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ON WEAK SOLUTIONS FOR SOME MODEL OF MOTION OF NONLINEAR VISCOUS-ELASTIC FLUID

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Dedicated to the memory of Juliusz P. Schauder

Abstract. We consider the statement of an initial boundary value problem for a generalized Oldroyd model describing both laminar and turbulent flows of a nonlinear visco-elastic fluid. The operator interpretation of a posed problem is presented. The properties of operators forming the corresponding equation are investigated. We introduce approximating operator equations and prove their solvability. On that base the existence theorem for the operator equation equivalent to the stated initial boundary value problem is proved.

Introduction

The system of equations of fluid motion in Cauchy form is well known [1] in hydrodynamics. Formally speaking, it describes the motion of all kinds of fluids. However this system contains the tensor of tangent pressure that is not explicitly expressed in terms of variables of the system. To get such an expression, as a rule, one involves various hypotheses on the relation between the tensor of tangent pressure and the tensor of velocities of deformation assuming that for specific fluids and specific motions those hypotheses should be verified in experiments. Such a hypothesis, describing both laminar and turbulent motions of a nonlinear

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viscous fluid, was suggested by O. A. Ladyzhenskaya in [2], [3]. Another one, taking into account the results of certain experiments, is contained in [4].

A lot of models for describing the motion of fluids are based on the so called constitutive Oldroyd equations [5] (see e.g. [6]–[8] where these equations were studied and modified). The models, based on those equations take into account effects of relaxation of pressure after stop and of the delay of deformations.

In this paper we present a certain modification of Oldroyd equations that unifies the approaches of [4] and [9]. The initial-boundary value problem with mixed boundary conditions, the most natural from physical viewpoint, is investigated. On a part of the boundary the velocities, and on the other one the surface forces are given. The similar problems for various versions of Navier–Stokes equations were investigated in [7] and [10].

Following Ladyshenskaya, we consider the problem of weak solutions. The existence theorem for a weak solution of the above mentioned initial-boundary value problem is established. The method of this paper is analogous to that of [9]. The problem is formulated in terms of a special operator equation, whose solvability is proved on the basis of a priori estimates and degree theory.

The paper consists of four sections. In the first one we introduce the main notations and concepts. We describe the formulations of the problem of weak solutions and of our initial-boundary value problem and consider the constitutive equation for our model. The functional spaces and operators, used in the paper, as well as operator equations, equivalent to the problem under consideration, are also introduced. In the second section the properties of operators involved in the above operator equation are investigated. In the third section we introduce some approximating operator equations. The existence results for solutions of those are obtained. In the last section the existence theorem for a solution of the operator equation, equivalent to the above-mentioned initial-boundary value problem, is formulated and proved.

1. Formulation of the evolution problem, equivalent operator equations

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$. In this paper we consider the motion of fluid filling the domain Ω, on the time interval $(0, T)$, $T > 0$.

1.1. Constitutive equation and formulation of the initial-boundary value problem. Let $v(t, x)$ be the velocity vector of a particle at the point x of the space at the time moment t and $v_1, \dots, v_n$ be components of v. Denote by $\mathcal{E}$ the tensor of velocities of deformations with components

$$\mathcal{E}_{ij} = \mathcal{E}_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

and by $\sigma = (\sigma_{ij})$ denote the tensor of tangent pressures (deviator of pressure tensor).

The character of motion of a fluid is determined by the choice of connection between $\mathcal{E}$ and σ. In [4] W. G. Litvinov considers the following constitutive relation:

$$\sigma_{ij} = 2[\varphi_1(U(v)) + \varphi_2(I(v), U(Av))]\mathcal{E}_{ij}(v), \tag{1.1}$$

where $I(v) = \sum_{i,j=1}^{n}(\mathcal{E}_{ij}(v))^2$. This relation contains the function $U(v)$ characterising the motion in domain Ω. If $U(v) < a$ for some positive constant a, then the motion is laminar. If $U(v) > a$, then the motion is turbulent. The level a determines the boundary, where motion becomes turbulent.

In the middle of the fifties Oldroyd [5] suggested a model of a fluid with constitutive equation

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\sigma = 2\nu\left(1 + æ\,\nu^{-1}\frac{\partial}{\partial t}\right)\mathcal{E}, \quad \lambda, \nu, æ > 0.$$

λ is called the time of relaxation, æ is the time of delay.

More general, nonlinear relations between σ and $\mathcal{E}$ are introduced in [6]:

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\sigma = \varphi(I_2(v))\mathcal{E} + æ\,\frac{\partial}{\partial t}(\psi(I_2(v))\mathcal{E}),$$

where $I_2(v) = (I(v))^{1/2}$. Expressing σ from this equation and using natural initial conditions, we obtain

$$\sigma = \frac{æ}{\lambda}\psi(I_2(v))\mathcal{E} + \int_0^t e^{-(t-\tau)/\lambda}\left(\frac{1}{\lambda}\varphi(I_2(v)) - \frac{æ}{\lambda^2}\psi(I_2(v))\right)\mathcal{E}\,d\tau.$$

If one denotes

$$\mu(I_2(v)) = \frac{1}{\lambda}\varphi(I_2(v)) - \frac{æ}{\lambda^2}\psi(I_2(v)),$$

the relation gets the following form

$$\sigma = \frac{æ}{\lambda}\psi(I_2(v))\mathcal{E} + \int_0^t e^{-(t-\tau)/\lambda}\mu(I_2(v))\mathcal{E}\,d\tau.$$

The first term corresponds to direct dependence of σ on $\mathcal{E}$, while the second one to indirect dependence via the effect of "memory" of a fluid. Taking into account such a form of dependence of σ on $\mathcal{E}$, the constitutive equation is naturally presented as follows

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\sigma = \frac{æ}{\lambda}\left(1 + \lambda\frac{\partial}{\partial t}\right)(\psi(I_2(v))\mathcal{E}) + \lambda\mu(I_2(v))\mathcal{E}.$$

Combining the above approach with relations (1.1), consider constitutive equation in the form

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma = \left(1+\lambda\frac{\partial}{\partial t}\right)(2(\varphi_1(U(v))+\varphi_2(I(v),U_A(v)))\mathcal{E}(v))-\lambda\widetilde{a}(t,x,v,D^1v).$$

Assuming that at the initial moment the fluid satisfies relations (1.1), one can derive σ from this equation:

$$\sigma = 2(\varphi_1(U(v))+\varphi_2(I(v),U_A(v)))\mathcal{E}(v) - \int_0^t e^{-(t-\tau)/\lambda}\,\widetilde{a}(\tau,x,v(\tau),D^1v(\tau))\,d\tau.$$

Introduce the notation

$$a(t,\tau,x,v(\tau),D^1v(\tau)) = e^{-(t-\tau)/\lambda}\,\widetilde{a}(\tau,x,v(\tau),D^1v(\tau)),$$

and rewrite the constitutive relation as follows

$$\sigma = 2(\varphi_1(U(v))+\varphi_2(I(v),U_A(v)))\mathcal{E}(v) - \int_0^t a(t,\tau,x,v(\tau),D^1v(\tau))\,d\tau. \tag{1.2}$$

The properties of functions, included in this equality, will be described below.

If the components $\sigma_{ij}(x)$ are differentiable in x, then by symbol $\operatorname{Div}\sigma$ we shall denote the vector

$$\left(\sum_{j=1}^n\frac{\partial\sigma_{1j}}{\partial x_j},\sum_{j=1}^n\frac{\partial\sigma_{2j}}{\partial x_j},\dots,\sum_{j=1}^n\frac{\partial\sigma_{nj}}{\partial x_j}\right),$$

whose coordinates are the divergences of rows of matrix $\sigma = (\sigma_{ij}(x))$.

Taking into account the constitutive relation (1.2), the fluid motion can be defined by means of the equation

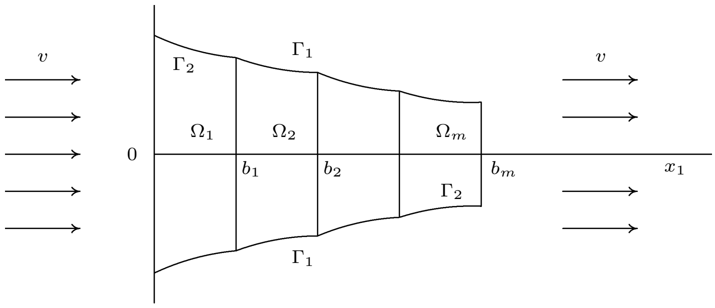
$$\rho\left(\frac{\partial v}{\partial t}+v_i\frac{\partial v}{\partial x_i}\right) = -\operatorname{grad}p+\operatorname{Div}\sigma+\varphi,\quad (t,x)\in(0,T)\times\Omega.$$

Here ρ is the fluid density, $p = p(t,x)$ is the pressure at the point x and time moment t, φ is the vector-function of volume force, acting on the fluid. Besides, here and further on we shall use the convention of summation on repeating indices.

The fluid is incompressible, therefore $\operatorname{div}v = 0$ for $(t,x)\in(0,T)\times\Omega$.

We suppose that the domain Ω is decomposed into open non-intersected subdomains Ω_i, $i=1,\dots,m$, such that $\overline{\Omega}=\bigcup_{i=1}^m\overline{\Omega}_i$, $\Omega_i\cap\Omega_j=\emptyset$ for $i\neq j$. Let the boundary Γ of domain Ω be Lipschitz continuous and Γ_1,Γ_2 be nonempty subsets of Γ such that $\Gamma=\overline{\Gamma}_1\cup\overline{\Gamma}_2$, $\Gamma_1\cap\Gamma_2=\emptyset$. Let also for each $i=1,\dots,m$ $(n-1)$-dimensional measure of intersection $\overline{\Omega}_i\cap\Gamma_1$ be positive.

The following example of domain Ω and its decomposition



satisfies all the above-mentioned conditions (see [4]). We suppose that on Γ_1 adhesion condition is valid

$$v\,|_{(0,T)\times\Gamma_1}=0,$$

and on Γ_2 a force, acting on a fluid surface is given

$$(-pE+\sigma)\nu\,|_{(0,T)\times\Gamma_2}=\Phi,$$

where ν is a unit external normal to Γ_2.

The functions $U(v)$ and $U_A(v)$ are defined as follows. Let

$$U(v)(x)=k_i\int_{\Omega_i} I(v)\,dy$$

for $x\in\Omega_i$ with positive constants k_i, $i=1,\dots,m$. Denote by P the operator of continuous prolongation on $(-\delta,T)\times\Omega$ of functions defined on $(0,T)\times\Omega$, where $\delta>0$. Choose a function $\omega\in C^\infty(\mathbb{R}_+)$ such that $\omega(y)\ge 0$ for $y\in\mathbb{R}_+$, $\omega(y)=0$ for $y\in[\delta,\infty)$. Let $h=(\int_0^\delta \omega(\tau)\,d\tau)^{-1}$ and

$$\rho_\delta(\tau)=\begin{cases} h\omega(\tau) & \text{for } \tau\ge 0,\\ 0 & \text{for } \tau<0.\end{cases}$$

Consider the averaging operator with rerspect to variable t

$$Y(v)(t,x)=\int_{-\delta}^{T}\rho_\delta(t-\tau)Pv(\tau,x)\,d\tau$$

and introduce the operator $U_A(v)=U(Y(v))$. The motion of a fluid in domain Ω is completely determined by the following initial-boundary value problem: the equation of motion

$$\rho\Big(\frac{\partial v}{\partial t}+v_i\frac{\partial v}{\partial x_i}\Big)=-\operatorname{grad}p+\operatorname{Div}\sigma+\varphi,\quad (t,x)\in(0,T)\times\Omega, \tag{1.3}$$

the constitutive relation

$$\sigma = 2\,(\varphi_1(U(v)) + \varphi_2(I(v), U_A(v)))\mathcal{E}(v) - \int_0^t a(t,\tau,x,v(\tau),D^1 v(\tau))\,d\tau, \tag{1.4}$$

the incompressibility equation

$$\operatorname{div} v = 0, \quad (t,x) \in (0,T)\times\Omega, \tag{1.5}$$

the boundary condition on Γ_1

$$v\,|_{(0,T)\times\Gamma_1} = 0, \tag{1.6}$$

the boundary condition on Γ_2

$$(-pE + \sigma)\nu\,|_{(0,T)\times\Gamma_2} = \Phi \tag{1.7}$$

and the initial condition

$$v(0,x) = v^0(x), \quad x \in \Omega. \tag{1.8}$$

Solution of problem (1.3)–(1.8) is a vector-function v and a scalar function p defined on $[0,T]\times\overline{\Omega}$ and satisfying (1.3)–(1.8).

We suppose, that

(1) the function $\varphi_1(y)$ satisfies the conditions

$$\varphi_1(y) \text{ is continuous on } \mathbb{R}_+ \text{ and } \varphi_1(y) \ge 0 \quad \text{for all } y \in \mathbb{R}_+, \tag{1.9}$$

$$\varphi_1(y_1) \ge \varphi_1(y_2) \quad \text{if } y_1 \ge y_2, \tag{1.10}$$

$$a_2 y \ge \varphi_1(y) \ge a_1 y \quad \text{for } y \in (a,\infty), \tag{1.11}$$

where a, a_1, a_2 are positive constants and $y_1, y_2 \in \mathbb{R}_+$;

(2) the function $\varphi_2(y_1,y_2)$ satisfies the conditions

$$\varphi_2(y_1,y_2) \text{ is continuous on } \mathbb{R}^2_+, \tag{1.12}$$

$$a_5 y_2 + a_4 \ge \varphi_2(y_1,y_2) \ge a_3 \quad \text{for all } (y_1,y_2) \in \mathbb{R}^2_+, \tag{1.13}$$

$$\varphi_2(y_1,y) \ge \varphi_2(y_2,y) \quad \text{if } y_1 \ge y_2, \tag{1.14}$$

where a_3, a_4, a_5 are positive constants and $y_1, y_2 \in \mathbb{R}_+$;

(3) the matrix-function $a(t,\tau,x,v,w)$ is defined for all $x \in \Omega$, $v \in \mathbb{R}^n$, $w \in \mathbb{R}^{n^2}$ and $(t,\tau) \in T_d$, where

$$T_d = \{(t,\tau) :\ t \in [0,T],\ t \ge \tau \ge 0\}.$$

$$\begin{cases} a \text{ is measurable in } t,\ \tau,\ x \text{ for all } v, w \\ \text{and is continuous in } v,\ w \text{ for almost all } t,\tau,x, \end{cases} \tag{1.15}$$

$$|a(t,\tau,x,v,0)| \le \mathcal{L}_1(t,\tau,x) + \mathcal{L}_2(t,\tau,x)|v|, \tag{1.16}$$

where $\mathcal{L}_2$ is an essentially bounded function and $\mathcal{L}_1$ belongs to class L_4 on $Q_d = T_d \times \Omega$;

$$|a(t,\tau,x,v,w) - a(t,\tau,x,v,\overline{w})| \le \mathcal{L}_2(t,s,x)|w - \overline{w}| \tag{1.17}$$

for any $(t,s,x) \in Q_d$, $v \in \mathbb{R}^n$, $w, \overline{w} \in \mathbb{R}^{n^2}$.

Note, that condition (1.14) differs from the monotonicity condition of [4]

$$[\varphi_2(y_1^2, y_2)y_1 - \varphi_2(y_3^2, y_2)y_3](y_1 - y_3) \ge a_2(y_1 - y_3)^2, \tag{1.18}$$

for all $y_1, y_2, y_3 \in \mathbb{R}_+$.

1.2. Principal notations and functional spaces. First we describe the spaces of functions on Ω used hereinafter:

- $L_2(\Omega)$ is the set of square integrable functions $w : \Omega \to \mathbb{R}$; the scalar product for $w, v \in L_2(\Omega)$ will be denoted by $(w, v)_{L_2(\Omega)}$,
- $W_2^1(\Omega)$ consists of functions from $L_2(\Omega)$ with partial derivative of the first order, belonging to $L_2(\Omega)$.

Introduce spaces of functions on Ω with values in space $\mathbb{R}^n$. Let now v, w be functions on Ω with values in $\mathbb{R}^n$.

- $L_2(\Omega)^n$ is the set of functions $w : \Omega \to \mathbb{R}^n$ with coordinates from $L_2(\Omega)$,
- $\|w\|_{L_2(\Omega)^n} = (\sum_{i=1}^n \int_\Omega w_i^2(x)\,dx)^{1/2}$ is the norm for $w \in L_2(\Omega)^n$,
- $W_2^1(\Omega)^n$ is the set of functions $w : \Omega \to \mathbb{R}^n$ with coordinates from $W_2^1(\Omega)$.

Following [4], introduce $V = \{v \in W_2^1(\Omega)^n : v|_{\Gamma_1} = 0,\ \operatorname{div} v = 0\}$. V is a Hilbert space with scalar product

$$(v, u)_V = \int_\Omega \mathcal{E}_{ij}(v)\mathcal{E}_{ij}(u)\,dx.$$

The corresponding norm is defined by equality

$$\|v\| = \left(\int_\Omega I(v)\,dx\right)^{1/2}.$$

From Korn's inequality and the fact, that $(n-1)$-dimensional measure of intersection $\overline{\Omega}_i \cap \Gamma_1$ is positive, it follows that this norm in the space V is equivalent to the norm induced from the space $W_2^1(\Omega)^n$.

Restrictions of functions from V on Ω_i form a space which will be denoted by V_i. The norm in V_i is defined by the equality

$$|||v|||_i = \left(\int_{\Omega_i} I(v)\,dx\right)^{1/2}.$$

Let H be the closure of V in the norm of space $L_2(\Omega)^n$, S be the set of step functions with constant values on each Ω_i, $i = 1, \dots, m$ and V^* be the space conjugate to V. Denote by (f, v) the action of functional f from V^* on a function v from V.

Introduce spaces of functions $v : [a, b] \to E$ with values in Banach space E:

- $L_p((a, b), E)$ is the space of functions integrable with degree $p \geq 1$, with the norm

$$\|v\|_{L_p((a,b),E)} = \bigg(\int_a^b \|v(t)\|_E^p \, dt \bigg)^{1/p},$$

- $L_\infty((a, b), E)$ is the space of essentially bounded functions with the norm

$$\|v\|_{L_\infty((a,b),E)} = \operatorname{vrai} \max_{(a,b)} \|v(t)\|_E,$$

- $C([a, b], E)$ is the space of continuous functions with the norm

$$\|v\|_{C([a,b],E)} = \max_{[a.b]} \|v(t)\|_E.$$

All spaces, mentioned above, are Banach ones. If the interval (a, b) is clear from context, then the symbols (a, b) in notations of spaces are omitted: $L_p(E)$, $L_\infty(E)$, $C(E)$. It is known, that the space $L_q((a, b), E^*)$ is conjugate to $L_p((a, b), E)$, $p > 1$, where $1/p + 1/q = 1$.

For a vector-function v from $L_p((a, b), V)$ denote by v_i the coordinate functions, by $\partial v/\partial t$, $\partial v/\partial x_i$ the first order partial derivatives and by $D^1 v$ the set of all derivatives $\partial v_i/\partial x_j$.

Now we can introduce the principal functional spaces used below.

$$\begin{aligned}
E_2 &= L_2((0, T), V) \text{ with the norm } \|v\|_{E_2} = \|v\|_{L_2((0,T),V)} \text{ for } v \in E_2,\\
E_2^* &= L_2((0, T), V^*) \text{ with the norm } \|f\|_{E_2^*} = \|f\|_{L_2((0,T),V^*)} \text{ for } f \in E_2^*,\\
E &= L_4((0, T), V) \text{ with the norm } \|v\|_E = \|v\|_{L_4((0,T),V)} \text{ for } v \in E,\\
E^* &= L_{4/3}((0, T), V^*) \text{ with the norm } \|f\|_{E^*} = \|f\|_{L_{4/3}((0,T),V^*)} \text{ for } f \in E^*,\\
W &= \{v : \ v \in E, \ v' \in E^*\} \text{ with the norm } \|v\|_W = \|v\|_E + \|v'\|_{E^*} \text{ for } v \in W.
\end{aligned}$$

The space W is Banach one and it is known (see [11, Theorem 1.17, p. 177]), that $W \subset C([0, T], H)$.

Denote by $\langle f, v\rangle$ the coupling of a functional f from E^* with a function v from E, and by $\langle f, v\rangle_2$ the coupling of a functional f from E_2^* with a function v from E_2.

1.3. Statement of the problem of weak solutions and equivalent operator equations. Let us introduce operators in functional spaces using the following equalities:

- $N_1 : V \to V^*$, $(N_1(u), h) = 2\int_\Omega \varphi_1(U(u))\mathcal{E}_{ij}(u)\mathcal{E}_{ij}(h)\,dx$;
- $N_2 : V \times S \to V^*$, $(N_2(u,s), h) = 2\int_\Omega \varphi_2(I(u), s)\mathcal{E}_{ij}(u)\mathcal{E}_{ij}(h)\,dx$;
- $K : V \to V^*$,

$$(K(u), h) = \rho \int_\Omega u_j \frac{\partial u_i}{\partial x_j} h_i\,dx,$$

where $u, h \in V$, $s \in S$;

- $B : T_d \times L_2(\Omega)^n \times V \to V^*$,

$$(B(t,\tau,u,v), h) = \int\limits_\Omega a_{ij}(t,\tau,x,u(x),D^1v(x))\frac{\partial h_i}{\partial x_j}(x)\,dx,$$

  where $u \in L_2(\Omega)^n$, $u, h \in V$.

Let $Q_T = (0,T) \times \Omega$ and $L_2(Q_T)^n = L_2(Q_T, \mathbb{R}^n)$:

- $C : L_2(Q_T)^n \times E_2 \to E_2^*$, $C(u,v) = \int_0^t B(t,\tau,u(\tau),v(\tau))\,d\tau$.

Note that $E \subset E_2$ and $E_2^* \subset E^*$ imply $C : L_2(Q_T)^n \times E \to E^*$.

Suppose that $n = 2$ or $n = 3$ and

$$\Phi \in L_{4/3}((0,T) \times \Gamma_2, \mathbb{R}^n), \quad \varphi \in L_{4/3}((0,T) \times \Omega, \mathbb{R}^n).$$

These functions define functionals $f, F \in L_{4/3}((0,T), V^*)$ on V by the equalities:

$$(F, h) = \int\limits_{\Gamma_2} \Phi h\,dx, \quad (f, h) = \int\limits_\Omega \varphi h\,dx,$$

for $h \in V$. The definitions are well-posed since $h \in W_2^1(\Omega)^n \subset L_4(\Gamma_2)^n$ and $h \in L_4(\Omega)^n$ for $n = 2, 3$.

The weak solution of problem (1.3)–(1.8) is a vector-function v such that

$$v \in L_4((0,T), V), \quad v' \in L_{4/3}((0,T), V^*), \tag{1.19}$$

$$\rho(v', h) + (N_1(v) + N_2(v, U_A(v)), h) + (K(v), h) - \Bigg(\int\limits_0^t B(t,\tau,v(\tau),v(\tau))\,d\tau, h\Bigg) = (F + f, h), \quad \text{for all } h \in V, \tag{1.20}$$

$$v(0) = v^0. \tag{1.21}$$

Condition (1.19) provides $v \in W \subset C([0,T], H)$. Therefore condition (1.21) is valid for $v^0 \in H$.

Using the Green formula it is possible to check that if v, p is a solution of problem (1.3)–(1.8), v satisfies conditions (1.19)–(1.21).

Equality (1.20) is equivalent to the operator equation

$$\rho v' + N_1(v) + N_2(v, U_A(v)) + K(v) - \int_0^t B(t,\tau,v(\tau),v(\tau))\,d\tau = F + f. \tag{1.22}$$

Let k be a positive number whose value will be defined in further arguments. Substitute $v(t) = e^{kt}\overline{v}(t)$ and multiple the equation by e^{-kt}. We obtain the equivalent operator equation

$$\rho\overline{v}' + \rho k\overline{v} + (N_1(e^{kt}\overline{v}) + N_2(e^{kt}\overline{v}, U_A(e^{kt}\overline{v})) + K(e^{kt}\overline{v}))e^{-kt} - \int_0^t e^{-kt}B(t,\tau,e^{k\tau}v(\tau),e^{k\tau}v(\tau))\,d\tau = \overline{F} + \overline{f},$$

where $\overline{F} = e^{-kt}F$, $\overline{f} = e^{-kt}f$.

To simplify the formulae we introduce the notations:

$$\begin{aligned} \overline{N}_1(\overline{u}) &= e^{-kt}N_1(e^{kt}\overline{u}), \\ \overline{N}_2(\overline{u}, s) &= e^{-kt}N_2(e^{kt}\overline{u}, s), \quad \text{where } s \in S, \\ \overline{B}(t,\tau,\overline{u},\overline{v}) &= e^{-k\tau}B(t,\tau,e^{k\tau}\overline{u}(\tau),e^{k\tau}\overline{v}(\tau)), \\ \overline{K}(\overline{u}) &= e^{-kt}K(e^{kt}\overline{u}), \\ \overline{C}(\overline{u},\overline{v}) &= \int_0^t e^{-k(t-\tau)}\overline{B}(t,\tau,e^{k\tau}\overline{v}(\tau),e^{k\tau}\overline{v}(\tau))\,d\tau \end{aligned} \tag{1.23}$$

Rewrite the operator equation as follows:

$$\rho\overline{v}' + \rho k\overline{v} + \overline{N}_1(\overline{v}) + \overline{N}_2(\overline{v}, U_A(e^{kt}\overline{v})) + \overline{K}(\overline{v}) - \overline{C}(\overline{v},\overline{v}) = \overline{F} + \overline{f}, \tag{1.24}$$

Then the problem of weak solution is equivalent to the existence problem for a solution $\overline{v} \in W$ of operator equation (1.24), and the solution should satisfy the initial conditions

$$\overline{v}(0) = v^0. \tag{1.25}$$

2. Studying the properties of operators

In this section the properties of operators from operator equation (1.24) are investigated.

2.1. Properties of operators $\overline{N}_1$ and $\overline{N}_2$.

Lemma 2.1. *If the function φ_1 satisfies conditions* (1.9)–(1.11)*, for any function $\overline{u}$ from E the function $\overline{N}_1(\overline{u})$ belongs to E^*. The map $\overline{N}_1 : E \to E^*$ is bounded, continuous, monotone and the following inequality holds:*

$$\langle \overline{N}_1(\overline{u}), \overline{u} \rangle \geq C_1 \|\overline{u}\|_E^4 - C_0 \tag{2.1}$$

with constants C_0, C_1, independent of $\overline{u}$.

Proof. To prove continuity and boundedness of the map $\overline{N}_1$ it is sufficient to establish the continuity and boundedness of $N_1 : E \to E^*$. By definition

$$\langle N_1(u), h \rangle = \int_0^T (N_1(u(t)), h(t))\, dt = 2 \int_0^T \int_\Omega \varphi_1(U(u(t))) \mathcal{E}_{ij}(u(t)) \mathcal{E}_{ij}(h(t))\, dx\, dt$$

for $u, h \in E$. As $\mathcal{E}(h) \in L_4((0,T), L_2(\Omega)^{n^2})$ and $\|\mathcal{E}(h)\|_{L_4((0,T),L_2(\Omega)^{n^2})} = \|h\|_E$,

$$\|N_1(u)\|_{E^*} \leq \|\varphi_1(U(u))\mathcal{E}(u)\|_{L_{4/3}((0,T),L_2(\Omega)^{n^2})}.$$

Therefore it is sufficient to show the continuity and boundedness of each map

$$\Phi_{ij} : u \mapsto \varphi_1(U(u))\mathcal{E}_{ij}(u) \quad \text{from } E \text{ to } \; L_{4/3}((0,T), L_2(\Omega)).$$

For $u \in E$ $U(u) \in L_2((0,T), S)$. Then, from conditions (1.10) and (1.11), it follows that $\varphi_1(U(u)) \in L_2((0,T), S)$ and the map $U \mapsto \varphi_1(U(u))$ from E to $L_2((0,T), S)$ is continuous as a superposition operator by the M. A. Krasnosel'skiĭ's theorem [12]. As $\mathcal{E}_{ij}(u) \in L_4((0,T), L_2(\Omega))$, by the Hölder inequality $\varphi_1(U(u))\mathcal{E}_{ij}(u) \in L_{4/3}((0,T), L_2(\Omega))$and the map Φ_{ij} is continuous as a product of continuous maps.

Conditions (1.10), (1.11) imply that

$$\|\varphi_1(U(u(t)))\mathcal{E}(u(t))\|_{L_2(\Omega)^{n^2}} \leq (a_2 \max_i k_i \|u(t)\|_V^2 + \varphi(a)) \|\mathcal{E}(u(t))\|_{L_2(\Omega)^{n^2}}.$$

Therefore $\|\varphi_1(U(u))\mathcal{E}(u)\|_{L_{4/3}((0,T),L_2(\Omega)^{n^2})} \leq C_0 \|u\|_E^3 + C_1$, and the map N_1 is continuous and bounded.

The monotonicity of map $\overline{N}_1$ follows from representation

$$\begin{aligned} \langle \overline{N}_1(\overline{u}) - \overline{N}_1(\overline{v}), \overline{u} - \overline{v} \rangle &= \int_0^T e^{-kt}(N_1(e^{kt}\overline{u}(t)) - N_1(e^{kt}\overline{v}(t)), \overline{u}(t) - \overline{v}(t)\, dt \\ &= \int_0^T e^{-2kt}(N_1(u(t)) - N_1(v(t)), u(t) - v(t))\, dt, \end{aligned}$$

where $u(t) = e^{kt}\overline{u}(t)$, $v(t) = e^{kt}\overline{v}(t)$, and from the monotonicity of map $N_1 : V \to V^*$, established in Lemma 4.1 [4].

Let us prove estimate (2.1). For $\overline{u} \in E$ and $u(t) = e^{kt}\overline{u}(t)$

$$\langle \overline{N}_1(\overline{u}), \overline{u}\rangle = \int_0^T e^{-2kt}(N_1(e^{kt}\overline{u}(t)), e^{kt}\overline{u}(t))\,dt = \int_0^T e^{-2kt}(N_1(u(t)), u(t))\,dt.$$

By definition

$$\begin{aligned}(N_1(u(t)), u(t)) &= 2\int_\Omega \varphi_1(U(u(t)))\mathcal{E}_{ij}(u(t))\mathcal{E}_{ij}(u(t))\,dx \\ &= 2\varphi_1(k_i|||u(t)|||_i^2)|||u(t)|||_i^2.\end{aligned}$$

Due to conditions (1.10), (1.11)

$$\varphi_1(k_i|||u(t)|||_i^2)|||u|||_i^2 \geq \begin{cases} 0 & \text{if } |||u(t)|||_i^2 \leq a/k_i, \\ a_1k_i|||u(t)|||_i^4 & \text{if } |||u(t)|||_i^2 > a/k_i, \end{cases}$$

and so

$$\varphi_1(k_i|||u(t)|||_i^2)|||u(t)|||_i^2 \geq a_1k_i|||u(t)|||_i^4 - \frac{a_1a^2}{k_i}$$

for each $i = 1, \ldots, m$. As $\|u(t)\| \leq \sum_{i=1}^m |||u(t)|||_i$, we get

$$(N_1(u(t)), u(t)) \geq 2a_1æ\,\|u(t)\|^4 - C$$

with $C = 2a_1a^2 \max_i k_i^{-1}$ and $æ = \min_i k_i$.

Coming back to estimate for $\langle \overline{N}_1(\overline{u}), \overline{u}\rangle$, we obtain

$$\begin{aligned}\langle \overline{N}_1(\overline{u}), \overline{u}\rangle &\geq \int_0^T e^{-2kt}(2a_1æ\,\|u(t)\|^4 - C)\,dt \\ &= \int_0^T 2a_1æ\,e^{2kt}\|\overline{u}(t)\|^4\,dt - C\int_0^T e^{-2kt}\,dt \geq 2a_1æ\,\|\overline{u}\|_E^4 - C\frac{1-e^{-2kT}}{2k}.\end{aligned}$$

The lemma follows. □

Lemma 2.2. *If the function φ_2 satisfies conditions* (1.12)–(1.14), *the map $\overline{N}_2 : E_2\times C([0,T], S) \to E_2^*$ is continuous and bounded. Besides, for any function $s \in C([0,T], S)$ the map $\overline{N}_2(\,\cdot\,, s) : E_2 \to E_2^*$ is monotone, coercive and the following inequality holds*

$$\langle \overline{N}_2(\overline{u}, s) - \overline{N}_2(\overline{v}, s), \overline{u} - \overline{v}\rangle_2 \geq 2a_3\|\overline{u} - \overline{v}\|_{E_2}^2 \tag{2.2}$$

for any $\overline{u}, \overline{v} \in E_2$.

Proof. Consider map $N_2 : E_2 \times C([0,T], S) \to E_2^*$. By definition

$$\langle N_2(u, s), h\rangle_2 = 2\int_0^T\int_\Omega \varphi_2(I(u), s)\mathcal{E}_{ij}(u)\mathcal{E}_{ij}(h)\,dx\,dt$$

for $u, h \in E_2$, $s \in C([0,T],S)$, therefore in order to prove the continuity and boundedness of N_2 it is necessary to show continuity and boundedness of the maps $(u,s) \mapsto \varphi_2(I(u),s)\mathcal{E}_{ij}(u)$ from $E_2 \times C([0,T],S)$ into $L_2((0,T),L_2(\Omega))$.

To prove the continuity we shall consider the map as the composition of continuous maps $u \mapsto I(u)$ from E_2 into $L_1(Q_T)$, $u \mapsto \mathcal{E}_{ij}(u)$ from E_2 into $L_2(Q_T)$, and continuous superposition operator $\Phi_s : (w,s,\overline{\mathcal{E}}) \mapsto \varphi_2(w,s)\overline{\mathcal{E}}$ from $L_1(Q_T) \times C([0,T],S) \times L_2(Q_T)$ into $L_2((0,T),L_2(\Omega))$. From (1.13) we derive the estimate:

$$|\varphi_2(w,s)\overline{\mathcal{E}}| \leq (a_5R + a_4)|\overline{\mathcal{E}}| \quad \text{for } |s| < R.$$

From this estimate and M. A. Krasnosel'skiĭ's theorem [12] we obtain that the superposition operator Φ_s is continuous. The boundedness of Φ_s also follows from this estimate.

Thus, we have established the continuity and boundedness of the map N_2 and so of the map $\overline{N}_2$.

Now establish estimate (2.2). Let $\overline{u}, \overline{v} \in E_2$ and $u(t) = e^{kt}\overline{u}(t)$, $v(t) = e^{kt}\overline{v}(t)$. By definition

$$\begin{aligned}
&\langle \overline{N}_2(\overline{u},s) - \overline{N}_2(\overline{v},s), \overline{u}-\overline{v}\rangle_2 \\
&\quad = \langle e^{-2kt}(N_2(u(t),s(t)) - N_2(v(t),s(t))), u(t)-v(t)\rangle_2 \\
&\quad = 2\int\limits_{Q_T}\int e^{-2kt}(\varphi_2(I(u(t)),s(t))\mathcal{E}_{ij}(u(t)) \\
&\qquad - \varphi_2(I(v(t)),s(t))\mathcal{E}_{ij}(v(t)))(\mathcal{E}_{ij}(u(t)) - \mathcal{E}_{ij}(v(t)))\,dx\,dt \\
&\quad = \int\limits_{Q_T}\int [e^{-2kt}(\varphi_2(I(u(t)),s(t)) + \varphi_2(I(v(t)),s(t)))(\mathcal{E}(u(t)) - \mathcal{E}(v(t)))^2 \\
&\qquad + e^{-2kt}(\varphi_2(I(u(t)),s(t)) - \varphi_2(I(v(t)),s(t)))(\mathcal{E}_{ij}(u(t)) + \mathcal{E}_{ij}(v(t))) \\
&\qquad \cdot\ (\mathcal{E}_{ij}(u(t)) - \mathcal{E}_{ij}(v(t)))]\,dx\,dt.
\end{aligned}$$

As $(\mathcal{E}_{ij}(u(t)) + \mathcal{E}_{ij}(v(t)))(\mathcal{E}_{ij}(u(t)) - \mathcal{E}_{ij}(v(t))) = I(u(t)) - I(v(t))$, by condition (1.18), the second term is nonnegative. Besides, by condition (1.13),

$$\varphi_2(I(u(t)),s(t)) + \varphi_2(I(v(t)),s(t)) \geq 2a_3,$$

therefore

$$\begin{aligned}
\langle \overline{N}_2(\overline{u},s) - \overline{N}_2(\overline{v},s), \overline{u}-\overline{v}\rangle_2 &\geq 2a_3 \iint\limits_{Q_T} e^{-2kt}(\mathcal{E}(u(t)) - \mathcal{E}(v(t)))^2\,dx\,dt \\
&= 2a_3 \iint\limits_{Q_T} (\mathcal{E}(\overline{u}(t) - \overline{v}(t)))^2\,dx\,dt = 2a_3\|\overline{u}-\overline{v}\|^2_{E_2},
\end{aligned}$$

as it is formulated in the assertion of lemma.

Note that $\overline{N}_2(0,s) = 0$. Then from inequality (2.2) for $\overline{v} = 0$ we get the coercive inequality

$$\langle \overline{N}_2(\overline{u}, s), \overline{u}\rangle_2 \geq 2a_0 \|\overline{u}\|^2_{E_2}. \tag{2.3}$$

□

As embeddings $E \subset E_2$ and $E_2^* \subset E^*$ are continuous, under the conditions of Lemma 2.2 the map $\overline{N}_2 : E \times C([0,T], S) \to E^*$ is continuous and for any function $s \in C([0,T], S)$ the map $\overline{N}_2(\,\cdot\,, s) : E \to E^*$ is monotone.

2.2. Properties of the operator U_A.

Lemmma 2.3. *The map $U_A : E_2 \to C([0,T], S)$ is completely continuous.*

Proof. Since any bounded closed set in S is compact, by the Arzéla–Askoli theorem in order to prove the compactness of map under consideration it is sufficient to establish equicontinuity and uniform boundedness of the set of functions $U_A(u)$ for any bounded set of functions u from E_2. For any $t_1, t_2 \in [0,T]$ and $x \in \Omega_l$

$$\begin{aligned}
&|U_A(u)(t_1) - U_A(u)(t_2)| \\
&\quad = K_l \int_{\Omega_l} \bigg[I\bigg(\int_{-\delta}^{T} \rho_\delta(t_1 - \tau) Pu(\tau, x)\, d\tau \bigg) - I\bigg(\int_{-\delta}^{T} \rho_\delta(t_2 - \tau) Pu(\tau, x)\, d\tau \bigg) \bigg]\, dx \\
&\quad = K_l \int_{\Omega_l} \bigg[\bigg(\int_{-\delta}^{T} \rho_\delta(t_1 - \tau) P\mathcal{E}_{ij}(u)(\tau, x)\, d\tau \bigg)^2 \\
&\qquad - \bigg(\int_{-\delta}^{T} \rho_\delta(t_2 - \tau) P\mathcal{E}_{ij}(u)(\tau, x)\, d\tau \bigg)^2 \bigg]\, dx \\
&\quad = K_l \int_{\Omega_l} \bigg(\int_{-\delta}^{T} (\rho_\delta(t_1 - \tau) - \rho_\delta(t_2 - \tau)) P\mathcal{E}_{ij}(u)(\tau, x)\, d\tau \\
&\qquad \cdot \int_{-\delta}^{T} (\rho_\delta(t_1 - \tau) + \rho_\delta(t_2 - \tau)) P\mathcal{E}_{ij}(u)(\tau, x)\, d\tau \bigg)\, dx \\
&\quad \leq k_l \int_{\Omega_l} \bigg(\int_{-\delta}^{T} |\rho_\delta(t_1 - \tau) - \rho_\delta(t_2 - \tau)| |P\mathcal{E}_{ij}(u)(\tau, x)|\, d\tau \\
&\qquad \cdot \max_{\tau} |\rho_\delta(t_1 - \tau) + \rho_\delta(t_2 - \tau)| \int_{-\delta}^{T} |P\mathcal{E}_{ij}(u)(\tau, x)|\, d\tau \bigg)\, dx.
\end{aligned}$$

As the function ρ_δ is bounded, applying the Hölder inequality we obtain

$$\begin{aligned}&|U_A(u)(t_1) - U_A(u)(t_2)|\\ &\quad\le C_1 k_l \Big(\int_{-\delta}^{T} |\rho_\delta(t_1-\tau) - \rho_\delta(t_2-\tau)|^2 d\tau\Big)^{1/2} \int_{\Omega_l}\int_{-\delta}^{T} |P\mathcal{E}_{ij}(u)(\tau,x)|^2\, d\tau\, dx\\ &\quad\le C_2 \Big(\int_{-\delta}^{T} |\rho_\delta(t_1-\tau) - \rho_\delta(t_2-\tau)|^2 d\tau\Big)^{1/2} \|Pu\|_{E_2}^2.\end{aligned}$$

It is easy to check that

$$\int_{-\delta}^{T} |\rho_\delta(t_1-\tau) - \rho_\delta(t_2-\tau)|^2 d\tau \to 0 \quad \text{at } |t_1-t_2| \to 0.$$

From this and from boundedness of the operator of prolongation P in $L_2(Q_T)$ it follows that the functions from the set $U_A(u)$ are uniformly continuous. Boundedness of the set $U_A(u)$ and continuity of the map U_A follow from continuity of U and Y whose composition forms U_A. The lemma follows. □

The embedding map $E \subset E_2$ is continuous, therefore the map $U_A : E \to C([0,T], S)$ is completely continuous.

2.3. Properties of the map $\overline{K}$.

Lemma 2.4. *The map $\overline{K} : E \to E^*$ is continuous and bounded. The map $\overline{K} : E_2 \cap L_4(Q_T)^n \to E^*$ is continuous. The map $\overline{K} : W \to E^*$ is completely continuous. Besides, for any function $\overline{u} \in E$ the following estimate holds:*

$$|\langle \overline{K}(\overline{u}), \overline{u}\rangle| \le C \|\overline{u}\|_{E_2} \|\overline{u}\|^2_{L_4(Q_T)^n} \tag{2.4}$$

with the constant $C = \rho e^{kT}$.

Proof. By definition of $\overline{K}$ for $\overline{u}, \overline{h} \in E$

$$\langle \overline{K}(\overline{u}), \overline{h}\rangle = \rho \int\int_{Q_T} e^{kt} \overline{u}_j(t) \frac{\partial \overline{u}_i(t)}{\partial x_j} \overline{h}_i(t)\, dx\, dt.$$

Then the continuity of the map $\overline{K} : E_2 \cap L_4(Q_T)^n \to E^*$ follows from that of the map

$$\overline{u} \mapsto \overline{u}_j \frac{\partial \overline{u}}{\partial x_j} \quad \text{from} \quad E_2 \cap L_4(Q_T)^n \quad \text{to} \quad L_{4/3}((0,T), L_{4/3}(\Omega)^n).$$

The continuity of embeddings $E \subset E_2$, $E \subset L_4(Q_T)^n$ causes the continuity of maps $\overline{K} : E \to E^*$ and $\overline{K} : W \to E^*$.

Applying the Hölder inequality we obtain the estimate

$$|\langle \overline{K}(\overline{u}), \overline{h}\rangle| \le \rho e^{kT} \int_0^T \|\overline{u}_j(t)\|_{L_4(\Omega)} \bigg\| \frac{\partial \overline{u}_i(t)}{\partial x_j} \bigg\|_{L_2(\Omega)} \|\overline{h}_i(t)\|_{L_4(\Omega)}\, dt$$
$$\le \rho e^{kT} \|\overline{u}_j\|_{L_4(Q_T)} \bigg\| \frac{\partial \overline{u}_i}{\partial x_j} \bigg\|_{L_2((0,T),L_2(\Omega))} \|\overline{h}_i\|_{L_4(Q_T)}.$$

From this it follows that the map $\overline{K}$ is bounded and that for $\overline{h} = \overline{u}$ the next estimate holds:

$$|\langle \overline{K}(\overline{u}), \overline{h}\rangle| \le \rho e^{kT} \|\overline{u}\|_{E_2} \|\overline{u}\|^2_{L_4(Q_T)^n}.$$

Let us prove the compactness of $\overline{K} : W \to E^*$. Choose an arbitrary bounded sequence $\{\overline{u}_l\}$, $\overline{u}_l \in W$ such that $\overline{u}_l \rightharpoonup \overline{u}_0$ weakly in E. As the embedding $V \subset L_4(\Omega)^n$ is completely continuous for $n = 2, 3$, by Theorem 2.1 ([13, p. 217]) the embedding $W \subset L_4(Q_T)^n$ is completely continuous. Therefore, without loss of generality, we can assume that

$$\overline{u}_l \to \overline{u}_0 \quad \text{strongly in } L_4(Q_T)^n.$$

Similarly, as the embedding $V \subset L_2(\Gamma_2)^n$ is completely continuous, the embedding $W \subset L_2((0,T), L_2(\Gamma_2)^n)$ is completely continuous. Therefore let us assume that

$$\|\overline{u}_l - \overline{u}_0\|_{L_2((0,T),L_2(\Gamma_2)^n)} \to 0 \quad \text{as } l \to \infty$$

and show that under these assumptions $\overline{K}(\overline{u}_l) \to \overline{K}(\overline{u}_0)$ strongly in E^*.

Using the Green formula we obtain from the definition of $\overline{K}$:

$$\begin{aligned}
\langle \overline{K}(\overline{u}_l) - \overline{K}(\overline{u}_0), \overline{h}\rangle &= \rho \iint_{Q_T} e^{kt} (\overline{u}_{lj} - \overline{u}_{0j})(t) \frac{\partial \overline{u}_{li}(t)}{\partial x_j} \overline{h}_i(t)\, dx\, dt \\
&\quad + \rho \iint_{Q_T} e^{kt} \overline{u}_{0j}(t) \bigg(\frac{\partial \overline{u}_{li}}{\partial x_j} - \frac{\partial \overline{u}_{0i}}{\partial x_j} \bigg)(t) \overline{h}_i(t)\, dx\, dt \\
&= \rho \iint_{Q_T} e^{kt} (\overline{u}_{lj} - \overline{u}_{0j})(t) \frac{\partial \overline{u}_{li}(t)}{\partial x_j} \overline{h}_i(t)\, dx\, dt \\
&\quad - \rho \iint_{Q_T} e^{kt} (\overline{u}_{li} - \overline{u}_{0i})(t) \frac{\partial \overline{u}_{0j}(t)}{\partial x_j} \overline{h}_i(t)\, dx\, dt \\
&\quad - \rho \iint_{Q_T} e^{kt} \overline{u}_{0j}(t) (\overline{u}_{li} - \overline{u}_{0i})(t) \frac{\partial \overline{h}_i(t)}{\partial x_j}\, dx\, dt \\
&\quad + \rho \int_0^T \bigg(\int_{\Gamma_2} \overline{u}_{0j}(t) (\overline{u}_{li} - \overline{u}_{0i})(t) \overline{h}_i(t) \nu_j\, d\tau \bigg) e^{kt}\, dt,
\end{aligned}$$

where ν is a unit external normal to Γ. Estimate each term in the above expression. For the first one we have:

$$\left|\rho \iint\limits_{Q_T} e^{kt}(\overline{u}_{lj}-\overline{u}_{0j})(t)\frac{\partial \overline{u}_{li}(t)}{\partial x_j}\overline{h}_i(t)\,dx\,dt\right| \le \rho e^{kT}\ |\overline{u}_{lj}-\overline{u}_{0j}\|_{L_4(Q_T)}\left\|\frac{\partial \overline{u}_{li}}{\partial x_j}\right\|_{L_2(Q_t)}\|\overline{h}_i\|_{L_4(Q_T)}.$$

The second one is equal to zero, as $\overline{u}_0(t)\in V$ and $\operatorname{div}\overline{u}_0(t)=0$.

For the third term we have the estimate:

$$\left|\rho \iint\limits_{Q_T} e^{kt}\overline{u}_{0j}(t)(\overline{u}_{li}-\overline{u}_{0i})(t)\frac{\partial \overline{h}_i(t)}{\partial x_j}\,dx\,dt\right| \le \rho e^{kT}\|\overline{u}_{0j}\|_{L_4(Q_T)}\|\overline{u}_{li}-\overline{u}_{0i}\|_{L_4(Q_T)}\left\|\frac{\partial \overline{h}_i}{\partial x_j}\right\|_{L_2(Q_T)}.$$

Estimate the fourth term:

$$\left|\rho\int_0^T\left(\int_{\Gamma_2}\overline{u}_{0j}(t)(\overline{u}_{li}-\overline{u}_{0i})(t)\overline{h}_i(t)\nu_j\,d\tau\right)e^{kt}\,dt\right| \le \rho e^{kT}\|\overline{u}_{0j}\|_{L_4((0,T),L_4(\Gamma_2))}\|\overline{u}_{li}-\overline{u}_{0i}\|_{L_2((0,T),L_2(\Gamma_2))}\|h_i\|_{L_4((0,T),L_4(\Gamma_2))}$$

due to continuity of embedding $E\subset L_4((0,T),L_4(\Gamma_2)^n)$.

All the norms in right-hand sides of the above estimates are uniformly bounded, then since we assume that we have chosen the sequence $\{\overline{u}_l\}$ tending in $L_4(Q_T)^n$ and $L_2((0,T),L_2(\Gamma_2)^n)$, each term tends to zero as $l\to\infty$ uniformly with respect to $\overline{h}$ with $\|\overline{h}\|_E\le 1$. This provides the strong convergence $\overline{K}(\overline{u}_l)\to\overline{K}(\overline{u}_0)$ in E^*. The lemma is proved. □

2.4. Properties of maps B and C.

Lemma 2.5. *Let a matrix-function a satisfy conditions* (1.15)–(1.17). *Then the maps $\overline{B}$ and $\overline{C}$, defined by* (1.23), *are continuous, bounded and for any $\overline{w}\in L_2(Q_T)^n$, $\overline{u}\in E_2$, the following estimate holds*

$$|\langle\overline{C}(\overline{w},\overline{u}),\overline{u}\rangle_2|\le C(1+\|\overline{w}\|_{L_2(Q_T)^n}+\|\overline{u}\|_{E_2})\|\overline{u}\|_{E_2} \tag{2.5}$$

with a constant C depending on characteristics $\mathcal{L}_1$, $\mathcal{L}_2$ and on T.

Proof. The operator $\overline{C}$ can be presented as superposition of continuous integral operator and $\overline{B}$. Therefore it is sufficient to establish continuity and boundedness of $\overline{B} : L_2(Q_T)^n \times E_2 \to L_2(T_d, V^*)$ defined by the equality

$$(e^{-k\tau} B(t,\tau, e^{k\tau}\overline{w}(\tau), e^{k\tau}\overline{u}(\tau)), \overline{h}) = (\overline{B}(t,\tau,\overline{w}(\tau),\overline{u}(\tau)),\overline{h}) = \int_\Omega a_{ij}(t,\tau,e^{k\tau}\overline{w}(\tau), e^{k\tau}D^1\overline{u}(\tau))\frac{\partial \overline{h}_i}{\partial x_j} e^{-k\tau}\,dx$$

for $\overline{w} \in L_2(Q_T)^n$, $\overline{u} \in E_2$ and $\overline{h} \in V$.

The continuity and the boundedness of $\overline{B}$ follows from the continuity and boundedness of the maps

$$\overline{a}_{ij} : (\overline{w},\overline{u}) \mapsto a_{ij}(t,\tau,e^{k\tau}\overline{w}(\tau), e^{k\tau}D^1\overline{u}(\tau))e^{-k\tau}$$

from $L_2(Q_T)^n \times E_2$ in $L_2(Q_d)$. Conditions (1.16) and (1.17) cause the estimate

$$|e^{-k\tau}a_{ij}(t,\tau,e^{k\tau}\overline{w}(\tau), e^{k\tau}D^1\overline{u}(\tau))| \le (e^{-k\tau}\mathcal{L}_1 + \mathcal{L}_2(|\overline{w}(\tau)| + |D^1\overline{u}(\tau)|)).$$

From this and from M. A. Krasnosel'skiĭ's theorem [12] of continuity of the superposition operator it follows that each map $\overline{a}_{ij}$ is continuous and bounded, and consequently this is valid for the map $\overline{B}$. Besides,

$$\|(\overline{B}(t,\tau,\overline{w}(\tau),\overline{u}(\tau)),\overline{h})| \le (\|\mathcal{L}_1\|_{L_2(\Omega)} + \|\mathcal{L}_2\|_{L_\infty(Q_d)}(\|\overline{w}(\tau)\|_{L_2(\Omega)^n} + \|D^1\overline{u}(\tau)\|_{L_2(\Omega)^n})\|D^1h\|_{L_2(\Omega)^n}.$$

From this fact and from the definition of $\overline{C}$ we obtain for $\overline{w} \in L_2(Q_T)^n$, $\overline{u} \in E_2$:

$$\begin{aligned}|\langle \overline{C}(\overline{w},\overline{u}),\overline{u}\rangle| &\le \int_0^T\int_0^t e^{-k(t-\tau)}|(\overline{B}(t,\tau,\overline{w}(\tau),\overline{u}(\tau)),\overline{u}(t))|\,d\tau\,dt \\ &\le \int_0^T\int_0^t (\|\mathcal{L}_1(t,\tau,\,\cdot\,)\|_{L_2(\Omega)} + \|\mathcal{L}_2\|_{L_\infty(Q_d)}(\|\overline{w}(\tau)\|_{L_2(\Omega)^n} \\ &\quad + \|D^1\overline{u}(\tau)\|_{L_2(\Omega)^n}))\,d\tau \|D^1u(t)\|_{L_2(\Omega)^n}\,dt \\ &\le C(1 + \|\overline{w}\|_{L_2(Q_T)^n} + \|\overline{u}\|_{E_2})\|\overline{u}\|_{E_2}\end{aligned}$$

with a constant C, depending only on $\|\mathcal{L}_1\|_{L_2(Q_d)}$, $\|\mathcal{L}_2\|_{L_\infty(Q_d)}$ and T. □

The following statement is a reformulation of Lemma 2.5 [9].

Lemma 2.6. *Let a matrix-function a satisfy conditions* (1.15)–(1.17), *then for any functions* $\overline{w} \in L_2(Q_T)^n$, $\overline{u},\overline{v} \in E_2$, *the following estimate holds*

$$\langle \overline{C}(\overline{w},\overline{u}) - \overline{C}(\overline{w},\overline{v}), \overline{u}-\overline{v}\rangle \le \frac{C}{\sqrt{2k}}\|\overline{u}-\overline{v}\|^2_{E_2} \tag{2.6}$$

with a constant C *independent of* k, $\overline{u}$, $\overline{v}$, $\overline{w}$.

3. Approximating equations and their solvability

In order to construct a family of approximating equations for (1.24) introduce the operator

$$N_0 : V \to V^*, \quad (N_0(\overline{u}), h) = \int_\Omega \|\overline{u}\|_V^2 \mathcal{E}_{ij}(\overline{u})\mathcal{E}_{ij}(h)\, dx,$$

and for $\varepsilon > 0$ consider the equation in the form:

$$\rho\overline{v}' + \rho k\overline{v} + \overline{N}_1(\overline{v}) + \overline{N}_2(\overline{v}, U_A(e^{kt}\overline{v})) + \varepsilon N_0(\overline{v}) + \overline{K}(\overline{v}) - \overline{C}(\overline{v}, \overline{v}) = \overline{F} + \overline{f}. \tag{3.1_ε}$$

In this section we show that for any k large enough each approximating equation (3.1_ε) with $\varepsilon > 0$ has a solution in W satisfying the initial conditions

$$\overline{v}(0) = v^0. \tag{3.2}$$

3.1. Properties of the operator N_0.

Lemma 3.1. *The map $N_0 : E \to E^*$ is continuous, d-monotone and the following inequality holds*

$$\|N_0(\overline{u})\|_{E^*} \leq \|\overline{u}\|_E^3 \quad \text{for } \overline{u} \in E. \tag{3.3}$$

By definition [11] the map $N_0 : E \to E^*$ is called d-monotone, if for any $\overline{u}, \overline{v} \in E$ the following inequality takes place

$$\langle N_0(\overline{u}) - N_0(\overline{v}), \overline{u} - \overline{v}\rangle \geq (\alpha(\|\overline{u}\|_E) - \alpha(\|\overline{v}\|_E))(\|\overline{u}\|_E - \|\overline{v}\|_E)$$

for some strongly increasing function α on $[0, \infty)$.

Proof. Estimate (3.3) follows from the estimate $\|N_0(\overline{u})\|_{V^*} \leq \|\overline{u}\|^3$ for $\overline{u} \in V$. To prove the continuity of maps N_0 it is sufficient to show that the maps $\Phi_{ij} : u \mapsto \|\overline{u}\|_V \mathcal{E}_{ij}(u)$ from E in $L_{4/3}((0,T), L_2(\Omega))$ are continuous. The map Φ_{ij} is continuous as a product of continuous maps

$$\overline{u} \mapsto \|\overline{u}\|^2 \quad \text{from } E \text{ into } L_2((0,T)) \quad \text{and}$$
$$\overline{u} \mapsto \mathcal{E}_{ij}(\overline{u}) \quad \text{from } E \text{ into } L_4((0,T), L_2(\Omega)).$$

Hence N_0 is continuous.

Now let us show d-monotonicity of N_0. For $\overline{u},\overline{v}\in V$ by the Hölder inequality we get

$$\begin{aligned}(N_0(\overline{u})-N_0(\overline{v}),\overline{u}-\overline{v}) &= \int_\Omega (\|\overline{u}\|^2\mathcal{E}_{ij}(\overline{u})-\|\overline{v}\|^2\mathcal{E}_{ij}(\overline{u}))(\mathcal{E}_{ij}(\overline{u})-\mathcal{E}_{ij}(\overline{v}))\,dx\\ &= \int_\Omega (\|\overline{u}\|^2\mathcal{E}^2(\overline{u})+\|\overline{v}\|^2\mathcal{E}^2(\overline{v})-\|\overline{u}\|^2\mathcal{E}_{ij}(\overline{u})\mathcal{E}_{ij}(\overline{v})\\ &\qquad -\|\overline{v}\|^2\mathcal{E}_{ij}(\overline{u})\mathcal{E}_{ij}(\overline{v}))\,dx\\ &\geq \|\overline{u}\|^4+\|\overline{v}\|^4-\|\overline{u}\|^3\|\overline{v}\|-\|\overline{v}\|^3\|\overline{u}\|.\end{aligned}$$

Hence, for $\overline{u},\overline{v}\in E$ by the Hölder inequality,

$$\begin{aligned}&\langle N_0(\overline{u})-N_0(\overline{v}),\overline{u}-\overline{v}\rangle\\ &\quad\geq \int_0^T (\|\overline{u}(t)\|^4+\|\overline{v}(t)\|^4-\|\overline{u}(t)\|^3\|\overline{v}(t)\|-\|\overline{v}(t)\|^3\|\overline{u}(t)\|)\,dt\\ &\quad= \|\overline{u}\|_E^4+\|\overline{v}\|_E^4-\int_0^T\|\overline{u}(t)\|^3\|\overline{v}(t)]\,dt-\int_0^T\|\overline{v}(t)\|^3\|\overline{u}(t)\|\,dt\\ &\quad\geq \|\overline{u}\|_E^4+\|\overline{v}\|_E^4-\|\overline{u}\|_E^3\|\overline{v}\|_E-\|\overline{v}\|_E^3\|\overline{u}\|_E\\ &\quad= (\|\overline{u}\|_E^3-\|\overline{v}\|_E^3)(\|\overline{u}\|_E-\|\overline{v}\|_E).\end{aligned}$$

The inequality

$$\langle N_0(\overline{u})-N_0(\overline{v}),\overline{u}-\overline{v}\rangle\geq(\|\overline{u}\|_E^3-\|\overline{v}\|_E^3)(\|\overline{u}\|_E-\|\overline{v}\|_E) \tag{3.4}$$

for $\overline{u},\overline{v}\in E$ proves d-monotonicity of operator N_0. □

3.2. The auxiliary problem. Specify the functions $s\in C([0,T],S)$, $\overline{w}\in L_2(Q_T)^n$ and consider the auxiliary problem

$$\begin{gathered}c\rho\overline{v}'+\rho k\overline{v}+\overline{N}_1(\overline{v})+\overline{N}_2(\overline{v},s)+\varepsilon N_0(\overline{v})-\overline{C}(\overline{w},\overline{v})=\overline{g},\\ \overline{v}(0)=v^0,\end{gathered} \tag{3.5}$$

where $\overline{g}\in E^*$, $v^0\in H$, $\varepsilon>0$. Denote by V_k the map

$$\begin{gathered}V_k:E\times C([0,T],S)\times\ L_2(Q_T)^n\to E^*,\\ V_k(\overline{v},s,w)=\rho k\overline{v}+\overline{N}_1(\overline{v})+\overline{N}_2(\overline{v},s)+\varepsilon N_0(\overline{v})-\overline{C}(\overline{w},\overline{v}).\end{gathered}$$

Then equation (3.5) is equivalent to

$$\rho\overline{v}'+V_k(\overline{v},s,\overline{w})=\overline{g}. \tag{3.6}$$

Lemma 3.2. *If conditions* (1.9)–(1.17) *are fulfilled, the map* V_k *is continuous and bounded. Besides, the operator* $V_k(\,\cdot\,, s, \overline{w}) : E \to E^*$ *is* d*-monotone and coercive. Problem* (3.6), (3.2) *has a unique solution* $\overline{v}$ *in* W*, and the correspondence* $v^0 \mapsto \overline{v}$ *is continuous as a map from* H *into* $C([0,T], H)$.

Proof. The map V_k is continuous and bounded since all the maps, whose sum forms V_k, are continuous and bounded. Now show d-monotonicity of operator $V_k(\,\cdot\,, s, \overline{w})$. Let $\overline{u}$, $\overline{v}$ be arbitrary functions from E. Then

$$\begin{aligned}(3.7)\quad & \langle V_k(\overline{u}, s, \overline{w}) - V_k(\overline{v}, s, \overline{w}), \overline{u} - \overline{v}\rangle \\ &= \rho k \int_0^T \|\overline{u}(t) - \overline{v}(t)\|^2_{L_2(\Omega)^n}\, dt + \langle \overline{N}_1(\overline{u}) - \overline{N}_1(\overline{v}), \overline{u} - \overline{v}\rangle \\ &\quad + \langle \overline{N}_2(\overline{u}, s) - \overline{N}_2(\overline{v}, s), \overline{u} - \overline{v}\rangle + \varepsilon\langle \overline{N}_0(\overline{u}) - \overline{N}_0(\overline{v}), \overline{u} - \overline{v}\rangle \\ &\quad - \langle \overline{C}(\overline{u}, \overline{w}) - \overline{C}(\overline{v}, \overline{w}), \overline{u} - \overline{v}\rangle.\end{aligned}$$

By estimate (2.2)

$$\langle \overline{N}_2(\overline{u}, s) - \overline{N}_2(\overline{v}, s), \overline{u} - \overline{v}\rangle \geq 2a_3 \|\overline{u} - \overline{v}\|^2_{E_2},$$

and from estimate (2.6) it follows that

$$|\langle \overline{C}(\overline{w}, \overline{u}) - \overline{C}(\overline{w}, \overline{v}), \overline{u} - \overline{v}\rangle| \leq \frac{C}{\sqrt{2k}} \|\overline{u} - \overline{v}\|^2_{E_2}.$$

Therefore, choosing k such that $C/\sqrt{2k} < a_3$, we obtain

$$(3.8)\quad \langle \overline{N_2}(\overline{u}, s) - \overline{N_2}(\overline{v}, s), \overline{u} - \overline{v}\rangle - \langle \overline{C}(\overline{w}, \overline{u}) - \overline{C}(\overline{w}, \overline{v}), \overline{u} - \overline{v}\rangle \geq a_3 \|\overline{u} - \overline{v}\|^2_{E_2}.$$

Note that the choice of k does not depend on s. Nonnegativity of the second and fourth summands in (3.7) follows from monotonicity of $\overline{N}_1$ and N_0. Thus we obtain the following estimate

$$\langle V_k(\overline{u}, s, \overline{w}) - V_k(\overline{v}, s, \overline{w}), \overline{u} - \overline{v}\rangle \geq a_3 \|\overline{u} - \overline{v}\|^2_{E_2}.$$

Besides, from estimate (3.4), it follows that

$$(3.9)\qquad \langle V_k(\overline{u}, s, \overline{w}) - V_k(\overline{v}, s, \overline{w}), \overline{u} - \overline{v}\rangle \geq \varepsilon(\|\overline{u}\|^3_E - \|\overline{v}\|^3_E)(\|\overline{u}\|_E - \|\overline{v}\|_E)$$

and that the operator $V_k(\,\cdot\,, s, \overline{w})$ is d-monotone. To prove coercivity of $V_k(\,\cdot\,, s, \overline{w})$ notice that $N_0(0) = \overline{N}_1(0) = \overline{N}_2(0, s) = 0$. Repeating the above estimates for $\overline{u} = 0$ and using estimates (2.1), (2.3), (2.6), (3.3) we obtain the inequality

$$\langle V_k(\overline{v}, s, \overline{w}), \overline{v}\rangle \geq C_1 \|\overline{v}\|^4_E - C_0 - C(1 + \|\overline{v}\|_{E_2}) \|\overline{v}\|_{E_2}.$$

Thus the coercivity inequality takes place

$$(3.10)\qquad \langle V_k(\overline{v}, s, \overline{w}), \overline{v}\rangle \geq C_1 \|\overline{v}\|^4_E - C(1 + \|\overline{v}\|_E + \|\overline{v}\|^2_E)$$

with some constant C.

The existence and uniqueness statement for the solution of problem (3.6), (3.2) and the continuous dependence of this solution on the initial conditions v^0 follow from Theorem 1.1 ([11, p. 239]). □

Let us introduce the map

$$L : W \times C([0,T],S) \times L_2(Q_T)^n \to E^* \times H,$$
$$L(\overline{u}, s, \overline{w}) = (\rho\overline{u}' + V_k(\overline{u}, s, \overline{w}), \overline{u}(0)).$$

Problem (3.6), (3.2) is equivalent to the equation

$$L(\overline{v}, s, \overline{w}) = (\overline{g}, v^0). \tag{3.11}$$

By Lemma 3.2 the last equation has a unique solution $\overline{v}$ for fixed s, $\overline{w}$, $\overline{g}$, v^0. This means that the map L is invertible in variable $\overline{v}$ for fixed s, w. Now we can formulate a statement describing properties of the inverse map.

Theorem 3.1. *If conditions* (1.9)–(1.17) *are fulfilled, then for any functions* $s \in C([0,T],S)$, $\overline{w} \in L_2(Q_T)^n$ *the map*

$$L(\,\cdot\,, s, \overline{w}) : W \to E^* \times H$$

is invertible. The inverse map

$$(\overline{g}, v^0) \mapsto L^{-1}(\overline{g}, v^0, s, \overline{w})$$

is continuous as a map from $E^* \times H \times C([0,T],S) \times L_2(Q_T)^n$ *into* W.

Proof. As it is mentioned above, the existence of inverse map L^{-1} follows from the assertion of Lemma 3.2. We need to show continuity of L^{-1}. With this aim we choose arbitrary sequences

$$\{\overline{g}_l\} : \overline{g}_l \in E^*,\ \overline{g}_l \to \overline{g}_0 \quad \text{strongly in } E^*,$$
$$\{v^l\} : v^l \in H,\ v^l \to v^0 \quad \text{strongly in } H,$$
$$\{s_l\} : s_l \in C([0,T],S),\ s_l \to s_0 \quad \text{strongly in } C([0,T],S),$$
$$\{\overline{w}_l\} : \overline{w}_l \in L_2(Q_T)^n,\ \overline{w}_l \to \overline{w}_0 \quad \text{strongly in } L_2(Q_T)^n.$$

Denote by $\overline{v}_l$ a solution of equation

$$L(\overline{v}, s_l, \overline{w}_l) = (\overline{g}_l, v^l). \tag{3.11$_l$}$$

Then $\overline{v}_l = L^{-1}(\overline{g}_l, v^l, s_l, \overline{w}_l)$. It is necessary to prove that $\{\overline{v}_l\}$ converges in W to a solution of equation (3.11$_0$).

Show that the sequence $\{\overline{v}_l\}$ is bounded in the norm of space W. By definition $\overline{v}_l$ is a solution of equation (3.11$_l$), therefore

$$\rho\overline{v}_l' + V_k(\overline{v}_l, s_l, \overline{w}_l) = \overline{g}_l. \tag{3.12}$$

Functionals from the equality can be applied to the function $\overline{v}_l$. We obtain

$$\frac{1}{2}\rho\|\overline{v}_l(T)\|_H^2 - \frac{1}{2}\rho\|\overline{v}_l(0)\|_H^2 + \langle V_k(\overline{v}_l, s_l, \overline{w}_l), \overline{v}_l\rangle = \langle \overline{g}_l, \overline{v}_l\rangle.$$

From estimate (3.10) and the equality $\overline{v}_l(0) = v^l$ it follows that

$$\frac{1}{2}\rho\|\overline{v}_l(T)\|_H^2 + C_1\|\overline{v}_l\|_E^4 \le \frac{1}{2}\rho\|v^l\|_H^2 + C(1 + \|\overline{v}_l\|_{E_2} + \|\overline{v}_l\|_{E_2}^2) + \|\overline{g}_l\|_{E^*}\|\overline{v}_l\|_E.$$

As $\|\overline{g}_l\|_{E^*}$, $\|v^l\|_H$ are jointly bounded, we get in a routine way that

$$\|\overline{v}_l\|_E \le C \tag{3.13}$$

with a constant C, independent of l. From equality (3.12) it follows that

$$\overline{v}_l' = \frac{1}{\rho}(\overline{g}_l - V_k(\overline{v}_l, s_l, \overline{w}_l)),$$

therefore the boundedness of $\|\overline{v}_l'\|_{E^*}$ follows from estimate (3.13) and the boundedness of operator V_k. Hence, the sequence $\{\overline{v}_l\}$ is bounded in the norm of space W. Furthermore, without loss of generality we shall assume, that

$$\overline{v}_l \rightharpoonup \overline{v}_0 \quad \text{weakly in } E, \qquad \overline{v}_l' \rightharpoonup \overline{v}_0' \quad \text{weakly in } E^*.$$

Denote by $\overline{g}$ the function defined by the equality

$$\rho\overline{v}_l' + V_k(\overline{v}_0, s_0, \overline{w}_0) = \overline{g}.$$

Subtract this equality from (3.12); apply the functionals from this difference to the function $\overline{v}_l - \overline{v}_0$:

$$\rho\langle\overline{v}_l' - \overline{v}_0', \overline{v}_l - \overline{v}_0\rangle + \langle V_k(\overline{v}_l, s_l, \overline{w}_l) - V_k(\overline{v}_0, s_0, \overline{w}_0), \overline{v}_0 - \overline{v}_0\rangle = \langle\overline{g}_l - \overline{g}, \overline{v}_l - \overline{v}_0\rangle.$$

Transform the equality to the form:

$$\begin{aligned}\frac{1}{2}\rho\|\overline{v}_l(T)-\overline{v}_0(T)\|_H^2-\frac{1}{2}\rho\|\overline{v}_l(0)-\overline{v}_0(0)\|_H^2+\langle V_k(\overline{v}_l, s_l, \overline{w}_l)-V_k(\overline{v}_0, s_l, \overline{w}_l), \overline{v}_l-\overline{v}_0\rangle \\ = \langle V_k(\overline{v}_0, s_0, \overline{w}_0) - V_k(\overline{v}_0, s_l, \overline{w}_l), \overline{v}_l - \overline{v}_0\rangle + \langle\overline{g}_l - \overline{g}, \overline{v}_l - \overline{v}_0\rangle.\end{aligned}$$

Using inequality (3.9) we get

$$\begin{aligned}\frac{1}{2}\rho\|\overline{v}_l(T) - \overline{v}_0(T)\|_H^2 &+ \varepsilon(\|\overline{v}_l\|_E^3 - \|\overline{v}_0\|_E^3)(\|\overline{v}_l\|_E - \|\overline{v}_0\|_E) \\ &\le \frac{1}{2}\rho\|\overline{v}_l(0) - \overline{v}_0(0)\|_H^2 + \langle V_k(\overline{v}_0, s_0, \overline{w}_0) \\ &\quad - V_k(\overline{v}_0, s_l, \overline{w}_l), \overline{v}_l - \overline{v}_0\rangle + \langle\overline{g}_l - \overline{g}, \overline{v}_l - \overline{v}_0\rangle.\end{aligned}$$

Show that each term in the right-hand side of the inequality tends to zero. This implies $\|\overline{v}_l\|_E \to \|\overline{v}_0\|_E$ as $l \to \infty$, hence $\overline{v}_l \to \overline{v}_0$ strongly in E.

Consider $\|\overline{v}_l(0) - \overline{v}_0(0)\|_H$. Let $\chi(t) \in C^1([0,T])$ and $h \in V$. From the formula of integration by parts we get

$$\langle (\overline{v}_l - \overline{v}_0)', \chi(t)h\rangle + \langle \overline{v}_l - \overline{v}_0, \chi'(t)h\rangle \\ = (\overline{v}_l(T) - \overline{v}_0(T), \chi(T)h) - (\overline{v}_l(0) - \overline{v}_0(0), \chi(0)h).$$

Each term in left-hand side of the equality tends to zero: the first one due to the assumption that $\overline{v}_l' \rightharpoonup \overline{v}_0'$ weakly in E^*, the second one due to the assumption, that $\overline{v}_l \rightharpoonup \overline{v}_0$ weakly in E. Choosing a function $\chi(t)$ such that $\chi(T) = 0$ and $\chi(0) = 1$, we get: $(\overline{v}_l(0) - \overline{v}_0(0), h) \to 0$ as $l \to \infty$. This means that $\overline{v}_l(0) \rightharpoonup \overline{v}_0(0)$ weakly in V^*. But $\overline{v}_l(0) = \overline{v}^l$, and $\overline{v}^l \to v^0$. Hence, $\overline{v}_0(0) = v^0$ and $\overline{v}_l(0) \to \overline{v}_0(0)$ strongly in V^* and in H as $l \to \infty$.

As the map V_k is continuous, $V_k(\overline{v}_0, s_l, \overline{w}_l) \to V_k(\overline{v}_0, s_0, \overline{w}_0)$ strongly in E^*. From this fact and from weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ in E it follows that

$$\langle V_k(\overline{v}_0, s_0, \overline{w}_0) - V_k(\overline{v}_0, s_l, \overline{w}_l), \overline{v}_l - v_0\rangle \to 0 \quad \text{as } k \to \infty.$$

Similarly, the strong convergence $\overline{g}_l \to \overline{g}_0$ in E^* and weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ in E provide convergence to zero of the expression $\langle \overline{g}_l - \overline{g}, \overline{v}_l - \overline{v}_0\rangle$. So, it is sufficient to present this expression in the form:

$$\langle \overline{g}_l - \overline{g}, \overline{v}_l - \overline{v}_0\rangle = \langle \overline{g}_l - \overline{g}_0, \overline{v}_l - \overline{v}_0\rangle + \langle \overline{g}_0 - \overline{g}, \overline{v}_l - \overline{v}_0\rangle.$$

Thus, it is proved that $\overline{v}_l \to \overline{v}_0$ strongly in E. To finish the proof of the theorem, it is necessary to show that $\overline{v}_l' \to \overline{v}_0'$ strongly in E^*.

Due to the continuity of map V_k the right-hand sides of equalities

$$\overline{v}_l' = \frac{1}{\rho}(\overline{g}_l - V_k(\overline{v}_l, s_l, \overline{w}_l))$$

converge to $(\overline{g}_0 - V_k(\overline{v}_0, s_0, \overline{w}_0))/\rho$ strongly in E^*. Hence, the left-hand sides of equalities $\overline{v}_l'$ tend strongly in E^* too, and their limit is equal to the weak limit $\overline{v}_0'$. □

3.3. Solvability of approximating equations. Equation (3.1_ε) can be written in form

$$\rho\overline{v}' + V_k(\overline{v}, U_A(e^{kt}\overline{v}), \overline{v}) + \overline{K}(\overline{v}) = \overline{F} + \overline{f}. \tag{3.14}$$

Consider the family of operator equations

$$\rho\overline{v}' + V_k(\overline{v}, \eta U_A(e^{kt}\overline{v}), \eta\overline{v}) + \eta\overline{K}(\overline{v}) = \overline{F} + \overline{f}, \quad \eta \in [0,1]. \tag{3.14_η}$$

For $\eta = 1$ the equation of the family coincides with (3.14).

Let us show that the set of solutions of the family of problems (3.14_η), (3.2) is bounded if k is large enough.

Lemma 3.3. *Let conditions* (1.9)–(1.17) *be fulfilled and k be large enough. Then there exists a constant C, such that each solution $\overline{v}$ of problem* (3.14_η), (3.2) *satisfies the estimate*

$$\|\overline{v}\|_W \le C, \tag{3.15}$$

and the constant C does not depend on $\varepsilon \in [0,1]$ and $\eta \in [0,1]$.

Proof. Let $\overline{v}$ be a solution of equation (3.14_η) for some $\eta \in [0,1]$. Consider actions of functionals from equality (3.14_η) on $\overline{v}$:

$$\rho\langle \overline{v}', \overline{v}\rangle + \langle V_k(\overline{v}, \eta U_A(e^{kt}\overline{v}), \eta\overline{v}), \overline{v}\rangle + \eta\langle \overline{K}(\overline{v}), \overline{v}\rangle = \langle \overline{F} + \overline{f}, \overline{v}\rangle.$$

From this and from estimate (3.10) we get:

$$\frac{1}{2}\rho\|\overline{v}(T)\|_H^2 - \frac{1}{2}\rho\|\overline{v}(0)\|_H^2 + C_1\|\overline{v}\|_E^4 - C(1 + \|\overline{v}\|_{E_2} + \|\overline{v}\|_{E_2}^2) \le \|\overline{F} + \overline{f}\|_{E^*}\|\overline{v}\|_E - \eta\langle \overline{K}(\overline{v}), \overline{v}\rangle.$$

Due to estimate (2.4) and the condition $\overline{v}(0) = v^0$ we have:

$$\frac{1}{2}\rho\|\overline{v}(T)\|_H^2 + C_1\|\overline{v}\|_E^4 \le \frac{1}{2}\rho\|v^0\|_H^2 + C_0 + C(1 + \|\overline{v}\|_{E_2})\|\overline{v}\|_{E_2} + \|\overline{F} + \overline{f}\|_{E^*}\|\overline{v}\|_E + \eta C_2\|\overline{v}\|_{E_2}\|\overline{v}\|_{L_4(Q_T)^n}^2.$$

As embeddings $E \subset E_2 \subset L_2(Q_T)^n$, $E \subset L_4(Q_T)^n$ are continuous, transform the inequality to the form:

$$\frac{1}{2}\rho\|\overline{v}(t)\|_H^2 + C_1\|\overline{v}\|_E^4 \le \frac{1}{2}\rho\|v^0\|_H^2 + C(1 + \|\overline{v}\|_E + \|\overline{v}\|_E^2 + \|\overline{v}\|_E^3).$$

Thus it is easy to get the following inequality $\|\overline{v}\|_E \le C$ with constant C depending on k and $\|\overline{F} + \overline{f}\|_{E^*}$ and independent of η and ε.

The estimate for $\|\overline{v}'\|_{E^*}$ follows from the equality

$$\overline{v}' = \frac{1}{\rho}(\overline{F} + \overline{f} - V_k(\overline{v}, \eta U_A(e^{kt}\overline{v}), \eta\overline{v}) - \eta\overline{K}(\overline{v}))$$

and boundedness of maps V_k, U_A, $\overline{K}$ in E. □

Now let us formulate the main statement of this section.

Theorem 3.2. *Let conditions* (1.9)–(1.17) *be fulfilled and k be large enough. Then for any functions $\overline{F}, \overline{f} \in E^*$, $v^0 \in H$ and arbitrary $\varepsilon \in (0,1]$ the problem* (3.1_ε), (3.2) *has at least one solution $\overline{v} \in W$, and this solution satisfies the estimate* (3.15).

Proof. Replace the investigation of problem (3.1_ε), (3.2) by that of equivalent operator equation

$$L(\overline{v}, U_A(e^{kt}\overline{v}), \overline{v}) = (\overline{F} + \overline{f} - \overline{K}(\overline{v}), v^0).$$

Apply the map, inverse to L, to both parts of the equality:

$$\overline{v} = L^{-1}(\overline{F} + \overline{f} - \overline{K}(v), v^0, U_A(e^{kt}\overline{v}), \overline{v}). \tag{3.16}$$

Note that due to Lemma 2.4 the map $\overline{v} \mapsto \overline{F} + \overline{f} - \overline{K}(\overline{v})$ from W into E^* is completely continuous. Due to Lemma 2.3 the map $\overline{U}_A : \overline{v} \mapsto U_A(e^{kt}\overline{v})$ from W into $C([0,T], S)$ is completely continuous. Besides, the embedding $W \subset L_2(Q_T)^n$ is completely continuous. Then the map

$$G_1 : W \to W, \quad G_1(\overline{v}) = L^{-1}(\overline{F} + \overline{f} - \overline{K}(v), v^0, U_A(e^{kt}\overline{v}), \overline{v})$$

is completely continuous as the superposition of above-mentioned completely continuous maps and continuous map L^{-1}.

Represent equation (3.16) in the form

$$\overline{v} - G_1(\overline{v}) = 0. \tag{3.17}$$

To investigate its solvability we apply the Leray–Schauder degree theory. Consider the auxiliary family of problems (3.14_η), (3.2) and the family of equivalent operator equations

$$L(\overline{v}, \eta U_A(e^{kt}\overline{v}), \eta\overline{v}) = (\overline{F} + \overline{f} - \eta\overline{K}(v), v^0), \quad \eta \in [0,1].$$

Transform it to the form

$$\overline{v} = L^{-1}(\overline{F} + \overline{f} - \eta\overline{K}(v), v^0, \eta U_A(e^{kt}\overline{v}), \eta\overline{v}), \quad \eta \in [0,1].$$

This family generates the completely continuous homotopy

$$G : [0,1] \times W \to W, \quad G(\eta, \overline{v}) = L^{-1}(\overline{F} + \overline{f} - \eta\overline{K}(v), v^0, \eta U_A(e^{kt}\overline{v}), \eta\overline{v})$$

and can be written in the form

$$\overline{v} - G(\eta, \overline{v}) = 0, \quad \eta \in [0,1]. \tag{3.17_η}$$

Due to Lemma 3.3 all solutions of equations of the family satisfy a priori estimate (3.15). Therefore all equations of the family have no solutions on the boundary of the ball $B_{C+1} \subset W$ of radius $C+1$ with centre at zero. Hence, for any $\eta \in [0,1]$, the degree of map $\deg(I - G(\eta, \cdot\,), \overline{B}_{C+1}, 0)$ is well-posed.

As the degree of map is constant under completely continuous homotopies,

$$\deg(I - G(1, \cdot\,), \overline{B}_{C+1}, 0) = \deg(I - G(0, \cdot\,), \overline{B}_{C+1}, 0).$$

Note, that $G(0, \overline{v}) = L^{-1}(\overline{F} + \overline{f}, v^0, 0, 0)$ does not depend on $\overline{v}$. Denote this function by $\overline{u}_0$. Then

$$\deg(I - G(0, \cdot\,), \overline{B}_{C+1}, 0) = \deg(I - \overline{u}_0, \overline{B}_{C+1}, 0) = \deg(I, \overline{B}_{C+1}, \overline{u}_0).$$

As $\overline{u}_0$ is a solution of (3.17_0), $\overline{u}_0$ satisfies a priori estimate (3.15) and so $\overline{u}_0 \in B_{C+1}$. Therefore $\deg(I, \overline{B}_{C+1}, \overline{u}_0) = 1$ and

$$\deg(I - \Phi(1, \cdot\,), \overline{B}_{C+1}, 0) = 1.$$

Since this degree is not zero, there exists a solution of operator equation (3.17_1) or (3.17), and so there exists a solution of problem (3.1_ε), (3.2) for any $\varepsilon \in (0,1]$.□

4. Of existence of a weak solution of the evolution problem

This section contains the main result of the paper, namely, the statement of existence of solution of the problem (1.24)–(1.25). This solution is a limit of solutions of approximating equations (3.1_ε) as $\varepsilon \to 0$.

Theorem 4.1. *Let conditions* (1.9)–(1.17) *be fulfilled, then for k large enough problem* (1.24), (1.25) *has at least one solution in W.*

Proof. Let ε_l be any sequence of numbers $\varepsilon_l \in (0,1]$ tending to zero. Denote by $\overline{v}_l$ the solution of approximating equation (3.1_{ε_l}) with initial conditions (3.2). Due to Lemma 3.3 the set of solutions $\{\overline{v}_l\}$ is bounded. Therefore without loss of generality we may suppose that

$$\overline{v}_l \rightharpoonup \overline{v}_0 \quad \text{weakly in } E, \qquad \overline{v}_l' \rightharpoonup \overline{v}_0' \quad \text{weakly in } E^*.$$

Besides, suppose that

$$\begin{aligned} &\overline{v}_l \to \overline{v}_0 \quad \text{strongly in } L_4(Q_T)^n;\\ &\overline{K}(\overline{v}_l) \to y_0 \quad \text{strongly in } E^*;\\ &U_A(e^{kt}\overline{v}_l) \to y_1 \quad \text{strongly in } C([0,T],S), \end{aligned}$$

since the embedding $W \subset L_4(Q_T)^n$ and the maps $\overline{K}$, U_A are completely continuous. Repeating the arguments of the proof of Theorem 3.1 we get that $\overline{v}_0(0) = v^0$.

Denote by $\overline{g}$ the function defined by the equality:

$$\rho\overline{v}_0' + \rho k\overline{v}_0 + \overline{N}_1(\overline{v}_0) + \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{C}(\overline{v}_0, \overline{v}_0) + \overline{K}(\overline{v}_0) = \overline{g}.$$

Subtract this equality from the equality for $\overline{v}_l$:

$$\rho\overline{v}_l' + \rho k\overline{v}_l + \overline{N}_1(\overline{v}_l) + \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) - \overline{C}(\overline{v}_l, \overline{v}_l) + \overline{K}(\overline{v}_l) + \varepsilon_l N_0(\overline{v}_l) = \overline{F} + \overline{f},$$

then we get

$$\begin{aligned}\rho(\overline{v}_l' - \overline{v}_0') &+ \rho k(\overline{v}_l - \overline{v}_0) + \overline{N}_1(\overline{v}_l) - \overline{N}_1(\overline{v}_0) + \overline{K}(\overline{v}_l) - \overline{K}(\overline{v}_0) + \varepsilon_l N_0(\overline{v}_l)\\ &+ \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{C}(\overline{v}_l, \overline{v}_l) + \overline{C}(\overline{v}_0, \overline{v}_0)\\ &= \overline{F} + \overline{f} - \overline{g}.\end{aligned}$$

Consider the action of obtained functionals on the function $\overline{v}_l - \overline{v}_0$:

$$\begin{aligned}\rho\langle \overline{v}_l' - \overline{v}_0', \overline{v}_l - \overline{v}_0\rangle + \rho k\langle \overline{v}_l - \overline{v}_0, \overline{v}_l - \overline{v}_0\rangle + \langle \overline{N}_1(\overline{v}_l) - \overline{N}_1(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)), \overline{v}_l - \overline{v}_0\rangle \\ - \langle \overline{C}(\overline{v}_l, \overline{v}_l) - \overline{C}(\overline{v}_0, \overline{v}_0), \overline{v}_l - \overline{v}_0\rangle + \langle \overline{K}(\overline{v}_l) - \overline{K}(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle \\ + \varepsilon_l\langle N_0(\overline{v}_l), \overline{v}_l - \overline{v}_0\rangle = \langle \overline{F} + \overline{f} - g, \overline{v}_l - \overline{v}_0\rangle.\end{aligned}$$

Transform the equality as follows:

$$\begin{aligned}\frac{1}{2}\rho\|(\overline{v}_l - \overline{v}_0)(T)\|_H^2 + \rho k\|\overline{v}_l - \overline{v}_0\|_H^2 + \langle \overline{N}_1(\overline{v}_l) - \overline{N}_1(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_l)), \overline{v}_l - \overline{v}_0\rangle \\ - \langle \overline{C}(\overline{v}_l, \overline{v}_l) - \overline{C}(\overline{v}_l, \overline{v}_0), \overline{v}_l - \overline{v}_0\rangle = \langle \overline{F} + \overline{f} - g, \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{K}(\overline{v}_0) - \overline{K}(\overline{v}_l), \overline{v}_l - \overline{v}_0\rangle + \varepsilon_l\langle \overline{N}_0(\overline{v}_l), \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_l)), \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{C}(\overline{v}_l, \overline{v}_0) - \overline{C}(\overline{v}_0, \overline{v}_0), \overline{v}_l - \overline{v}_0\rangle.\end{aligned} \tag{4.1}$$

Estimate the left-hand side of the equality. By Lemma 2.1 the map $\overline{N}_1$ is monotone, therefore the third summand is nonnegative. Due to estimate (3.8)

$$\begin{aligned}\langle \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_l)), \overline{v}_l - \overline{v}_0\rangle \\ - \langle \overline{C}(\overline{v}_l, \overline{v}_l) - \overline{C}(\overline{v}_l, \overline{v}_0), \overline{v}_l - \overline{v}_0\rangle \geq a_3\|\overline{v}_l - \overline{v}_0\|_{E_2}^2\end{aligned}$$

under the condition $C/\sqrt{2k} < a_3$ for C from estimate (2.6). Thus the left-hand side of (4.1) is not less than $a_3\|\overline{v}_l - \overline{v}_0\|_{E_2}^2$.

Show that each term in the right-hand side of (4.1) converges to zero as $l \to \infty$. This will imply that $\overline{v}_l \to \overline{v}_0$ strongly in E_2.

The first term converges to zero by definition of weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ in E. Rewrite the second one as follows

$$\langle \overline{K}(\overline{v}_0) - \overline{K}(\overline{v}_l), \overline{v}_l - \overline{v}_0\rangle = \langle y_0 - \overline{K}(\overline{v}_l), \overline{v}_l - \overline{v}_0\rangle + \langle \overline{K}(\overline{v}_0) - y_0, \overline{v}_l - \overline{v}_0\rangle.$$

The convergence to zero of each obtained term is provided by the assumptions that $\overline{K}(\overline{v}_l) \to y_0$ strongly in E^* and $\overline{v}_l \rightharpoonup \overline{v}_0$ weakly in E.

In the third term in the right-hand side of (4.1) the factors $\langle N_0(\overline{v}_l), \overline{v}_0 - \overline{v}_l\rangle$ are bounded, therefore as $\varepsilon_l \to 0$ the terms tend to zero.

Represent the fourth summand in the form

$$\begin{aligned}\langle \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_l)), \overline{v}_l - \overline{v}_0\rangle \\ = \langle \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{N}_2(\overline{v}_0, y_1), \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{N}_2(\overline{v}_0, y_1) - \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_l)), \overline{v}_l - \overline{v}_0\rangle.\end{aligned}$$

The first term here converges to zero by the definition of weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ in E. Note, that from the assumption of strong convergence $U_A(e^{kt}\overline{v}_l) \to y_1$ in $C([0,T],S)$ we have

$$\overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_l)) \to \overline{N}_2(\overline{v}_0, y_1) \quad \text{strongly in } E^*$$

due to continuity of map $\overline{N}_2$. From this and from the assumption of weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ in E we get the convergence to zero of the second of obtained terms. So the convergence to zero of the fourth term in the first part of (4.1) is proved. The last term tends to zero due to the continuity of the map C. Hence the right-hand side converges to zero. Thus we have established that

$$\overline{v}_l \to \overline{v}_0 \quad \text{strongly in } E_2. \tag{4.2}$$

Transform equality (4.1) to the form:

$$\begin{aligned}\frac{1}{2}\rho\|(\overline{v}_l - \overline{v}_0)(T)\|_H^2 + \rho k\|\overline{v}_l - \overline{v}_0\|_H^2 + \langle \overline{N}_1(\overline{v}_l)\overline{N}_1(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle \\ = \langle \overline{F} + \overline{f} - g, \overline{v}_l - \overline{v}_0\rangle + \langle \overline{K}(\overline{v}_0) - \overline{K}(\overline{v}_l), \overline{v}_l - \overline{v}_0\rangle \\ + \varepsilon_l\langle N_0(\overline{v}_l), \overline{v}_0 - \overline{v}_l\rangle \\ + \langle \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)), \overline{v}_l - \overline{v}_0\rangle \\ + \langle \overline{C}(\overline{v}_l, \overline{v}_l) - \overline{C}(\overline{v}_0, \overline{v}_0), \overline{v}_l - \overline{v}_0\rangle.\end{aligned} \tag{4.3}$$

Show that each term in the right hand side of this equality converges to zero as $l \to \infty$. This will imply

$$\langle \overline{N}_1(\overline{v}_l) - \overline{N}_1(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle \to 0 \quad \text{at } l \to \infty. \tag{4.4}$$

Consider only the fourth and the fifth terms in the right-hand side of (4.3).

From convergence (4.2) and continuity of U_A in E_2 it follows that

$$U_A(e^{kt}\overline{v}_l) \to U_A(e^{kt}\overline{v}_0) \quad \text{strongly in } C([0,T],S).$$

Then the continuity of map $\overline{N}_2$ provides the convergence

$$\overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) \to \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) \quad \text{strongly in } E^*. \tag{4.5}$$

From the continuity of map $\overline{C}$ we get the convergence

$$\overline{C}(\overline{v}_l, \overline{v}_l) \to \overline{C}(\overline{v}_0, \overline{v}_0) \quad \text{strongly in } E^*. \tag{4.6}$$

These convergences and the weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ in E provide convergence to zero of the fourth and the fifth terms in the right-hand side of (4.3) and so of the entire right-hand side. So the convergence (4.4) is established.

Show that

$$\overline{N}_1(\overline{v}_l) \to \overline{N}_1(\overline{v}_0) \quad \text{weakly in } E^*. \tag{4.7}$$

The sequence $\{\overline{v}_l\}$ and the map $\overline{N}_1$ are bounded, therefore the sequence $\{\overline{N}_1(\overline{v}_l)\}$ is also bounded. Without loss of generality let us suppose that

$$\overline{N}_1(\overline{v}_l) \rightharpoonup y_2 \quad \text{weakly in } E^*. \tag{4.8}$$

Consider the difference

$$\langle \overline{N}_1(\overline{v}_l), \overline{v}_l\rangle - \langle y_2, \overline{v}_0\rangle = \langle \overline{N}_1(\overline{v}_l) - \overline{N}_1(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle + \langle \overline{N}_1(\overline{v}_0), \overline{v}_l - \overline{v}_0\rangle + \langle \overline{N}_1(\overline{v}_l) - y_2, \overline{v}_0\rangle.$$

In the right-hand side of the equality each term converges to zero: the first one due to (4.4), the second one by the definition of weak convergence $\overline{v}_l \rightharpoonup \overline{v}_0$ as $l \to \infty$, the third one by the assumption (4.8). Hence,

$$\lim_{l\to\infty} \langle \overline{N}_1(\overline{v}_l), \overline{v}_l\rangle = \langle y_2, \overline{v}_0\rangle.$$

Then, due to Lemma 1.3(c) ([11, p. 85]), we have $N_1(\overline{v}_0) = y_2$ and so (4.7) holds. Note also that from convergence (4.2) and the continuity of $\overline{K}$ in E_2 it follows that

$$\overline{K}(\overline{v}_l) \to \overline{K}(\overline{v}_0) \quad \text{strongly in } E^*. \tag{4.9}$$

By definition $\overline{v}_l$ the following equality

$$\rho\overline{v}_l' + \rho k\overline{v}_l + \overline{N}_1(\overline{v}_l) + \overline{N}_2(\overline{v}_l, U_A(e^{kt}\overline{v}_l)) - \overline{C}(\overline{v}_l, \overline{v}_l) + \overline{K}(\overline{v}_l) + \varepsilon_l N_0(\overline{v}_l) = \overline{F} + \overline{f}$$

is fulfilled. Pass to the limit in the sense of weak convergence in E^* in each term of this equality. Taking into account (4.5)–(4.7) and (4.9), we receive

$$\rho\overline{v}_0' + \rho k\overline{v}_0 + \overline{N}_1(\overline{v}_0) + \overline{N}_2(\overline{v}_0, U_A(e^{kt}\overline{v}_0)) - \overline{C}(\overline{v}_0, \overline{v}_0) + \overline{K}(\overline{v}_0) = \overline{F} + \overline{f}.$$

As $\overline{v}(0) = v^0$, $\overline{v}$ is a required solution of problem (1.24)–(1.25). □

Note, that if the terms responsible for the memory effect of a fluid were omitted, we would get in equality (1.2) W. G. Litvinov's constitutive relations from [4]. However, these relations do not include the averaging operator in variable x. The question whether it is possible to omit this operator was arisen in [4]. The suggested methods allow us to obtain the existence theorem of a weak solution of this problem also with replacing condition (1.14) by (1.18).

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