{a-\varepsilon}{\mu}\frac{1-e^{-\mu}}{\mu}\\&-\frac{a+\varepsilon}{\mu}(t-(1+\Delta))\end{aligned}$$

$$
\begin{aligned}
&+\left((r-1)\frac{a-\varepsilon}{\mu}e^{-\mu}+\frac{1}{\mu}(a-\varepsilon)+\frac{1}{\mu}(a+\varepsilon)\right)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))})\\
=&-\beta-\frac{3(a+\varepsilon)}{\mu}\Delta+\frac{a-\varepsilon}{\mu}\left(1-(1-r)\frac{1-e^{-\mu}}{\mu}\right)-\frac{a+\varepsilon}{\mu}(t-(1+\Delta))\\
&+\left((r-1)\frac{a-\varepsilon}{\mu}e^{-\mu}+\frac{2a}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))})\\
=&\,p(\beta,\varepsilon,\mu,r,t-(1+\Delta)).
\end{aligned}
$$

Step 2. The function

$$
s\mapsto\partial_5 p(\beta,\varepsilon,\mu,r,s)=-\frac{a-\varepsilon}{\mu}+\left((r-1)\frac{a-\varepsilon}{\mu}e^{-\mu}+\frac{2a}{\mu}\right)e^{-\mu s}
$$

is strictly decreasing since

$$
0<(r-1)\frac{a-\varepsilon}{\mu}e^{-\mu}+\frac{2a}{\mu}.
$$

Recall (4) and (5). It follows that for all $t\in[1+\Delta,2+\Delta]$, $\beta<p(\beta,\varepsilon,\mu,r,t-(1+\Delta))$. Using Step 1 of the proof, one gets $\beta<x(t)$ for these t, which implies $2+\Delta<B$. □

The subsequent upper estimate for B is a digression which is not necessary for solutions to reach $-A$ but will be employed in Section 3 below.

Proposition 2.6. *If* (3) *holds then for every $f\in F_{\beta\varepsilon}$ and for every $(\phi,u)\in A_{\beta\varepsilon\mu r}$ the quantities $B=B(\phi,u,f,\beta,\varepsilon,\mu,r)$ and $\Delta=\Delta(\beta,\varepsilon,\mu,r)$ satisfy*

$$
B<1+\Delta+\frac{a+\varepsilon}{a-\varepsilon}\left(1+\Delta+\frac{2}{\mu}\right).
$$

Proof. Let $(x,v)=(x,v)^{(\phi,u)}$. For every $t_e\geq 1+\Delta$ with $x(t-1)\geq\beta$ for all $t\in[1+\Delta,t_e]$ the inequality $f(x(t-1))<-a+\varepsilon$ yields

$$
\begin{aligned}
v(t)&=e^{-\mu(t-(1+\Delta))}v(1+\Delta)+\int_{1+\Delta}^{t}e^{-\mu(t-s)}f(x(s-1))\,ds\\
&<e^{-\mu(t-(1+\Delta))}\frac{a+\varepsilon}{\mu}+\frac{\varepsilon-a}{\mu}(1-e^{-\mu(t-(1+\Delta))})=e^{-\mu(t-(1+\Delta))}\frac{2a}{\mu}-\frac{a-\varepsilon}{\mu}.
\end{aligned}
$$

Recall that (3) implies (2). Integration and the last estimate of Proposition 2.2 yield

$$
\begin{aligned}
x(t)&=x(1+\Delta)+\int_{1+\Delta}^{t}v(s)\,ds\\
&<-\beta+\frac{a+\varepsilon}{\mu}(1+\Delta)+\int_{1+\Delta}^{t}\left(e^{-\mu(s-(1+\Delta))}\frac{2a}{\mu}-\frac{a-\varepsilon}{\mu}\right)ds
\end{aligned}
$$

$$\leq -\beta + \frac{a+\varepsilon}{\mu}(1+\Delta) - \frac{a-\varepsilon}{\mu}(t-(1+\Delta)) + \frac{2a}{\mu^2}(1-e^{-\mu(t-(1+\Delta))})$$
$$\leq -\beta + \frac{1}{\mu}\bigg((a+\varepsilon)\bigg(1+\Delta+\frac{2}{\mu}\bigg) - (a-\varepsilon)(t-(1+\Delta))\bigg).$$

For $t = B$,

$$\beta = x(B) < -\beta + \frac{1}{\mu}\bigg((a+\varepsilon)\bigg(1+\Delta+\frac{2}{\mu}\bigg) - (a-\varepsilon)(B-(1+\Delta))\bigg),$$

hence

$$(a-\varepsilon)(B-(1+\Delta)) < -2\mu\beta + (a+\varepsilon)\bigg(1+\Delta+\frac{2}{\mu}\bigg),$$

which implies the desired estimate. □

Next, conditions are given which guarantee $v(B) < -r(a-\varepsilon)/\mu$.

PROPOSITION 2.7. *If $\beta > 0$, $0 < \varepsilon < a$, $\mu > 0$, $0 < r < 1$ and if* (3)–(5) *and*

$$r \leq 1 - \bigg(\frac{2a}{a-\varepsilon} + 2(a+\varepsilon)\Delta\frac{\mu}{a-\varepsilon}\bigg)e^{-\mu} \tag{6}$$

with $\Delta = \Delta(\beta,\varepsilon,\mu,r)$ hold then for every $f \in F_{\beta\varepsilon}$ and each $(\phi,u) \in A_{\beta\varepsilon\mu r}$ the solution $(x,v) = (x,v)^{(\phi,u)}$ and $B = B(\phi,u,f,\beta,\varepsilon,\mu,r)$ satisfy

$$-\frac{a+\varepsilon}{\mu} < v(B) < -r\frac{a-\varepsilon}{\mu}.$$

PROOF. For the lower estimate of $v(B)$, see Proposition 2.1. In order to derive the upper estimate of $v(B)$, note first that as in the proof of Proposition 2.5,

$$v(1+\Delta) < v(1) + 2(a+\varepsilon)\Delta.$$

Proposition 2.4, the preceding estimate, and the upper estimate of $v(1)$ from Proposition 2.2 combined yield

$$\begin{aligned}
v(B) &\leq \bigg(v(1+\Delta) + \frac{a-\varepsilon}{\mu}\bigg)e^{-\mu(B-(1+\Delta))} - \frac{a-\varepsilon}{\mu}\\
&< \bigg(v(1) + 2(a+\varepsilon)\Delta + \frac{a-\varepsilon}{\mu}\bigg)e^{-\mu(B-(1+\Delta))} - \frac{a-\varepsilon}{\mu}\\
&\leq \bigg(ue^{-\mu} + \frac{1}{\mu}(a+\varepsilon)(1-e^{-\mu}) + 2(a+\varepsilon)\Delta + \frac{a-\varepsilon}{\mu}\bigg)e^{-\mu(B-(1+\Delta))}\\
&\quad - \frac{a-\varepsilon}{\mu}\\
&\leq \bigg(\frac{a+\varepsilon}{\mu}e^{-\mu} + \frac{1}{\mu}(a+\varepsilon)(1-e^{-\mu}) + 2(a+\varepsilon)\Delta + \frac{a-\varepsilon}{\mu}\bigg)e^{-\mu(B-(1+\Delta))}\\
&\quad - \frac{a-\varepsilon}{\mu}\\
&= \bigg(\frac{2a}{\mu} + 2(a+\varepsilon)\Delta\bigg)e^{-\mu(B-(1+\Delta))} - \frac{a-\varepsilon}{\mu}.
\end{aligned}$$

By Proposition 2.5, $1 < B - (1+\Delta)$. Hence

$$v(B) < \left(\frac{2a}{\mu} + 2(a+\varepsilon)\Delta\right)e^{-\mu} - \frac{a-\varepsilon}{\mu}.$$

Multiplication of (6) by $(a-\varepsilon)/\mu$ and rearrangement of terms yield the upper estimate of the assertion. □

It has not yet been shown that (3)–(6) are compatible.

PROPOSITION 2.8. *Let $\mu > \log 2$. Then there exist $\beta_\mu > 0$, $\varepsilon_\mu \in (0,a)$, and an analytic function $r_\mu\colon (-\beta_\mu,\beta_\mu) \times (-\varepsilon_\mu,\varepsilon_\mu) \to (0,1)$ so that for every $\beta \in (0,\beta_\mu)$ and $\varepsilon \in (0,\varepsilon_\mu)$ the parameters β, ε, μ and $r = r_\mu(\beta,\varepsilon)$ satisfy (3)–(6).*

PROOF. Let $\mu > \log 2$. Fix $R_\mu = 1 - 2e^{-\mu} \in (0,1)$. Recall the definition of Δ and observe that (6) is equivalent to

$$0 \le \frac{a-\varepsilon}{\mu}(1-r) - \left(\frac{2a}{\mu} + \frac{4(a+\varepsilon)\mu}{r(a-\varepsilon)}\beta\right)e^{-\mu}.$$

The function

$$\begin{aligned} q_\mu\colon (0,1)\times\mathbb{R}\times(-\infty,a) \ni (r,\beta,\varepsilon)& \\ \mapsto \frac{a-\varepsilon}{\mu}(1-r) &- \left(\frac{2a}{\mu} + \frac{4(a+\varepsilon)\mu}{r(a-\varepsilon)}\beta\right)e^{-\mu} \in \mathbb{R}\end{aligned}$$

satisfies

$$q_\mu(R_\mu,0,0) = \frac{a}{\mu}(1-R_\mu) - \frac{2a}{\mu}e^{-\mu} = \frac{a}{\mu}(1-R_\mu-2e^{-\mu}) = 0.$$

Solve the equation $q_\mu(r,\beta,\varepsilon) = 0$ for r. There exist $\beta_{\mu 0} > 0$, $\varepsilon_{\mu 0} \in (0,a)$, and an analytic real function $r_{\mu 0}\colon (-\beta_{\mu 0},\beta_{\mu 0}) \times (-\varepsilon_{\mu 0},\varepsilon_{\mu 0}) \to (0,1)$ so that

$$r_{\mu 0}(0,0) = R_\mu \in (0,1)$$

and

$$q_\mu(r_{\mu 0}(\beta,\varepsilon),\beta,\varepsilon) = 0 \quad \text{for all } (\beta,\varepsilon) \in (-\beta_{\mu 0},\beta_{\mu 0}) \times (-\varepsilon_{\mu 0},\varepsilon_{\mu 0}).$$

In particular, (6) holds as an equation for $r = r_{\mu 0}(\beta,\varepsilon)$ and $(\beta,\varepsilon) \in (0,\beta_{\mu 0}) \times (0,\varepsilon_{\mu 0})$.

It is obvious that (3) and (4) hold for $\beta = 0 = \varepsilon$, μ, and $r = R_\mu$. (5) holds as well since

$$\begin{aligned} p(0,0,\mu,R_\mu,1) &= \frac{a}{\mu}\left(1 + (R_\mu - 1)\frac{1-e^{-\mu}}{\mu}\right) - \frac{a}{\mu} \\ &\quad + \left((R_\mu - 1)\frac{a}{\mu}e^{-\mu} + \frac{2a}{\mu}\right)\frac{1}{\mu}(1-e^{-\mu}) \\ &= \frac{a}{\mu}\frac{1-e^{-\mu}}{\mu}(R_\mu - 1 + 2 + (R_\mu - 1)e^{-\mu}) > 0. \end{aligned}$$

It follows that there exist $\beta_\mu \in (0,\beta_{\mu 0})$, $\varepsilon_\mu \in (0,\varepsilon_{\mu 0})$ so that the restriction r_μ of $r_{\mu 0}$ to $(-\beta_\mu,\beta_\mu)\times(-\varepsilon_\mu,\varepsilon_\mu)$ has the asserted properties. $\square$

The following result summarizes what has been achieved.

COROLLARY 2.1. *Let $\mu > \log 2$, $0<\beta<\beta_\mu$, $0<\varepsilon<\varepsilon_\mu$, $r = r_\mu(\beta,\varepsilon) \in (0,1)$, $f \in F_{\beta\varepsilon}$. For every $(\phi,u)\in A = A_{\beta\varepsilon\mu r}$ there exist positive reals $b = b(\phi,u,f,\beta,\varepsilon,\mu,r)$ and $B = B(\phi,u,f,\beta,\varepsilon,\mu,r) > b$ so that for $\Delta = \Delta(\beta,\varepsilon,\mu,r)$,*

$$b<\Delta, \quad \Delta+2<B,$$

and the solution $(x,v) = (x,v)^{(\phi,u)}$ of (1) satisfies

$$x(b)=\beta,\ \beta<x(t) \quad \textit{for all } t\in(b,B),\ x(B)=\beta$$

and

$$-\frac{a+\varepsilon}{\mu} < v(B) < -r\frac{a-\varepsilon}{\mu}.$$

In particular, $(x_B, v(B)) \in -A$.

For every fixed point $(\overline{\phi},\overline{u})$ of the map

$$R = R_{\beta\varepsilon\mu r f}\colon A \ni (\phi,u)\mapsto -(x_B, v(B)) \in A$$

there exists a periodic solution $(\overline{x},\overline{v})$ on $\mathbb{R}$ of (1) with $(\overline{x}_0,\overline{v}(0)) = (\overline{\phi},\overline{u})$. For $\overline{B} = B(\overline{\phi},\overline{u},f,\beta,\varepsilon,\mu,r)$,

$$\overline{x}(t+\overline{B}) = -\overline{x}(t),\ \overline{v}(t+\overline{B}) = -\overline{v}(t) \quad \textit{for all } t\in\mathbb{R},$$

and $(\overline{x},\overline{v})$ has minimal period $2\overline{B}$.

REMARK. It is easy to show that the return map R of the preceding corollary is continuous and maps bounded sets into sets with compact closure. Therefore Schauder's theorem guarantees the existence of fixed points. This is not pursued here as the objective are *attracting* fixed points and periodic orbits.

The following lower estimate of B which improves Proposition 2.5 will be important in the sequel.

PROPOSITION 2.9. *Let $\eta\in(0,1)$. Then there exists $\mu_\eta > \log 2$ so that for each $\mu>\mu_\eta$ there are $\beta_{\mu\eta}\in(0,\beta_\mu)$ and $\varepsilon_{\mu\eta}\in(0,\varepsilon_\mu)$ with the following property: For all $\beta\in(0,\beta_{\mu\eta})$, $\varepsilon\in(0,\varepsilon_{\mu\eta})$, for $r=r_\mu(\beta,\varepsilon)$, for all $f\in F_{\beta\varepsilon}$ and $(\phi,u)\in A_{\beta\varepsilon\mu r}$, $B = B(\phi,u,f,\beta,\varepsilon,\mu,r)$ and $\Delta=\Delta(\beta,\varepsilon,\mu,r)$ satisfy*

$$B-2-\Delta > \frac{2-\eta}{\mu}.$$

PROOF. *Step 1.* Let $\eta\in(0,1)$ be given. Choose $\eta_0\in(0,\eta)$. Recall from the proof of Proposition 2.8 the equation

$$p(0,0,\mu,r_\mu(0,0),1) = \frac{1-e^{-\mu}}{\mu^2}a(-2e^{-\mu}(1+e^{-\mu})+2)$$

for $\mu > \log 2$. It follows that there exists $\mu_\eta > \log 2$ so that for all $\mu \geq \mu_\eta$,

$$p(0,0,\mu,r_\mu(0,0),1) > \frac{(2-\eta_0)a}{\mu^2}.$$

Consider $\mu \geq \mu_\eta$. Choose $\overline{\beta}_\mu \in (0,\beta_\mu)$ and $\overline{\varepsilon}_\mu \in (0,\varepsilon_\mu)$ so small that for all $\beta \in (0,\overline{\beta}_\mu)$ and all $\varepsilon \in (0,\overline{\varepsilon}_\mu)$,

$$p(\beta,\varepsilon,\mu,r_\mu(\beta,\varepsilon),1) > \frac{(2-\eta_0)a}{\mu^2}.$$

Step 2. Let $\beta \in (0,\overline{\beta}_\mu)$, $\varepsilon \in (0,\overline{\varepsilon}_\mu)$, $r = r_\mu(\beta,\varepsilon)$, $f \in F_{\beta\varepsilon}$, $(\phi,u) \in A_{\beta\varepsilon\mu r}$, $\Delta = \Delta(\beta,\varepsilon,\mu,r)$, $B = B(\phi,u,f,\beta,\varepsilon,\mu,r)$. The second component of the solution $(x,v) = (x,v)^{(\phi,u)}$ of (1) is bounded by $(a+\varepsilon)/\mu$. This bound and the inequality $B > 2+\Delta$ (Proposition 2.5) combined give

$$\beta = x(B) = x(2+\Delta) + \int_{2+\Delta}^{B} v(t)\,dt \geq x(2+\Delta) - \frac{a+\varepsilon}{\mu}(B-(2+\Delta)).$$

Estimates as in the beginning of the proof of Proposition 2.5 yield

$$x(2+\Delta) > p(\beta,\varepsilon,\mu,r_\mu(\beta,\varepsilon),1).$$

The preceding inequalities combined with the choice of $\overline{\beta}_\mu$ and $\overline{\varepsilon}_\mu$ imply

$$\beta > \frac{(2-\eta_0)a}{\mu^2} - \frac{a+\varepsilon}{\mu}(B-(2+\Delta))$$

which is equivalent to

$$B - 2 - \Delta > -\frac{\mu\beta}{a+\varepsilon} + \frac{(2-\eta_0)a}{(a+\varepsilon)\mu}.$$

Now it becomes obvious that there exist $\beta_{\mu\eta} \in (0,\overline{\beta}_\mu)$ and $\varepsilon_{\mu\eta} \in (0,\overline{\varepsilon}_\mu)$ such that for all $\beta \in (0,\beta_{\mu\eta})$, $\varepsilon \in (0,\varepsilon_{\mu\eta})$, $r = r_\mu(\beta,\varepsilon)$, $f \in F_{\beta\varepsilon}$, and $(\phi,u) \in A_{\beta\varepsilon\mu r}$,

$$B - 2 - \Delta > \frac{2-\eta}{\mu}$$

for $B = B(\phi,u,f,\beta,\varepsilon,\mu,r)$ and $\Delta = \Delta(\beta,\varepsilon,\mu,r)$. □

Remark. The estimate

$$B - 1 - \frac{a+\varepsilon}{a-\varepsilon} - \Delta \leq \frac{a+\varepsilon}{a-\varepsilon}\left(\Delta + \frac{2}{\mu}\right)$$

of Proposition 2.6 implies that for certain sequences $\beta_n \to 0$, $\varepsilon_n \to 0$, $\mu_n \to \infty$, with $r_n = r_{\mu_n}(\beta_n,\varepsilon_n)$, $A_n = A_{\beta_n\varepsilon_n\mu_n r_n}$, $F_n = F_{\beta_n\varepsilon_n}$,

$$\limsup_{n\to\infty} \sup_{(\phi,u)\in A_n, f\in F_n} B(\phi,u,f,\beta_n,\varepsilon_n,\mu_n,r_n) \leq 2.$$

This shows in which sense the lower estimate of the preceding proposition is optimal.

3. Contracting return maps

For $\mu > \log 2$, $0 < \beta < \beta_\mu$, $0 < \varepsilon < \varepsilon_\mu$, $r = r_\mu(\beta, \varepsilon)$, $f \in F_{\beta\varepsilon}$, $A = A_{\beta\varepsilon\mu r}$ consider the return map $R : A \to A$ from Corollary 2.1. The first objective is to find an upper estimate of $\mathrm{Lip}\,(R)$ in terms of β, ε, μ, $L = \mathrm{Lip}\,(f)$ and

$$L_\beta = \mathrm{Lip}\,(f \mid [\beta, \infty)) = \mathrm{Lip}\,(f \mid (-\infty, -\beta]).$$

Notice already here that necessarily $L \geq (a - \varepsilon)/\beta$ becomes large for β small, while on the other hand each set $F_{\beta\varepsilon}$ contains functions f with L_β arbitrarily small and zero.

Let $\Delta = \Delta(\beta, \varepsilon, \mu, r)$. It is convenient to write $R = Q \circ P$ as composition of the map $P = P_{\beta\varepsilon\mu r f} = S_{\mu f}(1 + \Delta, \cdot)|A$ with the map $Q = Q_{\beta\varepsilon\mu r f}$ from $P(A)$ to $-A$ given by $Q(\psi, w) = S_{\mu f}(T(\psi, w), (\psi, w))$ where the map $T = T_{\beta\varepsilon\mu r f}$ from $P(A)$ to $(1, \infty)$ is defined by

$$T(\psi, w) = B(\phi, u, f, \beta, \varepsilon, \mu, r) - (1 + \Delta) \quad \text{for all } (\phi, u) \in A \text{ with } (\psi, w) = P(\phi, u).$$

(Observe that indeed all such $B(\phi, u, f, \beta, \varepsilon, \mu, r)$ coincide – the first argument in $[1 + \Delta, \infty)$ where x reaches the level β depends on $x \mid [\Delta, 1 + \Delta]$ and $v(1 + \Delta)$ but not on values at smaller arguments.)

The following estimates control the deviation of solutions from each other.

PROPOSITION 3.1. *Let (x, v), $(\overline{x}, \overline{v})$ be solutions of (1) on $[0, \infty)$. For every $t \in [0, 1]$, the following estimates hold.*

$$|v(t) - \overline{v}(t)| \leq |v(0) - \overline{v}(0)| e^{-\mu t} + L \frac{1 - e^{-\mu t}}{\mu} \|x_0 - \overline{x}_0\|,$$
$$|x(t) - \overline{x}(t)| \leq |v(0) - \overline{v}(0)| \frac{1 - e^{-\mu t}}{\mu} + \left(1 + L \frac{t}{\mu}\right) \|x_0 - \overline{x}_0\|.$$

In case $\beta \leq x(s)$, $\beta \leq \overline{x}(s)$ for all $s \in [-1, 0]$ the analogues of the previous estimates with L_β instead of L hold.

In case $x(s) \leq -\beta$, $\overline{x}(s) \leq -\beta$ for all $s \in [-1, 0]$ and $x(0) = -\beta = \overline{x}(0)$,

$$|v(t) - \overline{v}(t)| \leq |v(0) - \overline{v}(0)| e^{-\mu t} + L_\beta \frac{1 - e^{-\mu t}}{\mu} \|x_0 - \overline{x}_0\|,$$
$$|x(t) - \overline{x}(t)| \leq |v(0) - \overline{v}(0)| \frac{1 - e^{-\mu t}}{\mu} + L_\beta \frac{t}{\mu} \|x_0 - \overline{x}_0\|.$$

PROOF. The first estimate follows by means of the variation-of-constants formula and using the Lipschitz constant $L \leq \infty$ of f in the integrand. The second estimate follows from the first one using

$$|x(t) - \overline{x}(t)| = \left| x(0) - \overline{x}(0) + \int_0^t (v(s) - \overline{v}(s))\, ds \right| \leq \|x_0 - \overline{x}_0\| + \int_0^t |v(s) - \overline{v}(s)|\, ds.$$

The remaining assertions are obtained similarly. $\square$

It is convenient to restrict the range of parameters from here on as follows. Choose $\mu_0 > \log 2$ so that for all $\mu > \mu_0$,

$$e^{-\mu} + \frac{1-e^{-\mu}}{2\mu} \leq 1.$$

For each $\mu > \mu_0$ choose $\beta_{\mu 0} \in (0, \beta_\mu)$, $\varepsilon_{\mu 0} \in (0, \varepsilon_\mu)$ so that

$$\Delta(\beta, \varepsilon, \mu, r_\mu(\beta, \varepsilon)) = \frac{2\mu\beta}{r_\mu(\beta,\varepsilon)(a-\varepsilon)} < 1$$

for all $\beta \in (0, \beta_{\mu 0})$, $\varepsilon \in (0, \varepsilon_{\mu 0})$.

The next result contains an estimate of $\mathrm{Lip}\,(P)$.

PROPOSITION 3.2. *Let $\mu > \mu_0$, $\beta \in (0, \beta_{\mu 0})$, $\varepsilon \in (0, \varepsilon_{\mu 0})$, $f \in F_{\beta\varepsilon}$, $r = r_\mu(\beta, \varepsilon)$, $A = A_{\beta\varepsilon\mu r}$, $\Delta = \Delta(\beta, \varepsilon, \mu, r)$, $P = P_{\beta\varepsilon\mu r f}$. Let (ϕ, u), $(\overline{\phi}, \overline{u})$ in A be given. Then $(x, v) = (x, v)^{(\phi,u)}$ and $(\overline{x}, \overline{v}) = (x, v)^{(\overline{\phi},\overline{u})}$ satisfy*

$$\|x_{1+\Delta} - \overline{x}_{1+\Delta}\| \leq \left(\frac{2}{\mu} + \frac{L\Delta}{\mu^2}\right)|u - \overline{u}| + \frac{L_\beta}{\mu}\left(1 + \Delta\left(1 + L\frac{\Delta^2}{2}\right)\right)\|\phi - \overline{\phi}\|,$$
$$|v(1+\Delta) - \overline{v}(1+\Delta)| \leq \left(e^{-\mu} + \frac{L\Delta}{\mu}\right)|u - \overline{u}| + \left(\frac{L_\beta}{\mu} + \frac{LL_\beta}{\mu}\frac{\Delta^2}{2}\right)\|\phi - \overline{\phi}\|.$$

In particular,

$$\mathrm{Lip}\,(P) \leq e^{-\mu} + \frac{L\Delta}{\mu} + \frac{1}{\mu} + \frac{L\Delta}{2\mu^2} + \frac{L_\beta}{\mu}\left(1 + \Delta\left(1 + \frac{L\Delta^2}{2}\right)\right) + \frac{2L_\beta}{\mu} + \frac{2LL_\beta}{\mu}\frac{\Delta^2}{2}.$$

PROOF. *Step 1.* For $0 \leq t \leq 1$,

$$|v(t) - \overline{v}(t)| \leq e^{-\mu t}|u - \overline{u}| + L_\beta \frac{1 - e^{-\mu t}}{\mu}\|\phi - \overline{\phi}\| \leq e^{-\mu t}|u - \overline{u}| + \frac{L_\beta}{\mu}\|\phi - \overline{\phi}\|,$$

hence

$$\begin{aligned}|x(t) - \overline{x}(t)| &= \left| -\beta - (-\beta) + \int_0^t v(s)\,ds - \int_0^t \overline{v}(s)\,ds \right| \leq \int_0^t |v(s) - \overline{v}(s)|\,ds \\ &\leq \frac{1}{\mu}(1 - e^{-\mu t})|u - \overline{u}| + \frac{L_\beta}{\mu}\|\phi - \overline{\phi}\| t \leq \frac{1}{\mu}|u - \overline{u}| + \frac{L_\beta}{\mu}\|\phi - \overline{\phi}\|.\end{aligned}$$

Step 2. For $1 \le t \le 1+\Delta < 2$ it follows that

$$\begin{aligned}
|v(t)-\overline{v}(t)| &\le e^{-\mu(t-1)}|v(1)-\overline{v}(1)| + \int_1^t e^{-\mu(t-s)}|f(x(s-1))-f(\overline{x}(s-1)|\,ds \\
&\le e^{-\mu t}|u-\overline{u}| + e^{-\mu(t-1)}\frac{L_\beta}{\mu}\|\phi-\overline{\phi}\| \\
&\quad + \int_1^t e^{-\mu(t-s)} L\left(\frac{1}{\mu}(1-e^{-\mu(s-1)})|u-\overline{u}| + \frac{L_\beta}{\mu}\|\phi-\overline{\phi}\|(s-1)\right)ds \\
&\le e^{-\mu t}|u-\overline{u}| + \frac{L_\beta}{\mu}\|\phi-\overline{\phi}\| + \int_1^t L|u-\overline{u}|\frac{1}{\mu}(e^{-\mu(t-s)}-e^{-\mu(t-1)})\,ds \\
&\quad + L\frac{L_\beta}{\mu}\frac{(t-1)^2}{2}\|\phi-\overline{\phi}\| \\
&\le e^{-\mu t}|u-\overline{u}| + \frac{L}{\mu}|u-\overline{u}|\left(\frac{1}{\mu}(1-e^{-\mu(t-1)})-(t-1)e^{-\mu(t-1)}\right) \\
&\quad + \frac{L_\beta}{\mu}\left(1+L\frac{\Delta^2}{2}\right)\|\phi-\overline{\phi}\|.
\end{aligned}$$

By

$$\frac{1}{\mu}(1-e^{-\mu(t-1)})-(t-1)e^{-\mu(t-1)} \le \frac{1}{\mu}(1-e^{-\mu(t-1)}) \le t-1,$$

for $t = 1+\Delta > 1$,

$$|v(1+\Delta)-\overline{v}(1+\Delta)| \le \left(e^{-\mu}+\frac{L\Delta}{\mu}\right)|u-\overline{u}| + \frac{L_\beta}{\mu}\left(1+L\frac{\Delta^2}{2}\right)\|\phi-\overline{\phi}\|.$$

Step 3. Recall $\|x_{1+\Delta}-\overline{x}_{1+\Delta}\| = \max_{t\in[-1,0]}|x(1+\Delta+t)-\overline{x}(1+\Delta+t)|$. For $t\in[-1,0]$ with $0\le 1+\Delta+t\le 1$, Step 1 yields

$$|x(1+\Delta+t)-\overline{x}(t+\Delta+t)| \le \frac{1}{\mu}|u-\overline{u}| + \frac{L_\beta}{\mu}\|\phi-\overline{\phi}\|.$$

In case $1 < 1+\Delta+t \le 1+\Delta$,

$$|x(1+\Delta+t)-\overline{x}(1+\Delta+t)| \le |x(1)-\overline{x}(1)| + \int_1^{1+\Delta+t}|v(s)-\overline{v}(s)|\,ds.$$

The last estimate in Step 1 at $t=1$, the estimate of $v(s)-\overline{v}(s)$ for $1\le s\le 1+\Delta$ in Step 2, and the inequality $-e^{-\mu(s-1)}/\mu-(s-1)e^{-\mu(s-1)}\le 0$ combined imply

$$\begin{aligned}
|x(1+\Delta+t)-\overline{x}(1+\Delta+t)| &\le \frac{1}{\mu}|u-\overline{u}| + \frac{L_\beta}{\mu}\|\phi-\overline{\phi}\| \\
&\quad + \int_1^{1+\Delta+t}\left(e^{-\mu s}|u-\overline{u}| + \frac{L}{\mu^2}|u-\overline{u}| + \frac{L_\beta}{\mu}\left(1+L\frac{\Delta^2}{2}\right)\|\phi-\overline{\phi}\|\right)ds \\
&\le \frac{1}{\mu}|u-\overline{u}| + \frac{L_\beta}{\mu}\|\phi-\overline{\phi}\| + \frac{1}{\mu}(e^{-\mu}-e^{-\mu(1+\Delta+t)})|u-\overline{u}| \\
&\quad + (\Delta+t)\frac{L}{\mu^2}|u-\overline{u}| + (\Delta+t)\frac{L_\beta}{\mu}\left(1+L\frac{\Delta^2}{2}\right)\|\phi-\overline{\phi}\|.
\end{aligned}$$

By means of $e^{-\mu} - e^{-\mu(1+\Delta+t)} \le e^{-\mu} \le 1$,

$$|x(1+\Delta+t) - \overline{x}(1+\Delta+t)| \le \left(\frac{2}{\mu} + \frac{L\Delta}{\mu^2}\right)|u-\overline{u}| \\ + \frac{L_\beta}{\mu}\left(1+\Delta\left(1+L\frac{\Delta^2}{2}\right)\right)\|\phi-\overline{\phi}\|,$$

and the asserted estimate of $\|x_{1+\Delta} - \overline{x}_{1+\Delta}\|$ follows. Consequently,

$$\begin{aligned}\|P(\phi,u) &- P(\overline{\phi},\overline{u})\| \\ &= \frac{1}{2}\|x_{1+\Delta} - \overline{x}_{1+\Delta}\| + |v(1+\Delta) - \overline{v}(1+\Delta)| \\ &\le \left(\frac{1}{\mu} + \frac{L\Delta}{2\mu^2} + e^{-\mu} + \frac{L\Delta}{\mu}\right)|u-\overline{u}| + \frac{L_\beta}{\mu}\left(1+\Delta\left(1+L\frac{\Delta^2}{2}\right)\right)\frac{1}{2}\|\phi-\overline{\phi}\| \\ &\quad + \left(\frac{2L_\beta}{\mu} + \frac{2LL_\beta}{\mu}\frac{\Delta^2}{2}\right)\frac{1}{2}\|\phi-\overline{\phi}\|,\end{aligned}$$

which shows the asserted estimate of $\mathrm{Lip}\,(P)$. □

Remark. It is the term $L\Delta/\mu$ in the estimate of $\mathrm{Lip}\,(P)$ which presents most of the difficulties on the way to contracting return maps.

The next result prepares the proof of an estimate of $\mathrm{Lip}\,(T)$, which will be needed for the derivation of an estimate of $\mathrm{Lip}\,(Q)$.

Proposition 3.3. *Let $\mu > \mu_0$, $\beta \in (0, \beta_{\mu 0})$, $\varepsilon \in (0, \varepsilon_{\mu 0})$, $f \in F_{\beta\varepsilon}$, $r = r_\mu(\beta,\varepsilon)$, $A = A_{\beta\varepsilon\mu r}$, $\Delta = \Delta(\beta,\varepsilon,\mu,r)$. Let (ϕ,u) and $(\overline{\phi},\overline{u})$ in X be given. Set $(x,v) = (x,v)^{(\phi,u)}$ and $(\overline{x},\overline{v}) = (x,v)^{(\overline{\phi},\overline{u})}$.*

(a) *If $\beta \le \phi(t)$ and $\beta \le \overline{\phi}(t)$ for all $t \in [-1,0]$ then*

$$\|S_{\mu f}(1,(\phi,u)) - S_{\mu f}(1,(\overline{\phi},\overline{u}))\| \le \left(1 + \frac{3L_\beta}{\mu}\right)\|(\phi,u) - (\overline{\phi},\overline{u})\|$$

and, for every $t \in [0,1]$,

$$|v(t) - \overline{v}(t)| \le \left(1 + \frac{2L_\beta}{\mu}\right)\|(\phi,u) - (\overline{\phi},\overline{u})\|.$$

(b) *Suppose $(\phi,u) \in A$, $(\overline{\phi},\overline{u}) \in A$. Let $(\psi,w) = (x_{1+\Delta},\, v(1+\Delta))$, $(\overline{\psi},\overline{w}) = (\overline{x}_{1+\Delta}, \overline{v}(1+\Delta))$, $B = B(\phi,u,f,\beta,\varepsilon,\mu,r) \le \overline{B} = B(\overline{\phi},\overline{u},f,\beta,\varepsilon,\mu,r)$.*

Then, for $t \in [1+\Delta, B]$,

$$|v(t) - \overline{v}(t)| \le \left(1 + \frac{3L_\beta}{\mu}\right)^{1+[(a+\varepsilon)/(a-\varepsilon)](1+\Delta+2/\mu)}\|(\psi,w) - (\overline{\psi},\overline{w})\|$$

and

$$|x(t) - \overline{x}(t)| \le \left(2 + \frac{a+\varepsilon}{a-\varepsilon}\left(1+\Delta+\frac{2}{\mu}\right) \times \left(1+\frac{3L_\beta}{\mu}\right)^{1+[(a+\varepsilon)/(a-\varepsilon)](1+\Delta+2/\mu)}\right) \\ \times \|(\psi,w) - (\overline{\psi},\overline{w})\|.$$

PROOF. (a) By Proposition 3.1,

$$\begin{aligned}\|(x_1, v(1)) - (\overline{x}_1, \overline{v}(1))\| &= \frac{1}{2}\|x_1 - \overline{x}_1\| + |v(1) - \overline{v}(1)| \\ &\leq \left(\frac{1-e^{-\mu}}{2\mu} + e^{-\mu}\right)|u - \overline{u}| + \left(\frac{2L_\beta}{\mu} + 1 + \frac{L_\beta}{\mu}\right)\frac{1}{2}\|\phi - \overline{\phi}\|.\end{aligned}$$

By the choice of μ,

$$\frac{1-e^{-\mu}}{2\mu} + e^{-\mu} < 1 < 1 + \frac{3L_\beta}{\mu}.$$

The asserted Lipschitz estimate of $S_{\mu f}(1, \cdot\,)$ follows. Similarly one obtains the estimate of $|v(t) - \overline{v}(t)|$.

(b) Recall Proposition 2.5 and consider the integer $j \geq 1$ given by $1+\Delta+j \leq B < 1 + \Delta + j + 1$. By Proposition 2.6,

$$j < \frac{a+\varepsilon}{a-\varepsilon}\left(1 + \Delta + \frac{2}{\mu}\right).$$

If $t \in [1+\Delta, B] \subset [1+\Delta, \overline{B}]$ then $\beta \leq x(t-1)$, $\beta \leq \overline{x}(t-1)$. Therefore the first estimate in part (a) of the proposition and induction yield

$$\operatorname{Lip}\,(S_{\mu f}(k, \cdot\,) \mid S_{\mu f}(1+\Delta, A)) \leq \left(1 + \frac{3L_\beta}{\mu}\right)^k \quad \text{for } k = 0, \dots, j.$$

The second estimate in part (a) of the proposition shows that for every $k \in \{0, \dots, j\}$ and $t \in [1 + \Delta + k, 1 + \Delta + k + 1]$,

$$\begin{aligned}|v(t) - \overline{v}(t)| &\leq \left(1 + \frac{2L_\beta}{\mu}\right)\|S_{\mu f}(k, (\psi, w)) - S_{\mu f}(k, (\overline{\psi}, \overline{w}))\| \\ &\leq \left(1 + \frac{3L_\beta}{\mu}\right)^{1+k}\|(\psi, w) - (\overline{\psi}, \overline{w})\| \\ &\leq \left(1 + \frac{3L_\beta}{\mu}\right)^{1+j}\|(\psi, w) - (\overline{\psi}, \overline{w})\| \\ &\leq \left(1 + \frac{3L_\beta}{\mu}\right)^{1+[(a+\varepsilon)/(a-\varepsilon)](1+\Delta+2/\mu)}\|(\psi, w) - (\overline{\psi}, \overline{w})\|.\end{aligned}$$

The estimate of

$$\begin{aligned}|x(t) - \overline{x}(t)| &\leq |x(1+\Delta) - \overline{x}(1+\Delta)| + \int_{1+\Delta}^t |v(s) - \overline{v}(s)|\,ds \\ &= |\psi(0) - \overline{\psi}(0)| + \int_{1+\Delta}^t |v(s) - \overline{v}(s)|\,ds\end{aligned}$$

follows from the estimate of the integrand combined with

$$t - (1+\Delta) \leq B - (1+\Delta) \leq \frac{a+\varepsilon}{a-\varepsilon}\left(1 + \Delta + \frac{2}{\mu}\right)$$

and

$$|\psi(0) - \overline{\psi}(0)| \leq 2\|(\psi, w) - (\overline{\psi}, \overline{w})\|. \qquad \square$$

It is convenient to introduce

$$c = c(\beta,\varepsilon,\mu,\lambda) = 2 + \frac{a+\varepsilon}{a-\varepsilon}\left(1+\Delta+\frac{2}{\mu}\right)\left(1+\frac{3\lambda}{\mu}\right)^{1+[(a+\varepsilon)/(a-\varepsilon)](1+\Delta+2/\mu)}$$

for $\mu > \mu_0$, $|\beta| < \beta_{\mu 0}$, $|\varepsilon| < \varepsilon_{\mu 0}$, $r = r_\mu(\beta,\varepsilon)$, $\Delta = \Delta(\beta,\varepsilon,\mu,r)$ and $\lambda \in [0,\infty)$. Clearly, $c > 3$ and

$$\lim_{\mu\to\infty} c(0,0,\mu,0) = 3. \tag{7}$$

Proposition 3.4. *Let $\mu > \mu_0$, $\beta \in (0,\beta_{\mu 0})$, $\varepsilon \in (0,\varepsilon_{\mu 0})$, $f \in F_{\beta\varepsilon}$, $r = r_\mu(\beta,\varepsilon)$. If*

$$\frac{2a}{a-\varepsilon} < e^{\mu} \tag{8}$$

holds then

$$\mathrm{Lip}\,(T_{\beta\varepsilon\mu r f}) \le \frac{\mu}{|e^{-\mu}2a - a + \varepsilon|} c(\beta,\varepsilon,\mu,L_\beta).$$

Proof. *Step 1.* Let $\Delta = \Delta(\beta,\varepsilon,\mu,r)$, $(\psi,w) = S_{\mu f}(1+\Delta,(\phi,u))$ and $(\overline{\psi},\overline{w}) = S_{\mu f}(1+\Delta,(\overline{\phi},\overline{u}))$, with (ϕ,u) and $(\overline{\phi},\overline{u})$ in $A_{\beta\varepsilon\mu r}$. Set $(x,v) = (x,v)^{(\phi,u)}$ and $(\overline{x},\overline{v}) = (x,v)^{(\overline{\phi},\overline{u})}$, $B = B(\phi,u,f,\beta,\varepsilon,\mu,r)$, $\overline{B} = B(\overline{\phi},\overline{u},f,\beta,\varepsilon,\mu,r)$. Then

$$T_{\beta\varepsilon\mu r f}(\psi,w) - T_{\beta\varepsilon\mu r f}(\overline{\psi},\overline{w}) = B - \overline{B}.$$

Step 2. Recall

$$\psi(0) + \int_{1+\Delta}^{B} v(t)\,dt = x(B) = \beta = \overline{x}(\overline{B}) = \overline{\psi}(0) + \int_{1+\Delta}^{\overline{B}} \overline{v}(t)\,dt.$$

In case $\overline{B} \ge B$ it follows that

$$\begin{aligned}\|\psi - \overline{\psi}\| \ge |\psi(0) - \overline{\psi}(0)| &= \left|\int_{1+\Delta}^{B} (\overline{v} - v)(t)\,dt + \int_{B}^{\overline{B}} \overline{v}(t)\,dt\right| \\ &\ge \left|\int_{B}^{\overline{B}} \overline{v}(t)\,dt\right| - \int_{1+\Delta}^{B} |\overline{v}(t) - v(t)|\,dt.\end{aligned}$$

An application of Proposition 3.3 to the last integrand yields

$$\begin{aligned}&\|\psi - \overline{\psi}\| + (B - (1+\Delta)) \\ &\quad\cdot \left(1+\frac{3L_\beta}{\mu}\right)^{1+[(a+\varepsilon)/(a-\varepsilon)](1+\Delta+2/\mu)} \|(\psi,w) - (\overline{\psi},\overline{w})\| \ge \left|\int_{B}^{\overline{B}} \overline{v}(t)\,dt\right|.\end{aligned}$$

Proposition 2.6 gives

$$\begin{aligned}&\|\psi - \overline{\psi}\| + \frac{a+\varepsilon}{a-\varepsilon}\left(1+\Delta+\frac{2}{\mu}\right) \\ &\quad\cdot \left(1+\frac{3L_\beta}{\mu}\right)^{1+[(a+\varepsilon)/(a-\varepsilon)](1+\Delta+2/\mu)} \|(\psi,w) - (\overline{\psi},\overline{w})\| \ge \left|\int_{B}^{\overline{B}} \overline{v}(t)\,dt\right|.\end{aligned}$$

Step 3. Estimate of the last integrand: By Proposition 2.5, $2+\Delta \le t \le \overline{B}$ for $B \le t \le \overline{B}$, hence $\overline{x}(s-1)) \ge \beta$ for all $s \in [1+\Delta, t]$, and therefore $f(\overline{x}(s-1)) \le -a+\varepsilon$. By Proposition 2.1,

$$\overline{w} \le \frac{a+\varepsilon}{\mu}.$$

It follows that

$$\begin{aligned}\overline{v}(t) &= e^{-\mu(t-(1+\Delta))}\overline{w} + \int_{1+\Delta}^{t} e^{-\mu(t-s)} f(\overline{x}(s-1))\,ds \\ &\le e^{-\mu(t-(1+\Delta))}\frac{a+\varepsilon}{\mu} - (a-\varepsilon)\frac{1}{\mu}(1-e^{-\mu(t-(1+\Delta))}).\end{aligned}$$

As $2+\Delta \le t$,

$$\overline{v}(t) \le e^{-\mu}\frac{a+\varepsilon}{\mu} - (a-\varepsilon)\frac{1}{\mu}(1-e^{-\mu}) = \frac{1}{\mu}(e^{-\mu}2a - a + \varepsilon).$$

By hypothesis, the last term is negative.

Step 4. It follows that

$$\left|\int_B^{\overline{B}} \overline{v}(t)\,dt\right| = \int_B^{\overline{B}} |\overline{v}(t)|\,dt \ge (\overline{B}-B)\frac{1}{\mu}|e^{-\mu}2a - a + \varepsilon|.$$

Finally, the estimate of $\mathrm{Lip}\,(T_{\beta\varepsilon\mu rf})$ becomes obvious from the preceding estimate in combination with the results of Steps 1 and 2 and

$$\|\psi - \overline{\psi}\| \le 2\|(\psi,w) - (\overline{\psi},\overline{w})\|. \qquad \square$$

The proof of an estimate of $\mathrm{Lip}\,(Q)$ begins with upper estimates of $|\overline{v}(\overline{B}+s) - v(B+s)|$, $s \in [-1,0]$, in terms of $|\overline{v}(1+\Delta) - v(1+\Delta)|$, $\|(\overline{\psi},\overline{w}) - (\psi,w)\|$, and $|\overline{B} - B| = |T(\overline{\psi},\overline{w}) - T(\psi,w)|$.

PROPOSITION 3.5. *Let $\mu > \mu_0$, $\beta \in (0,\beta_{\mu 0})$, $\varepsilon \in (0,\varepsilon_{\mu 0})$, $f \in F_{\beta\varepsilon}$, $r = r_\mu(\beta,\varepsilon)$, $A = A_{\beta\varepsilon\mu r}$, $\Delta = \Delta(\beta,\varepsilon,\mu,r)$. Let $(\phi,u) \in A$, $(\overline{\phi},\overline{u}) \in A$, $(x,v) = (x,v)^{(\phi,u)}$, $(\overline{x},\overline{v}) = (x,v)^{(\overline{\phi},\overline{u})}$. Set $(\psi,w) = (x_{1+\Delta}, v(1+\Delta))$, $(\overline{\psi},\overline{w}) = (\overline{x}_{1+\Delta}, \overline{v}(1+\Delta))$. Suppose*

$$B = B(\phi,u,f,\beta,\varepsilon,\mu,r) \le \overline{B} = B(\overline{\phi},\overline{u},f,\beta,\varepsilon,\mu,r).$$

Then, for every $s \in [-1,0]$,

$$\begin{aligned}|\overline{v}(\overline{B}+s) - v(B+s)| \le{}& e^{-\mu(B+s-(1+\Delta))}|\overline{v}(1+\Delta) - v(1+\Delta)| \\ &+ \frac{L_\beta}{\mu}c(\beta,\varepsilon,\mu,L_\beta)\|(\overline{\psi},\overline{w}) - (\psi,w)\| \\ &+ 2\varepsilon|\overline{B}-B| + \frac{2a}{\mu}(e^{-\mu(B+s-(1+\Delta))} - e^{-\mu(\overline{B}+s-(1+\Delta))}).\end{aligned}$$

Proof. *Step 1.* Recall $B+s>1+\Delta$ (Proposition 2.5). Observe

$$\begin{aligned}|\overline{v}(\overline{B}+s)-v(B+s)| \le\, & e^{-\mu(B+s-(1+\Delta))}|\overline{v}(1+\Delta)-v(1+\Delta)| \\ & + \Big|\overline{v}(1+\Delta)(e^{-\mu(\overline{B}+s-(1+\Delta))}-e^{-\mu(B+s-(1+\Delta))}) \\ & + \int_{1+\Delta}^{\overline{B}+s} e^{-\mu(\overline{B}+s-t)} f(\overline{x}(t-1))\,dt \\ & - \int_{1+\Delta}^{B+s} e^{-\mu(B+s-t)} f(x(t-1))\,dt\Big|.\end{aligned}$$

Step 2. The last term is majorized by

$$\begin{aligned}&\left| -\int_{1+\Delta}^{B+s} e^{-\mu(B+s-t)}(f(x(t-1))-f(\overline{x}(t-1)))\,dt\right| \\ &+ \Big|\overline{v}(1+\Delta)(e^{-\mu(\overline{B}+s-(1+\Delta))}-e^{-\mu(B+s-(1+\Delta))}) + \int_{B+s}^{\overline{B}+s} e^{-\mu(\overline{B}+s-t)} f(\overline{x}(t-1))\,dt \\ &\qquad - \int_{1+\Delta}^{B+s}(e^{-\mu(B+s-t)}-e^{-\mu(\overline{B}+s-t)}) f(\overline{x}(t-1))\,dt\Big|.\end{aligned}$$

All arguments of f in the last integrands belong to $[\beta,\infty)$.

Step 3. It follows that

$$\begin{aligned}&\left| -\int_{1+\Delta}^{B+s} e^{-\mu(B+s-t)}(f(x(t-1))-f(\overline{x}(t-1)))\,dt\right| \\ &\qquad \le L_\beta \int_{1+\Delta}^{B+s} e^{-\mu(B+s-t)}|x(t-1))-\overline{x}(t-1)|\,dt \\ &\qquad \le \frac{L_\beta}{\mu}(1-e^{-\mu(B+s-(1+\Delta))}) \max_{t\in[\Delta,B-1]}|x(t)-\overline{x}(t)|.\end{aligned}$$

The estimate $|x(t)-\overline{x}(t)| \le \|\psi-\overline{\psi}\| \le 2\|(\psi,w)-(\overline{\psi},\overline{w})\|$ for all $t\in[\Delta,1+\Delta]$, the last estimate in Proposition 3.3, and the inequality $2 \le c(\beta,\varepsilon,\mu,L_\beta)=c$ combined imply

$$\left| -\int_{1+\Delta}^{B+s} e^{-\mu(B+s-t)}(f(x(t-1))-f(\overline{x}(t-1)))\,dt\right| \le \frac{L_\beta}{\mu} c\|(\psi,w)-(\overline{\psi},\overline{w})\|.$$

Step 4. Consider the second term of the sum in Step 2. Due to Proposition 2.1,

$$|\overline{v}(1+\Delta)| \le \frac{a+\varepsilon}{\mu},$$

and $-a-\varepsilon \le f(\overline{x}(t-1)) \le -a+\varepsilon$ for $1+\Delta \le t \le \overline{B}+s$. Therefore a lower bound for

$$\begin{aligned}I = &\overline{v}(1+\Delta)(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))})\\ &+ \int_{B+s}^{\overline{B}+s} e^{-\mu(\overline{B}+s-t)} f(\overline{x}(t-1))\,dt\\ &- \int_{1+\Delta}^{B+s} (e^{-\mu(B+s-t)} - e^{-\mu(\overline{B}+s-t)}) f(\overline{x}(t-1))\,dt\end{aligned}$$

is

$$\begin{aligned}&\frac{a+\varepsilon}{\mu}(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))}) - \frac{a+\varepsilon}{\mu}(1-e^{-\mu(\overline{B}-B)})\\ &\qquad + \frac{a-\varepsilon}{\mu}(e^{\mu(B+s)} - e^{\mu(1+\Delta)})(e^{-\mu(B+s)} - e^{-\mu(\overline{B}+s)})\\ &\quad= \frac{a+\varepsilon}{\mu}(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))}) - \frac{a+\varepsilon}{\mu}(1-e^{-\mu(\overline{B}-B)})\\ &\qquad + \frac{a-\varepsilon}{\mu}(1 - e^{-\mu(B+s-(1+\Delta))} - e^{-\mu(\overline{B}-B)} + e^{-\mu(\overline{B}+s-(1+\Delta))})\\ &\quad= \frac{2a}{\mu}(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))}) - \frac{2\varepsilon}{\mu}(1-e^{-\mu(\overline{B}-B)})\\ &\quad\ge \frac{2a}{\mu}(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))}) - 2\varepsilon(\overline{B}-B).\end{aligned}$$

Similarly one finds the upper bound

$$\begin{aligned}I \le &-\frac{a+\varepsilon}{\mu}(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))}) - \frac{a-\varepsilon}{\mu}(1-e^{-\mu(\overline{B}-B)})\\ &+ \frac{a+\varepsilon}{\mu}(e^{\mu(B+s)} - e^{\mu(1+\Delta)})(e^{-\mu(B+s)} - e^{-\mu(\overline{B}+s)})\\ = &-\frac{a+\varepsilon}{\mu}(e^{-\mu(\overline{B}+s-(1+\Delta))} - e^{-\mu(B+s-(1+\Delta))}) - \frac{a-\varepsilon}{\mu}(1-e^{-\mu(\overline{B}-B)})\\ &+ \frac{a+\varepsilon}{\mu}(1 - e^{-\mu(B+s-(1+\Delta))} - e^{-\mu(\overline{B}-B)} + e^{-\mu(\overline{B}+s-(1+\Delta))})\\ = &\frac{2\varepsilon}{\mu}(1-e^{-\mu(\overline{B}-B)}).\end{aligned}$$

It follows that

$$|I| \le 2\varepsilon|\overline{B}-B| + \frac{2a}{\mu}(e^{-\mu(B+s-(1+\Delta))} - e^{-\mu(\overline{B}+s-(1+\Delta))}).$$

Step 5. Steps 1–4 combined imply the desired estimate. □

COROLLARY 3.1. *Suppose the hypotheses of Proposition* 3.5 *are satisfied and* (8) *holds. Then* $\overline{v}$, v, $\overline{B}$, B, $\overline{\psi}$, ψ, $\overline{w}$, w *from Proposition* 3.5 *satisfy*

$$\begin{aligned}|\overline{v}(\overline{B}) - v(B)| \le &\left(e^{-\mu} + c(\beta,\varepsilon,\mu,L_\beta)\left(\frac{L_\beta}{\mu} + (2\varepsilon + 2ae^{-\mu})\frac{\mu}{|e^{-\mu}2a - a + \varepsilon|}\right)\right)\\ &\cdot \|(\overline{\psi},\overline{w}) - (\psi,w)\|.\end{aligned}$$

PROOF. Apply Proposition 3.5 for $s = 0$ and use the estimate $\overline{B} - B \leq \mathrm{Lip}\,(T_{\beta\varepsilon\mu rf})\|(\overline{\psi}, \overline{w}) - (\psi, w)\|$, Proposition 3.4, the inequality

$$
\begin{aligned}
e^{-\mu(B-(1+\Delta))} - e^{-\mu(\overline{B}-(1+\Delta))} &= \mu(\overline{B} - B)\frac{e^{-\mu(B-(1+\Delta))} - e^{-\mu(\overline{B}-(1+\Delta))}}{\mu(\overline{B} - B)} \\
&\leq \mu(\overline{B} - B)e^{-\mu(B-(1+\Delta))} \quad (\text{since } \overline{B} \geq B) \\
&\leq \mu(\overline{B} - B)e^{-\mu} \quad (\text{since } B - (1 + \Delta) > 1),
\end{aligned}
$$

and

$$|v(1 + \Delta) - \overline{v}(1 + \Delta)| \leq \|(\overline{\psi}, \overline{w}) - (\psi, w)\|. \qquad \square$$

Next $\|\overline{x}_{\overline{B}} - x_B\|$ is estimated in terms of $|v(1+\Delta) - \overline{v}(1+\Delta)|$, $\|\overline{x}_{1+\Delta} - x_{1+\Delta}\|$, and $B - 2 - \Delta > 0$.

PROPOSITION 3.6. *Suppose the hypotheses of Proposition* 3.5 *are satisfied and* (8) *holds. Then* $\overline{x}$, x, $\overline{B}$, B, $\overline{\psi}$, ψ, $\overline{w}$, w *from Proposition* 3.5 *satisfy*

$$
\begin{aligned}
\|\overline{x}_{\overline{B}} - x_B\| \leq \Bigg(&\frac{1}{\mu}(1 + L_\beta c(\beta, \varepsilon, \mu, L_\beta)) \\
&+ c(\beta, \varepsilon, \mu, L_\beta)\left(\frac{2\varepsilon\mu}{|e^{-\mu}2a - a + \varepsilon|} + \frac{2ae^{-\mu(B-2-\Delta)}}{|e^{-\mu}2a - a + \varepsilon|}\right)\Bigg) \\
&\cdot \|(\overline{\psi}, \overline{w}) - (\psi, w)\|.
\end{aligned}
$$

PROOF. For every $t \in [-1, 0]$,

$$
\begin{aligned}
|\overline{x}_{\overline{B}}(t) - x_B(t)| = |\overline{x}(\overline{B} + t) - x(B + t)| &= \left|\beta - \beta - \int_{\overline{B}+t}^{\overline{B}} \overline{v}(s)\,ds + \int_{B+t}^{B} v(s)\,ds\right| \\
&= \left|\int_t^0 (\overline{v}(\overline{B} + s) - v(B + s))\,ds\right| \leq \int_t^0 |\overline{v}(\overline{B} + s) - v(B + s)|\,ds.
\end{aligned}
$$

Proposition 3.5 is applied to the last integrand. Then integration gives

$$
\begin{aligned}
|\overline{x}_{\overline{B}}(t) - x_B(t)| \leq\ &\frac{1}{\mu}(e^{-\mu(B+t-(1+\Delta))} - e^{-\mu(B-(1+\Delta))})|\overline{v}(1 + \Delta) - v(1 + \Delta)| \\
&+ \frac{L_\beta}{\mu}c(\beta, \varepsilon, \mu, L_\beta)\|(\overline{\psi}, \overline{w}) - (\psi, w)\| + 2\varepsilon|\overline{B} - B| \\
&+ \frac{2a}{\mu^2}(e^{-\mu(B+t-(1+\Delta))} - e^{-\mu(\overline{B}+t-(1+\Delta))} \\
&- (e^{-\mu(B-(1+\Delta))} - e^{-\mu(\overline{B}-(1+\Delta))})).
\end{aligned}
$$

The inequalities (see Proposition 2.5) $e^{-\mu(B+t-(1+\Delta))} - e^{-\mu(B-(1+\Delta))} < 1$ and (since $\overline{B} \geq B$) $e^{-\mu(B-(1+\Delta))} - e^{-\mu(\overline{B}-(1+\Delta))} \geq 0$ show that the last term is majorized by

$$\frac{1}{\mu}|\overline{v}(1+\Delta)-v(1+\Delta)|+\frac{L_\beta}{\mu}c(\beta,\varepsilon,\mu,L_\beta)\|(\overline{\psi},\overline{w})-(\psi,w)\|$$
$$+2\varepsilon|\overline{B}-B|+\frac{2a}{\mu}\frac{1}{\mu}\mu|\overline{B}-B|\frac{e^{-\mu(B+t-(1+\Delta))}-e^{-\mu(\overline{B}+t-(1+\Delta))}}{\mu|\overline{B}-B|}.$$

The last quotient is bounded from above by $e^{-\mu(B+t-(1+\Delta))}\le e^{-\mu(B-2-\Delta)}$. It follows that

$$|x_{\overline{B}}(t)-x_B(t)|\le\frac{1}{\mu}|\overline{v}(1+\Delta)-v(1+\Delta)|+\frac{L_\beta}{\mu}c(\beta,\varepsilon,\mu,L_\beta)\|(\overline{\psi},\overline{w})-(\psi,w)\|$$
$$+\left(2\varepsilon+\frac{2a}{\mu}e^{-\mu(B-2-\Delta)}\right)|\overline{B}-B|.$$

The equation

$$|\overline{B}-B|=|T_{\beta\varepsilon\mu rf}(\overline{\psi},\overline{w})-T_{\beta\varepsilon\mu rf}(\psi,w)|,$$

Proposition 3.4 and the inequality

$$|\overline{v}(1+\Delta)-v(1+\Delta)|\le\|(\overline{\psi},\overline{w})-(\psi,w)\|$$

combined yield the desired estimate. □

For Lipschitz constants of R which become small for μ large and β, ε small it is necessary to control the term $B-2-\Delta$ in the estimate of Proposition 3.6.

COROLLARY 3.2. *Let $\eta\in(0,1)$ be given. Then there exists $\mu_\eta>\mu_0$ such that for every $\mu>\mu_\eta$ there are $\beta_{\mu\eta}\in(0,\beta_{\mu 0})$, $\varepsilon_{\mu\eta}\in(0,\varepsilon_{\mu 0})$ with the following property. For $0<\beta<\beta_{\mu\eta}$, $0<\varepsilon<\varepsilon_{\mu\eta}$, $f\in F_{\beta\varepsilon}$,*

$$\mathrm{Lip}\,(Q_{\beta\varepsilon\mu rf})\le e^{-\mu}+c(\beta,\varepsilon,\mu,L_\beta)\left(\frac{L_\beta}{\mu}+(2\varepsilon+2ae^{-\mu})\frac{\mu}{|e^{-\mu}2a-a+\varepsilon|}\right)$$
$$+\frac{1}{2}\left(\frac{1}{\mu}(1+L_\beta c(\beta,\varepsilon,\mu,L_\beta))\right.$$
$$\left.+c(\beta,\varepsilon,\mu,L_\beta)\left(\frac{2\varepsilon\mu}{|e^{-\mu}2a-a+\varepsilon|}+\frac{2ae^{-2+\eta}}{|e^{-\mu}2a-a+\varepsilon|}\right)\right).$$

PROOF. Proposition 2.9 guarantees the existence of $\mu_\eta>\mu_0$ such that for every $\mu>\mu_\eta$ there are $\beta_{\mu\eta}\in(0,\beta_{\mu 0})$, $\varepsilon_{\mu\eta}\in(0,\varepsilon_{\mu 0})$ with

$$B(\phi,u,f,\beta,\varepsilon,\mu,r_\mu(\beta,\varepsilon))-2-\Delta(\beta,\varepsilon,\mu,r_\mu(\beta,\varepsilon))\ge\frac{2-\eta}{\mu}$$

for all $\beta\in(0,\beta_{\mu\eta})$, $\varepsilon\in(0,\varepsilon_{\mu\eta})$, $(\phi,u)\in A_{\beta\varepsilon\mu r_\mu(\beta,\varepsilon)}$, $f\in F_{\beta\varepsilon}$. Clearly $\varepsilon_{\mu\eta}$ can be chosen so small that (8) holds for $\varepsilon\in(0,\varepsilon_{\mu\eta})$, too.

Use Corollary 3.1, the estimate from Proposition 3.6 multiplied by the weight 1/2, and the last inequality to deduce the asserted Lipschitz estimate. □

Notice that for every $\rho > 1$ and $\lambda > 0$ each set $F_{\beta\varepsilon}$ contains functions f satisfying

$$L < \rho\frac{a-\varepsilon}{\beta} \quad \text{and} \quad L_\beta < \lambda.$$

THEOREM 3.1. *Let $\rho \in (1, e^2/6)$ be given. Then there exists $\mu_\rho > \mu_0$ so that, for every $\mu > \mu_\rho$, there are $\lambda_\mu > 0$, $\beta_{\mu\rho} \in (0, \beta_{\mu 0})$, and $\varepsilon_{\mu\rho} \in (0, \varepsilon_{\mu 0})$ with the following property. For every $\beta \in (0, \beta_{\mu\rho})$, $\varepsilon \in (0, \varepsilon_{\mu\rho})$, $f \in F_{\beta\varepsilon}$ with*

$$\mathrm{Lip}\,(f) \le \rho\frac{a-\varepsilon}{\beta} \quad \textit{and} \quad \mathrm{Lip}\,(f \mid [\beta,\infty)) \le \lambda_\mu$$

and for $r = r_\mu(\beta,\varepsilon)$, $\mathrm{Lip}\,(R_{\beta\varepsilon\mu r f}) < 1$.

PROOF. Choose $\eta \in (0,1)$ so that $\rho e^\eta < e^2/6$, or equivalently,

$$6e^{-2+\eta}\rho < 1. \tag{9}$$

Choose $\mu_\eta > \mu_0$ according to Corollary 3.2. For $\mu > \mu_\eta$ choose $\beta_{\mu\eta} > 0$ and $\varepsilon_{\mu\eta} > 0$ according to Corollary 3.2. For $\mu > \mu_\eta$, $0 \le \beta < \beta_{\mu\eta}$, $0 \le \varepsilon < \varepsilon_{\mu\eta}$ and $\lambda \ge 0$ set

$$\begin{aligned} c_Q(\beta,\varepsilon,\mu,\lambda) = {} & e^{-\mu} + c(\beta,\varepsilon,\mu,\lambda)\left(\frac{\lambda}{\mu} + (2\varepsilon + 2ae^{-\mu})\frac{\mu}{|e^{-\mu}2a - a + \varepsilon|}\right) \\ & + \frac{1}{2}\left(\frac{1}{\mu}(1 + \lambda c(\beta,\varepsilon,\mu,\lambda))\right. \\ & \left. + c(\beta,\varepsilon,\mu,\lambda)\left(\frac{2\varepsilon\mu}{|e^{-\mu}2a - a + \varepsilon|} + \frac{2ae^{-2+\eta}}{|e^{-\mu}2a - a + \varepsilon|}\right)\right). \end{aligned}$$

An application of Corollary 3.2 shows that for $\mu > \mu_\eta$, $\lambda > 0$, $\beta \in (0, \beta_{\mu\eta})$, $\varepsilon \in (0, \varepsilon_{\mu\eta})$, $r = r_\mu(\beta,\varepsilon)$ and, for all $f \in F_{\beta\varepsilon}$, with $\mathrm{Lip}\,(f \mid [\beta,\infty)) \le \lambda$ the estimate $\mathrm{Lip}\,(Q_{\beta\varepsilon\mu r f}) \le c_Q(\beta,\varepsilon,\mu,\lambda)$ holds.

Recall Proposition 3.2. For $\mu > \mu_\eta$, $0 < \beta < \beta_{\mu\eta}$, $0 \le \varepsilon < \varepsilon_{\mu\eta}$, and $\lambda \ge 0$, set

$$\begin{aligned} c_P(\beta,\varepsilon,\mu,\lambda) = {} & e^{-\mu} + \frac{2\rho}{r_\mu(\beta,\varepsilon)} + \frac{1}{\mu} + \frac{\rho}{\mu r_\mu(\beta,\varepsilon)} \\ & + \frac{\lambda}{\mu}\left(1 + \frac{2\mu}{r_\mu(\beta,\varepsilon)(a-\varepsilon)}\beta\left(1 + \rho\frac{a-\varepsilon}{\beta}\frac{1}{2}\left(\frac{2\mu}{r_\mu(\beta,\varepsilon)(a-\varepsilon)}\beta\right)^2\right)\right) \\ & + \frac{2\lambda}{\mu} + \frac{2\lambda}{\mu}\rho\frac{a-\varepsilon}{\beta}\frac{1}{2}\left(\frac{2\mu}{r_\mu(\beta,\varepsilon)(a-\varepsilon)}\beta\right)^2. \end{aligned}$$

An application of Proposition 3.2 and the definition of Δ show that for $\mu > \mu_\eta$, $\lambda > 0$, $\beta \in (0, \beta_{\mu\eta})$, $\varepsilon \in (0, \varepsilon_{\mu\eta})$, $r = r_\mu(\beta,\varepsilon)$ and for all $f \in F_{\beta\varepsilon}$ with

$$\mathrm{Lip}\,(f) \le \rho\frac{a-\varepsilon}{\beta} \quad \text{and} \quad \mathrm{Lip}\,(f \mid [\beta,\infty)) \le \lambda$$

the estimate $\mathrm{Lip}\,(P_{\beta\varepsilon\mu rf}) \le c_P(\beta,\varepsilon,\mu,\lambda)$ holds. Consequently,

$$\mathrm{Lip}\,(R_{\beta\varepsilon\mu rf}) \le c_Q(\beta,\varepsilon,\mu,\lambda)c_P(\beta,\varepsilon,\mu,\lambda).$$

Dividing by β the function c_P is extended to arguments $(0,0,\mu,0)$ with $\mu > \mu_\eta$. Clearly $\lim_{\mu\to\infty} c_P(0,0,\mu,0) = 2\rho$.

Recall (7). It follows that $\lim_{\mu\to\infty} c_Q(0,0,\mu,0) = 3e^{-2+\eta}$. (9) permits to find $\mu_\rho \ge \mu_\eta$ so that, for each $\mu \ge \mu_\rho$,

$$c_Q(0,0,\mu,0)c_P(0,0,\mu,0) < 1.$$

For each $\mu \ge \mu_\rho$ there are $\lambda_\mu > 0$, $\beta_{\mu\rho} \in (0,\beta_{\mu\eta})$, $\varepsilon_{\mu\rho} \in (0,\varepsilon_{\mu\eta})$ so that, for all $\beta \in (0,\beta_{\mu\rho})$ and $\varepsilon \in (0,\varepsilon_{\mu\rho})$, $c_Q(\beta,\varepsilon,\mu,\lambda_\mu)c_P(\beta,\varepsilon,\mu,\lambda_\mu) < 1$, which completes the proof. $\square$

COROLLARY 3.3. *For μ, β, ε, f and $r = r_\mu(\beta,\varepsilon)$ as in Theorem 3.1 there exists a periodic solution $(\overline{x},\overline{v})\colon \mathbb{R} \to \mathbb{R}$ of (1) with $(\overline{x}_0,\overline{v}(0))$ equal to the fixed point $(\overline{\phi},\overline{u})$ of the contraction $R_{\beta\varepsilon\mu rf}$. For $\overline{B} = B(\overline{\phi},\overline{u},f,\beta,\varepsilon,\mu,r)$,*

$$(\overline{x},\overline{v})(t+\overline{B}) = -(\overline{x},\overline{v})(t) \quad \textit{for all } t \in \mathbb{R},$$

and $2\overline{B}$ is the minimal period of $(\overline{x},\overline{v})$.

4. Differentiable nonlinearities

Consider μ, β, ε, f and $r = r_\mu(\beta,\varepsilon)$ as in Theorem 3.1 and assume in addition that f is continuously differentiable. (The existence of such f in $F_{\beta\varepsilon}$ is obvious.) The present section shows that for such f the orbit

$$\{(\overline{x}_t,\overline{v}(t)) \in X : t \in \mathbb{R}\}$$

of the periodic solution $(\overline{x},\overline{v})$ from Corollary 3.3 is stable and hyperbolic. This means that for some associated Poincaré return map Π with fixed point $(\overline{\phi},\overline{u})$ the spectrum of $D\Pi(\overline{\phi},\overline{u})$ is contained in the open unit circle of the complex plane, compare e.g. Chapter XIV in [4].

Consider the closed hyperplane $Y = \{(\phi,u) \in X : \phi(0) = 0\}$ and the affine subspace $Y_\beta = \{(\phi,u) \in X : \phi(0) = -\beta\}$ of X. All tangent spaces of the C^1-submanifolds Y_β and $-Y_\beta$ coincide with Y, and $A = A_{\beta\varepsilon\mu r}$ is a subset of Y_β. Let $S = S_{\mu f}$, $\Delta = \Delta(\beta,\varepsilon,\mu,r)$, and write $B(\phi,u) = B(\phi,u,f,\beta,\varepsilon,\mu,r)$ for all $(\phi,u) \in A$.

PROPOSITION 4.1. *There are a bounded open neighbourhood U of $(\overline{\phi},\overline{u})$ in X and a continuously differentiable map $\tau\colon U \to (1,\infty)$ so that for all $(\phi,u) \in U$, $S(\tau(\phi,u),(\phi,u)) \in -A$, and, for all $(\phi,u) \in U \cap A$, $\tau(\phi,u) = B(\phi,u)$.*

PROOF. Let pr_1 denote the projection from X onto the first factor C, and recall the evaluation $ev_0\colon C \to \mathbb{R}$ from Section 1. Set $\overline{B} = B(\overline{\phi},\overline{u})$. The inequality

$\overline{v}(\overline{B}) < 0$ (Corollary 2.1) implies

$$D_1 S(\overline{B}, (\overline{\phi}, \overline{u}))1 \notin T_{-(\overline{\phi},\overline{u})}(-Y_\beta) = Y$$

since $D_1 S(\overline{B}, (\overline{\phi}, \overline{u}))1 = \dot{\overline{x}}_{\overline{B}}$ (see [4]) and $\dot{\overline{x}}_{\overline{B}}(0) = \dot{\overline{x}}(\overline{B}) = \overline{v}(\overline{B}) \neq 0$. An application of the Implicit Function Theorem as e.g. in Chapter XIV of [4] yields an open neighbourhood V of $(\overline{\phi}, \overline{u})$ in X and a continuously differentiable map $\tau_0 \colon V \to (1, \infty)$ with $\tau_0(\overline{\phi}, \overline{u}) = \overline{B}$ and $S(\tau_0(\phi, u), (\phi, u)) \in -Y_\beta$ for all $(\phi, u) \in V$, i.e., $x(\tau_0(\phi, u)) = \beta$ for $(x, v) = (x, v)^{(\phi,u)}$. Corollary 2.1 permits to find $\delta \in (0, 1)$ so that

$$\overline{x}(t) > \beta \quad \text{for } \Delta \leq t \leq \overline{B} - \delta, \ \beta > \overline{x}(\overline{B} + \delta),$$

and

$$\overline{v}(t) \in \left(-\frac{a+\varepsilon}{\mu}, -r\frac{a-\varepsilon}{\mu} \right) \quad \text{for } \overline{B} - \delta \leq t \leq \overline{B} + \delta.$$

The continuity of S implies that there exists an open neighbourhood $U \subset V$ of $(\overline{\phi}, \overline{u})$ in X so that for every solution $(x, v) = (x, v)^{(\phi,u)}$ with $(\phi, u) \in U$,

$$x(t) > \beta \quad \text{for } \Delta \leq t \leq \overline{B} - \delta, \ \beta > x(\overline{B} + \delta),$$
$$(\dot{x}(t) =) \ v(t) \in \left(-\frac{a+\varepsilon}{\mu}, -r\frac{a-\varepsilon}{\mu} \right) \quad \text{for } \overline{B} - \delta \leq t \leq \overline{B} + \delta,$$

and

$$\Delta + 1 < \tau_0(\phi, u) \in (\overline{B} - \delta, \overline{B} + \delta).$$

Set $\tau = \tau_0 \mid U$. Let $(\phi, u) \in U$, $(x, v) = (x, v)^{(\phi,u)}$. It follows that

$$x(\tau(\phi, u)) = \beta, \quad \beta < x(t) \quad \text{for } \Delta \leq t < \tau(\phi, u),$$
$$v(\tau(\phi, u)) \in \left(-\frac{a+\varepsilon}{\mu}, -r\frac{a-\varepsilon}{\mu} \right), \quad \Delta + 1 < \tau(\phi, u).$$

Consequently, $S(\tau(\phi, u), (\phi, u)) \in -A$. In case $(\phi, u) \in U \cap A$ the definition of B (Proposition 2.3) yields $\tau(\phi, u) = B(\phi, u)$. □

From here the proof of hyperbolic stability is completed exactly as in Section 4 of [8].

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