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FIXED POINT THEOREMS AND FIXED POINT INDEX FOR COUNTABLY CONDENSING MAPS

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Abstract. It is proved that there exists a fixed point index theory for operators which are condensing on the countable subsets of the space only. Even weaker compactness assumptions on countable subsets suffice, e.g. conditions with respect to classes of measures of noncompactness, or if measures of noncompactness of countable noncompact sets are not preserved (not necessarily decreased). As an application, we prove a generalization of the Fredholm alternative.

0. Introduction

The celebrated fixed point theorem of Darbo [9] states that a condensing operator in a Banach space has a fixed point if it maps a nonempty, closed, bounded, and convex set into itself. For applications to e.g. differential equations in Banach spaces it is desirable to have a fixed point theory for operators which are condensing on the countable subsets only. For example, for countably many uniformly bounded measurable functions $x_n : [0,1] \to X$ with a Hilbert space X

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the estimate

$$\chi\bigg(\bigg\{\int_0^1 x_n(s)\,ds : n = 1, 2, \dots\bigg\}\bigg) \le \int_0^1 \chi(\{x_n(s) : n = 1, 2, \dots\})\,ds$$

is known where χ denotes the Hausdorff measure of noncompactness in X (defined below). This was first proved in [28], see also [27]. (Be aware that the proof contains a minor mistake which however can be avoided, see [2], [36]. Different approaches may be found in [21], [25]). In contrast, an estimate as above need not hold for uncountable families of functions: without further assumptions one even runs into measurability problems; consider e.g. the family $x_{n,t} = e_n\chi_{\{t\}}$ indexed by $(n, t) \in \mathbb{N} \times M$ with a nonmeasurable set $M \subseteq [0, 1]$ and an orthonormal sequence $e_n \in X$.

The apparently first fixed point theorem for countably condensing operators was proved in [7] (see also [8]). Multivalued versions can be found in [22], [34]. However, all known fixed point theorems for countably condensing operators have the disadvantage that one has to know a priori a nonempty, closed, bounded and convex set which is mapped into itself. Only in [27] a fixed point theorem has been given which requires only the so-called Leray–Schauder boundary condition (defined below).

In particular, it is yet unknown whether there exists a fixed point index or at least a degree theory for countably condensing maps. It is the aim of this paper to show that such a fixed point index exists. More precisely, we will show that countably condensing maps are fundamentally restrictible (see below) which allows to define a fixed point index. The approach is even possible without further difficulties for multivalued maps in (locally convex) Fréchet spaces.

The plan of this paper is as follows: in Section 1, we define the fixed point index under certain compactness assumptions on countable sets. It will follow almost immediately from the definition that countably condensing operators (which we define in Section 2) satisfy these assumptions. However, the mentioned assumptions are even less restrictive: they lead to fixed point theorems if the operator does e.g. not preserve (not necessarily decrease) the measure of noncompactness. We will discuss such topics in Section 2. There we also lay the fundamentals to a different (more constructive) approach to the fixed point index for a slightly smaller class of maps. In Section 3, we show as an application a generalization of the Fredholm alternative.

1. The fixed point index by ultimately fundamental sets

It is well-known that a definition of a degree theory in infinite-dimensional Banach spaces is not possible without additional compactness assumptions, because all maps are homotopic. In particular, for a homotopy H, the map $H(0, \cdot\,)$ may have precisely one "essential" fixed point although $H(1, \cdot\,)$ has no fixed points. Roughly speaking, H can "loose" the "essential" fixed point by moving it "in infinite dimensions". One might suspect that this cannot happen, if one requires that all fixed points of H stay in some compact (and convex) set U. But this alone is not sufficient: think of a homotopy H with $U = \{x_0\}$; except for the fact that H has no fixed points outside U, we have no control over the behavior of H. In particular, x_0 might be an "essential" fixed point for $H(0, \cdot)$, but inessential for e.g. $H(1-\varepsilon, \cdot)$ and thus it might happen that x_0 is not a fixed point of $H(1, \cdot)$ anymore.

Thus, instead of the compactness of the convex hull of all fixed points of H, one needs the compactness of a "slightly larger" invariant convex set. It appears that the best definition for such a set is the following.

Throughout this paper, let X be a Fréchet space, and $K \subseteq X$ be closed and convex.

Definition 1.1. Let $D \subseteq K$, and $H : [0,1] \times D \to 2^K$. A closed convex set $U \subseteq K$ is called *fundamental* for H, if

(a) $H([0,1] \times (U \cap D)) \subseteq U$, and
(b) $x_0 \in \overline{\mathrm{conv}}(U \cup H([0,1] \times \{x_0\}))$ implies $x_0 \in U$.

The map H is called *fundamentally restrictible*, if $\overline{\mathrm{conv}} H([0,1] \times (U \cap D))$ is compact for some fundamental set U.

For fundamentally restrictible homotopies (more precisely, for the corresponding mappings $H(\lambda, \cdot)$), one may define a fixed point index (and a degree theory). Such a degree based on fundamental sets is given for single-valued maps in [24]. For multivalued maps, the corresponding theory was developed by V. V. Obukhovskiĭ and others, see e.g. the surveys [4]–[6] or the monograph [23].

We now intend to show that under certain compactness assumptions on countable sets on a homotopy H, this homotopy is fundamentally restrictible. It turns out that under reasonable assumptions on H any set U which satisfies the inclusion $U \subseteq \overline{\mathrm{conv}} H([0,1] \times (U \cap D))$ is automatically compact. However, this inclusion is converse to the inclusion in Definition 1.1. For this reason, we are interested in fundamental sets where we even have equality. These are the sets which we get for $V = \emptyset$ in the following definition.

Definition 1.2. Let $D \subseteq K$, and $H : [0,1] \times D \to 2^K$. Given $V \subseteq K$, we call a set $U \subseteq K$ *V-stably fundamental* for H if the following holds:

(a) we have the relation

$$U = \overline{\mathrm{conv}}(H([0,1] \times (U \cap D)) \cup V), \tag{1}$$

(b) $x_0 \in \overline{\mathrm{conv}}(U \cup H([0,1] \times \{x_0\}))$ implies $x_0 \in U$.

Thus, U is V-stably fundamental for H if and only if U is fundamental for H and (1) holds. We will soon see that such sets U always exist. Roughly speaking, the inclusion $H([0,1]\times(U\cap D))\subseteq U$ in the Definition 1.1 of fundamental sets will be needed to prove the existence of a fixed point index, but the converse inclusion $U\subseteq\overline{\mathrm{conv}}(H([0,1]\times(U\cap D))\cup V)$ from (1) will enable us to verify the compactness of such sets U easier.

For historical reasons, we recall a classical "construction" of an $\emptyset$-stably fundamental set. However, we will not make use of that construction, since it requires the axiom of choice: given a multivalued map $H:[0,1]\times D\to 2^K$, define a transfinite sequence of sets $U_\alpha\subseteq X$ by induction:

(a) $U_0=\overline{\mathrm{conv}}H([0,1]\times D)$,
(b) $U_{\alpha+1}=\overline{\mathrm{conv}}H([0,1]\times(U_\alpha\cap D))$,
(c) $U_\alpha=\bigcap_{\beta<\alpha}U_\beta$ if α is a limit ordinal.

A simple transfinite induction shows that U_α is a decreasing sequence of closed convex sets. The axiom of choice implies that the decreasing transfinite sequence U_α must stabilize. The limit set $U=\bigcap U_\alpha$ is called the *ultimate range* of H. It follows from the definition that U satisfies (1) (with $V=\emptyset$). One may verify by transfinite induction that each U_α (and so U) is a fundamental set for H.

Sadovskiĭ [33] (see also the text book [1]) has defined a degree theory for functions which are compact on their ultimate range. However, as we have remarked, it is even possible to define a degree theory for functions which are just fundamentally restrictible. The latter has not only the advantage that this condition is slightly less restrictive, but also the transfinite induction can be avoided in the proofs: in contrast to the above construction, the axiom of choice is not required to prove the existence of a V-stably fundamental set. This was first observed by Obukhovskiĭ (see e.g. [6], [23]).

Proposition 1.1. *For each multivalued map $H:[0,1]\times D\subseteq K\to 2^K$ and each $V\subseteq K$ there exists a V-stably fundamental set. Moreover, there is precisely one V-stably fundamental set U with the additional property that each fundamental set which contains V also contains U.*

Proof. Let $\mathfrak{U}$ be the collection of all fundamental sets for H which contain V. The family $\mathfrak{U}$ is not empty, because it contains K. Now $U=\bigcap\mathfrak{U}$ is the desired set. $\square$

Definition 1.3. We call the set U from Proposition 1.1 the *V-ultimately fundamental set* of H. When $V=\emptyset$, we say that U is the *ultimately fundamental set* of H and denote it by U_∞.

We say that the multivalued map H is *V-fundamentally restrictible* if there is some fundamental set $U\supseteq V$ such that $H([0,1]\times(U\cap D))$ is relatively compact.

Corollary 1.1. *H is V-fundamentally restrictible if and only if its V-ultimately fundamental set is compact. H is fundamentally restrictible if and only if its ultimately fundamental set is compact.*

Proof. If $U \supseteq V$ is fundamental such that $H([0,1] \times (U \cap D))$ is relatively compact, then the V-ultimately fundamental set $U_V \subseteq U$ is relatively compact by $U_V = \overline{\operatorname{conv}} H([0,1] \times (U_V \cap D)) \subseteq \overline{\operatorname{conv}} H([0,1] \times (U \cap D))$. For the second statement observe that by Proposition 1.1 the ultimately fundamental set is contained in U_V for any $V \subseteq K$. □

The crucial observation for us is that to check that H is V-fundamentally restrictible, it suffices to consider countable subsets of its V-ultimately fundamental set which satisfy (1) up to closures.

Theorem 1.1. *Let $D \subseteq K$, and $H : [0,1] \times D \to 2^K$ be upper semicontinuous and such that each value $H(\lambda, x)$ is separable. Let $V \subseteq K$ be separable, and $U \subseteq K$ satisfy* (1)*. Let $G \subseteq D$ be such that*

$$H([0,1] \times (U \cap D)) \subseteq \overline{\operatorname{conv}}(H([0,1] \times (U \cap G)) \cup V). \tag{2}$$

Suppose that for each countable subset $C \subseteq U$ the relations

$$\overline{C} = \overline{\operatorname{conv}}(H([0,1] \times (C \cap G)) \cup V) \tag{3}$$

and

$$G \cap \operatorname{conv}(H([0,1] \times (C \cap G)) \cup V) \subseteq \overline{C \cap G}$$

imply that C is precompact. Then U is compact.

Remark 1.1. Observe that the condition in Theorem 1.1 is trivially necessary for the compactness of U. If U is compact, then each countable subset $C \subseteq U$ is precompact, of course.

For the proof of Theorem 1.1, we need a simple lemma.

Lemma 1.1. *An upper semicontinuous multivalued map F in metric spaces with separable values $F(x)$ maps separable sets into separable sets.*

Proof. Without loss of generality, we may assume that $F : X \to 2^Y$ where X is a separable metric space, and Y is a metric space. Given some n, let U_x be the union of all open balls in Y with radius n^{-1} whose center belongs to $F(x)$. Since X is a separable metric space, its topology has a countable base $V_1, V_2, \ldots$. Let K denote the set of all indices k with the following property: there is some $x \in V_k$ such that $F(V_k) \subseteq U_x$. Since F is upper semicontinuous, each $x \in X$ is contained in some set V_k with this property, and so $X = \bigcup_{k \in K} V_k$. For any $k \in K$ choose some $x_k \in V_k$ with $F(V_k) \subseteq U_{x_k}$. Then $F(X) \subseteq \bigcup U_{x_k}$. Thus, if C_n is a countable and dense subset of $\bigcup F(x_k)$, we find for any $y \in F(X)$ some

$c \in C_n$ with $d(y,c) \leq 2n^{-1}$. Hence, the set $\bigcup C_n$ is a countable and dense subset of $F(X)$. $\square$

Proof of Theorem 1.1. Assume that U is not compact. Then there exists a sequence $x_n \in U$ without a Cauchy subsequence. Put $C_1 = \{x_1, x_2, \dots\}$.

We now define inductively countable sets $C_n \subseteq U$ with the properties:

$$C_n \subseteq C_{n+1}, \tag{5}$$

$$\overline{C}_{n+1} \supseteq \overline{\operatorname{conv}}(H([0,1] \times (C_n \cap G)) \cup V), \tag{6}$$

$$\overline{C_{n+1} \cap G} \supseteq G \cap \overline{\operatorname{conv}}(H([0,1] \times (C_n \cap G)) \cup V), \tag{7}$$

and

$$C_n \subseteq \overline{\operatorname{conv}}(H([0,1] \times (C_{n+1} \cap G)) \cup V). \tag{8}$$

This is indeed possible. If C_n is already defined, we have by (1) and (2) that $C_n \subseteq U = \overline{\operatorname{conv}}(H([0,1] \times (U \cap G)) \cup V)$. In particular, each $x \in C_n$ is the limit of a sequence of (finite) convex combinations of points from $H([0,1] \times (U \cap G)) \cup V$. We thus find a countable set $A_n \subseteq U \cap G$ with $C_n \subseteq \overline{\operatorname{conv}}(H([0,1] \times A_n) \cup V)$. In particular, any set $C_{n+1} \supseteq A_n$ satisfies (8). To fulfill also the other requirements, observe that $H([0,1] \times (C_n \cap G))$ is separable by Lemma 1.1. Thus, the sets $H_n = \overline{\operatorname{conv}}(H([0,1] \times (C_n \cap G)) \cup V)$ and $G \cap H_n$ are separable. Let $B_n \subseteq H_n$ be countable and dense in H_n, and $D_n \subseteq G \cap H_n$ be countable and dense in $G \cap H_n$. Then we may choose $C_{n+1} = A_n \cup B_n \cup C_n \cup D_n$. Observe that $C_{n+1} \subseteq U$, because $B_n \cup D_n \subseteq H_n \subseteq \overline{\operatorname{conv}}(H([0,1] \times (U \cap D)) \cup V) = U$ by (1).

Consider now the set $C = \bigcup C_n$. By (8), we have $C_n \subseteq \overline{\operatorname{conv}}(H([0,1] \times (C \cap G)) \cup V)$ for each n, and so

$$C \subseteq \overline{\operatorname{conv}}(H([0,1] \times (C \cap G)) \cup V). \tag{9}$$

Conversely, if $x \in \operatorname{conv}(H([0,1] \times (C \cap G)) \cup V)$, then x is the convex combination of finitely many points from $H([0,1] \times (C \cap G)) \cup V = \bigcup H([0,1] \times (C_n \cap G)) \cup V$. In view of (5), we find some index n such that x is the convex combination of finitely many points from $H([0,1] \times (C_n \cap G)) \cup V$, and so (6) implies $x \in \operatorname{conv}(H([0,1] \times (C_n \cap G)) \cup V) \subseteq \overline{C}$. Moreover, if additionally $x \in G$, we have by (7) that $x \in G \cap \operatorname{conv}(H([0,1] \times (C_n \cap G)) \cup V) \subseteq \overline{C \cap G}$. We thus have proved that C satisfies

$$\operatorname{conv}(H([0,1] \times (C \cap G)) \cup V) \subseteq \overline{C} \tag{10}$$

and (4). Taking the closures of both sides of the inclusions (9) and (10), we find that C also satisfies (3). Hence, the assumptions of the theorem imply that C is precompact which contradicts the fact that $C_1 \subseteq C$ contains a sequence x_n without a Cauchy subsequence. $\square$

Note that the assumption $U \subseteq \overline{\mathrm{conv}} H([0,1] \times (U \cap D))$ is essential in the above proof for the definition of the set A_n.

For the natural situation that D is closed, H takes compact values, and V is precompact, we may formulate the conditions in Theorem 1.1 in a way which is much easier to verify:

LEMMA 1.2. *Let X be a Fréchet space, $K \subseteq X$ closed and convex, $D \subseteq K$ closed, and $H : [0,1] \times D \to 2^K$ be upper semicontinuous with compact values $H(\lambda, x)$. Let $V \subseteq K$ be precompact. Then for any set $G \subseteq D$ and any $U \subseteq K$ with*

$$\mathrm{conv}(H([0,1] \times (G \cap U)) \cup V) \subseteq U \tag{11}$$

the following statements are equivalent:

(a) *Each countable set $C \subseteq U$ which satisfies (3) and (4) is precompact.*
(b) *Each countable set $C \subseteq U$ which satisfies (3) and (4) has the property that $C \cap G$ is precompact.*
(c) *Each countable set $C \subseteq G \cap U$ which satisfies*

$$G \cap \mathrm{conv}(H([0,1] \times C) \cup V) \subseteq \overline{C} \subseteq \overline{G} \cap \overline{\mathrm{conv}}(H([0,1] \times C) \cup V) \tag{12}$$

is precompact.
(d) (*If G is open in K*) *Each countable set $C \subseteq G \cap U$ which satisfies*

$$\overline{C} = \overline{G \cap \mathrm{conv}(H([0,1] \times C) \cup V)} \tag{13}$$

is precompact.

PROOF. The equivalence of (a) and (b) follows from the fact that H maps compact sets into compact sets. If C satisfies (3), and $C \cap G$ is precompact, then $\overline{C \cap G}$ is a compact subset of D, and so $C \subseteq \overline{\mathrm{conv}}(H([0,1] \times (\overline{C \cap G})) \cup V)$ is contained in a compact set.

Let (c) be satisfied. If $C \subseteq U$ is countable and (3) and (4) hold, then $C_0 = C \cap G$ satisfies

$$G \cap \mathrm{conv}(H([0,1] \times C_0) \cup V) \subseteq \overline{C}_0 \subseteq \overline{G} \cap \overline{C} = \overline{G} \cap \overline{\mathrm{conv}}(H([0,1] \times C_0) \cup V).$$

Hence, condition (c) implies that C_0 is compact, and so condition (b) holds.

Now, let condition (a) be satisfied. Let $C \subseteq G \cap U$ be countable and satisfy (12). By Lemma 1.1, the set $M = \mathrm{conv}(H([0,1] \times C) \cup V)$ is separable. Let $A \subseteq M \setminus G$ be countable and dense in $M \setminus G$, and put $C_0 = A \cup C$. Then $\overline{C}_0 = (\overline{M \setminus G}) \cup \overline{C}$ implies in view of (12) that

$$M \subseteq (\overline{M \setminus G}) \cup (G \cap M) \subseteq \overline{C}_0 \subseteq (\overline{M \setminus G}) \cup (\overline{G} \cap \overline{M}) \subseteq \overline{M},$$

and so we have $\overline{C}_0 = \overline{M}$ which in view of $C_0 \cap G = C$ means that

$$\overline{C}_0 = \overline{\operatorname{conv}}(H([0,1] \times (C_0 \cap G)) \cup V).$$

The relations $C_0 \cap G = C$ and (12) imply also

$$G \cap \operatorname{conv}(H([0,1] \times (C_0 \cap G)) \cup V) = G \cap M \subseteq \overline{C} = \overline{C_0 \cap G}.$$

We have $C_0 \subseteq U$, because (11) implies $A \subseteq M \subseteq U$. By condition (a), the set C_0 thus is precompact, and so the subset $C \subseteq C_0$ is also precompact.

Since condition (c) evidently implies (d), it only remains to show that condition (d) is sufficient, if G is open in K. But if G is open in K, we have

$$\overline{\overline{A} \cap G} = \overline{A \cap G} \qquad (A \subseteq K). \tag{14}$$

Indeed, since $A \subseteq (A \cap G) \cup (K \setminus G)$ and $K \setminus G$ is closed, we have $\overline{A} \subseteq (\overline{A \cap G}) \cup (K \setminus G)$, and so $\overline{A} \cap G \subseteq \overline{A \cap G}$ which implies (14).

By (14) we have in particular, that if $C \subseteq U$ is countable and satisfies (3), then $C_0 = C \cap G$ satisfies

$$\begin{aligned}\overline{C_0} = \overline{C \cap G} &= \overline{\overline{C} \cap G} = \overline{\overline{\operatorname{conv}}(H([0,1] \times C_0) \cup V) \cap G} \\ &= \overline{\operatorname{conv}(H([0,1] \times C_0) \cup V) \cap G}.\end{aligned}$$

By condition (d), the set $C \cap G = C_0$ is compact. Hence, condition (d) implies that condition (b) is satisfied. □

Now we can summarize the main result of this section.

Theorem 1.2. *Let X be a Fréchet space, $K \subseteq X$ closed and convex, $D \subseteq K$ be closed, and $H : [0,1] \times D \to 2^K$ be upper semicontinuous with compact values $H(\lambda, x)$. Let $V \subseteq K$ be precompact, and $U \subseteq K$ be the corresponding V-ultimately fundamental set. Let $G \subseteq D$ be such that* (2) *holds. Then the following statements are equivalent:*

(a) *H is V-fundamentally restrictible.*
(b) *U is compact.*
(c) *Each countable set $C \subseteq U$ which satisfies* (3) *and* (4) *is precompact.*
(d) *Each countable set $C \subseteq U$ which satisfies* (3) *and* (4) *has the property that $C \cap G$ is precompact.*
(e) *Each countable set $C \subseteq G \cap U$ which satisfies* (12) *is precompact.*
(f) (*If G is open in K*) *Each countable set $C \subseteq G \cap U$ which satisfies* (13) *is precompact.*

Proof. The equivalence of the first three statements follows from Corollary 1.1 and Theorem 1.1. The equivalence of the last four statements follows from Lemma 1.2. □

We will see in Section 2 that the equivalent statements in Theorem 1.1 are all satisfied for countable condensing homotopies H.

The definition of the fixed point index for fundamentally restrictible maps is based on a result on the extension of continuous functions in Fréchet spaces. More precisely, we need a special case of Dugundji's extension theorem [12] (see also [13, Chapter IX, Theorem 6.1]). However, Dugundji's extension theorem relies essentially on the axiom of choice. But one may give a simpler and more constructive proof without the (uncountable) axiom of choice, if one supposes a separability assumption which is satisfied for our applications. The construction of the following proof is well-known in finite-dimensional spaces for maps with bounded images (see e.g. [11, Proposition 1.1]). However, it seems that it has never been explicitly carried out in Fréchet spaces and without any boundedness assumption. So let us provide some details.

Lemma 1.3. *Let X be a metric space, Y be a Fréchet space, $D \subseteq X$ be closed and separable, and $f : D \to Y$ be continuous. Then there exists an extension of f to a continuous map $F : X \to \overline{\mathrm{conv}}(f(D))$.*

Proof. Let the metric in Y be generated by the countable family $\|\cdot\|_k$ of seminorms. Recall that a sequence converges (resp. is bounded or a Cauchy sequence) in Y if and only if it converges (resp. is bounded or a Cauchy sequence) with respect to each seminorm $\|\cdot\|_k$.

Since D is separable, there exists a dense subset $\{d_1, d_2, \dots\} \subseteq D$. Choose a sequence of numbers $a_n > 0$ such that $\sum a_n \max(\{1\} \cup \{\|f(d_k)\|_k : k = 1, \dots, n\})$ converges. For $x \in D$, put $F(x) = f(x)$, and for $x \notin D$, put $\lambda_n(x) = \max\{2 - d(x, d_n)/\mathrm{dist}(x, D), 0\}$ and

$$F(x) = \frac{\sum_{n=1}^{\infty} a_n \lambda_n(x) f(d_n)}{\sum_{n=1}^{\infty} a_n \lambda_n(x)}.$$

Observe that we do not divide by 0 since $\{d_1, d_2, \dots\}$ is dense in D. Moreover, the series converges in Y, since the partial sums form a Cauchy sequence with respect to each seminorm $\|\cdot\|_k$ (since λ_n is bounded by 2, and $\sum a_n \|f(d_n)\|_k$ converges). Let us prove now that F is continuous at each $x_0 \in X$, i.e. that for each $\varepsilon > 0$ and each k we find some $\delta > 0$ such that $d(x, x_0) \le \delta$ implies $\|F(x) - F(x_0)\|_k \le \varepsilon$.

In case $x_0 \notin D$, this follows immediately from the continuity of the functions λ_n, the uniform boundedness of the sequence $\lambda_n(x)$, and the fact that $\sum a_n$ and $\sum a_n \|f(d_n)\|_k$ converge.

Since the continuity of F at interior points x_0 of D is trivial, it remains to consider the case $x_0 \in \partial D$. Thus, let $\varepsilon > 0$ and some k be given. By the continuity of f, we find some $\delta > 0$ such that $d_0 \in D$ and $d(x_0, d_0) \le 3\delta$ implies $\|f(d_0) - f(x_0)\|_k \le \varepsilon$.

If $x \in X \setminus D$ and n are such that $\lambda_n(x) \neq 0$, then $d(x, d_n) \leq 2\mathrm{dist}(x, D) \leq 2d(x, x_0)$, and so $d(x_0, d_n) \leq d(x_0, x) + d(x, d_n) \leq 3d(x, x_0)$. Hence, we have for each $x \in X \setminus D$ with $d(x, x_0) \leq \delta$ that for each n the estimate

$$\lambda_n(x)\|f(d_n) - f(x_0)\|_k \leq \lambda_n(x)\varepsilon$$

holds, and so

$$\|F(x) - F(x_0)\|_k = \left\| \frac{\sum_{n=1}^{\infty} a_n \lambda_n(x)(f(d_n) - f(x_0))}{\sum_{n=1}^{\infty} a_n \lambda_n(x)} \right\|_k \leq \varepsilon.$$

Since also $\|F(x) - F(x_0)\|_k \leq \varepsilon$ for each $x \in D$ with $d(x, x_0) \leq \delta$, we may conclude that F is continuous at x_0. □

Let us now recall the definition of the fixed point index for fundamentally restrictible maps. For simplicity, we will restrict ourselves to the case of maps with convex values (but we indicate at the end of Section 2 how a generalization is possible).

Definition 1.4. Let X be a Fréchet space, $K \subseteq X$ closed and convex, and $\Omega \subseteq K$ be open in K. We call a multivalued map $H : [0,1] \times \overline{\Omega} \to 2^K \setminus \emptyset$ an *admissible homotopy*, if the following holds:

(a) H is upper semicontinuous, and each value $H(\lambda, x)$ is convex and compact.
(b) For each $x \in \partial\Omega := \overline{\Omega} \setminus \Omega$ we have $x \notin H([0,1] \times \{x\})$.
(c) H is fundamentally restrictible on $\overline{\Omega}$. By our above results, it is equivalent to require that there is some precompact $V \subseteq K$ with the following properties: If $C \subseteq \overline{\Omega}$ is countable and contained in the V-ultimately fundamental set of H, then the relation

$$\overline{\Omega} \cap \mathrm{conv}(H([0,1] \times C) \cup V) \subseteq \overline{C} \subseteq \overline{\Omega} \cap \overline{\mathrm{conv}}(H([0,1] \times C) \cup V) \tag{15}$$

implies that C is precompact.

If $F : \overline{\Omega} \to 2^K \setminus \emptyset$ is such that the constant homotopy $H(\lambda, x) = F(x)$ $(0 \leq \lambda \leq 1)$ is admissible, we call the pair (F, Ω) *admissible*.

The definition immediately implies:

Proposition 1.2. *If H is an admissible homotopy, and $\lambda : [0,1] \to [0,1]$ is continuous, then $\widetilde{H}(t, x) = H(\lambda(t), x)$ is an admissible homotopy. In particular, each of the pairs $(H(\lambda, \cdot), \Omega)$ is admissible.*

The definition of the fixed point index is as follows: let (F, Ω) be an admissible pair, and U_∞ denote the compact ultimately fundamental set of the constant homotopy $H(\lambda, x) = F(x)$. If $U_\infty = \emptyset$, then F has no fixed points, and we put $\mathrm{ind}_K(F, \Omega) = 0$. Otherwise, let U be a convex and compact set containing U_∞

such that $F(U \cap \overline{\Omega}) \subseteq U$ (for example, one may put $U = U_\infty$). Let R be some retraction from X onto U (such a retraction exists by Lemma 1.3). Then we define

$$\mathrm{ind}_K(F,\Omega) = \deg(\mathrm{id} - FR, R^{-1}(\Omega), 0),$$

where the right-hand side denotes the degree for multivalued upper semicontinuous compact maps with convex values. That this definition is independent of the particular choice of U and R, can be proved analogously to [15, Proposition 2.1]. Observe that the proof of this fact (see also [31, Lemma 2.2]) uses the convexity of the values of H.

Actually, the proofs in the cited papers [15], [31] use the ultimate range in place of the ultimately fundamental set, but an inspection of the proofs shows that the above claim is true. Moreover, we get analogously to [15, Theorem 2.1 and Remark 2.1] the following Theorem 1.3.

We remark that an alternative (although similar and actually equivalent) definition of the fixed point index can be found in [23] (in that monograph only Banach spaces are considered but the results which are essential for us hold also for Fréchet spaces). From that reference, we also take the restriction property:

Theorem 1.3. *Let X be a Fréchet space, $K \subseteq X$ closed and convex, and (F,Ω) be admissible. Then* $\mathrm{ind}_K(F,\Omega)$ *has the following properties:*

(a) (Fixed point property) *If* $\mathrm{ind}_K(F,\Omega) \neq 0$*, then F has a fixed point $x \in \Omega$, i.e. $x \in F(x)$.*
(b) (Homotopy invariance) *If H is an admissible homotopy, then*

$$\mathrm{ind}_K(H(0,\,\cdot\,),\Omega) = \mathrm{ind}_K(H(1,\,\cdot\,),\Omega).$$

(c) (Additivity) *If $\Omega_1, \Omega_2 \subseteq \Omega$ are disjoint and open in K such that $\Omega_1 \cup \Omega_2$ contains all fixed points of F in $\overline{\Omega}$, then*

$$\mathrm{ind}_K(F,\Omega) = \mathrm{ind}_K(F,\Omega_1) + \mathrm{ind}_K(F,\Omega_2).$$

(d) (Normalization) $\mathrm{ind}_K(F_0,\Omega) = 1$ *if* $F_0(x) \equiv \{x_0\} \subseteq \Omega$.
(e) (Restriction) *If $K_0 \subseteq K$ is closed and convex with $F(\overline{\Omega}) \subseteq K_0$, then*

$$\mathrm{ind}_K(F,\Omega) = \mathrm{ind}_{K_0}(F,\Omega \cap K_0).$$

The additivity in [15] is formulated less generally than in Theorem 1.3, but the proof shows (see the proof of [31, Theorem 2.3]) that the above formulation is correct (even a more general result holds, see [23]).

A generalization of Borsuk's theorem for multivalued fundamentally restrictible maps can be found in [6, Theorem 2.3.16]. We mention only the following special case of that result:

THEOREM 1.4. *Let X be a Fréchet space, $K = X$, and (F,Ω) be admissible. If Ω is a symmetric neighbourhood of* 0 *and F is odd with $0 \in F(0)$, then* $\operatorname{ind}_X(F,\Omega)$ *is odd.*

PROOF. By [6, Theorem 2.3.16], we only have to check that there is a symmetric fundamental set U with $U \cap \Omega \neq \emptyset$ and such that $F(U \cap \overline{\Omega})$ is relatively compact. It thus suffices to prove that the ultimate fundamental set U_∞ is nonempty and symmetric. But since 0 is a fixed point by assumption, we have $0 \in U_\infty$. Moreover, since F is odd also $U_0 = U_\infty \cap (-U_\infty)$ is fundamental and so $U_\infty \subseteq U_0$. This implies $U_\infty = U_0$, i.e. U_∞ is symmetric. □

The unsymmetry in the relation (15) is rather dissatisfying. It would be much more natural to consider (13) with $G = \Omega$ instead. However, for the choice $G = \Omega$, the relation (2) need not hold, which is required for Theorem 1.2. Recall that fundamental sets U have to satisfy the inclusion $H([0,1] \times (U \cap \overline{\Omega})) \subseteq U$. Let us for a moment call the set U *weakly fundamental*, if we require only

$$H([0,1] \times (\overline{U \cap \Omega})) \subseteq U$$

instead. Using a similar reasoning as before, we can prove that under an assumption of the type (13) with $G = \Omega$, the map H has a weakly fundamental compact set. However, we do not know whether the latter is sufficient to define a fixed point index. At least, this property is sufficient to prove the homotopy invariance of certain essential fixed points. We employ this idea in a more general setting in the forthcoming paper [35].

2. The fixed point index for countably condensing maps

We now formulate special cases of the results in Section 1 for countably condensing maps. We use the following definitions which are similar to those from [1] (see also [33]).

DEFINITION 2.1. Let X be a Fréchet space, and $K \subseteq X$ be closed and convex. A *measure of noncompactness* on K is a map γ from the system of bounded subsets of K into a partially ordered set which satisfies

$$\gamma(M) = \gamma(\overline{\operatorname{conv}} M).$$

Such a function γ is called

(a) *monotone*, if $M \subseteq N$ implies $\gamma(M) \leq \gamma(N)$ (for bounded $N \subseteq K$),
(b) *V-stable* (for fixed bounded $V \subseteq K$), if $\gamma(M \cup V) = \gamma(M)$ for each bounded set $M \subseteq K$.

If $D \subseteq K$, we call a function $F : D \to 2^K$ *condensing* on D (with respect to γ), if the relation

$$\gamma(F(C)) \not\geq \gamma(C) \qquad (16)$$

holds for each bounded but not precompact subset $C \subseteq D$ (in particular, $F(C)$ is bounded for such sets C). We call a homotopy $H : [0,1] \times D \to 2^K$ *condensing* on D, if

$$\gamma(H([0,1] \times C)) \not\geq \gamma(C) \tag{17}$$

holds for each bounded but not precompact subset $C \subseteq D$. We call F (resp. H) *countably condensing* on D, if (16) (resp. (17)) holds for each subset $C \subseteq D$ which is countable and bounded but not precompact. Moreover, we say that F (resp. H) is (*countably*) *condensing with respect to a class* Γ of measures of noncompactness, if for each (countable) bounded but not precompact subset $C \subseteq D$ one can find some $\gamma \in \Gamma$ with (16) (resp. (17)).

In particular, each operator which is condensing with respect to some γ is condensing with respect any class $\Gamma \supseteq \{\gamma\}$. Typical examples for γ are:

(a) The Hausdorff measure $\gamma = \chi_A$ of noncompactness for some $A \supseteq K$: $\chi_A(M)$ is the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in A.
(b) The Kuratowski measure $\gamma = \alpha$ of noncompactness: $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that M has a finite covering of sets with diameter smaller than ε.

χ_A and α are monotone and V-stable for each precompact $V \subseteq X$.

In Fréchet spaces, it is important that we allow γ to take also values in sets $R \neq [0,\infty)$: for example if the metric in X is generated by the countable family of seminorms $\|\cdot\|_k$, one may define $\gamma(M) = (\alpha_1(M), \alpha_2(M), \dots)$ where $\alpha_k(M)$ denotes the Kuratowski measure of noncompactness with respect to the seminorm $\|\cdot\|_k$. This choice is more natural (and usually provides better results) than the Kuratowski measure of noncompactness with respect to the metric. For more details we refer to [33].

Theorem 1.2 immediately implies:

Corollary 2.1. *Assume that $D \subseteq K$ is bounded and closed, and $H : [0,1] \times D \to 2^K$ is upper semi-continuous and takes compact values. If $V \subseteq K$ is precompact and H is countably condensing on D with respect to a class Γ of monotone V-stable measures of noncompactness on K, then H is V-fundamentally restrictible.*

In particular, if H is countably condensing on D with respect to a class of monotone measures of noncompactness on K, then H is fundamentally restrictible.

Proof. Let a countable set $C \subseteq D$ be given which is not precompact. Since C is bounded (because D is bounded), we must have (17) for some $\gamma \in \Gamma$. But if C satisfied (12), we would have $\gamma(C) \leq \gamma(H([0,1] \times C) \cup V) = \gamma(H([0,1] \times C))$, a contradiction. $\square$

Theorem 1.3 implies the following fixed point theorem.

THEOREM 2.1. *Let X be a Fréchet space, $K \subseteq X$ be closed and convex, and $\Omega \subseteq K$ be nonempty and open in K. Assume that there exists an upper semicontinuous homotopy $H : [0,1] \times \overline{\Omega} \to 2^K \setminus \emptyset$ with convex and compact values such that the following holds:*

(a) *The range of $H(0, \cdot)$ is contained in a compact and convex set $V_0 \subseteq \overline{\Omega}$.*
(b) *None of the mappings $H(\lambda, \cdot)$ $(0 \leq \lambda \leq 1)$ has a fixed point in $\partial\Omega := \overline{\Omega} \setminus \Omega$.*
(c) *H is fundamentally restrictible, i.e. there is a precompact $V \subseteq K$ with the following property: any countable $C \subseteq \overline{\Omega}$ which is contained in the V-ultimately fundamental set of H and satisfies* (15) *is precompact.*

    This condition is satisfied if Ω is bounded and H is countably condensing on $\overline{\Omega}$ with respect to a class of monotone measures of noncompactness on K.

Then $\operatorname{ind}_K(H(1, \cdot), \Omega) = 1$*, and $H(1, \cdot)$ has a fixed point in Ω.*

PROOF. H is an admissible homotopy. Put $F = H(0, \cdot)$. Then (F, Ω) is admissible. The homotopy invariance and the restriction property of the fixed point index imply

$$\operatorname{ind}_K(H(1, \cdot), \Omega) = \operatorname{ind}_K(F, \Omega) = \operatorname{ind}_{V_0}(F, \Omega \cap V_0).$$

Let $\Omega_0 \subseteq K$ be open in K with $V_0 \subseteq \Omega_0$, and R be a retraction of $\overline{\Omega}_0$ onto V_0. Then (FR, Ω_0) is admissible, and the restriction property of the fixed point index gives

$$\operatorname{ind}_K(FR, \Omega_0) = \operatorname{ind}_{V_0}(FR, \Omega_0 \cap V_0) = \operatorname{ind}_{V_0}(F, \Omega_0 \cap V_0) = \operatorname{ind}_{V_0}(F, \Omega \cap V_0).$$

For the last equality we have used the additivity of the fixed point index, since by assumption, all fixed points of F are contained in $\Omega \cap V_0$. Fix some $x_0 \in V_0$. Since V_0 is convex, the homotopy $H_0(\lambda, x) = \lambda x_0 + (1 - \lambda) FR(x)$ takes values in the compact set $V_0 \subseteq \Omega_0$ and thus is admissible on Ω_0. The homotopy invariance and normalization of the fixed point index thus imply

$$\operatorname{ind}_K(FR, \Omega_0) = \operatorname{ind}_K(H_0(1, \cdot), \Omega_0) = 1.$$

Combining the above formulas, the statement follows. □

We note that the proof of the previous result could be simplified, if we assumed the slightly more restrictive condition that $V_0 \subseteq \Omega$. Then the homotopy H_0 in the proof is even admissible on Ω, and we do not have to pass to a larger set Ω_0.

Observe that V need not be closed or convex, and even the choice $V = \emptyset$ is allowed.

As a consequence of Theorem 2.1, we get the following multivalued variant of the main fixed point theorem from [27] (see also [11, Theorem 18.1]):

COROLLARY 2.2. *Let X be a Fréchet space, $K \subseteq X$ closed and convex, and $\Omega \subseteq K$ be nonempty and open in K. Let $F : \overline{\Omega} \to 2^K \setminus \emptyset$ be upper semicontinuous with convex and compact values. Assume there is some $x_0 \in \Omega$ with the following properties:*

(a) *The Leray–Schauder boundary condition holds on $\partial\Omega := \overline{\Omega} \setminus \Omega$:*

$$F(x) - x_0 \not\ni \lambda(x - x_0) \quad (x \in \partial\Omega,\ \lambda \geq 1).$$

(b) *If $C \subseteq \overline{\Omega}$ is countable and satisfies*

$$\overline{\Omega} \cap \operatorname{conv}(F(C) \cup \{x_0\}) \subseteq \overline{C} \subseteq \overline{\Omega} \cap \overline{\operatorname{conv}}(F(C) \cup \{x_0\}), \tag{18}$$

then C is precompact.

This holds if Ω is bounded and F is countably condensing on $\overline{\Omega}$ with respect to a class of monotone $\{x_0\}$-stable measures of noncompactness on K.

Then $\operatorname{ind}_K(F, \Omega) = 1$*, and F has a fixed point in Ω.*

PROOF. Put $V = \{x_0\}$, and $H(\lambda, x) = \lambda F(x) + (1 - \lambda)x_0$ in Theorem 2.1, and observe that $\operatorname{conv} H([0, 1] \times C) = \operatorname{conv}(F(C) \cup \{x_0\})$ for each $C \subseteq \overline{\Omega}$. □

Observe that for the choice $\Omega = K$ the Leray–Schauder boundary condition is satisfied by definition. In this case, F maps the nonempty, closed, bounded, and convex set $\Omega = K$ into itself, and so Corollary 2.2 contains Darbo's fixed point theorem as a special case.

But if we have such an invariant set $\Omega = K$, Corollary 2.2 implies even more: if F is not condensing but, quite the opposite, if it is "expanding", we also have a fixed point theorem. F may even be condensing on some sets and expanding on others:

DEFINITION 2.2. Let $D \subseteq K$. We say that a map $F : D \to 2^K$ is *countably unpreserving* with respect to a class Γ of measures of noncompactness on K, if for each countable bounded but not precompact $C \subseteq D$ with bounded image $F(C)$, there is some $\gamma \in \Gamma$ such that

$$\gamma(F(C)) \neq \gamma(C).$$

COROLLARY 2.3. *Let X be a Fréchet space, $K \subseteq X$ nonempty, closed, convex, and bounded, and $F : K \to 2^K \setminus \emptyset$ be upper semi-continuous with convex and compact values. Fix some $x_0 \in K$. If F is countably unpreserving on K*

with respect to a class Γ *of* $\{x_0\}$*-stable measures of noncompactness on* K*, then* $\mathrm{ind}_K(F,K)=1$*, and* F *has a fixed point.*

Proof. Apply Corollary 2.2 with $\Omega = K$: if $C \subseteq K$ is countable and satisfies (18), then the sets C and $F(C)\cup\{x_0\}$ have the same convex hull. Since C is bounded, we find $\gamma(C)=\gamma(F(C))$ for each $\gamma\in\Gamma$ which implies that C is precompact. □

Recall the induction process used to define the ultimate range of a homotopy H. Let us stop this induction process already at the first limit ordinal ω $(=\mathbb{N})$: we have $U_0 = \overline{\mathrm{conv}}H([0,1]\times D)$, $U_{n+1} = \overline{\mathrm{conv}}H([0,1]\times(U_n\cap D))$, and $U_\omega = \bigcap_{n=0}^\infty U_n$. Since U_ω is a fundamental set for H, the compactness of U_ω allows to define the fixed point index (this is the first approach to the fixed point index for noncompact maps and was developed in [29]).

We now present a result which shows that U_ω is compact for countable condensing homotopies. The approach from Section 1 can not be used to this end, since it is not clear whether U_ω also satisfies the converse inclusion $U_\omega \subseteq \overline{\mathrm{conv}}H([0,1]\times(U_\omega\cap D))$. We need some auxiliary definitions:

Definition 2.3. We call a measure γ of noncompactness on K

(a) *finitely stable*, if $\gamma(M\cup\{x\})=\gamma(M)$ (i.e. if γ is V-stable for any finite set $V\subseteq X$),
(b) *almost lower semicontinuous*, if for any bounded set $M\subseteq K$ the following holds: if $B\subseteq K$ is bounded and $M_n\subseteq B$ satisfy

$$\lim_{n\to\infty}\sup_{x\in M}\mathrm{dist}(x,M_n)=0,$$

and if $c=\gamma(M_n)$ is independent of n, then $\gamma(M)\le c$.

The Hausdorff and Kuratowski measures of noncompactness are even *lower semicontinuous* in the sense that one may drop the additional assumption $c=\gamma(M_n)$ and may conclude instead $\gamma(M)\le\sup_n\gamma(M_n)$. Observe that for general measures of noncompactness not even the existence of a supremum is trivial.

Recall that a partially ordered set R is called *super Dedekind complete*, if each nonempty order bounded from above subset $M\subseteq R$ has a supremum s and a countable subset $M_0\subseteq M$ with $s=\sup M_0$.

Theorem 2.2. *Let* D *be bounded, and* $H:[0,1]\times D\to 2^K$ *have the property that* $H([0,1]\times A)$ *is precompact for any precompact* $A\subseteq D$*. Assume that* H *is countably condensing with respect to a countable family* Γ *of monotone, finitely stable, and almost lower semicontinuous measures of noncompactness on* K*. Assume that each* $\gamma\in\Gamma$ *takes its values in a super Dedekind complete set* (*which* *may depend on γ). Then each set $M \subseteq X$ with the property that $M \setminus U_n$ is finite for each n is precompact. In particular, U_ω is compact.*

Proof. Let $\mathcal{F}$ denote the family of all sets $M \subseteq X$ such that each of the sets $M \setminus U_n$ is finite, and $\mathcal{F}_D$ denote the family of all countable $C \in \mathcal{F}$ with $C \subseteq D$.

Step 1. Let us first show that there is some $B \in \mathcal{F}_D$ with $\gamma(B) \geq \gamma(C)$ for each $C \in \mathcal{F}_D$ and each $\gamma \in \Gamma$:

Let $\Gamma = \{\gamma_1, \gamma_2, \dots\}$. Since γ_n is monotone, and $\bigcup \mathcal{F}_D \subseteq D$, the set $M_n = \{\gamma_n(C) : C \in \mathcal{F}_D\}$ is order bounded from above by $\gamma_n(D)$. Consequently, $s_n = \sup M_n$ exists, and there is a countable subset $\{C_{1,n}, C_{2,n}, \dots\} \subseteq \mathcal{F}_D$ with $s_n = \sup_k \gamma_n(C_{k,n})$. Put $C_m = \bigcup_{k,n \leq m} C_{k,n}$, and $B_m = C_m \cap U_m$. Since $C_{k,n} \in \mathcal{F}_D$, the set $C_m \setminus U_m$ is finite, and so $\gamma(C_m) = \gamma(B_m \cup (C_m \setminus U_m)) = \gamma(B_m)$ for each $\gamma \in \Gamma$ and each m, since γ is finitely stable. The set $B = \bigcup B_m$ belongs to $\mathcal{F}_D$, because for $m \geq n$ we have $U_m \subseteq U_n$, and so the set

$$B \setminus U_n \subseteq \bigcup_{m=1}^{n-1} (C_m \setminus U_n) \cup \bigcup_{m=n}^{\infty} (B_m \setminus U_n) = \bigcup_{m=1}^{n-1} (C_m \setminus U_n)$$

is finite for each n. For each n and k, we have $\gamma_n(B) \geq \gamma_n(B_{\max\{k,n\}}) = \gamma_n(C_{\max\{k,n\}}) \geq \gamma_n(C_{k,n})$, i.e. $\gamma_n(B) \geq \sup_k \gamma_n(C_{k,n}) = s_n$, and so $\gamma_n(B) = \max M_n$.

Step 2. We show now the following: let $M \in \mathcal{F}$, and $x_n \in M$ such that no element in the sequence occurs infinitely many often. Then there is some $A \in \mathcal{F}_D$ and a sequence $y_n \in \operatorname{conv}(H([0,1] \times A))$ with $d(x_n, y_n) \to 0$.

To see this, observe first that $M \setminus U_1$ is finite by assumption and thus contains x_n only for finitely many n. Hence, it is no loss of generality to assume that $x_n \in U_1$ for all n.

Given some n, let $k_n \geq 1$ be the largest index with $x_n \in U_{k_n}$; if no largest index with this property exists, put $k_n = n$. For any k, the set $M \setminus U_k$ is finite by assumption and thus the set $J_k = \{n : x_n \notin U_k\}$ is finite. The relation $U_1 \supseteq U_2 \supseteq \dots$ implies that $I_k = \{n : k_n \leq k\}$ is contained in $J_{k+1} \cup \{1, 2, \dots, k\}$ and thus finite for any k.

We have $x_n \in U_{k_n} = \overline{\operatorname{conv}} H([0,1] \times (U_{k_n - 1} \cap D))$ for any n. Hence, there is some y_n which is the convex combination of (finitely many) elements from $H([0,1] \times (U_{k_n-1} \cap D))$ such that $d(x_n, y_n) < n^{-1}$. In particular, we find some finite $A_n \subseteq U_{k_n - 1} \cap D$ with $y_n \in \operatorname{conv} H([0,1] \times A_n)$. Now $A = \bigcup A_n$ is the required set. We have to check that $A \in \mathcal{F}_D$. Since $U_0 \supseteq U_1 \supseteq \dots$ implies $A_n \subseteq U_k$ for $k \leq k_n - 1$, we have $A \setminus U_k = \bigcup_n (A_n \setminus U_k) = \bigcup_{n : k > k_n - 1} (A_n \setminus U_k) \subseteq \bigcup_{n \in I_k} A_n$. But the last set is finite for each k, since I_k and A_n are finite.

Step 3. We prove now that all sets in $\mathcal{F}_D$ are precompact. Let $C \in \mathcal{F}_D$ be arbitrary. We have to prove that C is precompact. If C is finite, we are done already. Thus assume that C is infinite. Moreover, replacing C by the set $C \cup B \in \mathcal{F}_D$ with B from Step 1 if necessary, it is no loss of generality to assume that $\gamma(C) \geq \gamma(F)$ for each $F \in \mathcal{F}_D$ and each $\gamma \in \Gamma$. In particular, if x_n denotes an enumeration of the elements of $M := C$, and if A and y_n are chosen as in Step 2, we have $A \cup C \in \mathcal{F}_D$ and so $\gamma(C) \geq \gamma(A \cup C)$ $(\gamma \in \Gamma)$. Put $C_n = \{x_1, \dots, x_n, y_{n+1}, y_{n+2}, \dots\}$, and $C_0 = \{y_1, y_2, \dots\}$. Since each $\gamma \in \Gamma$ is finitely stable, we have $\gamma(C_n) = \gamma(C_0)$ for each n. Moreover, $d(x_n, y_n) \to 0$ implies $\sup_{x\in C} \operatorname{dist}(x, C_n) \to 0$. Since γ is almost lower semicontinuous, we may conclude that $\gamma(C) \leq \gamma(C_0)$. Hence,

$$\gamma(A \cup C) \leq \gamma(C) \leq \gamma(C_0) \leq \gamma(\operatorname{conv} H([0,1] \times A)) \leq \gamma(H([0,1] \times (A \cup C)))$$

for each $\gamma \in \Gamma$. Since H is countably condensing and $A \cup C$ is countable and bounded (since D is bounded) this implies that $A \cup C$ is precompact. Hence, C is precompact, as claimed.

Step 4. Now we prove that any set $M \in \mathcal{F}$ is precompact. We prove that any sequence $x_n \in M$ contains a Cauchy subsequence. If one element of this sequence occurs infinitely many often, we are done already. Otherwise, choose y_n and $A \in \mathcal{F}_D$ as in Step 2 of the proof. By Step 3, the set A is precompact. This implies by assumption that $\operatorname{conv} H([0,1] \times A)$ is precompact The sequence y_n belongs to this precompact set and thus contains a Cauchy subsequence. By $d(x_n, y_n) \to 0$, this implies that also x_n contains a Cauchy subsequence, as claimed. □

REMARK 2.1. The assumption that $H([0,1] \times A)$ be precompact for precompact $A \subseteq D$ is satisfied, if D is closed and H is upper semicontinuous and all values $H(x, \lambda)$ are compact. Indeed, since $\overline{A}$ is a compact subset of D, the image $H([0,1] \times \overline{A})$ is compact.

The idea for the proof of Theorem 2.2 is taken from the proof of [22, Theorem 3.1] (be aware that in the proof of [22, Theorem 3.1] there is a mistake, since it need not necessarily be the case that the sequence M considered there belongs to Z; however the problem can be avoided as can be seen in Step 1 from our above proof).

For deeper results in the fixed point index theory of fundamentally restrictible maps, a role is played by so-called 1- and 2-completely fundamentally restrictible maps, see e.g. [6], [23]. These are maps which are V-fundamentally restrictible for each set $V \subseteq \overline{\Omega}$ which consists of 1 resp. 2 points. Of course, Theorem 1.2 may be used to verify this property. In particular, Corollary 2.1 implies that maps which are countably condensing with respect to a class of monotone and finitely stable measures of noncompactness are 1- and 2-completely fundamentally restrictible.

We note that there exists a fixed point theory for the case then $H(\lambda, x)$ is not convex but only acyclic (with respect to the Čech cohomology with rational coefficients). This theory was initiated with the fixed point theorem in [14] and is now rather developed (see in particular [17]); for historical surveys see [4], [6], see also [20]. Such a theory exists even if H is generalized acyclic in the sense from [6], in particular if $H=\varphi\circ h$ where $h:[0,1]\times D\to 2^Y\setminus\emptyset$ is upper semicontinuous with compact acyclic values, and $\varphi:Y\to K$ is continuous. The latter is e.g. of interest for the existence of periodic solutions of differential inclusions, since the translation operator usually has this property, see e.g. [6], [10].

The definition of the fixed point index in Section 1 does not transfer to (almost) acyclic maps. However, for 2-completely fundamentally restrictible such maps in Banach spaces a degree theory was introduced in [30]; see also [23]. The degree was generalized to locally convex spaces in [37]; see also [6]. Since Theorem 1.2 can be used to verify that a map is 2-completely fundamentally restrictible (see above), one can combine these results to define a fixed point index for countably condensing acyclic homotopies. One has to pay for this approach by more restrictive compactness assumptions (compared to the theory for convex-valued homotopies).

Another approach to define a fixed point index for maps with nonconvex values is rather new and consists in approximating such maps by single-valued maps (which is somewhat analogous to the convex-valued case), see e.g. [3], [18], [19]. We do not know whether it is possible to take advantage of this approach in connection with fundamentally restrictible maps.

3. An application

In this section, we consider only single-valued maps. Recall that an operator is called *proper*, if preimages of compact sets are compact. Any continuous proper operator F in metric spaces maps closed sets into closed sets. Indeed, let M be closed, and $y_n\in F(M)$ with $y_n\to y$. There are $x_n\in M$ with $y_n=F(x_n)$. Since the preimage of the compact set $\{y,y_1,y_2,\dots\}$ is compact, the sequence $x_n\in M$ has a convergent subsequence; the limit x belongs to M, because M is closed. The continuity of F implies $Fx=y$, and so $F(M)$ is closed.

We say that a function γ defined on the system of countable bounded subsets of $K\subseteq X$ with values in a partially ordered set is (algebraic) *countably compactly stable*, if $\gamma(A+C)=\gamma(A)$ for any countable bounded subsets $A,C\subseteq K$ with precompact C.

For example, the Hausdorff and Kuratowski measures of noncompactness are countably compactly stable, because they are monotone, algebraic semi-additive (i.e. $\gamma(A+C)\le\gamma(A)+\gamma(C)$) and vanish on precompact sets.

Proposition 3.1. *Let X be a Fréchet space, $K\subseteq X$, $D\subseteq K$ bounded and closed in X, and Γ be a class of monotone and countably compactly stable* *functions on K. If $F : D \to K$ is countably unpreserving with respect to Γ, then* $\mathrm{id} - F$ *is proper.*

Proof. Let $M \subseteq X$ be compact. Then $B = (\mathrm{id} - F)^{-1}(M)$ is closed in X (since D is closed), and we only have to prove that B is precompact. Otherwise, B contains a sequence $x_n \in B$ without a Cauchy subsequence. Then $A = \{x_1, x_2, \dots\} \subseteq D$ is bounded but not precompact, and $y_n = x_n - Fx_n \in M$. Hence, $C = \{y_1, y_2, \dots\}$ is precompact, and we have $A \subseteq F(A) + C$ and $F(A) \subseteq A + (-C)$. In particular, $F(A)$ is bounded, and we have $\gamma(A) \leq \gamma(F(A))$ and $\gamma(F(A)) \leq \gamma(A)$ for each $\gamma \in \Gamma$. This is not possible, since F is countably unpreserving. $\square$

We are now in a position to prove the following generalization of the Fredholm alternative.

Theorem 3.1. *Let X be a Banach space, and $A : X \to X$. Assume that A is continuous and countably condensing on a neighbourhood of* 0 *with respect to a class Γ of monotone and countably compactly stable measures of noncompactness on X. Then the following holds:*

(a) *If $F = \mathrm{id} - A$ is odd and $F^{-1}(\{0\}) = \{0\}$, then F maps any neighbourhood of* 0 *onto a neighbourhood of* 0*. In particular, if A is linear or at least homogeneous, then $F^{-1}(\{0\}) = \{0\}$ implies that F is onto.*

(b) *If A is linear, then the operator $F = \mathrm{id} - A$ has a finite-dimensional null space and a closed range.*

Proof. Let $K = X$, and $\Omega_0 \subseteq K$ be an open neighbourhood of 0 such that A is continuous and countably condensing on $\overline{\Omega}_0$. Without loss of generality, let Ω_0 be bounded (X is a Banach space).

(a) Let a neighbourhood U of 0 be given. Then $\Omega = (U \cap \Omega_0) \cap (-(U \cap \Omega_0))$ is a symmetric neighbourhood of 0 which is contained in Ω_0. By Proposition 3.1, the set $M = F(\partial\Omega)$ is closed. Since $0 \notin M$ by assumption, we find a convex open neighbourhood V of 0 with $V \cap M = \emptyset$. We claim that $V \subseteq F(\Omega) \subseteq F(U)$.

Let $y \in V$ be given. Consider the homotopy $H(\lambda, x) = A(x) + \lambda y$. None of the maps $H(\lambda, \cdot\,)$ $(0 \leq \lambda \leq 1)$ has a fixed point $x \in \partial\Omega$, for otherwise $\lambda y = x - Ax \in M$ in contradiction to $\lambda y \in V$. Let $C \subseteq \overline{\Omega}$ be countable but not precompact. If $\Lambda \subseteq [0,1]$ is countable and dense, then $H(\Lambda \times C)$ is dense in $H([0,1] \times C)$, and so these sets have the same closed and convex hull. Consequently, we have for each $\gamma \in \Gamma$ that

$$\gamma(H([0,1] \times C)) = \gamma(H(\Lambda \times C)) = \gamma(A(C) + \Lambda y) = \gamma(A(C)).$$

For the last equality we have used the fact that γ is countably compactly stable. We may conclude that $\gamma(H([0,1] \times C)) \not\geq \gamma(C)$ for some $\gamma \in \Gamma$. We thus have proved that H is countably condensing with respect to Γ. In view of Corollary 2.1, the homotopy H is admissible, and so $\mathrm{ind}_X(H(1,\cdot),\Omega) = \mathrm{ind}_X(A,\Omega)$. By Theorem 1.4, $\mathrm{ind}_X(A,\Omega)$ is odd, and so $H(1,x) = x$ for some $x \in \Omega$. But this means $F(x) = y$, and so $y \in F(\Omega)$, as claimed.

(b) Let $D = \overline{\Omega}_0$. By Corollary 2.1, the constant homotopy $H(t,x) = A(x)$ is fundamentally restrictible (on D), i.e. its ultimate fundamental set U_∞ is compact. But each fundamental set contains all fixed points of A in D. In particular, U_∞ contains $F^{-1}(\{0\}) \cap D$. Hence, $U_\infty \cap F^{-1}(\{0\})$ is a compact neighbourhood of 0 in the space $N = F^{-1}(\{0\})$. But this space has only compact neighbourhoods if it has finite dimension, see e.g. [32, Theorem 1.22].

With the following reasoning we can prove that F has closed range without appealing to the theorem of Hahn–Banach (and thus avoiding the (uncountable) axiom of choice). There is some $\rho > 0$ such that $M = \{x \in X : \|x\| \le 2\rho\}$ is contained in D. For $x \in X$, let $[x]$ denote the equivalence class of X in the factor space X/N, i.e. $\|[x]\| = \inf\{\|x+y\| : y \in N\}$. Let

$$S_0 = \{[x] \in X/N : \|[x]\| = \rho\} \quad \text{and} \quad S = \{x \in M : [x] \in S_0\}.$$

Since N has a finite dimension, the set S is closed: indeed, if $x_n \in S$ converges to x, then $x \in M$, and

$$\rho \le \|x_n + y\| \to \|x + y\| \quad \text{for each } y \in N.$$

Moreover, if $y_n \in N$ are such that $\|x_n + y_n\| \to \rho$, then the triangle inequality implies that y_n is bounded, and so a subsequence of y_n converges to some $y_0 \in N$; hence we must have $\|x + y_0\| = \rho$, and so $x \in S$.

Now, let $F_0 : X/N \to X$ be defined by $F_0[x] = Fx$. Observe that for each $[x] \in S_0$, we find some $y \in N$ with $x + y \in M$, and so $x + y \in S$. Hence, $F_0(S_0) \subseteq F(S)$. Since $S \subseteq M$ is bounded and closed, Proposition 3.1 implies that $F(S)$ is closed. In view of $S \cap F^{-1}(\{0\}) = S \cap N = \emptyset$, we have $0 \notin F(S) = \overline{F(S)}$, and so

$$d = \mathrm{dist}(0, F_0(S_0)) \ge \mathrm{dist}(0, F(S)) > 0.$$

We may conclude that the inverse of F_0 is bounded by $d^{-1}\rho$, and so the range of F_0 (which is the range of F) is complete and thus closed. $\square$

Considering the remark in the above proof, note that the Hahn–Banach extension theorem can be proved without the (uncountable) axiom of choice for a large class of spaces, e.g. for separable spaces [16, p. 183], but not for all spaces (see e.g. [26]).

For linear operators A under more restrictive countable compactness assumptions on A, the Fredholm property of $\mathrm{id} - A$ follows from [1, Theorem 2.3.7] (which is proved by purely linear arguments and needs the Hahn–Banach extension theorem for the proof).

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