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SEMIDISCRETIZATION SCHEMES FOR THE AUTONOMOUS DIFFERENTIAL EQUATIONS WITH NONCOMPACT SEMIGROUPS USING THE FUNCTIONALIZING PARAMETER METHOD

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Dedicated to the memory of Juliusz P. Schauder

Abstract. In the case of the abstract autonomous semilinear equation in a Banach space we provide conditions which ensure that the approximate cycles given by a semidiscretization method converge to the exact cycle.

1. Introduction

Many mathematical models decribing auto-oscillations can be reduced to an existence problem for a cycle of the autonomous semilinear equation

$$x' = Ax + f(x), \tag{1}$$

where the linear operator A is the infinitesimal generator of a C_0-semigroup $\exp\{At\}$ that acts on a Banach space E, and f is a nonlinear contiuous operator acting from E to E. In this paper we study the semidiscretisation method for an approximate computation of the cycles of the equation (1). This method consists of an aproximation of the operators A and f by appropriety chosen approximate operators A_h and f_h, acting in the spaces E_h (in applications those spaces are usualy finite dimensional spaces) without change of the time derivative. We

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suppose that the approximate cycles can be found exactly and we want to suggest the conditions which provide the convergence of approximate cycles to the exact cycle of equation (1). We take as a semidiscretization scheme the following equations

$$x_h' = A_h x_h + f_h(x_h), \tag{2}$$

where h is a parameter of the semidiscretization, the operators A_h are infinitisimal operators of C_0-semigroups of linear operators $\exp\{A_h t\}$ acting in the Banach spaces E_h, and f_h are continous operators from E_h to E_h.

We suppose that $h \in H = \{h_n : h_n > 0,\ h_n \downarrow 0\} \cup \{0\}$, and we identify the operators $A_0,\ f_0$ with the operators $A,\ f$, and the space E_0 with the space E accordingly. Thus for $h = 0$ equation (2) becomes equation (1).

In the paper [6], for semilinear parabolic equations conditions for the convergence of the approximate cycles to the exact cycle were given. By the method proposed in [4], the problem of periodic solutions of equation (2) is replaced by the functional equation

$$u = F(h, T, u), \tag{3}$$

where T is the unknown period and F is an equivalent (see [9]) compact continuous operator. After this the parameter T is functionalized (see [2]) in such a way that after replacing the parameter T by the the functional $T(u)$ the following inequality

$$\operatorname{ind}(z_0, F(0, T(\,\cdot\,), \,\cdot\,)) \neq 0 \tag{4}$$

holds. In the inequality (4) z_0 is a periodic solution corresponding to the cycle of equation (1). This solution is selected by the functionalization of the parameter T. Recall (see [2]) that without the functionalization of the parameter the periodic sollution corresponding to the cycle is not isolated and the topological index of the set of all periodic solutioins corresponding to the same cycle is equal to zero.

The compactness of the operator F permits to study the equation (3) by the methods of topological degree theory for compact vector fields in infinite dimensional spaces (see, for example, [8]). In [6] the main assumptions which provided the compactness of F were the compactness of the resolvant of the operator A and uniform strict positiveness (see [9]) of the operators A_h. Under those conditions the semigroups considered in [6] are analytic compact semigroups. But with such conditions it's not possible to investigate mathematical models which contain hyperbolic equations because none of the mentioned conditions are satisfied. In this case the semigroups are C_0-semigroups and are neither analytic nor compact. As it is shown below for a wide class of equations (1), where $\exp\{At\}$ is non analytic and is non compact (in particular for hyperbolic equation with dissipative members, see, for example, [3]), this difficulty can be surmounted since we can establish that the operator F is a (q,χ)-bounded operator, where $q<1$ and χ is a Hausdorff measure of noncompactness. Consequently (see, [1]) F is a condensing operator with respect to the Hausdorff measure of noncompactness. This result permits to leave almost without changes the proof of the abstract theorem on the functionalization of the parameter from [6], simply changing the word compact by the words (q,χ)-bounded with a constant $q<1$, and certainly remplacing the topological degree theory for compact vector fields by this theory for condensing vector fields (see [1]). It is possible since the derivative of a (q,χ)-bounded operator is (q,χ)-bounded too (see [1]). Therefore 1 as a point of spectrum of this derivative can be only an eigenvalue of finite multiplicity and the functional $T(u)$ functionalizing the parameter T in [6] must not be changed. Comparing this abstract theorem with the Theorem 4.4.11 from [1], let us remark that we don't suppose that 1 is a simple eigenvalue. This condition is remplaced as in [6] by an other one which is adopted to varify in the case of integral operator.

2. Main assumptions

At this point we also give the main conditions on the operators A_h and f_h. In order to state these conditions we need auxiliary operators connecting the spaces E_h and E.

We assume that for $h\in H\setminus\{0\}$, there exist linear uniformly bounded operators $Q_h : E_h\to E$, $P_h : E\to E_h$. We set $Q_0=I$, $P_0=I$ and we suppose that these operators satisfy the following conditions:

$$P_hQ_h = I_h \quad \text{for } h\in H, \tag{5}$$

where I_h is identity operator on the space E_h;

$$Q_hP_hx\to x \quad \text{as } h\to 0 \text{ for all } x\in E. \tag{6}$$

Now let us state the assumptions on the aproximate operators A_h. We assume that the semigroups $\exp\{A_ht\}$ approximate the semigroup $\exp\{At\}$, which means that

(A_1) For every $x\in E$, $Q_h\exp\{A_ht\}P_hx\to\exp\{At\}x$, as $h\to 0$, uniformly with respect to t from any bounded segment $[0,d]$, $d>0$.

In addition we suppose that the semigroups $\exp\{A_ht\}$ are uniformly strictly contractive:

(A_2) There exists a constant $\gamma>0$, such that $\|Q_h\exp\{A_ht\}P_hx\|\le e^{-\gamma t}$ for $h\in H$, $t\in[0,\infty)$.

Before giving the conditions on the operators f_h, recall that for a bounded set $\Omega \subset E$ the Hausdorff measure of noncompactness $\chi(\Omega)$ is given by the following formula

$$\chi(\Omega) = \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

For properties of the measure of noncompactness χ see, for example, [1].

Let us set $\varphi(h,x) = Q_h f_h(P_h x)$. In the sequel we suppose also that the following two assumptions hold.

(A_3) The operator φ is continuous with respect to all its variables and is bounded on bounded sets.

(A_4) There exists a constant $k < \gamma$ such that, for any bounded set $\Omega \subset E$,

$$\chi\bigg(\bigcup_{h\in H} \varphi(h,\Omega)\bigg) \le k\chi(\Omega).$$

Remark 1. If the constant k in the assumption (A_4) is equal to zero (it means that the operator φ is compact) then we need not assume A_2).

Below we always suppose that the space E is separable.

3. Main result

Let $C_T(E_h)$ be a space of continuous T-periodic functions with values in E_h endowed the usual uniform norm. Following [9], as T-periodic solutions of the equation (2), we take solutions in the space $C_T(E_h)$ of the equivalent integral equation

$$u(t) = \exp\{A_h t\}(I - \exp\{A_h T\})^{-1} \int_0^T \exp\{A_h(T-s)\} f_h(u(s))\, ds + \int_0^t \exp\{A_h(t-s)\} f_h(u(s))\, ds. \tag{7}$$

Assume that

(A_5) The equation (1) has a twice continuously differentiable T_0-periodic solution z_0 and the nonlinear operator f is uniformly differentiable in the points $z_0(t)$, i.e.

$$f(z_0(t)+w) - f(z_0(t)) = f'(z_0(t))w + \omega(t,w),$$

where $\|\omega(t,w)\|/\|w\| \to 0$ as $\|w\| \to 0$ uniformly with respect to $t \in [0,T_0]$.

Since equation (1) is autonomous, z_0' is a T_0-periodic solution of the linearized equation

$$y' = Ay + f'(z_0(t))y. \tag{8}$$

We suppose that

(A_6) The equation (8) has no T_0-periodic solution that is lineary independent of z_0' and has no Floquet solutions adjoint to z_0' i.e. there is no solution having the form

$$y(t) = y_0(t) + \frac{t}{T_0} z_0'(t),$$

where y_0 is a T_0-periodic function.

THEOREM 1. *Let conditions* (5), (6) *and assumptions* (A_1)–(A_6) *hold. Then, for sufficiently small h, equations* (2) *have T_h-periodic solutions z_h, such that $T_h \to T_0$ as $h \to 0$ and*

$$\left\| Q_h z_h \left(\frac{T_h}{T_0} t \right) - z_0(t) \right\| \to 0 \quad \text{as } h \to 0$$

uniformly with respect to $t \in [0, T_0]$.

4. Auxiliary propositions

Below we will use the measure of noncompactness ν defined on bounded subsets Ω of the space $C_{T_0}(E)$ by the following formula $\nu(\Omega) = \sup_t \chi(\Omega(t))$.

DEFINITION 1. The continuous operator

$$F : H \times [T_0 - \Delta, T_0 + \Delta] \times C_{T_0}(E) \to C_{T_0}(E)$$

is (q, χ, ν)-bounded (see [1]) if, for every bounded set $\Omega \in C_{T_0}(E)$, the inequality

$$\chi(F(H \times [T_0 - \Delta, T_0 + \Delta] \times \Omega)) \leq q\nu(\Omega) \tag{9}$$

holds.

Recall that, following [1], if the last inequality has the form

$$\chi(F(H \times [T_0 - \Delta, T_0 + \Delta] \times \Omega)) \leq q\chi(\Omega)$$

we say that the operator F is (q, χ)-bounded. We need also one result from [4] which we give in a form convenient for the sequel.

LEMMA 1. *Let E be a separable Banach space, $\{y_m\} \subset L^1([0, T_0], E)$ be a sequence of summable functions and there exist $p, r \in L^1([0, T_0], \mathbb{R})$ such that*

$$\|y_m(t)\| \leq p(t), \quad \text{for a.a. } t \in [0, T_0], \text{ and all } m = 1, 2, \ldots$$

and $\chi(\{y_m(t)\}) \leq r(t)$ for a.a. $t \in [0, T_0]$. Then for all $\varepsilon > 0$ there exist $e_\varepsilon \subset [0, T]$, compact set $K_\varepsilon \subset E$ and a sequence $\{g_m\} \subset L^1([0, T_0], E)$ such that

$$\operatorname{meas}(e_\varepsilon) < \varepsilon, \tag{10}$$

$$g_m(t) \in K_\varepsilon \quad \text{for all } t \in [0, T_0],\ m = 1, 2, \ldots \tag{11}$$

and

$$\|y_m(t) - g_m(t)\| \le r(t) + \varepsilon, \quad \text{for a.a. } t \in [0,T] \setminus e_\varepsilon. \tag{12}$$

The following theorem is a version of Theorem 2 from [6] for the case of a (q,χ)-bounded operator (see also Theorem 4.4.11 in [1])

THEOREM 2. *Let $F : H \times [T_0 - \Delta, T_0 + \Delta] \times B_{C_{T_0}}(u_0, r) \to C_{T_0}(E)$ be (q,χ)-bounded continuous with respect to its all variables and $q < 1$. Let the following conditions hold:*

(1) $F(0, T_0, z_0) = z_0$,
(2) *the operator $F(0,T,u)$ is differentiable with respect to u at the points (T, u_0),*

$$F(0,T,z_0 + w) - F(0,T,z_0) = F'_u(0,T,z_0)w + \omega_1(T,w), \tag{13}$$

where $\|\omega_1(T,k)\|/\|k\| \to 0$ as $\|k\| \to 0$ uniformly with respect to T, and the operator $F'_u(0,T,z_0)$ is strongly continuous with respect to T,
(3) *the function $F(0,T,z_0)$ is differentiable with respect to T at the point T_0, i.e.*

$$F(0,T+s,z_0) - F(0,T,z_0) = F'_T(0,T_0,z_0)s + \omega_2(s), \tag{14}$$

where $\|\omega_2(s)\|/|s| \to 0$ as $|s| \to 0$,
(4) *the operator $F'_u(0,T_0,z_0)$ satisfies the following conditions*
(a) $1 \in \sigma(F'_u(0,T_0,z_0))$,
(b) *the subspace of eigenvectors corresponding to the eigenvalue* 1 *is one-dimensional,*
(c) *the equation*

$$F'_T(0,T_0,z_0) = w - F'_u(0,T_0,z_0)w \tag{15}$$

has no solutions.

Then for all sufficiently small h, there exist T_h, u_h such that (h, T_h, u_h) satisfies the equation (3) *and $T_h \to T_0$, $u_h \to u_0$ as $h \to 0$.*

Proof of the Theorem 1. In the equation (2) let us change the variables $u_h(\tau) = x_h(T\tau/T_0)$, and for the equation obtained after this change of variables let us construct the integral operator equivalent to the T_0-periodic solution problem. So we obtain the operator

$$F(h,T,u)(t) = Q_h \frac{T}{T_0} \exp\left\{\frac{T}{T_0} A_h t\right\}(I - \exp\{A_h T\})^{-1} \cdot \int_0^{T_0} \exp\left\{\frac{T}{T_0} A_h (T_0 - s)\right\} P_h \varphi(h, u(s))\, ds + \frac{T}{T_0} \int_0^t Q_h \exp\{A_h(t-s)\} P_h \varphi(h, u(s))\, ds. \tag{16}$$

The equivalence of the fixed point problem for the operator $F(h,T,\cdot\,)$ to the problem of T-periodic solution of the equation (2) follows from the conditions (5). The details can be found in [5]. Chose $\Delta > 0$ such that the following inequality $q = k(T_0+\Delta)/\gamma(T_0-\Delta) < 1$ holds. We want to demonstrate that the operator $F : H \times [T_0-\Delta, T_0+\Delta] \times C_{T_0}(E) \to C_{T_0}(E)$, defined by the equality (16), satisfies the conditions of Theorem 2.

In order to prove that F is (q,χ)-bounded, we prove the following statement.

Proposition 1. *Let the conditions* (5), (6) *and the assumptions* (A_1)–(A_4) *hold. Then the operator F, defined by equality* (7)*, is continuous with respect to all its variables and is (q,χ,ν)-bounded.*

Proof. The conditions (5), (6) and the assumptions (A_1), (A_3) imply the continuity of the operator F, defined by the formula (7), with respect to all its variables.

Let us demonstrate now that F is (q,χ,ν)-bounded. Let estimate

$$\chi_{C_{T_0}(E)}(F(H,[T_0-\Delta,T_0+\Delta],\Omega)).$$

Since E is separable, then the space $C_{T_0}(E)$ is separable and every subset of $C_{T_0}(E)$ is separable too. Therefore there exist sequences $\{h_m\} \subset H, \{T_m\} \subset [T_0-\Delta, T_0+\Delta], \{u_m\} \subset \Omega$ such that

$$\chi_{C_{T_0}(E)}(F(H,[T_0-\Delta,T_0+\Delta],\Omega)) = \chi_{C_{T_0}(E)}(\{w_m\}), \tag{17}$$

where $w_m = F(h_m, T_m, u_m)$. Using now the assumption (A_4) and properties of the Hausdorff measure of noncompactness (see [1]), we obtain the following estimates

$$\begin{aligned}\chi_E\left(\left\{\frac{T_m}{T_0}\varphi(h_m,u_m(t))\right\}\right) &\le \frac{T_0+\Delta}{T_0}\chi_E(\{\varphi(h_m,u_m(t))\}) \\ &\le \frac{T_0+\Delta}{T_0} k \chi_E(\{u_m(t)\}) \\ &\le \frac{T_0+\Delta}{T_0} k \sup_t \chi_E(\{u_m(t)\}) \le \frac{T_0+\Delta}{T_0} k\nu(\Omega).\end{aligned}$$

From the assumption (A_3) we conclude that the functions

$$y_m(t) = \frac{T_m}{T_0}\varphi(h_m, u_m(t)), \quad m = 1, 2, \dots$$

are continuous and uniformly bounded by a constant M. Therefore we can apply Lemma 1 to the sequence $\{y_m\}$, in which

$$p(t) \equiv M, \qquad q(t) \equiv \frac{T_0 + \Delta}{T_0} k\nu(\Omega).$$

So we have e_ε, K_ε and $\{g_m\}$ satisfying the relations (10)–(12). Let us take the T_0-periodic extension of g_m. We preserve the same notation for such extension. Let

$$z_m(t) = Q_{h_m} \exp\left\{\frac{T_m}{T_0} A_{h_m} t\right\}(I - \exp\{A_{h_m} T_m\})^{-1} \cdot \int_0^{T_0} \exp\left\{\frac{T_m}{T_0} A_{h_m}(T_0 - s)\right\} P_{h_m} g_m(s)\, ds + \int_0^t Q_{h_m} \exp\{A_{h_m}(t-s)\} P_{h_m} g_m(s)\, ds. \tag{18}$$

It follows from (A_1) and (11) that the sequence $\{z_m\}$ is relativly compact in the space $C_{T_0}(E)$. Let us evaluate $\|w_m - z_m\|_{C_{T_0}}$. Since the functions w_m and z_m are T_0-periodic, it is sufficient to estimate $\|w_m(t) - z_m(t)\|_E$ for $t \in [0, T_0]$. Using (A_2) we have

$$\begin{aligned}\|w_m(t) - z_m(t)\| \le{}& e^{-\gamma(T_0-\Delta)t/T_0}(1 - e^{-\gamma(T_0-\Delta)})^{-1} \\ &\cdot \int_{[0,T_0]\setminus e_\varepsilon} e^{-\gamma(T_0-\Delta)(T_0-s)/T_)} \left(k\frac{T_0 - \Delta}{T_0}\nu(\Omega)\right) + \varepsilon)\, ds \\ &+ \int_{[0,t]\setminus e_\varepsilon} e^{-\gamma(T_0-\Delta)(t-s)/T_)} \left(k\frac{T_0 - \Delta}{T_0}\nu(\Omega) + \varepsilon\right) ds \\ &+ (1 - e^{-\gamma(T_0-\Delta})^{-1} M\varepsilon \\ \le{}& q(\nu(\Omega) + \varepsilon) + (1 - e^{-\gamma(T_0-\Delta)})^{-1} M\varepsilon.\end{aligned}$$

So the functions $\{z_m\}$ represent a relativly compact $(q\nu(\Omega) + C\varepsilon)$-net of $\{w_m\}$. Since ε is arbitrary, using (17) we have the estimate (9). □

Evidently $\nu(\Omega) \le \chi(\Omega)$. Therefore we have the following statement.

COROLLARY 1. *Let the conditions* (5), (6) *and the assumptions* (A_1)–(A_4) *hold. Then the operator F, defined by the equality* (7)*, is continuous with respect to all its variables and is a (q, χ)-bounded operator.*

We return now to the proof of Theorem 1. Condition (1) follows from the assumption that a T_0-periodic solution z_0 exists. Assumption (A_5) gives the condition (2). Since z_0 is twice differentiable we have condition (3).

Now let us verify condition (4). Since z_0' is a solution of the linearized equation (8) we obtain (4)(a). As we remarked by Theorem 1.5.9 (see [1]) the operator $F_u'(0, T_0, z_0)$ is (q, χ)-bounded too. Therefore (see [1]) 1 as a point of the spectrum of the operator $F_u'(0, T_0, z_0)$ can be only an eigenvalue of finite mutiplicity. If there exits an eigenvector v_0 linearly independent with z_0', then v_0 would be a solution of (8) contradicting the assumption. Therefore we have (4)(b). We pass now to condition (4)(c). Les us remark that the function

$$\widehat{w}(t) = \frac{t}{T_0} z_0'(t)$$

is a solution of the differential equation

$$w' = Aw + f'(z_0(t))w + \frac{1}{T_0} z_0'(t).$$

Therefore

$$\widehat{w}(t) = \int_0^t \exp\{A(t-s)\}\widehat{w}(s)\,ds + \frac{1}{T_0}\int_0^t \exp\{A(t-s)\}z_0'(s)\,ds. \tag{19}$$

If we suppose now that there exists a solution $\widetilde{w} = \widetilde{w}(t)$ of equation (15), then as it is shown in [7]

$$\begin{aligned} \widetilde{w}(t) = {} & \exp\{At\}(I - \exp\{AT\})^{-1} \int_0^{T_0} \exp\{A(T_0 - s)\} f'(z_0(s))\widetilde{w}(s) ds \\ & + \int_0^t \exp\{A(t-s)\} f'(z_0(s))\widetilde{w}(s) ds \\ & + \exp\{At\}(I - \exp\{AT\})^{-1} \int_0^{T_0} \exp\{A(T_0 - s)\} f'(z_0(s)) z_0'(s)\,ds \\ & + \int_0^t \exp\{A(t-s)\} f'(z_0(s)) z_0'(s)\,ds. \end{aligned} \tag{20}$$

Thus from (20) we have

$$\begin{aligned} \widetilde{w}(t) = {} & \exp\{At\}\widetilde{w}(0) + \int_0^t \exp\{A(t-s)\} f'(z_0(s))\widetilde{w}(s)\,ds \\ & + \int_0^t \exp\{A(t-s)\} f'(z_0(s)) z_0'(s)\,ds. \end{aligned} \tag{21}$$

Substracting (21) from (19) we obtain that the function $y(t) = -\widetilde{w}(t) + \widehat{w}(t)$ satisfies the equality

$$y(t) = \exp\{At\}y(0) + \int_0^t \exp\{A(t-s)\}f'(z_0(s))y(s)\,ds.$$

Therefore

$$y(t) = -\widetilde{w}(t) + \frac{t}{T_0} z_0'(t)$$

is a solution of equation (8), contradicting the assumption. □

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