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LERAY–SCHAUDER DEGREE: A HALF CENTURY OF EXTENSIONS AND APPLICATIONS

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Dedicated to the memory of Juliusz Schauder and Jean Leray

Abstract. The Leray–Schauder degree is defined for mappings of the form $I - C$, where C is a compact mapping from the closure of an open bounded subset of a Banach space X into X. Since the fifties, a lot of work has been devoted in extending this theory to the same type of mappings on some nonlinear spaces, and in extending the class of mappings in the frame of Banach spaces or manifolds. New applications of Leray–Schauder theory and its extensions have also been given, specially in bifurcation theory, nonlinear boundary value problems and equations in ordered spaces. The paper surveys those developments.

1. Introduction

The *algebraic topology of Banach spaces,* and its application to nonlinear equations, has started with the work of Juliusz Schauder in the five years period 1927–1932 ([73]–[77]). Schauder identified an important class of nonlinear operators in a Banach space, the *completely continuous perturbations of identity,* for which he could generalize two important results of Brouwer in finite-dimensional topology: the fixed point and invariance of domain theorems. Schauder applied the first extension – nowadays called the *Schauder fixed point theorem* [73], [78], [76] — to the existence of solutions of differential equations for which uniquenes does not necessarily hold. Schauder applied his *invariance of domain theorem*

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[75], [77] to exhibit nonlinear elliptic problems for which uniqueness implies existence.

In 1933, Schauder got the opportunity to meet Leray in Paris (an indirect consequence of Nazi's epuration in Göttingen that Schauder was supposed to visit), and a second important period in infinite dimensional topology started from their collaboration. Leray and Schauder immediately realized that the topology of completely continuous perturbations of identity in a Banach space was the right setting to develope Leray's continuation method for nonlinear integral equations (called by him the *Arzelá-Schmidt's method*), introduced in his thesis [43] of 1933, and in particular to liberate it from unnecessary uniqueness and regularity assumptions. Leray said to Schauder:

> I have read your paper on the relationship between existence and uniqueness of the solution of a nonlinear equation. I know now that existence is independent of uniqueness. I admire your topological methods. In my opinion they ought to be useful for establishing an existence theorem independent of the whole question of uniqueness and assuming only some a priori estimates.

Schauder replied:

> Das wäre ein Satz[1].

This became a theorem in a few days and a fundamental joint paper in a few weeks [53]. To prove it, the topology of Banach spaces was developed by extending Brouwer's *topological degree* to the completely continuous perturbations of identity in a Banach space, providing fundamental *continuation theorems* successfully applied to nonlinear elliptic boundary value problems.

Schauder did not make any further applications of the continuation theorems (concentrating his interests to hyperbolic partial differential equations), in contrast to Leray, who used it in problems of wakes and bows in hydrodynamics, in fully nonlinear elliptic problems of Bernstein type, and extended it to some nonlinear spaces.

The influence of [53] on contemporary mathematics is considerable. The quick search in the Mathematical Reviews disclosing 591 references to papers that make use of it, mentioned by Peter Lax in [42], is surely underestimated, and the real figure should be much larger than one thousand. The reader can consult the monographs [174], [126], [141], [81] and their references to get a first idea of the tremendous bibliography related to the consequences and extensions of [53]. The bibliography of this paper includes a (surely uncomplete) list of some one hundred and twenty monographs dealing with Leray–Schauder theory and its applications, published between 1948 and 1999.

To keep the paper at a reasonable length and to remain within a minimum of competence of the author, we exclude important aspects of the development of

[1] That would be a theorem.

Leray–Schauder theory, like the many variants and extensions of Schauder fixed point theorem, the case of mappings between spaces of different "dimensions", the theory of multi-valued mappings, the case of equivariant mappings, Nielsen's fixed point theory, asymptotic fixed point theorems, the computation of Leray–Schauder degree, and its use in critical point theory.

In the whole paper, if X is a metric space, $I = [0,1]$, $A \subset X \times I$, and $\lambda \in I$, we write $A_\lambda = \{x \in X : (x,\lambda) \in A\}$. For $a \in X$ and $r > 0$, $B(a,r)$ denotes the open ball of center a and radius r.

2. The Leray–Schauder's paper revisited

Leray and Schauder define a *completely continuous* mapping from a metric space A into a metric space B as a continuous mapping on A which takes bounded subsets of A into relatively compact ones of B. When a continuous mapping takes A into a relatively compact subset, it is nowadays said to be *compact* on A.

Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity in a Banach space X. If $U \subset X$ is an open bounded set, $f : \overline{U} \to X$ is compact, and $z \notin (I - f)(\partial U)$, the *Leray–Schauder degree* $\deg_{LS}[I - f, U, z]$ of $I - f$ in U over z is constructed from the Brouwer degree by approximating the compact mapping f over $\overline{U}$ by mappings f_ε with range in a finite-dimensional subspace X_ε (containing z) of X, and showing that the Brouwer degrees $\deg_B[(I - f_\varepsilon)|_{X_\varepsilon}, U \cap X_\varepsilon, z]$ stabilize for sufficiently small positive ε to a common value defining $\deg_{LS}[I - f, U, z]$. This topological degree "algebraically counts" the number of fixed points of $f(\,\cdot\,) - z$ in U, and, for f of class C^1, and $I - f'(a)$ invertible for each fixed point a of $f(\,\cdot\,) - z$ in U, Leray and Schauder show that

$$\deg_{LS}[I - f, U, z] = \sum_{a \in (I-f)^{-1}(z)} (-1)^{\sigma_j(a)},$$

where $\sigma_j(a)$ is the sum of the algebraic multiplicities of the eigenvalues of $f'(a)$ contained in $]1, \infty[$.

The Leray–Schauder degree conserves the basic properties of Brouwer degree, as listed in [53].

Theorem 2.1. *The Leray–Schauder degree has the following properties.*

(i) (Additivity) *If $U = U_1 \cup U_2$, where U_1 and U_2 are open and disjoint, and if $z \notin (I - f)(\partial U_1) \cup (I - f)(\partial U_2)$, then*

$$\deg_{LS}[I - f, U, z] = \deg_{LS}[I - f, U_1, z] + \deg_{LS}[I - f, U_2, z].$$

(ii) (Existence) *If $\deg_{LS}[I - f, U, z] \neq 0$, then $z \in (I - f)(U)$.*

(iii) (Homotopy invariance) *Let $\Omega \subset X \times I$ be a bounded open set, and let $F : \overline{\Omega} \to X$ be compact. If $x - F(x, \lambda) \neq z$ for each $(x, \lambda) \in \partial\Omega$, then $\deg_{LS}[I - F(\cdot, \lambda), \Omega_\lambda, z]$ is independent of λ.*

In [53], property (iii) is stated under slightly more restrictive assumptions.

Denote by Σ the (possibly empty) set defined by

$$\Sigma = \{(x, \lambda) \in \overline{\Omega} : x = F(x, \lambda)\}.$$

An important consequence of the existence and homotopy invariance properties of Leray–Schauder's degree is the celebrated *Leray–Schauder continuation theorem.*

Theorem 2.2. *Assume that $F : \overline{\Omega} \to X$ is completely continuous, and that the following conditions hold.*

(i) $\Sigma \cap \partial\Omega = \emptyset$ (*a priori estimate*),
(ii) $\deg_{LS}[I - F(\cdot, 0), \Omega_0, 0] \neq 0$ (*degree condition*),

then Σ contains a continuum $\mathcal{C}$ along which λ takes all values in I.

In other words, under the above assumptions, Σ contains a compact connected subset $\mathcal{C}$ connecting Σ_0 to Ω_1. In particular, the equation $x = F(x, 1)$ has a solution in Ω_1.

Leray and Schauder use instead of (ii) a slightly less general condition requiring in addition that Σ_0 *is a finite nonempty set* $\{a_1, \dots, a_\mu\}$, and observe furthermore that, by refining Assumption (ii), one can obtain a more precise conclusion, which already anticipates subsequent results in bifurcation theory and alternative theorems described later. Let a is an *isolated* fixed point of f, for $r > 0$ small, define its *local Leray–Schauder index* by

$$\operatorname{ind}_{LS}[I - f, a] := \deg_{LS}[I - f, B(a, r), 0].$$

Theorem 2.3. *If a_1 is an isolated fixed point of $F(\cdot, 0)$ and if*

$$\operatorname{ind}_{LS}[I - F(\cdot, 0), a_1] \neq 0,$$

then $(a_1, 0)$ belongs to a continuum in Σ containing one of the points $(a_2, 0), \dots, (a_\mu, 0)$, or to a continuum in Σ along which λ takes all the values in I.

Leray and Schauder emphasize the following important special case of condition (iii), whose statement does not involve explicitely the concept of degree:

Corollary 2.1. *If condition* (i) *holds, and if*

(ii') $\quad 0 \in \Omega_0, \quad F(\cdot, 0) = 0,$

then Σ contains a continuum $\mathcal{C} \ni (0,0)$ along which λ takes all values in I.

They also mention the special case of condition (ii), where Σ_0 is a finite nonempty set $\{a_1, \dots, a_\mu\}$ with μ odd and $I - F(\cdot, 0)$ one-to-one on a neighbourhood of each a_j.

The assumption of complete continuity for F restricts the structure of the equation to be considered, but is satisfied by many abstract formulations of differential and integral equations.

Condition (i) requires the *a priori* knowledge of some properties of the solution set Σ and is in general the most difficult one to check. An important special case, already emphasized by Leray and Schauder, corresponds to the situation where $\Omega = X \times I$ and the set Σ is bounded. In their own words [53]:

> Soit une famille d'équations [...] qui dépendent continûment du paramètre k ($k_1 \leq k \leq k_2$)
>
> $$x - \mathcal{F}(x,k) = 0.$$
>
> L'une des conséquences de notre théorie est la suivante: il suffit de savoir majorer *a priori* toutes les solutions que possèdent ces équations et de vérifier, *pour une valeur particulière* k_0 *de* k, une certaine condition d'unicité, pour avoir le droit d'affirmer que l'équation $x - \mathcal{F}(x,k) = 0$ possède au moins une solution quel que soit k.[2]

The precise formulation of this important special case goes as follows.

COROLLARY 2.2. *Assume that F is completely continuous on $X \times I$, and that the following conditions hold.*

(i) $F(\cdot, 0) = 0$.
(ii) *There exists $r > 0$ such that*

$$\Sigma := \{(x,\lambda) \in X \times I : x = F(x,\lambda)\} \subset B(r) \times I.$$

Then Σ contains a continuum $\mathcal{C} \ni (0,0)$ along which λ takes all values in I.

Motivated by applications to nonlinear elliptic problems, Leray and Schauder have formulated continuation theorems for *more general equations,* which are reducible to some compact perturbations of identity. Essentially, they consider equations which can be written in the form

$$G(x,y,\lambda) = 0, \tag{1}$$

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[2]Consider a family of equations [...] depending continuously on the parameter k ($k_1 \leq k \leq k_2$)

$$x - \mathcal{F}(x,k) = 0.$$

One of the consequences of our theory is the following one: it suffices to be able to find an *a priori* estimate for all the solutions of those equations and to check, *for one particular value* k_0 *of* k, some uniqueness condition to have the right to affirm that equation $x - \mathcal{F}(x,k) = 0$ has at least one solution for each k.

where X, X_0, Y are Banach spaces with X_0 compactly imbedded in X, $G : X \times X_0 \times I \to Y$ is Fréchet differentiable and such that, near each solution (x^*, λ^*) of (1), the equation in y

$$G(x, y, \lambda) = 0$$

has a unique solution $y = F(x, \lambda)$ close to x^*, which defines locally a completely continuous mapping F. Thus, near (x^*, x^*, λ^*), equation (1) is equivalent to the fixed point problem

$$x = F(x, \lambda).$$

Leray and Schauder assume that the equation $x = F(x, 0)$ has a finite number of solutions and the sum of their indices is not zero. Then they conclude that there exists in $X \times I$ a continuum of solutions along which λ takes all the values of I.

Extensions and developments of this idea, including the construction of a degree for mappings having *diagonal* or *intertwined representation,* have been made in 1951 by Cronin [19], in 1967 by Zabreĭko-Krasnosel'skiĭ [82], and in 1968 by Browder–Nussbaum [14] and Browder [11], [129]. Extensions of the Leray–Schauder theory to other classes of mappings between Banach spaces or manifolds, which took place from the sixties, have been fundamental for the development of nonlinear functional analysis in the second half of the century. They will be described in subsequent sections.

References to further applications of the Leray–Schauder continuation theorem to nonlinear elliptic boundary value problems can be found in [94], [144], [135], [95], [84], [184], to nonlinear parabolic boundary value problems in [94], [99], to Navier–Stokes equation in [88], [158], and to ordinary differential equations in [144], [145], [134], [141], [92], [104], [115].

The reader can consult [30], [52], [37] for interesting first hand informations about the genesis of the Leray–Schauder's paper. For their contribution, Leray and Schauder were jointly awarded the Malaxa Prize in 1938.

3. Some surveys of Leray and of Schauder

In 1936, Leray [46] and Schauder [78] each published a survey of their joint work, which contain both interesting general remarks and which, together with a further survey of Leray in 1950 [48], paved the way for several developments of their theory achieved in the second half of the century.

Leray's paper [46] is the text of a lecture given at the *Conférences internationales des Sciences mathématiques* devoted to *Equations aux dérivées partielles. Conditions propres à déterminer les solutions,* organized in June 1935 at the University of Genève. The part devoted to the general theory expresses Leray's concept of "solving an equation":

> Pour pouvoir affirmer que l'équation $x + \mathcal{F}(x) = 0$ est résoluble, il suffit de démontrer qu'elle ne présente pas de solution *arbitrairement grande* quand on la réduit continûment à une équation telle que $x = 0$. Démontrer qu'une équation fonctionnelle a des solutions revient donc à résoudre le problème suivant: assigner des majorantes aux solutions qu'elle possède éventuellement. Il serait d'ailleurs inimaginable qu'on puisse résoudre une équation par un procédé qui ne fournisse pas de renseignements sur l'ordre de grandeur des inconnues. Pour nous, résoudre une équation, c'est majorer les inconnues et préciser leur allure le plus possible; ce n'est pas en construire, par des développements compliqués, une solution dont l'emploi pratique sera presque toujours impossible.
>
> On peut se permettre de considérer ce théorème d'existence comme étant une généralisation au cas non linéaire de l'alternative de Fredholm: soit une équation de Fredholm $x + \mathcal{L}(x) = b$, (où $\mathcal{L}(x) = \int K(s, s')x(s')\, ds$ est complètement continue); cette équation possède sûrement une solution, sauf si l'équation $x+\mathcal{L}(x) = 0$ en possède une; or ce cas est justement celui où l'équation proposée admettrait des solutions *arbitrairement grandes*[3].

One should mention also an unnoticed lecture of Leray [45] at Julia's *Séminaire de mathématiques* devoted in 1935–36 to *Topology,* and published in a mimeographic form only. Leray's lecture, delivered on December 18, 1935, surveys essentially finite dimensional simplicial homology, Brouwer degree, the Jordan–Brouwer theorem and Alexandroff's theorem. It ends with a section entitled "Topologie des espaces abstraits"[4] in which Leray first explains the difficulty of the problem:

> Nous envisageons des espaces abstraits de Banach; ce sont ceux qu'on rencontre le plus fréquemment en analyse [...]. Un domaine borné d'un espace de Banach, en général, n'est pas compact et ne peut pas être assimilé à un complexe. Il semble d'abord que les propriétés de la topologie combinatoire y tombent en défaut.
>
> Certes il est facile de définir le groupe d'homologie d'un domaine appartenant à un espace de Banach et de définir l'homomorphisme qu'engendre une transformation continue opérant dans un tel espace. Mais dans l'espace de Hilbert, la correspondance qui associe au point $(x_1, x_2, \dots)$ le point $(0, x_1, x_2, \dots)$ transforme l'un en l'autre un hyperplan de l'espace et l'espace entier. On contredit aisément le théorème de Jordan. On construit aisément des transformations pour lesquelles il est absurde d'admettre qu'il existe un degré topologique possédant les propriétés usuelles.
>
> [Considérons] les transformations du type $y = x + \mathcal{F}(x)$ [...], $\mathcal{F}(x)$ étant un point qui dépend continûment de x, et qui décrit un domaine de définition. [...] Le degré topologique d'une telle transformation existe; et le théorème d'Alexandroff vaut quand l'homéomorphisme entre les deux ensembles fermés F_1 et F_2 est une correspondance

[3]To be able to claim that equation $x + \mathcal{F}(x) = 0$ is solvable, it is sufficient to prove that it has no *arbitrarily large* solution when one reduces it continuously to an equation like $x = 0$. To prove that a functional equation has solutions is reduced to solving the following problem: assign bounds to its possible solutions. It should be indeed unimaginable that one could solve an equation through a method which does not provide information of the order of magnitude of the unknowns. For us, to solve an equation consists in bounding its unknowns and precise their shape as much as possible; it is not to construct, through complicated developments, a solution whose practical use will be almost always impossible. One can consider this existence theorem as being a generalization to the nonlinear case of the Fredholm alternative: let a Fredholm equation $x+\mathcal{L}(x) = b$, (where $\mathcal{L}(x) = \int K(s, s')x(s')\, ds$ is completely continuous); this equation surely has a solution, except if the equation $x + \mathcal{L}(x) = 0$ has one; and this case is exactly the one where the proposed equation would have *arbitrarily large* solutions.

[4]Topology of abstract spaces.

> de ce type. (Le premier théorème de cette nature est dû à J. Schauder; il s'agissait de "l'invariance du domaine")[5].

Schauder's survey [78] is the text of his lecture at the *Conference on Topology* held in Moscow in 1935. After summarizing the main abstract results contained in [53], Schauder writes, as a footnote to the special case of the continuation theorem where $F(\,\cdot\,,0)=0$:

> Es ist unmöglich, aus diesem einfachen Sonderfall, der übringens, — wie man leicht beweisen kann, — mit dem Fixpunktsatze in linearen, normierten und vollständigen Räumen (vgl. Schauder, Studia Math. 2 (1930), s. 171–180; Satz 2) äquivalent ist, rückwärts irgendein tieferliegendes Resultat zu erzielen (z.B. die algebraische Additionsformel)[6].

After mentioning the recent extension by Leray of the Jordan–Brouwer theorem to completely continuous perturbations of identity in Banach spaces [44], Schauder traces a program for the further development of infinite-dimensional algebraic topology:

> Auf änliche weise, wie hier für den Abbildungsgrad dargelegt wurde, lassen sich auch weitere topologische Invarianten (Bettische Gruppen, Zahlen usw.) (wie mir soeben Herr Leray brieflich ankündigt) für den unendlichdimensionalen Raum definieren. Doch weder Herr Leray, noch ich sehen vorläufig irgendeine Anwendungen dieser anderen Invarianten für den Fall des Funktionalraumes in der Analysis[7].

Important contributions to the *algebraic topology in infinite-dimensional spaces* has been made by Boltjanskiĭ, Švarc, Mitjagin, Gęba, Granas, Eells, Namioka, Mukherjea, Morava, Weinberg, and others (see references in [112], [125], [107]).

Finally, Schauder concludes his survey as follows:

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[5] We consider abstract Banach spaces; they are the most frequent in analysis [...]. A bounded domain of a Banach space is not in general compact and cannot be assimitated to a complex. It seems first that the properties of combinatorial topology fail. However it is easy to define the homology group of a domain belonging to a Banach space and to define the homomorphisms induced by a continuous transformation acting in such a space. But, in Hilbert space, the transformation which associates to the point $(x_1,x_2,\dots)$ the point $(0,x_1,x_2,\dots)$ transforms a hyperplane of the space into the whole space. One easily contradict Jordan's theorem. One easily constructs transformations for which it is absurd to admit the existence of a topological degree having the usual properties. [Consider] the transformations of the type $y=x+\mathcal{F}(x)$ [...] $\mathcal{F}(x)$ being a point depending continuously on x, and which describes a domain of definition [...]. The topological degree of such a transformation exists; and Alexandroff theorem holds when the homeomorphism between the two closed sets F_1 and F_2 is a transformation of this type. (The first theorem of this nature is due to J. Schauder; it was the "invariance of domain" theorem).

[6] It is impossible from this simple special case which, indeed — as one can prove it easily — is equivalent to the fixed point theorem in linear, normed and complete spaces (cf. Schauder, Studia Math. 2 (1930), p. 171–180; Theorem 2), to obtain some of the deep results (for example the algebraic addition formula).

[7] In a way similar to the one made here for the degree of an application, it will be possible to define other topological invariants (Betti groups, numbers, etc.) (as Mr Leray recently briefly anounced to me) for the infinite dimensional spaces. However, neither Mr. Leray nor me see for the moment significant applications of those other invariants in the case of function spaces of analysis.

> Anderseits kann ich eine ähnliche Theorie der Abbildungsgrades auch dann entwickeln, wenn es sich um allgemeinere Räume handelt, etwa um lineare metrisches Räume, in welchen es beliebig kleine, konvexe Umgebungen der Null gibt. Ein gemeinsame Arbeit von Hernn Leray und mir über verschiedene topologische Invarianten in möglichst allgemeinen Räumen wird erscheinen. Auch nichtlineare Räume könnten betrachtet werden. Etwa metrische Räume, deren Umgebungen, z.B. den Umgebungen in Banachschen Räumen homöomorph sind[8].

Thus, the way is traced for the extension of Leray–Schauder fixed point theory to locally convex spaces and to Banach manifolds. Schauder's tragical death during the Second World War did not allow him to contribute to those planned developments, and the legacy remained in the sole hands of Leray.

Leray never delivered an invited address or an invited lecture in section at any *International Congress of Mathematicians.* However, volume II of the *Proceedings of the ICM* of 1950 held in Cambridge, Massachussets, contains an interesting paper of him [48]. A closer look shows that this volume II consists in the proceedings of three conferences, respectively on Algebra, Analysis and Applied Mathematics, held during the ICM to supplement the regular program. The *Conference on Analysis* is divided in three parts: *Algebraic tendencies in analysis, Analysis in the large* and *Analysis and geometry in the large,* and Leray's paper is one of the four contributions in this last direction. Leray's survey is dedicated

> à la mémoire du profond mathématicien polonais Jules Schauder, victime des massacres de 1940[9].

It describes the extension of the Leray–Schauder theory, mentioned by Schauder in [78], to completely continuous perturbations of identity in a *locally convex topological vector space* and Leray's *fixed point theory of continuous mappings of some compact topological spaces into itself* developed during war time.

After some pioneering work of Rothe in 1939 [69], the extension of Leray–Schauder theory to completely continuous perturbations of identity in locally convex spaces was also worked out in details by Nagumo in 1951 [60], but the number of its applications to differential equations has been rather limited. It is still an open problem to know if the extension works for arbitrary topological vector spaces, even if Klee, Kaballo, Kayser, Krauthausen, Riedrich, Hahn, Alex, Kaniok, Van der Bijl, Dobrowolski, Hart, Van der Mill, Pötter, Okoń and others have shown that it can be done in some more general classes than the locally convex ones (see [81] for references).

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[8] On the other way, one can also develope a theory of the topological degree when one deals with more general spaces, namely linear metric spaces in which arbitrarily small convex neighbourhoods of the origin are given. A joint work of Mr Leray and myself on various topological invariants in spaces as general as possible will appear. Also, nonlinear spaces can be treated, namely metric spaces whose neighbourhoods are homeomorphic to Banach spaces.

[9] Dedicated to the memory of the deep Polish mathematician Jules Schauder, victim of the massacres of 1940.

Concerning the existence of fixed points for a continuous mapping f from a compact space X into itself, Leray first observes in [48] that if there exist a *retraction* r of an open subset V of a locally convex space E onto X, (i.e. an continuous map $r : V \to X$ such that $r|_X = I$), then the fixed points of f in X are those of the completely continuous mapping $f \circ r$, and a *fixed point index* $\mathrm{ind}_X[f, U]$ for the fixed points of f in $U \subset X$ can be defined as the Leray–Schauder degree $\deg_{LS}[I - f \circ r, r^{-1}(U), 0]$.

For E a Hilbert space, such a space X is an *absolute neighbourhood retract* (ANR). Recall, following K. Borsuk, that a compact metric space X is a *compact* ANR if, for any subspace A of a separable metric space Y which is homeomorphic to X, there exists an open set $U \supset A$ of Y and a retraction of U onto A. If $f : X \to X$ is continuous, the *Lefschetz number* $\Lambda_X[f]$ of f is the alternate sum of the traces of the homology endomorphisms induced by f. The *Lefschetz fixed point theorem* states that *if X is a compact metric* ANR*, any continuous mapping $f : X \to X$ with non-zero Lefschetz number has a fixed point.* First anounced by Lefschetz in 1923, for X a finite polyhedron or a manifold without boundary, this result was proved for finite polyhedra by H. Hopf in 1928, and extended by Lefschetz to compact ANR's, implicitely in 1930 and explicitely in 1937.

One of the main problems for fixed point theory in infinite dimensional spaces is to "localize" the Lefschetz fixed point theorem, i.e. to construct a theory which contains both the Leray–Schauder and the Lefschetz theory a special case. The corresponding tool is called a *fixed point index theory* and we describe its development in the next section.

4. Fixed point index in compact ANR's

In a series of papers which constitute a course in algebraic topology teached in captivity in the Oflag XVIIA, in Austria, Leray [47] developes new tools in algebraic topology (the future *spectral sequences* and *sheaf homology*) in order to extend the concepts and results of [53] to some nonlinear spaces and unify his theory with Lefschetz one. He introduces the *convexoid spaces,* a class of compact spaces which contains the finite polyhedra and finite unions of compact convex sets in locally convex topological vector spaces. The following assertion summarizes some of Leray's results in his own terms, but in modern notations.

Let X be a convexoid space; let f be a continuous mapping from a closed subset of X into X, and let us consider the equation

$$x = f(x). \tag{2}$$

Assume the set of its solutions is compact. The total index $\mathrm{ind}_X[f, U]$ (Leray writes $i(U)$) of the solutions of (2) belonging to an open subset U of X is an integer, defined when $\overline{U}$ belongs to the domain of f and ∂U does not meet the set of solutions of (2). This index has the following properties:

1. $\mathrm{ind}_X[f, U]$ remains constant, as long as it remains defined, when f varies continuously according to a parameter belonging to a connected topological space.
2. If f is defined over X, $\mathrm{ind}_X[f, X] = \Lambda_X[f]$.
3. If f is defined over $\overline{U}$ and $\overline{U}$ contains no solution of (2), then

$$\mathrm{ind}_X[f, U] = 0.$$

We refer to [47] for the involved explicit definition of this index.

A partial unification is obtained in this way between Lefschetz and Leray–Schauder theory, but some open problems remain concerning convexoid spaces: for example, it is not clear whether an arbitrary Euclidian manifold is convexoid. In [48], Leray mentions that

> Ce théorème [de Lefschetz] est une conséquence de la théorie précédente [de 1945]; mais il s'applique à certains espaces compacts auxquels cette théorie n'a pas été étendue. Le problème est ouvert de savoir si cette théorie est un cas particulier d'une théorie plus générale, applicable à tout espace compact[10].

This open problem raised by Leray has motivated a substantial activity in the development of fixed point theory in the second half of the century, and we describe its main lines.

In order to construct a theory of the fixed point index in a context similar to that in which Lefschetz had proved his fixed point theorem in 1942, F. E. Browder, in his unpublished Princeton University PhD thesis written under Lefschetz and Hurewicz [7], constructs in 1948 a *fixed point index for compact* ANR*'s,* using as a basic tool Leray's theory as applied to finite polyhedra.

In 1950, Hanner proves that any ANR is ε-dominated by polyhedra, which implies that the Lefschetz theorem for compact ANR's follows from the one for finite polyhedra. One year later, Dugundji proves that any convex subset of a normed linear space is an ANR, giving an elegant and powerful way to connect topology and functional analysis. As observed by R. F. Brown in [16]:

> Leray worked with convexoid spaces, rather than with a more usual generalization of polyhedra such as absolute neighbourhood retracts (ANR's) because he wanted to use his theory to obtain new results in analysis. There were certain kinds of subsets of function spaces which he needed for this purpose and which he could prove to be convexoid but which were not known to be ANR's. However, in 1951, Dugundji showed that these subsets are indeed ANR's, so the motivation for the further study of convexoid spaces was gone.

[10] This [Lefschetz] theorem is a consequence of the preceding theory [of 1945]; but it applies to some compact spaces to which this theory has not been extended. The problem is open to know if this theory is a special case of a more general one, applicable to any compact space.

O'Neill rederives the principal results of Browder's thesis for the special case of *finite polyhedra* X in his M.I.T. PhD thesis of 1953 (written under Hurewicz) [64]. If U and V are open sets of X whose boundaries contain no fixed points of $f : X \to X$, O'Neill proves the existence of a fixed point index $\operatorname{ind}_X[f, U]$ having the following properties:

(1) *If* U *contains no fixed point of* f*, then* $\operatorname{ind}_X[f, U] = 0$.
(2) $\operatorname{ind}_X[f, U] + \operatorname{ind}_X[f, V] = \operatorname{ind}_X[f, U \cup V] + \operatorname{ind}_X[f, U \cap V]$.
(3) *There is a neighbourhood* N *of* f (*in the compact-open topology*) *such that* $g \in N$ *implies that* $\operatorname{ind}_X[g, U]$ *exists and is equal to* $\operatorname{ind}_X[f, U]$.
(4) *If* $\overline{U}$ *is a subpolyhedron of* X *and* $f(U) \subset U$*, then* $\operatorname{ind}_X[f, U] = \Lambda_{\overline{U}}[f]$.
(5) *If* $U \cup f(U)$ *is contained in a subpolyhedron of* X *isomorphic under a map* h *to a subpolyhedron of the finite polyhedron* Y *and if* $g : Y \to Y$ *is continuous and such that* $gh = hf$ *on* U*, then* $\operatorname{ind}_X[f, U] = \operatorname{ind}_Y[g, h(U)]$.

He then shows that the fixed point index is uniquely determined in the class of finite polyhedra by properties (2) to (5), giving the first *axiomatic characterization* of the fixed point index. Using the results of O'Neill's paper, Bourgin [6] re-establishes in 1955 the theory of the fixed point index for *compact* ANR*'s,* along lines similar to those of Browder's thesis.

In 1959, Leray [49] introduces a concept of *generalized trace* which allows an extension of the link between the fixed point index and the the corresponding *generalized Lefschez number.* He proves that if X is convexoid, $U \subset X$ open and $f : \overline{U} \to X$ continuous without fixed points in ∂U, then

$$K = \bigcap_{n \geq 1} f^n(\overline{U}) = \lim_{n \to \infty} f^n(\overline{U})$$

is compact and invariant for f. Furthermore, if $K \subset U$, $\operatorname{ind}_X[f, U]$ and $\Lambda_K[f]$ are defined and equal. More generally, if F is a compact such that

$$K \subset f(F) \subset F \subset \overline{U},$$

then $\Lambda_F[f]$ is defined and equal to $\operatorname{ind}_X[f, U]$.

The same year, Deleanu [23] carries out in detail the extension of Leray's theory to *retracts of convexoid spaces* (which include the ANR's), using the sharpening form of Leray's results given in 1959.

One year later, F. E. Browder [8] goes outside the frame of reference of ANR's or of retraction properties in general. He takes up the theory of the fixed point index at the combinatorial or homology level on which it was treated by Leray in 1945, but for more general spaces, the *compact semi-complexes* similar in their nature to those made by Lefschetz in his treatment of his fixed point theorem (1942). Given a category of compact topological spaces X and of permissible continuous mappings h, Browder introduces as follows a concept of fixed point index in this category. If U is an open subset of X, and $f : \overline{U} \to X$ a continuous mapping such that f has no fixed point on ∂U, the *fixed point index* $\mathrm{ind}_X[f, U]$ is an integer which has the following four properties:

(a) (Invariance under homotopy) *If f_λ, $0 \le \lambda \le 1$, is a homotopy of f_0 to f_1, where all the f_λ are mappings of $\overline{U}$ into X and none have any fixed points on ∂U, then* $\mathrm{ind}_X[f_0, U] = \mathrm{ind}_X[f_1, U]$.

(b) (Additivity) *If U contains a finite family of mutually disjoint open sets U_j, $1 \le j \le s$, and if $\overline{U} \setminus \bigcup_{j=1}^{s} U_j$ contains no fixed point of $f : \overline{U} \to X$, then*

$$\mathrm{ind}_X[F, U] = \sum_{j=1}^{s} \mathrm{ind}_X[f, U_j].$$

In particular, if $\overline{U}$ itself contains no fixed point of f, then $\mathrm{ind}_X[f, U] = 0$.

(c) (Normalization) *If $U = X$, then* $\mathrm{ind}_X[f, U] = \Lambda_U[f]$.

(d) (Commutativity) *Let X_1 and X_2 be two spaces in our category, h a permissible mapping of X_1 into X_2, U_2 an open subset of X_2, f a continuous mapping of $\overline{U_2}$ into X_1. Let $U_1 = h^{-1}(U_2)$. Suppose that hf has no fixed point on ∂U_2. Then*

$$\mathrm{ind}_{X_2}[hf, U_2] = \mathrm{ind}_{X_1}[fh, U_1].$$

Browder proves that if the category consists of ANR's and all continuous mappings, and if a fixed point index $\mathrm{ind}_X[f, U]$ can be defined to satisfy properties (a) to (d), it is uniquely characterized by those properties.

The same year, Browder [9] states and proves the following very general version of the continuation theorem in the frame of the fixed point index.

THEOREM 4.1. *Let X be a Hausdorff space, U an open subset of $X \times I$, F a continuous mapping of $\overline{U}$ into a compact space Y lying in a category A for which a fixed point index is defined.* (*Thus Y may be an* ANR *a neighbourhood retract of a convexoid space, or an* HLC* *space*). *Let G be a continuous mapping of $Y \times I$ into X, H the mapping of $\overline{U}$ into X given by $H(x, \lambda) = G(F(x, \lambda), \lambda)$. Let Ψ be the natural injection of X into $X \times I$, $\Psi_\lambda(x) = (x, \lambda)$, $U_\lambda = \Psi_\lambda^{-1}(U)$, $h_\lambda = H\Psi_\lambda$ mapping $\overline{U_\lambda}$ into X. Suppose that h_λ has no fixed points on ∂U_λ for $\lambda \in I$. Let $U' = G^{-1}(U)$, $U'_0 = \Psi_0^{-1}(U')$, $F_0 = F\Psi_0$, $G_0 = G\Psi_0$. Suppose that* $\mathrm{ind}_Y[F_0G_0, U'_0] \ne 0$. (*In the case in which X itself lies in A, we make the simpler assumption that* $\mathrm{ind}_X[h_0, U_0] \ne 0$). *Then there exists a connected subset C_1 in U intersecting both $X \times \{0\}$ and $X \times \{1\}$ such that for all (x, λ) in C_1, $h_\lambda(x) = x$.*

The last paper of Leray on the fixed point index, published in 1972 [50],

> describes as shortly as possible the main features of the fixed-point theory itself [...] and makes accessible the part of [47] which is not presented by D. G. Bourgin [91].

Further contributions to fixed point index theory on compact spaces are due to Thompson, Knill and others. See [93], [146], [112] for references.

5. Fixed point index for non-compact ANR's

With Schauder fixed point theorem in mind, one can ask if the Lefschetz fixed point theorem remains true for *compact* maps on *arbitrary* ANR's. An affirmative answer is supplied by Granas in 1967 [34].

In 1969, F. E. Browder [9] shows that if X is a Banach space, U an open subset of X, and f a compact mapping of U into X whose fixed point set S is a compact subset of U, then, for any compact ANR $R \supset f(S)$,

$$\mathrm{ind}_R[f|_{U\cap R}, U \cap R] = \deg_{LS}[I - f, U, 0].$$

This allows a definition of a fixed point index in a natural geometric way for any compact mapping of an open subset U of a non-compact ANR X into X.

The same year, if X is an ANR and $f : U \to X$ an admissible compact map, V open in a normed space E which r-dominates X, and $s : X \to V$, $r : V \to X$ are such that $rs = I$, Granas [33] defines the index of f in U by

$$\mathrm{ind}_X[f, U] = \deg_{LS}[s \circ f \circ r, r^{-1}(U), 0].$$

Some extensions of Schauder fixed point theorem proved in 1955 by Darbo [22] and Krasnosel'skiĭ [41] suggest that the fixed point index and degree theories could be extended to a more general class of perturbations of identity.

In 1971, Nussbaum [61] defines a fixed point index $\mathrm{ind}_X[f, U]$ for mappings f having some type of asymptotic compactness property and defined on open subsets U of certain "nice" class $\mathcal{F}$ of ANR's. Let X be a closed subset of a Banach space B whose norm induces the metric on X. We say that $X \in \mathcal{F}$ if there exists a locally finite cover $\{C_i : i \in I\}$ of X by closed, convex sets $C_i \subset X$. Let A be a subset of a Banach space B and let $g : A \to B$ be a continuous map. Let $K_1(g, A) = \overline{\mathrm{co}}\, g(A)$ be the convex closure of $g(A)$, $K_n(g, A) = \overline{\mathrm{co}}\, g(A \cap K_{n-1}(g, A))$, $n > 1$, and $K_\infty(g, A) = \bigcap_{n\geq 1} K_n(g, A)$.

Let U be an open set of $X \in \mathcal{F}$ and $f : \overline{U} \to X$ be continuous. Assume that $f(x) \neq x$ for $x \in \partial U$, and that $K_\infty(f, U)$ is compact. If $K_\infty^* = X \cap K_\infty(f, U)$, Nussbaum defines the generalized fixed point index $\mathrm{ind}_X[f, U]$ by $\mathrm{ind}_{K_\infty^*}[f, U \cap K_\infty^*]$.

The generalized index satisfies properties like those of the classical fixed point index.

(i) *Let* $S = \{x \in U : f(x) = x\}$ *and assume that* $S \subset U_1 \cup U_2$*, where* U_1 *and* U_2 *are two disjoint open subsets of* U*. Then* $\mathrm{ind}_X[f, U_i]$ *is defined* $(i = 1, 2)$ *and* $\mathrm{ind}_X[f, U] = \mathrm{ind}_X[f, U_1] + \mathrm{ind}_X[f, U_2]$.

(ii) *Let* Ω *be an open set of* $X \times I$, $X \in \mathcal{F}$. *Let* $F : \Omega \to X$ *be a continuous map and assume* $\Sigma = \{(x, \lambda) \in \Omega : F(x, \lambda) = x\}$ *is compact. Suppose there exists an open subset* O *of* Ω *such that* $\Sigma \subset O$ *and such that* $K_\infty(F, O)$ *is compact. Then the fixed point index* $\operatorname{ind}_X[F(\cdot, \lambda), \Omega_\lambda]$ *is defined and constant for* $\lambda \in I$.

There is also an analogous of the commutativity property, and, when X is a Banach space and $g : \overline{U} \to X$ is compact,

$$\operatorname{ind}_X[f, U] = \deg_{LS}[I - f, U, 0].$$

Special case are given by the so called *k-set-contractions* ($k < 1$). Recall that the Kuratowski's *measure of noncompactness* of a set A of finite diameter in a metric space is defined by

$$\gamma(A) = \inf\left\{r > 0 : A = \bigcup_{i=1}^{m} A_i,\ d(A_i) \le r\right\},$$

and that, for the metric spaces X and Y, $f : X \to Y$ is a *k-set-contraction* if $\gamma(f(A)) \le k\gamma(A)$ for every $A \subset X$ with $\gamma(A)$ finite. A standard example when $X = Y$ is a Banach space is the sum of a strict contraction and a completely continuous mapping. A more general class is that of *condensing* maps $f : X \to Y$ such that $f(A)$ is bounded and $\gamma(f(A)) < \gamma(A)$ for all bounded sets A such that $\gamma(A) > 0$.

The above homotopy invariance property specializes as follows.

(ii') *Let* Ω *be an open set of* $X \times I$, $X \in \mathcal{F}$. *Let* $F : \Omega \to X$ *be a continuous map and assume that* F *is a local strict-set-contraction in the following sense: given* $(x, \lambda) \in \Omega$, *there exists an open neighbourhood of* (x, λ) *in* Ω, $N_{(x,\lambda)}$, *such that for any subset* A *of* X,

$$\gamma(F(N_{(x,\lambda)} \cap (A \times I))) \le k_{(x,\lambda)}\gamma(A), \quad k_{(x,\lambda)} < 1.$$

Assume that $\Sigma = \{(x, \lambda) \in \Omega : F(x, \lambda) = x\}$ *is compact. Then the fixed point index* $\operatorname{ind}_X[F(\cdot, \lambda), \Omega_\lambda]$ *is defined and constant for* $\lambda \in I$.

Further contributions to the fixed point index of non-compact mappings are due to Brown, Eells, Fournier, Fenske, Peitgen, Steinlein, and others.

Independently of Nussbaum's work, the theory of rotation (equivalent to degree) for the related condensing or *limit compact* perturbations of identity was introduced and developed in Voronezh, starting in 1967 with Sadovskiĭ's extension of Schauder's fixed point theorem to condensing mappings [70]. In 1968, Vainniko and Sadovskiĭ define a degree for condensing operators in separable Banach spaces, Borisovich and Sapronov reduce the degree for condensing mappings to the relative degree of some associated compact perturbation of identity, and Sadovskiĭ defines a degree for limit compact and condensing operators in locally convex vector spaces. All this is described, with a large bibliography, in Sadovskiĭ's survey paper [71] and in the monograph [169], and further contributions can be found in the proceedings of Voronezh seminar [159], [170], [175], [178], [183].

The applications of the fixed point index in infinite dimensional space to nonlinear differential equations started essentially in the seventies and concern mostly the use of a fixed point index in *cones* or in *wedges,* initiated by M. A. Krasnosel'skiĭ (see [180]). Specially noticeable is Nussbaum's work in functional differential equations [61] and Amann and Dancer's contributions to nonlinear equations in ordered spaces [1], [21]. See respectively [186] and [2] for further references.

The axiomatic characterization of the fixed point index and its connection with Leray–Schauder degree naturally raises the question of an *axiomatic characterization of the Leray–Schauder degree.* The following result is proved in 1973 by Amann and Weiss [3], showing in particular that the commutativity property is not necessary to characterize axiomatically the degree.

Theorem 5.1. *Let X be a locally convex linear space, and let ω_X denote either the set τ_X of all open subsets of X, or, when nontrivial, the set β_X of bounded open subsets of X. Then, for the family of mappings $I - f$ wich f compact on $\overline{U}$, and $0 \notin (I-f)(\partial U)$, there exists exactly one topological degree* $\deg[I-f, U, 0]$ *satisfying the following properties:*

- (D1) (Normalization) *For every $U \in \omega_X$ with $0 \in U$,* $\deg[I, U, 0] = 1$.
- (D2) (Additivity) *For every nonempty $U \in \omega_X$, every pair of disjoint subsets $U_1, U_2 \in \omega_X$, and every compact $f : \overline{U} \to X$ with $0 \notin (I-f)(\overline{U} \setminus (U_1 \cup U_2))$,* $\deg[I-f, U, 0] = \deg[I-f, U_1, 0] + \deg[I-f, U_2, 0]$.
- (D3) (Homotopy invariance) *For every nonempty $U \in \omega_X$, and for every compact $F : \overline{U} \times I \to X$ such that $x \neq F(x, \lambda)$ for $(x, \lambda) \in \partial U \times [0,1]$,* $\deg[I - F(\,\cdot\,, \lambda), U, 0]$ *is constant.*

Nussbaum [62] has extended this uniqueness property to the case of k-set contractions ($k < 1$) and condensing maps in a Banach space. In the same paper, he gives an alternate definition of the degree when f is a k-set contraction ($k < 1$) in a Banach space, namely $\deg[I - f, U, 0] = \deg_{LS}[I - f \circ \rho, V, 0]$, where ρ is a retraction onto $K_\infty(f, U)$ and $V = U \cap \rho^{-1}(U \cap K_\infty(f, U))$.

6. Continuation theorems in the form of alternatives

In 1955, Schaefer [72] formulates a special case of Leray–Schauder continuation in the form of an alternative, and proves it as a consequence of Schauder fixed point theorem.

Theorem 6.1. *Let X be a Banach space and $f : X \to X$ be completely continuous. Then either there exists for each $\lambda \in [0,1]$ at least one $x \in X$ such that $x = \lambda f(x)$, or the set $\{x \in X : x = \lambda f(x),\ 0 < \lambda < 1\}$ is unbounded in X.*

In 1972, Rabinowitz [67] formulates a continuation theorem when the parameter is the whole real line.

Theorem 6.2. *Consider*

$$x = F(x,\lambda)$$

where $F : X \times \mathbb{R} \to X$ is completely continuous and $F(\,\cdot\,,0) = 0$. Let S denote the closure of the set of solutions of this equation. Then the component of S to which $(0,0)$ belongs is unbounded in $X \times \mathbb{R}^+$ and in $X \times \mathbb{R}^-$.

In the middle eighties Fitzpatrick, Massabo and Pejsachowicz [26], [55] extend this result to the *multi-parameter* case $\lambda \in \mathbb{R}^k$ and to the level of generality of Theorem 2.2.

Theorem 6.3. *Let $\mathcal{O} \subset X \times \mathbb{R}^k$ be an open set and $F : \overline{\mathcal{O}} \to X$ a completely continuous mapping. Assume that, for some $\lambda_0 \in \mathbb{R}^k$ such that $\mathcal{O}_{\lambda_0} \neq \emptyset$, the Leray–Schauder degree $\deg_{LS}[I - F(\,\cdot\,,\lambda_0), \mathcal{O}_{\lambda_0}, 0]$ is well defined and non-zero. Then there exists a connected subset $\mathcal{C}$ of $\Sigma^{\mathcal{O}}$ such that $\mathcal{C} \cap (\mathcal{O}_{\lambda_0} \times \{\lambda_0\}) \neq \emptyset$, the (covering) dimension of $\mathcal{C}$ at each point is at least k, and either $\mathcal{C}$ is unbounded or $\mathcal{C} \cap \partial\mathcal{O} \neq \emptyset$. Moreover, when $\mathcal{O} = X \times \mathbb{R}$, and when $(x_n) \subset X$ is bounded whenever $(\lambda_n) \subset \mathbb{R}$ is bounded and $(x_n,\lambda_n) \in \mathcal{C}$, $\mathcal{C}$ covers $\mathbb{R}$, in the sense that, for each $\lambda \in \mathbb{R}$, there exists some $x \in X$ such that $(x,\lambda) \in \mathcal{C}$.*

Capietto, Mawhin and Zanolin [18], in the early nineties, show how to use suitable continuous functionals to eliminate one of the alternatives and get useful continuation theorems in the absence of a priori bounds.

Theorem 6.4. *Let $\Omega \subset X \times I$ be an open set and let $F : \overline{\Omega} \to X$ be a completely continuous mapping such that Σ_0 is bounded and*

$$\deg_{LS}[I - F(\,\cdot\,,0), \Omega_0, 0] \neq 0,$$

with Ω_0 an open bounded neighbourhood of Σ_0. Assume moreover that there exists a continuous function $\varphi : X \times I \to \mathbb{R}$, proper on Σ, and two real numbers c_-, c_+ such that

$$c_- < \min_{x\in\Sigma_0} \varphi(x,0) \le \max_{x\in\Sigma_0} \varphi(x,0) < c_+,$$

and $\varphi(\Sigma) \cap \{c_-, c_+\} = \emptyset$. Then Σ contains a continuum $\mathcal{C}$ along which λ takes all values in I.

A rather direct consequence of Theorem 6.4, which is more easy to apply, is the following one.

Corollary 6.1. *Let $\Omega \subset X \times I$ be an open set and let $F : \overline{\Omega} \to X$ be a completely continuous mapping such that Σ_0 is bounded and*

$$\deg_{LS}[I - F(\,\cdot\,, 0), \Omega_0, 0] \neq 0,$$

with Ω_0 an open bounded neighbourhood of Σ_0. Assume moreover that there exist a continuous mapping $\varphi : X \times I \to \mathbb{R}_+$, an unbounded increasing sequence $(c_k)_{k\in\mathbb{N}}$ and $R > 0$ such that $\varphi(u,\lambda) \neq c_k$ for all $k \in \mathbb{N}$ and $(u,\lambda) \in \Sigma$ with $\|u\| \geq R$, and $\varphi^{-1}([0, c_n[) \cap \Sigma$ is bounded for each $n \in \mathbb{N}$. Then there exist a continuum $\mathcal{C} \subset \Sigma$ along which λ takes all values in I.

Extensions, variants and applications of those results to nonlinear boundary value problems has been made by Furi, Pera, Martelli, Henrard, Precup, Garcia–Huidobro, Manásevich, Mawhin, Zanolin, Capietto, Dambrosio and others (see [186], [194], [59] for references).

7. Essential maps and continuation theorems

From 1959, Granas [89], [35], [152] has developed continuation theorems in normed spaces based upon the concept of essential map, which avoids the explicit use of degree.

Let C be a convex set of a normed vector space E, X an arbitrary subset of C, $A \subset X$ closed in X, and denote by $\mathcal{K}_A(X,C)$ the set of all compact maps $F : X \to C$ such that the restriction $F|_A : A \to C$ is fixed-point free. Call F, $G \in \mathcal{K}_A(X,C)$ *homotopic* provided there is a compact homotopy $H : X \times [0,1] \to C$ which is fixed-point free on A and such that $H(\,\cdot\,,0) = F$ and $H(\,\cdot\,,1) = G$. Call $F \in \mathcal{K}_A(X,C)$ *essential* provided every $G \in \mathcal{K}_A(X,C)$ such that $F|_A = G|_A$ has a fixed point.

Then one has the so-called *topological transversality theorem*.

Theorem 7.1. *Let F and G be two maps in $\mathcal{K}_A(X,C)$, such that F and G are homotopic in $\mathcal{K}_A(X,C)$. Then F is essential if and only if G is essential.*

For example, if U is an open subset of C, and if we take $X = \mathrm{cl}_C U$, $A = \partial_C U$, the constant map $F(x) = u_0$ is essential in $\mathcal{K}_{\partial_C U}(\overline{U}, C)$ for any $u_0 \in U$. This provides an extension of Corollary 2.2 to a convex subset of a normed space. If U is a convex open bounded symmetric neighbourhood of 0 in E, any compact map $F \in \mathcal{K}_{\partial U}(\overline{U}, E)$ that is antipodal-preserving on ∂U is essential.

Further extensions (zero-epi and zero-essential maps), variants and applications of this results have been given by Furi, Martelli, Vignoli, Ize, Massabó, Pejschachowicz, Pera [31], [54], Gęba, Granas, Kaczyński, Krawcewicz, Guenther, Lee [32], [163], [36], Frigon [179], O'Regan [192], [208] and others.

8. Bifurcation theory

One of the most spectacular application of Leray–Schauder theory in the second half of the century deals with *bifurcation theory.* Early traces of a topological theory of bifurcation can already be found in Poincaré's work on the figures of equilibrium of rotating bodies [66], and the word is mentioned in Leray–Schauder's paper [53]:

> Le problème de Dirichlet étudié ci-dessus peut admettre plusieurs solutions, peut-être même des faisceaux de solutions; quand on fait varier les données, des "bifurcations" peuvent se produire[11].

But the general theory in its essentially definitive form is initiated around 1950 by M. A. Krasnosel'skiĭ [39], [40], [177].

The problem consists in studying an equation of the form

$$x = T(x, \lambda), \tag{3}$$

in a real Banach space X, when $T : X \times \mathbb{R} \to X$ is completely continuous and

$$T(0, \lambda) = 0$$

for each $\lambda \in \mathbb{R}$. Thus $x = 0$ is a solution to (3) for each $\lambda \in \mathbb{R}$ (the *trivial solution*). $(\lambda^*, 0)$ is a *bifurcation point* for (3) if there exist a sequence (λ_k, x_k) of solutions of (3) in $(\mathbb{R}, X \setminus \{0\})$ which converges to $(\lambda^*, 0)$. The problem consists in finding conditions for the existence of bifurcation points.

Krasnosel'skiĭ considers the special case of equations of the form

$$x = \lambda L x + R(x, \lambda), \tag{4}$$

where $L : X \to X$ and $R : X \times \mathbb{R} \to X$ are completely continuous, and

$$\lim_{x \to 0} \frac{\|R(x, \lambda)\|}{\|x\|} = 0,$$

uniformly on bounded λ-sets. It is not too difficult to prove that if $(\lambda^*, 0)$ is a bifurcation point for (3), then λ^* is a characteristic value of L (i.e. the reciprocal of an eigenvalue). Krasnosel'skii's theorem provides a sufficient conditions for a characteristic value of L to be a bifurcation point.

THEOREM 8.1. *For each a real characteristic value λ^* of L with odd multiplicity, $(\lambda^*, 0)$ is a bifurcation point of* (4).

Of fundamental importance in Krasnosel'skiĭ's proof of this theorem is the Leray–Schauder formula

$$\operatorname{ind}_{LS}[I - L, 0] = (-1)^{\sigma},$$

[11] The Dirichlet problem considered above may admit several solutions, maybe even sheaves of solutions; when the data vary, "bifurcations" may occur.

where $L : X \to X$ is linear, invertible and completely continuous, and σ is the sum of the multiplicities of the eigenvalues of L contained in $]1, \infty[$.

The same year, Krasnosel'skiĭ introduces the interesting and fruitful concept of *bifurcation from infinity.* The point (λ^*, ∞) is said to be a bifurcation point for (4) if there exists a sequence (λ_n, x_n) of solutions of (4) such that $\lambda_n \to \lambda^*$ and $\|x_n\| \to \infty$. He proves the following existence result.

Theorem 8.2. *If*

$$\lim_{\|x\|\to\infty} \frac{\|R(x,\lambda)\|}{\|x\|} = 0, \tag{5}$$

uniformly on bounded λ-sets, then, for each real characteristic value λ^* of L with odd multiplicity, (λ^*, ∞) *is a bifurcation point of* (4).

A global version of Krasnosel'skiĭ's Theorem 8.1 is given by Rabinowitz in 1971 [67]. Let $\mathcal{S}$ denote the closure in $\mathbb{R} \times X$ of the set of $(\lambda, x) \in \mathbb{R} \times (X \setminus \{0\})$ satisfying (4).

Theorem 8.3. *If λ^* is a real characteristic value of L with odd multiplicity, then $\mathcal{S}$ contains a component $\mathcal{C}$ which either is unbounded, or contains* $(\lambda^{**}, 0)$, *where $\lambda^{**} \neq \lambda^*$ is a characteristic value of L.*

The global version for bifurcation from infinity is due to Toland and to Rabinowitz (1973).

Theorem 8.4. *If* (5) *holds and if λ^* is a real characteristic value of L with odd multiplicity, then* (4) *possesses an unbounded component of solutions $\mathcal{D}$ which contains* (λ^*, ∞). *Moreover, if $\Lambda \subset \mathbb{R}$ is an interval which contains only the characteristic value λ^* of L, and $\mathcal{M}$ is a neighbourhood of* (λ^*, ∞) *whose projection on $\mathbb{R}$ lies in Λ and whose projection on X is bounded away from* 0, *then either:*

(a) *$\mathcal{D} \setminus \mathcal{M}$ is bounded in $\mathbb{R} \times X$, in which case $\mathcal{D} \setminus \mathcal{M}$ meets $\mathbb{R} \times \{0\}$, or*
(b) *$\mathcal{D} \setminus \mathcal{M}$ is unbounded.*

If (b) *occurs and $\mathcal{D} \setminus \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $\mathcal{D} \setminus \mathcal{M}$ contains* $(\widehat{\lambda}, \infty)$, *where $\widehat{\lambda} \neq \lambda^*$ is a characteristic value of L.*

The global version for bifurcation of mappings in cones is due to Turner (1971) and Dancer (1973).

The proof of Theorem 8.3 leads to a bifurcation theorem for (3) which reveals its deep relation with continuation techniques.

Theorem 8.5. *If $a < b$ are such that the Leray–Schauder indices*

$$\mathrm{ind}_{LS}[I - T(\,\cdot\,, a), 0] \quad \text{and} \quad \mathrm{ind}_{LS}[I - T(\,\cdot\,, b), 0]$$

exist and are different, then $(\lambda^*, 0)$ *is a bifurcation point of* (3) *for some* $\lambda^* \in [a, b]$. *Furthermore, if* $\mathcal{O} \supset \{0\} \times [a, b]$ *is open, then there is a connected set of nontrivial solutions of* (3) *whose closure* C *intersects* $\{0\} \times [a, b]$ *and either* C *is unbounded, intersects* $\partial\mathcal{O}$ *or contains a trivial solution not in* $\{0\} \times [a, b]$.

Many variants, extensions and generalizations of those results have been proved by Alexander–Yorke, Amann, Antman, Berestycki, Dancer, Fitzpatrick–Pejsachowicz, Gęba, Hetzer, Ize, Krawcewicz, Laloux, Magnus, Mawhin, Nussbaum, Schmitt, Smith, Stuart, Toland, Turner, Weistreich and others. For references see [130], [38], [206], [123], [174].

In the late eighties and early nineties, Fitzpatrick and Pejsachowicz [27], [187] consider the class of *quasilinear Fredholm mappings* $f : X \to Y$ between real Banach spaces, introduced in 1972 by Šnirel'man [80] and applied to nonlinear Hilbert problems by Efendiev, which have a representation of the form

$$f(x) = L(x)x + C(x), \quad x \in X.$$

Here $C : X \to Y$ is completely continuous and L is the restriction to X of a continuous map $\overline{L}$ from the complexification $\overline{X}$ of X to the set $\Phi_0(X, Y)$ of the continuous linear Fredholm operators of index zero between X and Y. They prove the following result.

Theorem 8.6. *Let* $f : \mathbb{R} \times X \to Y$ *be quasilinear Fredholm and* $\mathcal{O} \subset \mathbb{R} \times X$ *be open with* $f(\lambda, 0) = 0$ *if* $(\lambda, 0) \in \mathcal{O}$. *Suppose that* $[a, b] \times \{0\} \subset \mathcal{O}$, *that* $L_\lambda := D_x f(\lambda, 0)$ *exists as a Fréchet derivative, uniformly in* λ, *and depends continuously on* λ, *for* $\lambda \in [a, b]$. *Assume that* $L : [a, b] \to \Phi_0(X, Y)$ *has invertible end-points and that* $\sigma(L, [a, b]) = -1$. *Then there is a connected set of nontrivial solutions of* $f(\lambda, x) = 0$ *whose closure* $\mathcal{C}$ *intersects* $[a, b] \times \{0\}$ *and either* $\mathcal{C}$ *is unbounded, intersects* $\partial\mathcal{O}$ *or contains a trivial solution not in* $[a, b] \times \{0\}$.

In this theorem $\sigma(L, [a, b])$ denotes the *parity* [27] of the corresponding admissible path L, which belongs to $\{-1, 1\}$ and is defined by

$$\sigma(L, [a, b]) = \varepsilon(M_a \circ L_a) \cdot \varepsilon(M_b \circ L_b),$$

where $M : [a, b] \to GL(X, Y)$ (a *parametrix* for L) is such that $M_\lambda \circ L_\lambda$ is a linear compact perturbation of identity for each $\lambda \in [a, b]$, and, for $\lambda = a$ or b, $\varepsilon(M_\lambda \circ L_\lambda)$ is defined by $(-1)^{\sigma_\lambda}$, where σ_λ is the sum of the algebraic multiplicities of the negative eigenvalues of $M_\lambda \circ L_\lambda$.

9. *A*-proper mappings and mappings of monotone type

In 1969, motivated by the recent development of the theory of *monotone* and *accretive operators,* Browder and Petryshyn [15] construct a degree theory for a class of mappings between Banach spaces X and Y, that they call *A-proper.*

This requires the existence of a suitable admissible approximation scheme for the real *separable* Banach spaces X and Y. Let $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ be sequences of oriented finite dimensional subspaces such that $\dim X_n = \dim Y_n$ and let W_n be a linear map of Y onto Y_n for each $n \in \mathbb{Z}^+$. The scheme $\Gamma_A = \{X_n, Y_n, W_n\}$ is said to be *admissible* for (X, Y) if $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$ and $\{W_n\}$ is uniformly bounded. An important special case is the *projectionally complete* scheme $\{X_n, Y_n, Q_n\}$ where Q_n is a linear projection of Y onto Y_n such that $Q_n y \to y$ as $n \to \infty$ for each $y \in Y$.

One can now define the concept of *A-proper* map $T : D \subset X \to Y$ with respect to Γ_A. Letting $D_n = D \cap X_n$, such a map T is defined by the properties that $W_n T : D_n \subset X_n \to Y_n$ is continuous and the following condition holds: if $\{x_{n_j}\}$, with $x_{n_j} \in D_{n_j}$, is any bounded sequence such that $W_{n_j}(T(x_{n_j}) - g) \to 0$ for some $g \in Y$, then there exists a subsequence $\{x'_{n_j}\}$ and $x \in D$ such that $x'_{n_j} \to x$ in X and $T(x) = g$. A-proper mappings can be naturally associated to some nonlinear boundary value problems and to various classes of monotone-like operators.

If now $D \subset X$ is a dense linear subspace, $G \subset X$ is open, bounded and such that $G_D = G \cap D \neq \emptyset$, and if $T : \overline{G_D} \to Y$ is A-proper with respect to the complete projectional scheme $\Gamma = \{X_n, Y_n, Q_n\}$ and such that $y \notin T(\partial G \cap D)$, the degree $\operatorname{Deg}[T, G_D, y]$ of T on G_D over y is the subset of $\mathbb{Z}' = \mathbb{Z} \cup \{\infty\} \cup \{-\infty\}$ made of the accumulation points of the set $\{\deg_B[T_n, G_n, Q_n y] : n \geq 1\}$.

Like in Leray–Schauder's approach, an approximation by mappings between spaces of the same finite dimension is used, but the approximation scheme, as in a Galerkin method, is the same for all mappings.

This (multivalued) degree conserves some properties of Leray–Schauder degree, and *A-proper homotopies* can be defined from which follow new continuation theorems for various classes of mappings. Further contributions have been made by Alexander, Dupuis, Fitzpatrick, Krawcewicz, Kröger, Kryszewski, Perne, Milojevič, Nowak, Pascali, Przedradzki, Rothe, Toland, Webb, Weinberg, Werenski, Willem, Wong and others. References can be found in [120], [180], [189], [198].

In the early seventies, single-valued degrees have been defined independently by Skrypnik [110], [118] and Browder [13] for various class of monotone-like operators between a Banach space X and its dual X^*, like the S_+-mappings $T : X \to X^*$ defined by the property that

$$x_n \rightharpoonup x \quad \text{and} \quad \limsup_{n \to \infty} \langle T x_n, x_n - x \rangle \leq 0 \Rightarrow x_n \to x,$$

which contain as special case suitable perturbations of the hemicontinuous strongly monotone mappings. Extensions in various directions have been made by

Berkovits, Browder, Hidirov, Kartsatos, Klimov, Krawcewicz, Mawhin, Milojevič, Mustonen, Pascali, Skrypnik, Willem and others. The corresponding continuation theorems have applications to various types of elliptic, parabolic and hyperbolic equations. See [182], [13], [180] for references.

10. Nonlinear Fredholm mappings between Banach manifolds

It is a classical result of F. Riesz that if $L : X \to X$ is a linear completely continuous operator in the Banach space X, then $F = I - L$ is a linear Fredholm operator of index zero.

Building on some ideas of Smale and his 1965 infinite-dimensional version of Sard's theorem [79], Elworthy and Tromba [25] develope in 1968 a degree theory for some nonlinear proper Fredholm operators of index zero between infinite dimensional Banach manifolds. Recall that a *nonlinear Fredholm operator of index zero* is a C^1 mapping whose differential at each point is a linear Fredholm mapping of index zero. This is the case for a C^1 completely continuous perturbation of identity. A map is *proper* is the inverse image of any compact set is compact.

A pioneering work in this direction was already made in 1936 par Caccioppoli [17], using, in contrast to Smale, Elworthy and Tromba, some finite-dimensional global Lyapunov–Schmidt reduction instead of a Sard-type lemma. Caccioppoli's approach is developed by Sapronov in 1972–73, who treats compact perturbations of nonlinear Fredholm mappings (see the survey [5] and the Voronezh seminars [159], [170], [175], [178], 183] for an accessible reference).

Formally, taking for simplicity M and N as Banach spaces, this degree theory for nonlinear Fredholm operators is based upon the fact that any such operator can be locally written in the form $\Phi = H + C$, where $H : M \to N$ is an homeomorphism and C is completely continuous.

The obstacle in using this degree for multiplicity and bifurcation results is that its absolute value only is invariant under proper suitable homotopies, unless some orientations have been defined for the manifolds and/or the mappings. This is a delicate problem and various contributions in this direction have been obtained by Ovchinnikov, Borisovich, Zvyagin, Sapronov, Isnard, Gęba, Fitzpatrick–Pejsachowicz–Rabier, Kielhofer, Bartsch, Li, Benevieri–Furi and others.

In the useful special case of some compactlike perturbations of *linear* Fredholm mappings of index zero, one can state continuation theorems which have been widely applied. Let X and Z be real normed vector spaces and let the (not necessarily continuous) linear mapping $L : D(L) \subset X \to Z$ be Fredholm of index zero. The set $\mathcal{F}(L)$ of linear continuous mapppings of finite rank $A : X \to Z$ such that $L + A : D(L) \to Z$ is a bijection is non empty. If E is a metric space and $G : E \to Z$ is such that $(L + A)^{-1}G$ is compact on E for some $A \in \mathcal{F}(L)$, the same is true for any $B \in \mathcal{F}(L)$, and G is then called *L-compact* on E. G is called *L-completely continuous* on E if it is compact on each bounded set of E.

Theorem 10.1. *Let $\Omega \subset X \times I$ be an open bounded set, $L : D(L) \subset X \to Z$ be linear Fredholm of index zero, $N : \overline{\Omega} \to Z$ L-compact, and*

$$\Sigma = \{(x, \lambda) \in (D(L) \times I) \cap \overline{\Omega} : Lx = N(x, \lambda)\}.$$

Assume that the following conditions are satisfied:

(i) $\Sigma \cap \partial\Omega \neq \emptyset$ (*a priori estimate*),
(ii) $N(\overline{\Omega}_0 \times \{0\}) \subset Y$, *$Y$ a direct summand of $R(L)$ in Z* (*transversality condition*),
(iii) $\deg_B[N(\cdot, 0)|_{\ker L}, \Omega_0 \cap \ker L, 0] \neq 0$ (*degree condition*).

Then Σ contains a continuum $\mathcal{C}$ along which λ takes all values in I.

This result can be viewed as the natural extension, to the case where ker $L \neq \{0\}$, of the following easy consequence of Corollary 2.2 when ker $L = \{0\}$, to which Theorem 10.1 formally reduces if one adopts the following degree theory in 0-dimensional space: $\deg_B[0, \{0\}, 0] = 1$, $\deg_B[0, \emptyset, 0] = 0$.

Theorem 10.2. *Let L be invertible, $L^{-1}N$ compact on $\overline{\Omega}$, and assume the following conditions hold:*

(i) $\Sigma \cap \partial\Omega \neq \emptyset$,
(ii) $N(\cdot, 0) = 0$,
(iii) $0 \in \Omega_0$.

Then Σ contains a continuum $\mathcal{C} \ni (0, 0)$ along which λ takes all values in I.

Theorem 10.1 can also be stated in the form of an alternative. We restrict to the following simple one of Schaefer type.

Theorem 10.3. *Let $L : D(L) \subset X \to Z$ be linear Fredholm of index zero, $f : X \to Z$ L-completely continuous, $Q : Z \to Z$ a continuous projector such that* $\ker Q = R(L)$. *Assume that there exists $r > 0$ such that the following conditions hold:*

(a) $Qf^{-1}(0) \cap \ker L \subset B(r) \cap \ker L$,
(b) $\deg_B[Qf|_{\ker L}, B(r) \cap \ker L, 0] \neq 0$.

Then either equation $Lx = \lambda f(x)$ has at least one solution for each $\lambda \in [0, 1]$, or the set

$$\{(x, \lambda) \in D(L) \times]0, 1[: Lx = \lambda f(x)\}$$

is unbounded.

Theorems 10.1 and 10.3 can be traced to [56]. Other proofs or variants are due to Furi, Gęba, Granas, Hetzer, Iannacci, Kaczyński, Krawcewicz, Martelli, Pejsachowicz, Pera, Vignoli, Volkmann, Rybakowski, Fečkan, Ward, Erbe, Wu and others. It has received a large number of applications to various boundary value problems for ordinary and partial differential equations, and in particular to problems of the Landesman–Lazer type (see the references in [144], [134], [141], [150], [166], [186], [194]).

11. Conclusion

In 1972, when he was awarded the Feltrinelli Prize, Jean Leray delivered a lecture on *Mathematics and its applications,* in which he said [51]

> Pour beaucoup d'autres problèmes très généraux, des théorèmes d'existence peuvent être établis; bien que l'intuition physique les suggère souvent, leurs preuves emploient des théories diverses et originales: celle des opérateurs, qui doit tant à David Hilbert; celle des contractions, qui remonte à Emile Picard; celle de l'inversion des applications fonctionnelles de votre compatriote R. Caccioppoli; celle des points fixes, que Jules Schauder réussit à appliquer aux espaces fonctionnels. J'aimerais pouvoir détailler le développement de celle-ci: dénombrer les points fixes d'une application d'un espace topologique en lui-même, comme le théorème de d'Alembert dénombre les zéros d'un polynome d'une variable, nécessite l'élaboration de la topologie algébrique des espaces topologiques; il apparaît alors que les propriétés topologiques fondamentales de l'espace euclidien s'étendent aux espaces topologiques; cette extension conduit à l'emploi de nouvelles méthodes: suite spectrale, cohomologie relative à un faisceau, trace généralisée; elles ont permis à divers mathématiciens de développer la théorie des fonctions de plusieurs variables complexes, la géométrie différentielle, la géométrie algébrique, la théorie des points fixes elle-même, enfin la théorie des hyperfonctions, qui prolonge celle des distributions[12].

In some letters he wrote to me later, Jean Leray says what follows:

> Je suis très touché de votre lettre si chaleureuse, rendant à la mémoire de Jules Schauder un hommage tellement mérité et manifestant tant d'enthousiasme pour ce qui est, peut-être, la meilleure part de mes écrits. [...] Je suis très heureux que celle-ci [la théorie du degré topologique] reste si vivante. [...] Je partage votre [...] admiration pour Jules Schauder, votre amour des points fixes[13].

This conclusion should be shared by many mathematicians.

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[12]For many other very general problems, existence theorems can be established; although physical intuition often suggests them, their proof use various and original theories: the theory of operators, which owes so much to David Hilbert; the theory of contractions, which goes back to Emile Picard; the inversion of functional applications of your fellow-countryman R. Caccioppoli; fixed point theory, that Jules Schauder succeeded to apply to function spaces. I would like to be able to describe its development: to count the fixed points of a mapping of a topological space into itself, like d'Alembert's theorem counts the zeros of a polynomial in one variable, requires the elaboration of algebraic topology of topological spaces; it appears then that the fundamental topological properties of Euclidian space can be extended to topological spaces; this extension leads to the use of new methods: spectral sequence, sheaf cohomology, generalized trace; they have allowed various mathematicians to develop the theory of functions of several complex variables, differential geometry, algebraic geometry, fixed point theory itself, and finally the theory of hyperfunctions, which extends the theory of distributions.

[13]I am very touched by your so warm letter, giving to the memory of Jules Schauder a fully deserved homage et expressing so much enthusiasm for what is, maybe, the best part of my writings[...]. I am very happy that the theory of topological degree remains so much alife. [...] I share your [...] admiration for Jules Schauder, your love of fixed points.

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