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DECAYING SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. The existence of radial solutions for some semilinear elliptic equations in an exterior domain is established under sublinearity or sign assumption imposed on the nonlinearity.

1. Introduction

The paper is concerned with the existence of decaying radial solutions for nonlinear elliptic equations in exterior domains.

We consider the following BVP:

$$\begin{aligned} -\Delta u &= f(\|x\|, u) && \text{for } \|x\| \geq 1,\ x \in \mathbb{R}^n,\ n \geq 3, \\ u(x) &= 0 && \text{for } \|x\| = 1, \end{aligned} \tag{1.1}$$

and assuming that f is sublinear with respect to the second variable and decays sufficiently quickly with respect to the first variable, by Schauder theorem we prove the existence of at least one radial, decaying solution. Assuming that f changes sign and relaxing the sublinearity assumption, with the use of Leray–Schauder degree theory, we also obtain the existence of at least one radial, decaying solution. Our result is meaningful only when $f(\,\cdot\,, 0) \not\equiv 0$. Similar problems but with nonlinearity with separated variables were considered in [20].

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The papers, in which f is superlinear with respect to the second variable, include [5], [6] (cone compression and expansion approach) and [17] (variational methods).

Similar BVP in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$:

$$\begin{aligned} -\Delta u &= f(x, u, \nabla u) && \text{for } x \in \Omega, \\ u(x) &= 0 && \text{for } x \in \partial\Omega, \end{aligned} \tag{1.2}$$

was considered in [9]. Under the sublinearity assumption imposed on f (with respect to the second an the third variable) and decay with respect to x the author proves (by sub- and supersolution method) the existence of at least one decaying solution. This result is incomparable with ours, since the solution obtained in [9] can be nonradial, even if nonlinearity f is radially symmetric with respect to x. Related nonradial problems were considered in [1], [8], [10]–[14].

2. Main results

Consider the following BVP:

$$\begin{aligned} -\Delta u &= f(\|x\|, u) && \text{for } \|x\| \geq 1,\ x \in \mathbb{R}^n,\ n \geq 3, \\ u(x) &= 0 && \text{for } \|x\| = 1, \\ \lim_{\|x\|\to\infty} u(x) &= 0 \end{aligned} \tag{2.1}$$

where $f : [1, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. Looking for radial solutions of (2.1) leads to the BVP on a half line

$$\begin{gathered} -v'' + \frac{1-n}{r} v' = f(r, v) \quad \text{for } r \in [1, \infty) \\ v(1) = 0, \quad \lim_{r\to\infty} v(r) = 0. \end{gathered} \tag{2.2}$$

One can readily find the Green functions for the above problem. Namely

$$G(r, s) = \begin{cases} \dfrac{1}{n-2} s(1 - r^{2-n}) & \text{for } s > r, \\ \dfrac{1}{n-2} r^{2-n}(s^{n-1} - s) & \text{for } s \leq r, \end{cases}$$

is the Green function for (2.2) since it is continuous (nonnegative) and satisfies:

1° for any $s \in [1, \infty)$, $G(\,\cdot\,, s)$ satisfies the homogenous equation, i.e. for any $r \neq s$

$$-\frac{\partial^2 G}{\partial r^2}(r, s) + \frac{1-n}{r} \frac{\partial G}{\partial r}(r, s) = 0,$$

2° for any $s \in (1, \infty)$

$$\lim_{r\to s^+} \frac{\partial G}{\partial r}(r, s) - \lim_{r\to s^-} \frac{\partial G}{\partial r}(r, s) = 1,$$

3° for any $s \in \mathbb{R}$ the function $G(\,\cdot\,, s)$ satisfies the boundary conditions (i.e. $G(1, s) = 0$ and $\lim_{r\to\infty} G(r, s) = 0$ for every $s \in [1, \infty)$).

In our case the Green function is not symmetric, since the differential operator defined by the left-hand side of (2.2) (with boundary conditions taken into account) is not (formally) selfadjoint. We have the following estimates:

$$|G(r,s)| \le \begin{cases} \dfrac{1}{n-2}s & \text{for } s > r, \\ \dfrac{1}{n-2}r^{2-n}s^{n-1} & \text{for } s \le r, \end{cases} \tag{2.3}$$

and

$$\left|\frac{\partial G}{\partial r}(r,s)\right| \le \begin{cases} r^{1-n}s & \text{for } s > r, \\ r^{1-n}s^{n-1} & \text{for } s \le r, \end{cases} \tag{2.4}$$

Therefore, integrating (2.2), one obtains the following integral equation in the space $\mathrm{BC}([1,\infty))$ (the space of bounded and continuous functions from $[1,\infty)$ to $\mathbb{R}$):

$$v(r) = Sv(r) \stackrel{\mathrm{df}}{=} \int_1^\infty G(r,s)f(s,v(s))\,ds$$

provided f satisfies the following asymptotic condition with respect to the first variable

$$|f(r,v)| \le g(v)r^\beta \tag{2.5}$$

where $\beta < -2$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Then for $\|v\|_\infty \le M$, denoting by $C = \sup_{|v|\le M}|g(v)|$ and using estimate (2.3), one can derive

$$\begin{aligned} |Sv(r)| &\le \int_1^\infty G(r,s)s^\beta|g(v(s))|\,ds \le \sup_{|v|\le M}|g(v)| \int_1^\infty G(r,s)s^\beta\,ds \\ &\le \frac{C}{n-2}r^{2-n}\left(\int_1^r s^{n+\beta-1}\,ds + \int_r^\infty s^{\beta+1}\,ds\right) \end{aligned}$$

whence

$$\begin{aligned} &|Sv(r)| \le Ch_{\beta,n}(r), \\ &h_{\beta,n}(r) \stackrel{\mathrm{df}}{=} \begin{cases} \dfrac{-1}{(n+\beta)(\beta+2)}r^{\beta+2} & \text{for } n+\beta > 0, \\ \dfrac{1}{n-2}r^{2-n}\left(\ln r - \dfrac{1}{\beta+2}\right) & \text{for } n+\beta = 0, \\ \dfrac{-(n+2\beta+2)}{(n-2)(n+\beta)(\beta+2)}r^{2-n} & \text{for } n+\beta < 0, \end{cases} \end{aligned} \tag{2.6}$$

The continuity of G implies

$$(Sv)'(r) = \int_1^\infty \frac{\partial G}{\partial r}(r,s)f(s,v(s))ds,$$

and using (2.4) we have the following estimate

$$
(2.7)\qquad \begin{aligned} &|(Sv)'(r)| \le Ck_{\beta,n}(r),\\ &k_{\beta,n}(r) \overset{\text{df}}{=} \begin{cases} \dfrac{2-n}{(n+\beta)(\beta+2)} r^{\beta+1} & \text{for } n+\beta>0,\\ r^{1-n}\left(\ln r - \dfrac{1}{\beta+2}\right) & \text{for } n+\beta=0,\\ \dfrac{-(n+2\beta+2)}{(n+\beta)(\beta+2)} r^{1-n} & \text{for } n+\beta<0,\end{cases}\end{aligned}
$$

Moreover,

$$
(2.8)\qquad \begin{aligned} \sup_{r\in[1,\infty)} \int_1^\infty |G(r,s)| s^\beta\,ds &\le \sup_{r\in[1,\infty)} h_{\beta,n}(r)\\ &= L_{\beta,n} \overset{\text{df}}{=} \begin{cases} \dfrac{-1}{(n+\beta)(\beta+2)} & \text{for } n+\beta>0,\\ 1 & \text{for } n+\beta=0,\\ \dfrac{-(n+2\beta+2)}{(n-2)(n+\beta)(\beta+2)} & \text{for } n+\beta<0,\end{cases}\end{aligned}
$$

Now we shall show that

$$
S : \mathrm{BC}([1,\infty)) \to \mathrm{BC}([1,\infty))
$$

is compact. Therefore we have to verify that for any ball $B(0,M)$ its image under opreator S is relatively compact in the space $\mathrm{BC}([1,\infty))$. Take any $M>0$. From the estimate (2.6) it follows that the functions from the set

$$
\{Sv \in \mathrm{BC}([1,\infty)) : \|v\|_\infty \le M\}
$$

are equibounded by the function $h_{\beta,n}$, defined in (2.6), which is decaying. Fix arbitrary $\varepsilon>0$. Then one can choose $r_0 \ge 1$ such that for $r \ge r_0$ we have $|Ch_{\beta,n}(r)| \le \varepsilon$ and consequently

$$
(2.9)\qquad |Sv(r)| \le \varepsilon \quad \text{for } \|v\|_\infty \le M,\ r \ge r_0.
$$

Since, for $\|v\|_\infty \le M$, the functions Sv are equibounded and, due to the equiboundedness of $(Sv)'$ (see (2.7)), also equicontinuous, therefore from Ascoli–Arzelá theorem on the interval $[1,r_0]$ the set

$$
\{(Sv)/[1,r_0] : \|v\|_\infty \le M\}
$$

is relatively compact. So it has a finite ε-net $\{Sv_1/[1,r_0],\dots,Sv_k/[1,r_0]\}$. But then by (2.9) the set $\{Sv_1,\dots,Sv_k\}$ constitutes an ε-net for $\{Sv \in \mathrm{BC}([1,\infty)) : \|v\|_\infty \le M\}$. Thus we have proved its relative compactness. Since $M>0$ was arbitrary chosen, the operator $S : \mathrm{BC}([1,\infty)) \to \mathrm{BC}([1,\infty))$ maps bounded sets of $\mathrm{BC}([1,\infty))$ into relatively compact ones so it is compact. For some compactness criteria in the space $\mathrm{BC}([1,\infty))$ (and more general ones) one can see [15] and [19].

THEOREM 2.1. *Assume that f satisfies asymptotic condition* (2.5), *where function g is continuous and sublinear i.e.*

$$\limsup_{|v|\to\infty} |g(v)|/|v| = N < 1/L_{\beta,n}, \tag{2.10}$$

where $L_{\beta,n}$ is defined in (2.8). *Then the* BVP (2.2) *admits at least one solution.*

PROOF. By assumption (2.10), for $\varepsilon = 1/L_{\beta,n} - N > 0$, choose $M > 0$ such that $|g(v)|/|v| < N + \varepsilon = 1/L_{\beta,n}$ for any $|v| > M$. Define $T = \sup_{|v|\le M} |g(v)|$. Hence, for $\|v\|_\infty \le \max\{TL_{\beta,n}, M\}$,

$$\begin{aligned}
\|Sv\|_\infty &\le \sup_{r\in[1,\infty)} \bigg(\int_1^\infty |G(r,s)f(s,v(s))|\,ds\bigg) \\
&\le \sup_{r\in[1,\infty)} \bigg(\int_{\{s:|v(s)|\le M\}} |G(r,s)s^\beta g(v(s))|\,ds \\
&\quad + \int_{\{s:|v(s)|>M\}} |G(r,s)s^\beta g(v(s))|\,ds\bigg) \\
&\le \sup_{r\in[1,\infty)} \bigg(\int_{\{s:|v(s)|\le M\}} |G(r,s)s^\beta| \max\{T, M/L_{\beta,n}\}\,ds \\
&\quad + \int_{\{s:|v(s)|>M\}} |G(r,s)s^\beta| \max\{TL_{\beta,n}, M\}/L_{\beta,n})\,ds\bigg) \\
&= \max\{T, M/L_{\beta,n}\} \sup_{r\in[1,\infty)} \int_1^\infty |G(r,s)s^\beta|\,ds \\
&\le \max\{T, M/L_{\beta,n}\}L_{\beta,n} = \max\{TL_{\beta,n}, M\}.
\end{aligned}$$

Then $S : B(0, \max\{TL_{\beta,n}, M\}) \to B(0, \max\{TL_{\beta,n}, M\})$ so by the Schauder theorem we obtain a fixed point of S in $\mathrm{BC}[1,\infty))$ i.e. $v_0 = Sv_0$. However, by (2.6), the function Sv_0 decays at infinity and so does v_0. Therefore it is a solution to our BVP (2.2).

EXAMPLE. In particular Theorem 2.1 holds for functions of the following form $f(r,v) = h(r)g(v)$, where $h : [1,\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy:

(i) $g(0) \ne 0$ and $\limsup_{|v|\to\infty} |g(v)|/|v| = N < 1/L_{\beta,n}$,
(ii) $|h(r)| \le Hr^\beta$ for some constants $H > 0$ and $\beta < -2$.

THEOREM 2.2. *If we assume that f satisfies the asymptotic condition* (2.5), *where the function g is continuous, and that there exists a positive constant D such that*

$$f(r,v)v < 0 \quad \text{for all } r \in [1,\infty) \text{ and } |v| > D, \tag{2.11}$$

then the BVP (2.2) *admits at least one solution.*

Proof. We consider the family of BVPs:

$$-v'' + \frac{1-n}{r}v' = \lambda f(r,v), \quad v(1)=0, \quad \lim_{r\to\infty} v(r) = 0, \tag{2.12}$$

which can be restated as:

$$v = \lambda S v, \quad \lambda \in [0,1]. \tag{2.13}$$

Now suppose that for some $\lambda \in [0,1]$ there exists a solution v_λ such that $\|v_\lambda\|_\infty > D$. Since $v_\lambda(1) = 0$ and $\lim_{r\to\infty} |v_\lambda(r)| = 0$ it attains either postive maximum or negative minimum at some point $r_\lambda \in [1,\infty)$. We shall limit ourselves only to the case of maximum, because the case of minimum can be considered in a similar way. In such a case we have $v_\lambda(r_\lambda) > D$, $v_\lambda''(r_\lambda) \le 0$, $v_\lambda'(r_\lambda) = 0$ and

$$f(r_\lambda, v_\lambda(r_\lambda)) = -v_\lambda''(r_\lambda) + \frac{1-n}{r_\lambda} v_\lambda'(r_\lambda) \ge 0$$

which contradicts assumption (2.11). Therefore $\|v_\lambda\|_\infty = v_\lambda(r_\lambda) \le D$. Thus we have obtained a priori bounds for the solutions of (2.12) or (2.13). The Leray–Schauder degree

$$\deg(\mathrm{id} - \lambda S, B(0,D), 0)$$

is therefore well defined for all $\lambda \in [0,1]$. By homotopy invariance of the degree we have

$$\deg(\mathrm{id} - S, B(0,D), 0) = \deg(\mathrm{id}, B(0,D), 0) = 1$$

so S has a fixed point $v_0 \in \mathrm{BC}[1,\infty))$. Since, by (2.6), for each $v \in \mathrm{BC}[1,\infty))$, the function Sv must decay, so does a fixed point. Then it is a solution for BVP (2.2).

Corollary 3.1. *By estimates* (2.6) *and* (2.7) *one also obtains the rate of convergence to* 0 *at* ∞ *for the solution of* (2.2) *and its derivative.*

Example. In particular Theorem 2.2 holds for functions with separated variables i.e. $f(r,v) = h(r)g(v)$, where $h : [1,\infty) \to (0,\infty)$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, and satisfy:

(i) $g(0) \ne 0$ and $g(v)v < 0$ for $|v| > M$, where M is some positive constant,
(ii) $h(r) \le H r^\beta$ for some constants $H > 0$ and $\beta < -2$.

Remark. Although assumption (2.11) implies the existence of constant upper and lower solutions, they are not very helpful in establishing the existence of a decaying solution for (2.2) but only of a bounded one (see [19]).

Remark. The existence result [9] for the BVP (1.2) holds true for example if the nonlinearity (sufficiently smooth) satisfies the condition:

$$|f(x,u,p)| \le A\|x\|^\alpha + B\|x\|^\beta |u|^\sigma + C\|x\|^\gamma \|p\|^\tau$$

where A, B, C are some positive constants, α, β, $\gamma < -2$ and σ, $\tau \in (0,1)$. Our result (in the sublinear case) cannot be deduced from the above, even if the domain has radial symmetry (is exterior of a ball) and the nonlinearity does not depend on ∇u and is radially symmetric in x. It follows from the fact that although the sub- and supersolutions for the problem (2.1) are radial ($\underline{u}(x) = c_- \|x\|^\theta$, $\overline{u}(x) = c_+ \|x\|^\theta$, respectively, $c_- < 0$, $c_+ > 0$, $\theta < 0$) the monotone iteration scheme used in [9] provides the solution which can be nonradial. However, if f is autonomous there are many results which guarantee that a positive solution to (1.1) must be necessarily radial e.g. [2]–[4], [7], [16]. In our approach the nonlinearity has to depend explicitly on $\|x\|$, since it justifies our compactness argument.

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