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# A FIXED POINT INDEX FOR EQUIVARIANT MAPS

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Abstract. The purpose of the paper is to define a fixed point index for equivariant maps of $G$-ENR's and to state and prove some of its properties, such as the compactly fixed $G$-homotopy property, the Lefschetz property, its converse, and the retraction property. At the end, some examples are given of equivariant self-maps which have a nonzero index (hence cannot be deformed equivariantly to be fixed point free) but have a zero $G$-Nielsen invariant.

## 1. Introduction

The aim of the paper is to define an equivariant fixed point index for compactly fixed $G$-maps $f : U \to X$, where $U$ is an open $G$-subset of the $G$-ENR $X$, and $G$ is a compact Lie group. The main point is to try to mix the ideas of the Hopf–Dold (equivariant) fixed point index $I$ (see e.g. [5], [7], [21]) with Nielsen fixed point theory (as done in [9]). While the Hopf–Dold fixed point index is invariant under suspension, i.e. it is defined up to stable equivalence of maps, the Nielsen number relies on a more truly homotopical approach, being invariant only up to homotopy. The motivation in introducing the Nielsen number is the following: if $I(f) \neq 0$ then each (compactly fixed) map $f'$ stably equivalent to $f$ has at least one fixed point, and this implies that every map $f''$ homotopic to $f$ has at least a fixed point; one wants to know when the converse of this last statement is true (i.e. when $I(f) = 0$ implies that $f$ can be homotopically

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deformed to be fixed point free), and to compute a lower bound of the number of its fixed points.

The (local) Nielsen number $N(f)$ of a compactly fixed map $f$ plays a central role whenever dealing with both these problems (for the definition and details please refer to [16], [2], [9] and [13]; for more recent surveys see [3], [12]): it is an integer, invariant up to (unstable) homotopy, with the following two main properties:

(1) $N(f)$ is a lower bound of the number of fixed points of $f$.
(2) (Wecken property): if $M$ is a manifold of dimension $\geq 3$ then there exists a compactly fixed homotopy $f_t$ such that $f_0 = f$ and $f_1$ has *exactly* $N(f)$ fixed points.

As a consequence, the Converse of the Lefschetz property holds whenever the dimension of $M$ is $\geq 3$ and $I(f) = 0$ implies $N(f) = 0$ (e.g. if $M$ is a Jiang space of dimension $\geq 3$). On the other hand it is not difficult to find examples of maps with $I(f) = 0$ and $N(f) \neq 0$. As the Hopf–Dold fixed point index is therefore the best index in the stable category, somehow the Nielsen number (and the related algebraic *generalized Lefschetz number*) is more adequate to the ordinary unstable homotopy category.

The same happens in the equivariant settings: the Lefschetz–Dold equivariant fixed point index is stable, and it is unique among all the indexes with this property (cf. e.g. [5], [7], [21]), as e.g. the equivariant Lefschetz class defined by Laitinen and Lück in [19], or Komiya [17], [18], while other invariants such as the equivariant Lefschetz number defined by Wilczyński in [25] or the $G$-Nielsen invariant and the equivariant Nielsen number by Wong in [26], [27] are unstable.

Here we define an unstable equivariant fixed point index (i.e. it is invariant up to compactly fixed $G$-homotopy but not up to stable $G$-equivalence), and we prove some of its properties, following the lines of [5]. It is very close to the equivariant Nielsen number of [26], [27], but it is not the same index, as it is readily seen from the definition. One of the differences is that it does not count the number of fixed points, as local Nielsen numbers do, but it only tries to detect when there must be at least one fixed point (the Lefschetz property in Proposition 4.13). Also, under some mild assumptions a kind of converse of the Lefschetz property holds, as shown in Section 4.4. Another difference is that it is invariant up to compactly fixed $G$-homotopy, whereas the equivariant Nielsen number needs the homotopy to be $G$-compactly fixed.

In Section 2 we start by defining the Reidemeister set; we follow a non-traditional approach, suggested by M. Citterio in [4], allows to simplify some proofs of its properties. Section 3 is devoted to the definition of the non-equivariant fixed point index, which is very close to the local Nielsen number of Fadell and Husseini in [9]. The equivariant fixed point index is then defined in Section 4, with some of its common properties. At the end, in Section 5, some examples are given, which can be of some interest by themselves. It is shown that even if for every isotropy group $H \subset G$ the $G$-map $f^H : X^H \to X^H$ can be deformed to be fixed point free, where $X$ is a smooth $G$-manifold and $G$ a finite group, the $G$-map $f$ may not be deformed equivariantly to be fixed point free.

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## 2. The Reidemeister set $\mathcal{R}(f)$

There are several ways to define the Reidemeister trace and the Nielsen number: using covering spaces, obstruction theory and others. Here we modify a little the standard definitions, in order to simplify some proofs. Instead of considering the Reidemeister classes in the fundamental group, or as classes of liftings to the universal covering space of the given map, we give an alternate equivalent definition after introducing the (equalizer) space $E(f)$. Further details can be found in [4].

**2.1. The space $E(f)$.** Let $X$ be a topological space, $U \subset X$ a subset of $X$ and $f : U \to X$ a continuous map. Let $I$ denote the unit interval $I = \{t \in \mathbb{R} : 0 \le t \le 1\}$ and $X^I$ the space of all the maps $\lambda : I \to X$ with the compact-open topology. Then the evaluation map $\varepsilon_{0,1} : X^I \to X \times X$ defined by $\varepsilon_{0,1}(\lambda) = (\lambda(0), \lambda(1))$ for all $\lambda \in X^I$ is a fibration, and we can look at the pull-back diagram

$$\begin{tikzcd}
E(f) \arrow[r] \arrow[d, "\overline{\varepsilon}_{0,1}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X^I \arrow[d, "\varepsilon_{0,1}"] \\
U \arrow[r, "{(i,f)}"'] & X \times X
\end{tikzcd}$$

where the map $(i, f)$ is defined by $(i, f)(x) = (x, f(x))$ for all $x \in U$ and $E(f) := U \sqcap X^I$ is homeomorphic to the space of all the paths $\lambda : I \to X$ such that $\lambda(0) \in U$ and $\lambda(1) = f(\lambda(0))$.

**2.2. Properties of $E(f)$.** The space $E(f)$ has a *quasi-functorial* behavior, in the following sense. Let $X$ and $Y$ spaces, $U \subset X$ and $V \subset Y$ subsets of $X$ and $Y$, and $f : U \to X$, $g : V \to Y$ two maps. Let $W \subset X$ be another subset of $X$, and $k : W \to Y$ a map. Then the composite maps $kf$ and $gk$ are defined resp. on $f^{-1}W$ and $k^{-1}V$. We write $kf = gk$ when $kf(x) = gk(x)$ for all $x \in f^{-1}W \cap k^{-1}V$.

Under these hypotheses, we are going to define a map

$$k^E : D(k^E) \subset E(f|f^{-1}W \cap k^{-1}V) \to E(g)$$

from a suitable domain $D(k^E)$. By definition, $f(f^{-1}W \cap k^{-1}V) \subset W$, hence there is a pull-back diagram

$$\begin{tikzcd} E(f : f^{-1}W \cap k^{-1}V \to W) \arrow[r] \arrow[d, "\overline{\varepsilon}_{0,1}"'] \arrow[dr, phantom, "\lrcorner"] & W^I \arrow[d, "\varepsilon_{0,1}"] \\ f^{-1}W \cap k^{-1}V \arrow[r, "(i,f)|f^{-1}W \cap k^{-1}V"'] & W \times W \end{tikzcd}$$

and an inclusion map $\phi : E(f : f^{-1}W \cap k^{-1}V \to W) \to E(f|f^{-1}W \cap k^{-1}W)$ induced by the inclusion $W \to X$. On the other hand, there is a map $\phi' : E(f : f^{-1}W \cap k^{-1}V \to W) \to E(g)$ induced by the map $k : W \to Y$. Let $D(k^E) := \mathrm{Im}(\phi)$. We define the map $k^E$ by setting $k^E := \phi'\phi^{-1} : D(k^E) \to E(g)$.

A particular case is e.g. when $U \subset V \subset X = Y$, and $f : V \to X$. Then the identity map $1_X : X \to X$ induces a map which we denote with $\rho_V := 1_X^E : E(f|U) \to E(f)$ (it might be not an inclusion). When $X \subset Y$ and $f : U = V \to X$, the inclusion $i : X \to Y$ induces an inclusion $i^E : E(f) \to E(if)$.

Let $\{f_t\} : U \times I \to X$ be a homotopy. Let $F : U \times I \to X \times I$ be the fat homotopy, defined by $F(x,t) = (f_t(x), t)$ for all $(x,t) \in U \times I$. Then for each $t \in I$ the inclusion $i_t : X \to X \times I$, which sends $x$ to $(x,t)$, induces a map $i_t^E : E(f_t) \to E(F)$.

Proposition 2.1. *For all $t \in I$, the map $i_t^E : E(f_t) \to E(F)$ is a homotopy equivalence.*

Proof. It is a consequence of the co-gluing lemma (see e.g. [20, p. 71]), applied to the pull-back diagrams defining $E(f_t)$ and $E(F)$, in which the vertical arrows are fibrations. □

Remark 2.2. As it will be seen in the next section, this approach is equivalent to the classical one in [9]. The main difference is that the notation here is slightly more compact (especially with non-connected spaces) and the homotopy property is a direct consequence of Proposition 2.1. Thus the functorial-type properties are easier to prove with such a categorical approach. For example, Proposition 3.13 can give a simpler proof of the pushout formula for generalized Lefschetz numbers of [8].

**2.3. Definition of $\mathcal{R}(f)$.** Given a map $f : U \to X$, where $U$ is a subset of $X$, the Reidemeister set $\mathcal{R}(f)$ is the set of connected components of $E(f)$, namely, $\pi_0(E(f))$. The elements of $E(f)$ are paths $w : I \to X$ such that $w(1) = fw(0)$ and $w(0) \in U$. It is easy to see that two paths $w_1$ and $w_2$ belong to the same component in $E(f)$ if and only if there is a path $\lambda : I \to U$ such that $\lambda w_2 = w_1 f(\lambda)$ (in our notation, given two paths $\lambda_1$ and $\lambda_2$, the path $\lambda_1\lambda_2$ is the path defined by $\lambda_1\lambda_2(t) = \lambda_1(2t)$ when $t \in [0, 1/2]$ and $= \lambda_2(2t-1)$ when $t \in [1/2, 1]$). Given an element $w$ of $E(f)$, we will denote with $[w]$ its component in $\mathcal{R}(f)$, also called class, as $[w]$.

For any path-connected component $U_i$ of $U$ such that $fU_i$ is in the same path-connected component of $U_i$ in $X$, let us choose a base-path $w_i$, i.e. a path in $X$ such that $w_i(0) \in U_i$ and $fw_i(0) = w_i(1)$. Let $A$ denote such a set of components of $U$. Then for every $i \in A$, the fundamental group $\pi_1(U, w_i(0))$ acts on $\pi_1(X, w_i(0))$ as follows (this action can be called fundamental action, or Reidemeister action): given the elements $g \in \pi_1(U, w_i(0))$ and $\alpha \in \pi_1(X, w_i(0))$,

$$g \cdot \alpha := g\alpha w_i f(g^{-1}) w_i^{-1}$$

where we denote a path and the corresponding homotopy class of paths with the same symbol.

Proposition 2.3. *There is a bijection between the Reidemeister set $\mathcal{R}(f)$ and the disjoint union of the orbit sets*

$$\mathcal{R}(f) \cong \coprod_{i \in A} \pi_1(X, w_i(0))/\sim .$$

Proof. Let us consider the following maps: for any point $w$ in $E(f)$, $w(0)$ belongs to exactly one $U_i$. Therefore there exists a path

$$\lambda : (I, 0, 1) \to (U_i, w_i(0), w(0)).$$

Let us define $\psi(w) = \lambda w f(\lambda^{-1}) w_i^{-1}$. It is easy to see that the orbit of this element in $\pi_1(X, w_i(0))$ does not depend upon the choice of $\lambda$; moreover, this map is locally constant for all $i$, hence induces a map $\psi : \mathcal{R}(f) \to \coprod_{i \in A} \pi_1(X, w_i(0))/\sim$.

On the other hand, for each $i \in A$, there is a map $\varphi' : \pi_1(X, w_i(0)) \to \mathcal{R}(f)$, which sends a loop $\alpha$ to the connected component of $\alpha w_i$ in $E(f)$. But for any $g \in \pi_1(U, w_i(0))$ the corresponding point $g \cdot \alpha w_i = g\alpha w_i f(g^{-1})$ is in the same path-connected component of $\alpha w_i$, and hence $\varphi'$ induces a map on $\coprod_{i \in A} \pi_1(X, w_i(0))/\sim$.

It is now easy to see that $\varphi$ is the inverse of $\psi$ and vice-versa. □

**2.4. Properties of $\mathcal{R}(f)$.** The properties of $E(f)$ give rise to properties of $\mathcal{R}(f)$ in a natural way. The first thing to do is to observe its *quasi-functorial* behavior. Let $U \subset X$, $V \subset Y$ and $f : U \to X$, $g : V \to Y$ be maps. If $W \subset X$ is a subspace and $k : W \to Y$ is a map such that $kf = gk$ (where it is defined, in $f^{-1}W \cap k^{-1}V$), then the map $k^E : D(k^E) \subset E(f|f^{-1}W \cap k^{-1}V) \to E(g)$, defined above, induces a function $k_* : D(k) \subset \mathcal{R}(f|f^{-1}W \cap k^{-1}V) \to \mathcal{R}(g)$, where $D(k)$ is the image of the map $\pi_0 E(f : f^{-1}W \cap k^{-1}V \to W) \to \pi_0 E(f|f^{-1}W \cap k^{-1}V)$

induced by the inclusion. The map $k_*$ is well-defined, because whenever $w_1$ and $w_2$ are points of $E(f : f^{-1}W \cap k^{-1}V \to W)$ in the same path-connected component of $E(f|f^{-1}W \cap k^{-1}V)$, their images $k(w_1)$ and $k(w_2)$ are in the same path-connected component of $E(g)$.

If $U \subset V \subset X = Y$ and $f : V \to X$, then we obtain the localization function $\rho_U := 1_{X*} : \mathcal{R}(f|U) \to \mathcal{R}(f)$.

PROPOSITION 2.4 (Localization). *If $U \subset V$ and $f : V \to X$ is a map, then there is a function $\rho_U : \mathcal{R}(f|U) \to \mathcal{R}(f)$.*

Let $\{f_t\} : U \times I \to X$ be a homotopy. Let $F : U \times I \to X \times I$ be the fat homotopy, defined by $F(x,t) = (f_t(x), t)$ for all $(x,t) \in U \times I$. Then for each $t \in I$ the inclusion $i_t : X \to X \times I$, which sends $x$ to $(x,t)$, induces a map $i_{t*} : \mathcal{R}(f_t) \to \mathcal{R}(F)$. Because of Proposition 2.1, the following proposition holds.

PROPOSITION 2.5 (Homotopy Invariance). *If $f_t : U \to X$, $t \in I$, is a homotopy, then $\mathcal{R}(f_0) \cong \mathcal{R}(f_1)$.*

The following two propositions are easy consequences of the definition.

PROPOSITION 2.6 (Additivity). *If $U = \coprod_i U_i$ is the disjoint union of some open subsets $U_i \subset X$, and $f : U \to X$, then $\mathcal{R}(f) \cong \coprod \mathcal{R}(f|U_i)$.*

PROPOSITION 2.7 (Multiplicativity). *Let $f : U \subset X \to X$ and $f' : U' \subset X' \to X'$ be given maps, and $f \times f' : U \times U' \to X \times X'$ the Cartesian product. Then the equality $\mathcal{R}(f \times f') \cong \mathcal{R}(f) \times \mathcal{R}(f')$ holds.*

## 3. The index $\mathcal{I}(f)$

Let $X$ be an ENR, $U \subset X$ an open subset of $X$ and $f : U \to X$ a compactly fixed map, i.e. a map such that $\mathrm{Fix}(f)$ is compact. We define an index of $f$ as an element of a suitable commutative ring $R$ with unity.

**3.1. The ring $R$.** Let $R^+$ be the set of all functions defined from finite sets to $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, i.e. functions with finite domain and non-zero integer values, with the following equivalence: when there is a bijection $b$ between the domains of two such functions $\phi$ and $\phi'$ such that $\phi b = \phi'$, then $\phi \cong \phi'$.

We use the symbol $z_j$ to denote the function $z_j : \{*\} \to j \in \mathbb{Z}^*$ for all $j \in \mathbb{Z}^*$, and the symbol 0 to denote the function with empty domain.

We can define a (commutative and associative) sum and a (commutative and associative) product on the set $R^+$: given two functions $\phi_1 : S_1 \to \mathbb{Z}^*$ and $\phi_2 : S_2 \to \mathbb{Z}^*$, which determine equivalence classes $\overline{\phi}_1$ and $\overline{\phi}_2$ in $R^+$, let $\overline{\phi}_1 + \overline{\phi}_2$ be the equivalence class of the function $\phi_1 \coprod \phi_2 : S_1 \coprod S_2 \to \mathbb{Z}^*$ defined on the disjoint union of $S_1$ and $S_2$, and let $\overline{\phi}_1 \cdot \overline{\phi}_2$ be the class of the function $\phi_1 \times \phi_2 : S_1 \times S_2 \to \mathbb{Z}^*$ defined by $\phi_1 \times \phi_2(s_1, s_2) := \phi_1(s_1)\phi_2(s_2)$ for all pairs $(s_1, s_2) \in S_1 \times S_2$.

Thus $R^+$ has a structure of (commutative) monoid with respect to both these operations, and the distributive law holds. The element 0 defined above is the (additive) zero and $1 := z_1$ is the (multiplicative) unity. Let us note that $z_i \cdot z_j = z_{ij}$.

Every element of $R^+$ can be written as a sum $\sum_{j\in\mathbb{Z}^*} k_j z_j$ where $k_j$ are positive integers, non-zero for a finite set of $j$'s.

Let $R$ be the Grothendieck ring with respect to the sum of $R^+$, i.e. the set of all the formal sums $\sum_{j\in\mathbb{Z}^*} k_j z_j$ where $k_j$ are integers, non-zero for a finite set of $j$. With the sum and the product induced by $R^+$ it is a commutative ring with unity $1 = z_1$.

It is worthwhile to note that $\mathbb{Z}^*$ is a multiplicative monoid, and thus the monoid-ring $\mathbb{Z}[\mathbb{Z}^*]$ is well-defined, and it isomorphic to $R$ in an obvious way.

Using prime factorization, we can see that $R$ is generated by the elements $\{z_p\}$ with $p$ prime $\geq 2$ and $p = \pm 1$.

Later we will see the topological meaning of such a representation of $R$: a map with 2 fixed point classes of fixed point index $-12$ and 7 fixed point classes with fixed point index 13 will give rise to an element of $R$ written as $2z_{-12} + 7z_{13} = 2z_{-1}z_2^2z_3 + 7z_{13}$.

There are two homomorphisms $t, N : R \to \mathbb{Z}$, the trivialization homomorphism $t$ defined by $t(\phi) = \sum_z \phi(z)$ where $z$ ranges on the domain of $\phi$, and the norm homomorphism $N$ defined by $N(\phi) = \#\mathrm{supp}(\phi)$, i.e. it is the cardinality of the support of $\phi$. The first preserves the additive structure of $R$, the second preserves also the multiplicative structure. In $\mathbb{Z}[\mathbb{Z}^*]$ they correspond to the homomorphisms given by $t : z_j \to j \in \mathbb{Z}$ and $N : z_j \to 1 \in \mathbb{Z}$.

Sometimes we will deal with functions with finite support, but with a larger domain. In this case, we just take the restriction of the function to its support, and associate to it an element in $R$.

**3.2. The definition of $\mathcal{I}(f)$.** Let $f : U \to X$ be as above. There exists a *coordinate* function $\mathrm{cd} : \mathrm{Fix} f \to \mathcal{R}(f)$ defined as follows. For each $x \in \mathrm{Fix}(f)$, let $c_x$ denote the constant path in $x$. It gives rise to a map $c : \mathrm{Fix}(f) \to E(f)$. If $q$ denotes the projection $q : E(f) \to \mathcal{R}(f)$, then $\mathrm{cd} := qc$. As $\xi$ ranges in $\mathcal{R}(f)$, the counter-images $\mathrm{cd}^{-1}\xi$ are compact subsets of $\mathrm{Fix}(f)$, both closed and open in $\mathrm{Fix}(f)$. Hence they are a finite number. They are the (Nielsen) fixed point classes of $f$ in $U$. Any class $\mathrm{cd}^{-1}\xi$ is compactly contained in a neighbourhood $W$ such that $\mathrm{Fix}(f) \cap W = \mathrm{cd}^{-1}\xi$. Therefore the index $I(f|W)$ of $f$ in $W$ is defined (see e.g. [6], [5]). Because it does not depend on the choice of $W$, but only on $\xi$, we write $I(\xi)$. The fixed point classes with non-zero index are called essential, the others inessential. The number of essential fixed point classes is the (local) Nielsen number of $f$. It is worthwhile to note that two fixed points

$x$ and $y$ belong to the same fixed point classes if and only if there is a path $c : (I, 0, 1) \to (U, x, y)$ which is homotopic to $fc$ rel. endpoints.

We now define the generalized index of $f$ as the index in $R$ given by the function

$$\mathcal{I}(f) : \mathcal{R}(f) \to \mathbb{Z}, \quad \xi \mapsto I(\xi)$$

restricted to its support. The support is finite because $\mathrm{Fix}(f)$ is compact, and so it gives rise to an element of $R$. The trivialization $t(\mathcal{I}(f))$ is the classical Hopf index $I(f)$, while the norm $N(\mathcal{I}(f))$ is the local Nielsen number.

It is easy to see that $\mathcal{I}(f)$ is a kind of *symmetrization* of the local generalized Lefschetz number (also known as Reidemeister trace) $\mathcal{L}(f) := \sum_{\xi \in \mathcal{R}(f)} I(\xi) \cdot \xi \in \mathbb{Z}\mathcal{R}(f)$, where $\mathbb{Z}\mathcal{R}(f)$ denotes the free Abelian group generated by the elements of $\mathcal{R}(f)$, which is isomorphic to the group of all functions $\phi : \mathcal{R}(f) \to \mathbb{Z}$ with finite support. In fact, let $s : \mathbb{Z}\mathcal{R}(f) \to R$ be the unique map which sends the function $\phi : \mathcal{R}(f) \to \mathbb{Z}$ to its equivalence class in $R$. Then by definition $s(\mathcal{L}(f)) = \mathcal{I}(f)$. For details on the generalized Lefschetz number, and its local version, the standard references are [13] and [11].

Remark 3.1. As suggested by the name itself, the generalized Lefschetz number (Reidemeister trace) is first defined as a trace-like quantity, also for local maps (see [11]). In our setting, for the sake of simplicity we don't use algebraic traces; the approach tries to be closer to the classical *local* fixed point indices of [5]. Of course in computations very often the traces are the only possible way to get actual computations.

**3.3. Properties of $\mathcal{I}(f)$.** The index $\mathcal{I}$ has some properties, which are analogous to the properties of the fixed point index $I$ of [5]. Let $f : U \subset X \to X$ be a compactly fixed map, $U$ open in $X$ and ENR (this is of course true if $X$ is an ENR).

Let $\phi \in R$ be an index defined on a finite set $S$. If $S'$ is a set and $\rho : S \to S'$ is a map, then it is possible to define an index $\rho\phi$ as follows: let $\phi' : S' \to \mathbb{Z}$ be the function $\phi'(s') := \sum_{s:\rho(s)=s'} \phi(s)$; it has a finite support, hence the restriction of $\phi'$ to the support is a unique element of $R$.

Definition 3.2. A graph $\widetilde{\Gamma}'_U$ in $U$ such that every connected component of $\mathrm{Fix}(f)$ has a vertex of $\widetilde{\Gamma}'_U$, and every two connected components of $\mathrm{Fix}(f)$ which are in the same Nielsen class are connected by an edge of $\widetilde{\Gamma}'_U$ is called the $U$-Nielsen abstract graph for $f$ (or, in short, Nielsen abstract graph). Each vertex of $\widetilde{\Gamma}'_U$ has an index, namely the fixed point index of the corresponding connected component of $\mathrm{Fix}(f)$. A connected component of $\widetilde{\Gamma}'_U$ is essential or inessential according to whether the sum of the indexes of its vertices is non-zero or zero, respectively. The essential $U$-Nielsen abstract graph $\widetilde{\Gamma}_U$ for $f$ is the union of the essential connected components of the $U$-Nielsen abstract graph $\widetilde{\Gamma}'_U$.

A map $\Lambda : \widetilde{\Gamma}_U \to X$ which sends edges of $\widetilde{\Gamma}_U$ to Nielsen paths in $X$ (i.e. paths which are homotopic to their images under $f$) and vertices to fixed points (that belong to the corresponding Nielsen class) is called a Nielsen graph $\Gamma_U$. (It is an abuse of notation, because actually it is not a graph.)

Let us note that there exists always at least one essential $U$-Nielsen graph, for any map $f : U \subset X \to X$.

Proposition 3.3 (Localization). *If $W$ is an open set such that $\mathrm{Fix}(f) \subset W \subset U$ then*

$$\mathcal{I}(f) = \rho_W \mathcal{I}(f|W)$$

*where $\rho_W : \mathcal{R}(f|W) \to \mathcal{R}(f)$ is the localization function of Proposition 2.4. Moreover,*

$$\mathcal{I}(f) = \mathcal{I}(f|W)$$

*if and only if $W$ contains an essential $U$-Nielsen graph $\Gamma_U$.*

Proof. If $\mathrm{cd}_W : \mathrm{Fix}(f) \to \mathcal{R}(f|W)$ and $\mathrm{cd} : \mathrm{Fix}(f) \to \mathcal{R}(f)$ are the coordinate functions, then $\rho_W \mathrm{cd}_W = \mathrm{cd}$; hence for all $\xi \in \mathcal{R}(f)$ we have that $\mathrm{cd}^{-1}\xi = \mathrm{cd}^{-1}\rho^{-1}\xi$, and because the index $I$ is additive,

$$I(\mathrm{cd}^{-1}\xi) = \sum_{\xi' \in \rho^{-1}\xi} I(\mathrm{cd}^{-1}\xi'),$$

hence $\mathcal{I}(f) = \rho_W \mathcal{I}(f_{|W})$.

If $W$ contains an essential $U$-Nielsen graph $\Gamma_U$, then it is easy to see that $\rho_W$ induces a bijection between the supports of $\mathcal{I}(f)$ and $\mathcal{I}(f|W)$. On the other hand, let us consider an essential $W$-Nielsen graph $\Gamma_W$ of $f|W$ in $W$. Then because $\mathcal{I}(f) = \mathcal{I}(f|W)$, $\Gamma_W$ is also an $U$-Nielsen graph. □

Corollary 3.4. *If $W$ is an open set such that $\mathrm{Fix}(f) \subset W \subset U$, and $\mathcal{I}(f) = \mathcal{I}(f|W)$, then for each open set $V$ such that $W \subset V \subset U$, the equality $\mathcal{I}(f) = \mathcal{I}(f|V)$ holds.*

Proof. By the previous proposition, there exists an essential $U$-Nielsen graph in $W$. Hence the same holds for each $V \supset W$ in $U$. □

Proposition 3.5 (Unity). *If $f$ is constant, then $\mathcal{I}(f) = 0$ if $fU \notin U$ and $\mathcal{I}(f) = 1$ if $fU \in U$.*

Proof. When $f$ is constant, there exists at most one essential fixed point class, and its index $I(f)$ is 0 or 1 according to whether $fU \notin U$ or $fU \in U$. □

Proposition 3.6 (Additivity). *If $U = \coprod_i U_i$ is the disjoint union of some open subsets $U_i \subset X$, then $\mathcal{I}(f) = \sum_i \mathcal{I}(f|U_i)$.*

Proof. This follows from Proposition 2.6 and the same property for the index $I$. Here the sum is the (disjoint) sum in $R$. □

Proposition 3.7 (Multiplicativity). *Let $f : U \subset X \to X$ and $f' : U' \subset X' \to X'$ be given compactly fixed maps, and $f \times f' : U \times U' \to X \times X'$ their Cartesian product. Then the equality $\mathcal{I}(f \times f') = \mathcal{I}(f) \times \mathcal{I}(f')$ holds.*

Proof. Again, in view of Proposition 2.7 and the same property for the index $I$, we get the result. Notice that the product on the right hand side is in $R$. $\square$

Proposition 3.8 (Homotopy Invariance). *If $f_t : U \subset X \to X$ is a compactly fixed deformation, with $t \in I$, then $\mathcal{I}(f_0) = \mathcal{I}(f_1)$.*

Proof. Let $F : U \times I \to X \times I$ be the fat homotopy, defined by $F(x,t) = (f_t(x), t)$ for all $(x,t) \in U \times I$. Then for each $t \in I$ the inclusion $i_t : X \to X \times I$, which sends $x$ to $(x,t)$, induces a bijection $i_{t*} : \mathcal{R}(f_t) \to \mathcal{R}(F)$, because of Proposition 2.5. Hence the fixed point classes of $f_t$ are exactly the intersections of the fixed point classes of $F$ with the slice $X \times \{t\}$. But by the Homotopy Invariance of $I$, the index of the $t$-fixed point class does not depend upon $t$, hence the thesis. $\square$

The following is an easy consequence of Propositions 3.8 and 3.3.

Corollary 3.9 (Lefschetz Property). *If $f_t : U \subset X \to X$, $t \in I$, is a compactly fixed deformation, such that $\mathrm{Fix}(f_1) = \emptyset$, then $\mathcal{I}(f_0) = 0$.*

Proposition 3.10 (Commutativity). *Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be open subsets, and $k_1 : U_1 \to X_2$, $k_2 : U_2 \to X_1$ be maps. Then $k_2k_1 : k_1^{-1}U_2 \to X_1$ and $k_1k_2 : k_2^{-1}U_1 \to X_2$ are defined. If $\mathcal{R}(k_1k_2|k_2^{-1}k_1^{-1}U_2) = \mathcal{R}(k_1k_2)$ and $\mathcal{R}(k_2k_1|k_1^{-1}k_2^{-1}U_1) = \mathcal{R}(k_2k_1)$ then*

$$\mathcal{I}(k_2k_1) = \mathcal{I}(k_1k_2).$$

Proof. Let us consider the following diagram,

$$\begin{tikzcd}
\mathcal{R}(k_2k_1|k_1^{-1}k_2^{-1}U_1) \arrow[r, "k_{1*}"] \arrow[d, "\rho_1"'] & \mathcal{R}(k_1k_2) \\
\mathcal{R}(k_2k_1) & \mathcal{R}(k_1k_2|k_2^{-1}k_1^{-1}U_2) \arrow[l, "k_{2*}"] \arrow[u, "\rho_2"']
\end{tikzcd}$$

where $\rho_1$ and $\rho_2$ are the localization functions, and $k_{1*}$, $k_{2*}$ are the functions induced by $k_1$ and $k_2$. Because we have assumed that

$$\mathcal{R}(k_2k_1|k_1^{-1}k_2^{-1}U_1) = \mathcal{R}(k_2k_1)$$

the function $\rho_1$ is a bijection between the supports of $\mathcal{I}(k_2k_1|k_1^{-1}k_2^{-1}U_1)$ and $\mathcal{I}(k_2k_1)$. The same holds for $\rho_2$ and $\mathcal{I}(k_1k_2)$. Therefore the functions $k_{1*}\rho_1^{-1}$ and $k_{2*}\rho_2^{-1}$ are well-defined. The composition $k_{2*}\rho_2^{-1}k_{1*}\rho_1^{-1}$ sends an element $[w]$ of $\mathcal{R}(k_2k_1)$ to $[k_2k_1(w)]$ in $\mathcal{R}(k_2k_1)$. But it is easy to see that $[k_2k_1(w)] = [w]$

for all $w$ in $E(k_2k_1)$, hence the composition is the identity map. In the same way $k_{1*}\rho_1^{-1}k_{2*}\rho_2^{-1}$ is the identity on $\mathcal{R}(k_1k_2)$, and hence also $k_{1*}$ and $k_{2*}$ are bijections. Now, because of commutativity property for the index $I$, $k_{1*}$ is index-preserving, i.e. for all $\xi \in \mathcal{R}(k_2k_1|k_1^{-1}k_2^{-1}U_1)$ the equality $I(\xi) = I(k_{1*}\xi)$; and the same holds for $k_{2*}$. Therefore the equalities

$$\mathcal{I}(k_1k_2) = \mathcal{I}(k_2k_1|k_1^{-1}k_2^{-1}U_1) = \mathcal{I}(k_2k_1) = \mathcal{I}(k_1k_2|k_2^{-1}k_1^{-1}U_2)$$

hold, i.e. the thesis. □

Unlike the index $I$, the commutativity property does not hold in general, for the index $\mathcal{I}$, without the hypotheses of the previous Proposition, as shown in the following example.

EXAMPLE 3.11. Let $X_1 = X_2 = X$ be the complex plane $\mathbb{C}$ and $U_1 = U_2 = U$ the square $\{\mathrm{Re}(z) \in [-2,2], \mathrm{Im}(z) \in [-2,2]\} \subset X$. Let $k_1 : U \to \mathbb{C}$ be any map such that

(1) $k_1(-1-i) = -1+i$,
(2) $k_1(1-i) = 1+i$,
(3) $k_1^{-1}U$ is the union of two disjoint neighbourhoods of $-1-i$ and $1-i$,
(4) The map $k_1$ is constant in some neighbourhoods of $-1-i$ and $1-i$.

It is easy to see that such a map exists. Let $k_2$ be the conjugation map, i.e. $k_2(z) = \overline{z}$. Without loss of generality we can also assume that $\mathrm{Fix}(k_2k_1) = \{-1-i, 1-i\}$, and by assumption the indexes are 1. But now, $k_1^{-1}U_2$ is the disjoint union of two neighbourhoods, hence $\mathcal{I}(k_2k_1) = 2$ (as an element of $R$). On the other hand, $k_2^{-1}U_1 = U$ and hence $\mathcal{I}(k_1k_2) = z_2$, therefore $\mathcal{I}(k_2k_1) \neq \mathcal{I}(k_1k_2)$.

PROPOSITION 3.12 (Retraction). *Let $f : U \subset X \to X$ be a compactly fixed map, $X \subset Y$ and $r : Y \to X$ a retraction. Then*

$$\mathcal{I}(f) = \mathcal{I}(f'|r^{-1}U)$$

*where $f'$ is the composition $f' = ifr : r^{-1}U \to Y$.*

PROOF. Because $\mathrm{Fix}(f) = \mathrm{Fix}(f')$, $f'$ is compactly fixed. Let us consider the functions $i_* : \mathcal{R}(f) \to \mathcal{R}(f')$ and $r_* : \mathcal{R}(f') \to \mathcal{R}(f)$ induced by the inclusion $i : X \to Y$ and the retraction $r : Y \to X$. The composition $r_*i_* : \mathcal{R}(f) \to \mathcal{R}(f)$ is the identity, as it is readily seen, hence $i_*$ is injective. On the other hand if $[w]$ is in $\mathcal{R}(f')$, then $[w] = [ifr(w)] \in \mathcal{R}(f')$, which is in the image $i_*\mathcal{R}(f)$; this implies that $i_*$ is surjective. Therefore $i_* : \mathcal{R}(f) \to \mathcal{R}(f')$ is a bijection. Because of the retract property for the index $I$, $i_*$ is also index-preserving, i.e. $I(\xi) = I(i_*\xi)$ for all $\xi \in \mathcal{R}(f)$, and hence $\mathcal{I}(f) = \mathcal{I}(f')$. □

Let us consider $\mathcal{R}(f)$, where $f : U \subset X \to X$. It is a set, hence it makes sense to define the Abelian group $\mathbb{Z}\mathcal{R}(f)$ of all the finite-support functions $\phi$ :

$\mathcal{R}(f) \to \mathbb{Z}$, with the sum $+$ given by $(\phi + \phi')(\xi) := \phi(\xi) + \phi'(\xi) \in \mathbb{Z}$ for all $\xi \in \mathcal{R}(f)$. The generalized (local) Lefschetz number $\mathcal{L}(f)$ is defined as the function $\mathcal{L}(f) : \xi \to I(\xi)$.

Proposition 3.13 (Union). *Let $f : U \subset X \to X$ be a compactly fixed map, and let $U_1$ and $U_2$ be open subsets such that $U = U_1 \cup U_2$. Assume that the restrictions $f|U_1$, $f|U_2$ and $f|U_1 \cap U_2$ are compactly fixed. Let $\rho_1 : \mathcal{R}(f|U_1) \to \mathcal{R}(f)$, $\rho_2 : \mathcal{R}(f|U_2) \to \mathcal{R}(f)$ and $\rho : \mathcal{R}(f|U_1 \cap U_2) \to \mathcal{R}(f)$ denote the localization functions. Then*

$$\mathcal{L}(f) = \rho_1\mathcal{L}(f|U_1) + \rho_2\mathcal{L}(f|U_2) - \rho\mathcal{L}(f|U_1 \cap U_2).$$

Proof. Let $\xi \in \mathcal{R}(f)$ be given. Then its value $(\rho_1\mathcal{L}(f|U_1) + \rho_2\mathcal{L}(f|U_2) - \rho\mathcal{L}(f|U_1 \cap U_2))(\xi)$ is by definition $\rho_1\mathcal{L}(f|U_1)(\xi) + \rho_2\mathcal{L}(f|U_2)(\xi) - \rho\mathcal{L}(f|U_1 \cap U_2)(\xi)$, where the sum is in $\mathbb{Z}$. Moreover, the equalities

$$\begin{aligned} \rho_1\mathcal{L}(f|U_1)(\xi) &= \sum_{\sigma_1 : \rho_1(\sigma_1) = \xi} \mathcal{L}(f|U_1)(\sigma_1), \\ \rho_2\mathcal{L}(f|U_2)(\xi) &= \sum_{\sigma_2 : \rho_2(\sigma_2) = \xi} \mathcal{L}(f|U_2)(\sigma_2), \end{aligned}$$

and

$$\rho\mathcal{L}(f|U_1 \cap U_2)(\xi) = \sum_{\sigma : \rho(\sigma) = \xi} \mathcal{L}(f|U_1 \cap U_2)(\sigma)$$

follow by the definition of the localization functions.

Let us consider in $U$ a neighbourhood $W$ of the fixed point class $\mathrm{cd}^{-1}\xi$ in $\mathrm{Fix}(f|U)$ corresponding to $\xi$, such that $W \cap \mathrm{Fix}(f) = \mathrm{cd}^{-1}\xi$. By definition, we have that $I(f|W) = \mathcal{L}(f|U)(\xi)$. Also, $I(f|W) = I(f|W \cap U_1) + I(f|W \cap U_2) - I(f|W \cap U_1 \cap U_2)$ by additivity of $I$. Therefore the conclusion follows from the following identities,

$$\begin{aligned} \sum_{\sigma_1 : \rho_1(\sigma_1) = \xi} \mathcal{L}(f|U_1)(\sigma_1) &= I(f|W \cap U_1), \\ \sum_{\sigma_2 : \rho_2(\sigma_2) = \xi} \mathcal{L}(f|U_2)(\sigma_2) &= I(f|W \cap U_2), \end{aligned}$$

and

$$\sum_{\sigma : \rho(\sigma) = \xi} \mathcal{L}(f|U_1 \cap U_2)(\sigma) = I(f|W \cap U_1 \cap U_2),$$

because if $x \in \mathrm{Fix}(f|U_1)$ has the coordinate such that $\rho_1\mathrm{cd}(x) = \xi$, then $x \in W$, and the same for $f|U_2$ and $f|U_1 \cap U_2$. $\square$

Remark 3.14. As noted above, with Proposition 3.13 it is easy to prove a push-out formula for the generalized Lefschetz number (see [8]).

## 4. The index $\mathcal{I}_G(f)$ of a $G$-map

Now we carry on the definition of $\mathcal{I}(f)$ to the case of equivariant maps. Let $G$ be a compact Lie group and $X$ a $G$-ENR. Let $U \subset X$ be an open $G$-subset of $X$, and let $f : U \to X$ be a compactly fixed $G$-map. In order to define the equivariant index $\mathcal{I}_G(f)$ of $f$, it is necessary to introduce the definition of *taut* maps and *weakly taut* maps.

**4.1. Taut maps and taut approximations.** Let $G$ be a compact Lie group. Let $Y$ be a topological $G$-space, and $A \subset Y$ a $G$-subspace. Let $V$ be a neighbourhood of $A$ in $Y$, and $\phi : V \to Y$ a map.

Definition 4.1. We say that a $G$-map $\phi$ is *taut* over $A$ if there exists a $G$-retraction $r : V \to A$ such that $\phi = \phi r$. If $\phi$ is defined on a $G$-set $W \supset V$, we say that it is taut over $A$ in $U$ when its restriction to $U$ is taut over $A$.

The definition makes sense only when $A$ is a $G$-neighbourhood retract in $Y$. We now see that if $Y$ is a compact $G$-ENR, and $A \subset Y$ is a neighbourhood retract, then any self-map can be approximated by maps which are taut over $A$ in a neighbourhood of $A$ in $Y$. Let d denote an invariant metric on $Y$. A homotopy $F : Y \times I \to Y$ is called an $\varepsilon$-homotopy (or $\varepsilon$-deformation) provided that $\mathrm{d}(F(x,t), F(x,t')) < \varepsilon$ for all $x \in Y$ and $t, t' \in I$.

Proposition 4.2. *Let $Y$ be a compact $G$-*ENR*, and $A \subset X$ a neighbourhood retract. Then for each $\varepsilon > 0$ there exists a $G$-neighbourhood $V$ of $A$ in $X$ and an equivariant $\varepsilon$-homotopy* rel. *$A$ from the identity $1_Y$ to a map which is taut over $A$ in $V$.*

Proof. It follows easily from Proposition 2.5 of [25], for example. □

Let $X$ be a $G$-ENR, $U \subset X$ a subset of $X$ and $f : U \to X$ a map. Let us recall that if $Y$ is a $G$-space, then for each subgroup $H$ of $G$, the fixed subspace of $H$ is $Y^H = \{y \in Y : Hy = y\}$ and the singular locus $Y_s^H$ is $Y_s^H = \{y \in Y : G_y \supset H, G_y \neq H\}$, where $G_y$ denotes the isotropy group of $y$, i.e. the subgroup of all $\{g \in G : gy = y\}$. The space $Y_H$ is given by the points which have isotropy exactly $H$, i.e. $Y_H = \{y \in Y : G_y = H\}$, and the equality $Y^H \setminus Y_H = Y_s^H$ holds. Let $WH$ denote the Weyl group of $H$, i.e. the quotient $N_G(H)/H$, where $N_G(H)$ is the normalizer of $H$ in $G$.

Definition 4.3. We say that an equivariant map $f : U \subset X \to X$ is *taut*, if for all $H \subset G$ the restriction $f^H : U^H \to X^H$ is taut over $U_s^H$ in a neighbourhood of $U_s^H$ in $U^H$. For each $\varepsilon > 0$, we say that a $G$-map $f'$ is a taut $\varepsilon$-approximation of $f$ if there exists an equivariant $\varepsilon$-deformation $f_t$, $t \in I$, such that $f_0 = f$, and $f_1 = f'$ is taut.

Proposition 4.4. *If $C$ is a compact $G$-ENR, and $f : C \to X$ is a $G$-map, then for any $\varepsilon > 0$ there exists a taut $\varepsilon$-approximation of $f$.*

Proof. Because $f$ is defined on the compact set $C$, it suffices to prove the proposition for the identity $1_C : C \to C$. Let $\tau_0 = 1_C$. We want to define an $\varepsilon$-approximation of $1_C$, i.e. to extend $\tau_0$ to $C \times I$ such that $\tau_1^H$ is taut over $C_s^H$. We use induction over orbit types. For details about this procedure, see e.g. [25], [21], [10]. Assume that $\tau_t$ is defined on $C^H \times \{0\} \cup C_s^H \times I$. Then we can extend to a $WH$-equivariant $\varepsilon$-homotopy $\tau_t'$ on the whole $C^H$ (see e.g. Proposition of [25]). Moreover, for any $\varepsilon' > 0$, because $C_s^H$ is a $WH$-neighbourhood retract in $C^H$, by applying Proposition 4.2, we can find a $WH$-equivariant $\varepsilon'$-deformation $\phi_t$ of $1_{C^H}$, rel. $C_s^H$, which is taut over $C_s^H$. The composition $\tau_t'\phi_t$ is then a $2\varepsilon$-homotopy, for a suitable choice of $\varepsilon'$, extending $\tau_t | C^H \times \{0\} \cup C_s^H \times I$ and $\tau_1'\phi_1$ is taut over $C_s^H$. Because $C$ has only a finite number of orbit types, say $k$, this procedure yields an $\varepsilon$-approximation of $1_C$, if we start the induction with an $\varepsilon/k$-approximation, and hence the thesis. □

Proposition 4.5. *Let $C$ be a compact $G$-ENR. For each $\varepsilon > 0$ there exists an $\varepsilon$-deformation $\tau_t$ of the identity $1_{C\times I} = \tau_0$ of $C \times I$ such that for each subgroup $H \subset G$ the restriction $\tau_1^H$ is taut over $C^H \times \partial I \cup C_s^H \times I$ and $\tau_t$ is the identity whenever restricted to $C \times \{0\}$ and $C \times \{1\}$.*

Proof. We can use again induction over orbit types, and the same argument of the proof of Proposition 4.4, by virtue of the fact that $C^H \times \partial I \cup C_s^H \times I$ is a $WH$-neighbourhood retract in $C^H$ for all $H$ (it is the product property of $WH$-cofibrations, because in a $G$-ENR the inclusion of a neighbourhood retract is always a $G$-cofibration — see e.g. [6, Exercise 3, p. 84], or Theorem 1.9 of [23, p. 27], or [22]). Moreover, by construction $\tau_t^H$ is an homotopy rel. $C^H \times \partial I \cup C_s^H \times I$, and hence $\tau_t$ is the identity when restricted to $C \times \{0\}$ and $C \times \{1\}$.□

**4.2. Weakly taut $G$-maps and the definition of $\mathcal{I}_G(f)$.** Let $G$ be a compact Lie group, $X$ a $G$-ENR, $U \subset X$ an open subset, and $f : U \to X$ a compactly fixed $G$-map (i.e. $\mathrm{Fix}(f)$ is compact). As shown e.g. in [26], it could be possible to define an equivariant index of such an $f$, just by taking the sum (in $R$) of all the indexes $\mathcal{I}(f|U_H)$, but the resulting element of $R$ should not be a compactly fixed $G$-homotopical invariant of $f$ (there are simple examples of compactly fixed $G$-deformations of a map which do not preserve such an index), but only a $G$-compactly fixed invariant. Nevertheless, there is a class of maps which behaves well with respect to compactly fixed $G$-deformations.

Definition 4.6. We say that a map $\varphi : U \to X$ is *weakly taut* if it is compactly fixed in $U$ and for each isotropy group $H \subset G$ there is an open neighbourhood $W_H$ of $U_s^H$ in $U^H$ such that $\varphi(W_H) \subset U_s^H$. We say that $\varphi : U \to X$ is *weakly taut around* $\mathrm{Fix}(f)$ if there exists a $G$-neighbourhood $V$ of $\mathrm{Fix}(f)$ in $U$ such that the restriction $\varphi|V$ is weakly taut.

It is easy to see that a compactly fixed taut $G$-map is weakly taut, while the converse need not be true. Now we start defining the index for maps which are weakly taut around the fixed point set, and later we will show how to extend the definition to all the compactly fixed $G$-maps.

We say that an isotropy group $H \subset G$ is w-finite when its Weyl group $WH$ is finite. Let us consider the set of conjugacy classes of w-finite isotropy groups in $G$ for $U \subset X$, $\mathrm{Iso}_w(G,U)$, and let we choose a given isotropy group $H$ in each conjugacy class $(H) \in \mathrm{Iso}_w(G,U)$.

Definition 4.7. Let $U \subset X$ be a $G$-ENR, subspace of the $G$-space $X$, and let $f : U \to X$ be a compactly fixed $G$-map, weakly taut around $\mathrm{Fix}(f)$. Then we define its equivariant index

$$\mathcal{I}_G(f) = \sum_{(H)\in \mathrm{Iso}_w(G,C)} \mathcal{I}(f^H|U_H) \in R,$$

where $f^H|U_H : U_H \to X^H$ is the restriction of $f^H : U^H \to X^H$ to $U_H$.

The definition makes sense, because there are only a finite number of elements in $\mathrm{Iso}_w(G,C)$, the index $\mathcal{I}(f^H|U_H)$ does not depend upon the choice of $H$ in the conjugacy class $(H)$, and $U_H$ is an open subset of $X^H$; moreover, because $f^H$ is assumed to be weakly taut around $\mathrm{Fix}(f)$, the restriction $f^H|U_H$ is compactly fixed for each $H \subset G$.

In order to extend the definition to a general compactly fixed $G$-map, we need the following Lemma, the homotopy property for weakly taut maps.

Lemma 4.8. *Let $U$ be a $G$-ENR in $X$, and $f_t : U \to X$ a $G$-deformation, compactly fixed in $U$, such that $f_0$ and $f_1$ are weakly taut around their fixed point sets $\mathrm{Fix}(f_0)$ and $\mathrm{Fix}(f_1)$. Then*

$$\mathcal{I}_G(f_0) = \mathcal{I}_G(f_1).$$

Proof. Let $O$ be a neighbourhood which retracts on $U$ in a $G$-Euclidean space $\mathbb{R}^n$ and $G$-retraction $r : O \to U$; let $i : U \to O$ denote the inclusion, and let us assume $U$ to be closed in $O$. Because $f_t$ is compactly fixed, there exists a compact $G$-set $K_1$ in $O$ such that $\mathrm{Fix}(f_t) \subset K_1$ for all $t \in I$. Because $O$ is open in $\mathbb{R}^n$, there exist $G$-neighbourhoods of $K_1$, say $C$ and $K$, in $O$, such that $K_1 \subset C \subset \mathrm{int}K$, $C$ is a compact $G$-ENR, and $\mathrm{d}(f_t r(x), x) > \varepsilon_0 > 0$ for a given $\varepsilon_0$ and for all $x \in K \setminus C$. It turns out that $C \cap U$ and $K \cap U$ are compact

$G$-neighbourhoods of $K_1$. Now we can apply Proposition 4.5 to $C$ in order to have a map $\tau_1 : C \times I \to C \times I$, $\varepsilon$-homotopic to $1_{C\times I}$ for any $\varepsilon > 0$ and such that for each subgroup $H \subset G$ the restriction $\tau_1^H$ is taut over $C^H \times \partial I \cup C_s^H \times I$ and $\tau_t$ is the identity whenever restricted to $C \times \{0\}$ and $C \times \{1\}$.

This map extends trivially to a map $\tau_1' : C \times I \cup O \times \partial I \to C \times I \cup O \times \partial I$ by defining it to be the identity on $O \times \partial I$.

If $\varepsilon$ is small enough, this map extends to a map $\Xi : O \times I \to O \times I$ which is the identity outside $K \times I$ and which adds no new fixed points (we only have to check in $K \times I \setminus C \times I$), by the Tietze–Gleason Theorem (for details see e.g. [25]).

Now let us consider the composition

$$U \times I \xrightarrow{i\times 1_I} O \times I \xrightarrow{\Xi} O \times I \xrightarrow{r\times 1_I} U \times I \xrightarrow{f_t} X$$

which we denote with $F_t : U \times I \to X$. By definition, $F_t$ is exactly $f_0$ and $f_1$ when $t = 0, 1$.

We need to show that the restriction $F_t^H$ to $U^H$ is always a compactly fixed deformation in $U_H$. We start by remarking that, because $F_t$ coincides with $f_t$ outside the compact set $K \cap U$, $F_t$ is still a compactly fixed deformation. Hence the restriction $F_t^H$ is a compactly fixed deformation in $U^H$ for all $H$.

Because $f_0$ and $f_1$ are weakly taut around $\mathrm{Fix}(f_0)$ and $\mathrm{Fix}(f_1)$, there are open neighbourhoods $V_0$ and $V_1$ of $\mathrm{Fix}(f_0^H)$ and $\mathrm{Fix}(f_1^H)$ in $U^H$ such that $f_0^H(V_0) \subset X_s^H$ and $f_1^H(V_1) \subset X_s^H$. Now, because $C \setminus V_0$ and $C \setminus C_1$ are compact, there exist positive real numbers $\varepsilon_1$ and $\rho_1$ and a neighbourhood $W$ of $U_s^H \cap C$ in $C \cap U$ such that for all $x \in W$, the implications

$$\mathrm{d}(f_0(x), x) \le \varepsilon_1 \Rightarrow B_{\rho_1}(x) \subset V_0,$$

$$\mathrm{d}(f_1(x), x) \le \varepsilon_1 \Rightarrow B_{\rho_1}(x) \subset V_1,$$

hold true, where $B_{\rho_1}(x)$ denotes the ball of radius $\rho_1$ and center $x$.

Now, we know that $U_s^H \times I \cup U^H \times \partial I$ is a neighbourhood retract in $U^H \times I$. Therefore there exists a neighbourhood $W_1$ of $U_s^H$ in $U^H$ and a retraction $k : W_1 \times I \to U_s^H \times I \cup U^H \times \partial I$. Let $V \subset W_1 \times I$ be the counter-image $V := k^{-1}(W \times I)$. It is a neighbourhood of $U_s^H \times I$, and hence there exists a neighbourhood $W_2$ of $U_s^H \cap C$ in $C \cap U$ such that $W_2 \times I \subset V$. But now it is easy to see that for given $\varepsilon_1$ and $\rho_1$, there exists an $\varepsilon$ small enough such that for each $x \in W_2$ either the image $(r^H \times 1_I) \circ \Xi(x,t)^H$ is in $U_s^H \times I$ or it is of the form $(r^H \times 1_I) \circ \Xi(x,t)^H = (x', i)$ with $i \in \{0, 1\}$ and $\mathrm{d}(x, x')$ is small. But by assumption, either $\mathrm{d}(f_i(x'), x') > \varepsilon_i$ or $B_{\rho_1}(x')$ is contained in $V_i$, with $i = 0, 1$. In both the latter cases, this property implies that $\mathrm{Fix}(F_t^H) \cap U_H \subset C \cap U \setminus W_2$ for all $t$. Because $C \cap U \setminus W_2$ is a compact set, it is equivalent to say that $F_t^H$ is a compactly fixed homotopy in $U_H$.

So, for each $H$ there is a compactly fixed homotopy from $f_0^H|U_H$ to $f_1^H|U_H$, and by the Homotopy Property the equality $\mathcal{I}(f_0^H|U_H) = \mathcal{I}(f_1^H|U_H)$ holds, and hence also the wanted equality $\mathcal{I}_G(f_0) = \mathcal{I}_G(f_1)$ holds true. □

Remark 4.9. Lemma 4.8 was first proved for compactly fixed $G$-maps on compact $G$-ENR's by D. Wilczyński in 1984 [25], and independently by K. Komiya in 1987 [17], [18] in a different form (on smooth $G$-manifolds). Here we give a modified proof, closer to our needs and notation, which gives the same result on an arbitrary $G$-ENR. The concept itself of *weakly taut* maps is implicit in the papers [25], [17], [18].

A similar result (using transverse foliations on manifolds) has been presented also by Balanov–Kushkuley in [1].

With the following proposition we show that every $G$-map can be approximated by weakly taut maps; this will be the key step in order to define the index $\mathcal{I}_G$ of an arbitrary map.

Proposition 4.10. *Let $f : U \to X$ be a compactly fixed $G$-map. Then for all $\varepsilon > 0$ there exists a compactly fixed $\varepsilon$-approximation $f'$ of $f$ which is weakly taut around* $\mathrm{Fix}(f')$.

Proof. As before, let $r : O \to U$ be a $G$-retraction, where $O$ is an open set of an Euclidean $G$-space. We can take compact neighbourhoods $C$ and $K$ of $\mathrm{Fix}(f)$ in $O$ such that $C$ is a $G$-ENR contained in the interior $\mathrm{int}(K)$. By applying Proposition 4.4 to the composition

$$C \xrightarrow{r} U \xrightarrow{f} X$$

and extending the resulting $\varepsilon$-deformation of $rf$ to $O$ relatively to $O \setminus K$, with $\varepsilon$ small enough, we find a map $h : O \to X$. It is easy to see that the map $f' := hi : U \to X$ now is weakly taut around $\mathrm{Fix}(f')$. □

Finally, now we can define the index $\mathcal{I}_G$ of an arbitrary compactly fixed $G$-map $f : U \to X$.

Definition 4.11. Let $X$ be a $G$-space, $U \subset X$ a $G$-ENR, open subset of $X$, and $f : U \to X$ a compactly fixed $G$-map. Let $f'$ any compactly fixed $G$-deformation of $f$ which is weakly taut around its fixed point set $\mathrm{Fix}(f')$. Then we define

$$\mathcal{I}_G(f) := \mathcal{I}_G(f').$$

The definition makes sense, because any compactly fixed map $f$ has at least one weakly taut approximation, via a compactly fixed $G$-deformation (by Proposition 4.10), and any two such deformations, say $f'$ and $f''$, have the same index $\mathcal{I}_G(f') = \mathcal{I}_G(f'')$ because of Lemma 4.8.

**4.3. Properties of $\mathcal{I}_G(f)$.** The properties of the equivariant index $\mathcal{I}_G$ are similar to the properties of the non-equivariant index $\mathcal{I}$. We begin with the following two propositions, which are the most important, and also the easiest to prove (it suffices to apply definitions).

Proposition 4.12 (Homotopy Invariance). *Let $U \subset X$ be as above, and let $f_t : U \to X$ be a $G$-deformation. Then*

$$\mathcal{I}_G(f_0) = \mathcal{I}_G(f_1).$$

Corollary 4.13 (Lefschetz Property). *If $f_t : U \subset X \to X$, $t \in I$, is a compactly fixed $G$-deformation, such that* $\mathrm{Fix}(f_1) = \emptyset$, *then* $\mathcal{I}(f_0) = 0$.

If $W \subset U$ is an open $G$-subset of $U$, and $f : U \to X$ a $G$-map, then for each $H \in \mathrm{Iso}_w(U,G)$ there is a localization function $\rho_{W^H} : \mathcal{R}(f^H|W^H) \to \mathcal{R}(f^H)$ as defined in Proposition 2.4. Therefore it makes sense to state the following proposition (where the localization function $\overline{\rho}_W$ is defined as $\rho_{W^H}$ on the component $\mathcal{R}(f^H|W^H)$ of the disjoint union of all the $\mathcal{R}(f^H|W^H)$'s, with $H$ ranging in $\mathrm{Iso}_w(U,G)$).

Proposition 4.14 (Localization). *If $W$ is an open set such that* $\mathrm{Fix}(f) \subset W \subset U$ *then*

$$\mathcal{I}_G(f) = \overline{\rho}_W \mathcal{I}_G(f|W).$$

Proof. If $f$ is weakly taut around its fixed point set, then the proposition is trivial. Otherwise, it is enough to see that any weakly taut approximation $f'$ of $f$ is also weakly taut when restricted to $W$, and that by homotopy property of $\mathcal{R}$ (Proposition 2.5), for each $H$ the equalities $\mathcal{R}(f'^H) \cong \mathcal{R}(f^H)$ and $\mathcal{R}(f'^H|W^H) \cong \mathcal{R}(f^H|W^H)$ hold true. □

Proposition 4.15 (Unity). *If $f$ is constant, then $\mathcal{I}_G(f) = 0$ if $fU \notin U$ and $\mathcal{I}_G(f) = 1$ if $fU \in U$.*

Proof. The proof is immediate, because $f$ is already weakly taut around its fixed point set. □

Proposition 4.16 (Additivity). *If $U = \coprod_i U_i$ is the disjoint union of some open subsets $U_i \subset X$, then $\mathcal{I}_G(f) = \sum_i \mathcal{I}_G(f|U_i)$.*

Proof. If $f$ is weakly taut around $\mathrm{Fix}(f)$, then the conclusion follows from Proposition 3.6. Otherwise, a weakly taut approximation of $f$ in $U$ induces weakly taut approximations of the restrictions $f|U_i$, and hence the thesis. □

Proposition 4.17 (Multiplicativity). *Let $f : U \subset X \to X$ and $f' : U' \subset X' \to X'$ be given compactly fixed $G$-maps, and $f \times f' : U \times U' \to X \times X'$ their Cartesian product. Then the equality $\mathcal{I}_G(f \times f') = \mathcal{I}_G(f) \times \mathcal{I}_G(f')$ holds.*

Proof. The proof follows directly from Proposition 3.7. □

Proposition 4.18 (Retraction). *Let $f : U \subset X \to X$ be a compactly fixed $G$-map, $X \subset Y$ and $r : Y \to X$ a $G$-retraction. Then*

$$\mathcal{I}_G(f) = \mathcal{I}_G(ifr|r^{-1}U).$$

Proof. For each $H \in \mathrm{Iso}_w(U,G)$, we have the restricted map $f^H|U_H : U_H \to X^H$, and the corresponding retraction $r^H : X^H \to Y^H$. Without loss of generality we may assume that $f$ is weakly taut around $\mathrm{Fix}(f)$ (otherwise we see that any weakly taut approximation of $f$ gives rise to a weakly taut approximation of $ifr$). It may happen that $(r^{-1}U)_H$ is not equal to $r^{-1}(U_H)$, but using Proposition 3.3 it is possible to see that

$$\mathcal{I}((ifr)^H|(r^{-1}U)_H) = \mathcal{I}((ifr)^H|r^{-1}(U_H)),$$

because each Nielsen graph can be contained in $U_H$. Therefore, because of Proposition 3.12, the equality $\mathcal{I}(f^H|U_H) = \mathcal{I}((i^H f^H r^H|r^{-1}(U_H))$ is true, and so for each $H$ we have that $\mathcal{I}(f^H|U_H) = \mathcal{I}((ifr)^H|(r^{-1}U)_H)$ which is the thesis.□

The union property (Proposition 3.13) and the commutativity property (Proposition 3.10) can also be easily extended to this equivariant settings. Obviously, the assumptions have to be checked for each restriction $f^H|U_H$, with $H \in \mathrm{Iso}_w(U,G)$.

**4.4. The converse of the Lefschetz property.** It is natural to ask when the converse of the Lefschetz property 4.13 holds. More precisely, let $G$ be a compact Lie group acting on a $G$-ENR $X$, and let $U \subset X$ be an open equivariant subspace. Given a compactly fixed $G$-map $f : U \to X$ such that $\mathcal{I}_G(f) = 0$ when does there exist a fixed point free $G$-map $f'$, compactly fixed $G$-homotopic to $f$? As far as I know, it is necessary to use the equivariant Hopf construction and an equivariant version of the Wecken–Jiang Theorem, as done in [26], [27]. Both these techniques need a $G$-simplicial structure on $U$, in the sense of Illman [14].

Definition 4.19. A locally finite simplicial complex $K$ is a W-complex if either

(a) it has no local cut points and each connected component is not a surface or
(b) it is a 1-manifold.

If $G$ is a compact Lie group and $K$ a $G$-complex, then $K$ is a W-$G$-complex if for all subgroups $H \subset G$ the subspace $K_H$ is a W-complex.

Proposition 4.20. *Let $X$ be a locally finite $G$-complex and $U \subset X$ an open $G$-subspace which is a W-$G$-complex. Let $f : U \to X$ be a compactly fixed $G$-map with index $\mathcal{I}_G(f) = 0$. Then there is a fixed point free $G$-map $f'$ which is compactly fixed $G$-homotopic to $f$.*

Proof. Let $h$ be a weakly taut approximation of $f$. Then $h$ is $G$-compactly fixed in the sense of [26], and it is easy to see that, because $\mathcal{I}_G(h) = 0$, the equivariant Nielsen number $N^c_G(h) = 0$, as defined in [26]. Now it is easy to modify the proof of Theorem A of [25], using Theorem 4.3 and Corollary 5.7 of [26], in order to get the result. We omit the details. $\square$

Remark 4.21. There is a point, that in [26] the assumption is that $U$ is a $G$-complex of type S, instead of a W-$G$-complex as we have done. But it can be seen from the paper [15] that it suffices to assume $U$ to be a W-$G$-complex to get the result. Furthermore, the modification of the proof of Theorem A of [25] to this Nielsen environment is not completely straightforward, but essentially it involves well-known techniques in Nielsen fixed point theory; this is the reason for which we have omitted the details.

Remark 4.22. Another point is that the equivariant Nielsen number $N^c_G(f)$ of Wong, whenever $f$ is *weakly taut* around its fixed point set, (and therefore a $G$-compactly fixed map), is zero if and only if the index $\mathcal{I}_G(f)$ is zero; one might suspect that they are equivalent. This is not the case. The main difference is that $\mathcal{I}_G(f)$ is defined for all compactly fixed $G$-maps, and it is a $G$-homotopy invariant, while the latter is defined only on $G$-compactly fixed maps, and the deformations must be assumed to be $G$-compactly fixed too, as pointed out e.g. in Remark 2.7 of [26]. Moreover, in [26], Example 8 exhibits a $G$-compactly fixed map with non-vanishing $N^c_G(f)$ but with $\mathcal{I}_G(f) = 0$ (of course $f$ is not weakly taut around its fixed point set, otherwise both indices would be zero). On the other hand, it is not difficult to define also a map $f$ with $N^c_G(f) = 0$ and $\mathcal{I}_G(f) \neq 0$. On the bouquet of three circles $X = S^1 \vee S^1 \vee S^1 = a \vee b \vee c$, with the action of $G := \mathbb{Z}_2 = \langle r \rangle$ (group of order 2) given by $r(b) = c$: just take a deformation of the identity with one single fixed point (the base point of the bouquet) and equivariant with respect to the $G$-action. Then $f$ is $G$-compactly fixed, and $N^c_G(f) = 0$, but $\mathcal{I}_G(f) \neq 0$.

Corollary 4.23. *Let $G$ be a compact Lie group, $X$ a smooth $G$-manifold, $U \subset X$ an open $G$-subset and $f : U \to X$ a compactly fixed $G$-map. Then if for each $H \subset G$ the dimension of any component of $U_H$ is different from 2, $f$ can be compactly fixed $G$-deformed to be fixed point free if and only if $\mathcal{I}_G(f) = 0$.*

Proof. Every such a $G$-manifold can be triangulated, by [14], hence we can apply Proposition 4.20. $\square$

## 5. Some examples

In this section we show how it may happen that $\mathcal{I}_G(f) \neq 0$ for a self-map $f : X \to X$ defined on a smooth $G$-manifold, even if the $G$-Nielsen invariant of $f$ on $M$ is zero, i.e. even if for each isotropy group $H$ the Nielsen number

$N(f^H) = 0$, where $G$ is a finite group (see [26], [27] for details on the $G$-Nielsen invariant).

The first example of this kind was given by A. Vidal, in the Workshop "Dynamical Zeta functions, Nielsen theory and Reidemeister torsion (Warsaw, 1996)" [24]. Here we show that the same phenomenon can happen also for (closed) manifolds.

In the following example we show that it may happen that $\mathcal{I}_G(f) \neq 0$ and for all subgroups $H$ of $G$ the index $\mathcal{I}(f^H)$ is zero, even if the map $f : X \to X$ is the identity, $X$ is a compact smooth closed connected $G$-manifold and $G$ is the non-Abelian group of order 6.

Example 5.1. Let $T$ be the 2-dimensional torus, and $C$ a closed simple curve bounding a disc $B$ in $T$. Let $V$ be the plane, and

$$G = \langle r_1, r_2 | r_1^2 = r_2^2 = (r_1 r_2)^3 = 1\rangle$$

be as above the dihedral group $D_3$, acting on $V$ as reflections $r_1$ and $r_2$ along two lines meeting at an angle of $\pi/3$. Let us consider the $G$-space $W$ given by the representation $V$ plus three times the trivial (real) representation, i.e. $W$ is the direct sum of $V$ and a 3-space on which $G$ acts trivially. The isotropy groups of this $G$-action are 1, $H_1$ and $G$ up to conjugacy, where $H_1 = \langle r_1\rangle$ is the subgroup generated by $r_1$ in $G$. Then we may embed $T$ in $W^{H_1} \cong \mathbb{R}^4$ and we may assume that $C = T \cap W^G$, because $W^G$ is a 3-dimensional linear subspace of $W^{H_1}$. Also, $W^G$ cuts $W^{H_1}$ into two components $W_+^{H_1}$ and $W_-^{H_1}$ and $r_1 W_+^{H_1} = W_-^{H_1}$. Without loss of generality we may assume that $T \cap W_-^{H_1}$ is the complement of $B$ and $T \cap W_+^{H_1}$ is the interior of $B$.

Now if we have a $G$-regular neighbourhood $X$ of the $G$-space given by $GT$ in $W$ we obtain a $G$-space $G$-deformable to $GT$. Moreover, $X$ can be chosen to be a compact smooth $G$-manifold with boundary, and obviously is of dimension $5 = \dim W$. Finally, let $Y = 2X$ be the $G$-space obtained by joining two copies of $X$ along the boundary $\partial X$, with the identity identification map on $\partial X$. It is a compact smooth $G$-manifold of dimension 5 without boundary.

Let us compute Euler characteristic $\chi(Y^H)$ for any isotropy group $H$. Using the addition formula for $\chi$, we know that $\chi(Y) = 2\chi(X) - \chi(\partial X)$; $\partial X$ is a closed manifold of dimension 4 and has $\chi(\partial X) = 0$ by Lefschetz duality, if $\chi(X) = 0$; but $\chi(X) = \chi(GT) = 0$, hence $\chi(Y) = 0$. The subspace $Y^{H_1}$ fixed by $H_1$ is the sum of two copies of $X^{H_1}$ joined along their boundary, and $X^{H_1}$ is a regular neighbourhood of $T$ in $W^{H_1} \cong \mathbb{R}^4$, hence $X^{H_1} = T \times B^2$ where $B^2$ denotes a closed 2-disc, and $Y^{H_1} = T \times S^2$, where $S^2$ is the 2-sphere. Moreover, $X^G = C \times B^2$ and $Y^G = C \times S^2 \subset T \times S^2 = Y^{H_1}$ for the same reason. Hence $\chi(Y^{H_1}) = \chi(Y^G) = 0$ because $\chi(T) = \chi(C) = 0$.

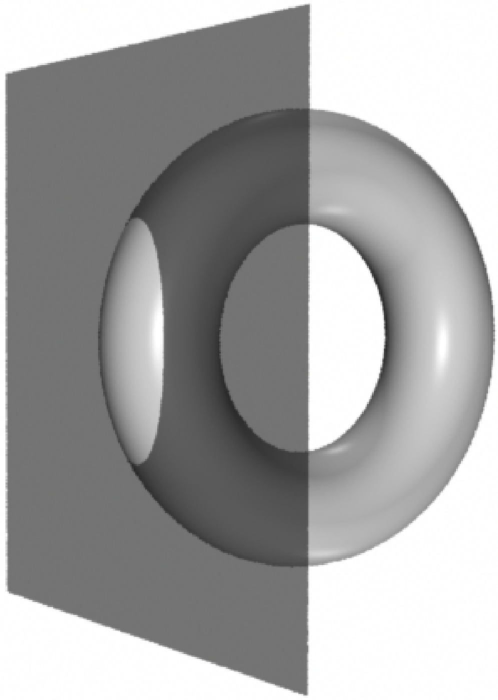

Figure 1. The torus $T$ included in the space $W^{H_1}$ and the plane $W^G$

Now we claim that $\mathcal{I}_G(1_Y) \neq 0$. This happens if and only if for some $H$ the index $\mathcal{I}(f|Y_H) \neq 0$ for a taut approximation $f$ of $1_Y$. Now consider $\mathcal{I}(f|Y_{H_1})$. The space $Y_{H_1}$ is exactly $Y^{H_1} \setminus Y^G$, hence it splits in the two connected components $Y \cap W_-^{H_1}$ and $Y \cap W_+^{H_1}$, which by assumption are $(T \setminus B) \times S^2$ and $\mathrm{int} B \times S^2$. It is now easy to see that $\mathcal{I}(f|\mathrm{int} B \times S^2) = z_2$ and $\mathcal{I}(f|(T \setminus B) \times S^2) = z_{-2}$ by multiplicativity of the index. Hence $\mathcal{I}_G(1_Y) \neq 0$.

By taking the product with $S^{2n}$ (trivial action of $G$ on $S^{2n}$) we can see that all the manifolds $Y \times S^{2n}$ have again nonzero index $\mathcal{I}_G(1_Y) \neq 0$ but $\chi(Y^H) = 0$ for all isotropy groups $H$. Hence there exist such closed smooth compact connected $G$-manifolds for all odd dimensions $\geq 5$.

In the previous example the property is essentially a group property of the group $G$ (the so-called gap condition in the dimension function of the representation ring is not fulfilled for the 3-dihedral group). Equivalently, what really happens geometrically is that $Y_{H_1}$ is not connected, because the codimension of $Y^G$ in $Y^{H_1}$ is 1, and at the same time $WH_1 = H_1$, so that the equivariant map must not be symmetric around $Y^G$.

In the following example we show what might happen even to the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ of order 2.

Example 5.2. Let $X_0$ be the wedge of three circles $a, b, c$ on a common point $x_0$ and $G$ the group $\mathbb{Z}/2\mathbb{Z}$ of order 2. The group $G$ acts by swapping $a$ and $b$, i.e. if $h : (a, x_0) \to (b, x_0)$ is a point-based homeomorphism between $a$ and $b$ then $g \neq 1$ in $G$ is defined by $gx := h(x)$ if $x \in a$, $gx = h^{-1}(x)$ if $x \in b$ and $gx := x$ if $x \in c$. We may as well embed $X_0$ in the Euclidean 3-space $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_i \in \mathbb{R}\}$ so that $G$ is the group generated by the reflection $x_3 \to -x_3$ and $c$ is contained in the plane $\{x_3 = 0\}$ (see Figure 2).

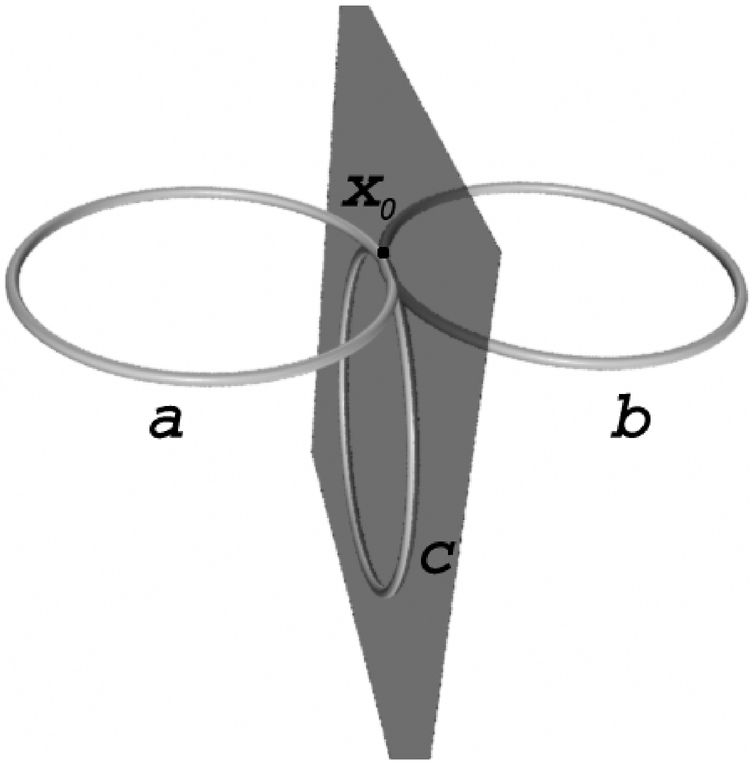


Figure 2. The bouquet of three circles

Let $f_0 : (a, x_0) \to (X_0, x_0)$ be the map defined by $f_0(a) := a^{-1}b^{-1}ab$ where with an abuse of notation we identify $a$ with the path $a : (I, \partial I) \to (a, x_0)$ and the equality is intended as an equality of homotopy classes rel. endpoints of paths. Moreover, let $f_0$ be the identity on $c$; there exists a unique $G$-map extending this map to the whole $X_0$ and it is still denoted by $f_0$. We may assume $f_0$ to be taut over $\{x_0\}$ in a small neighbourhood of $x_0$ and linear. It is easy to see that we can $\varepsilon$-deform $f_0$ in an equivariant way so that it has 4 fixed points $x_1, x_2$ in $a$

and $y_1, y_2 \in b$ of index respectively $+1, -1, +1, -1$ and $y_1 = gx_1, y_2 = gx_2$ (and with an abuse of notation we still call $f_0$ the deformed map).

Now by taking a small $G$-neighbourhood of $X_0$ in $\mathbb{R}^3$ we can find a compact smooth 3-manifold $M$ (with boundary) which retracts on $X_0$, as in Figure 3.

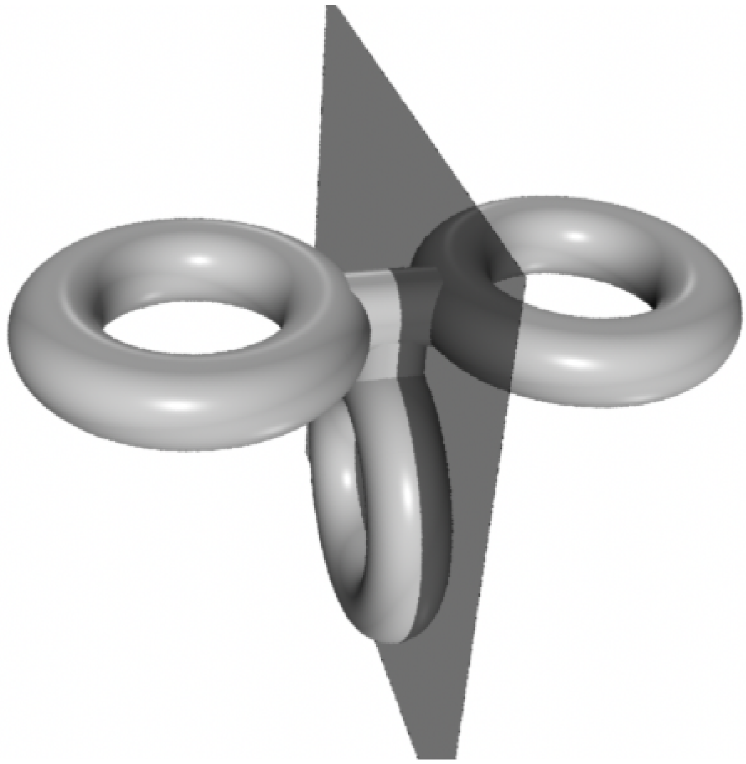

Figure 3. The thickened bouquet of circles

It is also easy to see that $X_0$ can be moved to the boundary of $M$ equivariantly, and we define $X$ to be the boundary of $M$. Let $r : X \to X_0$ be the $G$-retraction and $i : X_0 \to X$ the inclusion. The $G$-surface $X$ is closed compact and $G$-smooth. Let $f := if_0r$ be the composition of $f_0$ with $i$ and $r$.

We want to show now that the Nielsen numbers $N(f) = N(f^G) = 0$ (and hence $\mathcal{I}(f) = \mathcal{I}(f^G) = 0$) but $\mathcal{I}_G(f) \neq 0$.

First notice that $X^G = c \cup S^1$ and that there are no fixed points in $X^G$ (by definition of $f_0$). Hence $N(f^G) = 0$.

Let us now look at the fixed point classes of $f_0$ in $X_0$. There are at most two classes, $\{x_1, y_2\}$ and $\{x_2, y_1\}$, because there is a simple path $\lambda$ from $x_1$ to $y_2$ such that $\lambda \sim f(\lambda)$. But $x_1$ and $y_2$ have index $+1$ and $-1$ respectively, hence $N(f_0) = 0$. By commutativity, it follows that $N(f) = 0$.

We now have to compute $\mathcal{I}(f|D)$ for any connected component of $X_{(1)} = X \setminus X_s$. Let $D$ and $D'$ be the connected components of $X_{(1)}$. Then by equivariance $\mathcal{I}(f|D) = \mathcal{I}(f|D')$ where $g_* : \mathcal{R}(f;D) \cong \mathcal{R}(f;D')$. Let $D$ be the connected component where $a$ lies. We can take a small closed $G$-neighbourhood $Y$ of $X_0$ in $X$ and define a deformation $G$-retraction $R : X \to Y$. It can be shown that without loss of generality $R$ can be chosen such that $D = R^{-1}(E)$ where $E$ is the connected component of $Y \setminus Y_2$ containing $a$. Therefore because of the retract property of $\mathcal{I}$, $\mathcal{I}(f|D) = \mathcal{I}(f'|E)$ where $f' : Y \to Y$ is the $G$-map defined by $f'(x) = f_0(r(x))$ for all $x \in Y \subset X$.

So let us compute $\mathcal{I}(f'|E)$. Let $x_1$ be the base-point of $E$ and $Y$. If $w$ is a simple path in $X_0$ joining $w$ and $x_0$ which does not meet $x_2$, then $\pi_1(Y, x_1)$ is the free group generated by $\alpha := waw^{-1}$, $\beta := wbw^{-1}$ and $\gamma := wcw^{-1}$ and $\pi_1(D, X_1)$ is the free group on generators $\alpha'$ and $\gamma'$ where $j_\pi(\alpha') = \alpha$ and $j_\pi(\gamma') = \gamma$ if $j : E \to Y$ is the inclusion. The Reidemeister set $\mathcal{R}(f', E)$ is the set of orbits in $\pi_1(Y, x_1)$ under the action of $\pi_1(E, x_1)$ given by $\xi \cdot t := j_\pi(\xi) t f'_\pi(\xi^{-1})$ for all $\xi \in \pi_1(E, x_1)$ and $t \in \pi_1(Y, x_1)$.

By definition, the support of $\mathcal{I}(f'|E)$ is given by $\mathrm{cd}(x_1) \cup \mathrm{cd}(x_2)$ where $\mathrm{cd} : \mathrm{Fix}(f') \cap E \to \mathcal{R}(f'; E)$. More closely, $\mathrm{cd}(x_1) = [1]$ and $\mathrm{cd}(x_2) = [\beta^{-1}]$. The first equality is trivial. The second comes from the choice of the path $\lambda : (I, 0, 1) \to (E, x_1, x_2)$ given by $\lambda =$ the simple path in $X_0$ which does not meet $x_0$ from $x_1$ to $x_2$. We want to show that they belong to different Reidemeister orbits in $\mathcal{R}(f', E)$. As long as $f'_\pi(\alpha) = \alpha\beta\alpha^{-1}\beta^{-1}$ and $f'_\pi(\gamma) = \alpha\beta\gamma\beta^{-1}\alpha^{-1}$, the length of the word $\xi f'_\pi(\xi^{-1})$ can be $\leq 1$ if and only if $\xi = 1$, hence $b^{-1} \neq \xi f'_\pi(\xi^{-1})$ for all $\xi \in \pi_1(E, x_1)$ and the conclusion follows.

Therefore there are 2 distinct essential classes in $E$, and 2 in $gE$. By commutativity, the same assertion holds when $x_1$ and $x_2$ are thought as fixed point classes in $D$ of $f$, and hence $\mathcal{I}(f|X \setminus X_s) = 2z_1 + 2z_{-1} \neq 0$. Therefore $\mathcal{I}_G(f) \neq 0$.

Example 5.3. Let us consider again the dihedral group $G = D_3$ of Example 5.1 acting on the plane $V$, with reflections $r_1$ and $r_2$ along two lines meeting at an angle of $\pi/3$. Let us consider the $G$-space $W$ given by the representation $V$ plus $k \geq 1$ times the trivial (real) representation, i.e. $W$ is the direct sum of $V$ and a $k$-space on which $G$ acts trivially. The isotropy groups of this $G$-action are again 1, $H_1$ and $G$ up to conjugacy, where $H_1$ is as above. The action is orthogonal, hence it induces an action on the unit sphere

$$X = S^{k+1} = \{x \in W : |x|^2 = 1\} \subset W.$$

It is also easy to see that $X^G = S^{k-1}$, $X^{H_1} = S^k$, and $X^G$ is an equator of $X^{H_1}$ which is an equator of $X$ (see Figure 4).

Let us start now with the antipodal map $a_{k-1} : X^G \to X^G$, and its cone $Ca_{k-1} : CX^G \to CX^G$. Because $X^G$ cuts $X^{H_1}$ into two open $k$-balls $D^k_+$ and

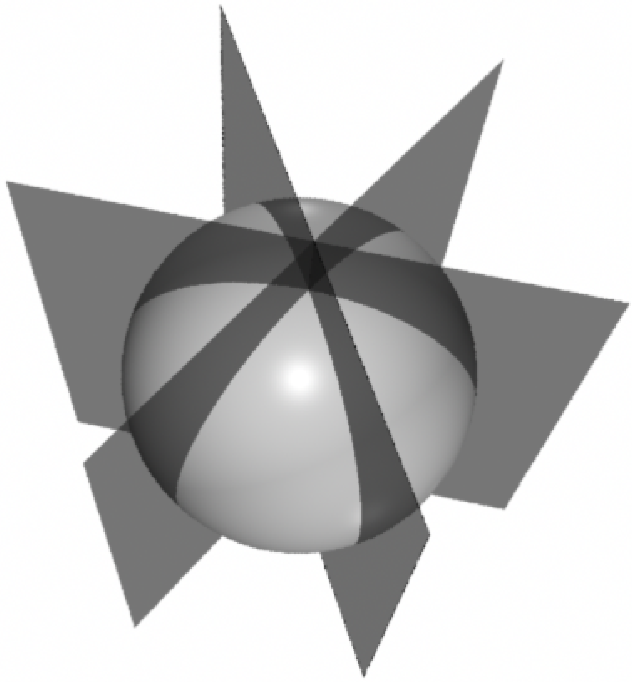

Figure 4. The sphere $X$ and the three planes for $(k = 1)$

$D^k_-$, we may identify $CX^G$ with the closure of $D^k_+$, and extend the map $a_{k-1}$ to $X^G \cup D^k_+$ as the cone map $Ca_{k-1}$. Now, because the inclusion of the closed ball $\overline{D}^k_+$ in $X^{H_1}$ is a cofibration, there is an extension of the map $Ca_{k-1} : \overline{D}^k_+ \to \overline{D}^k_+$ to the whole $X^{H_1}$, say $h : X^{H_1} \to X^{H_1}$ which is homotopic to the antipodal map $a_k : S^k = X^{H_1} \to X^{H_1}$. Also, without loss of generality, we can assume that $h$ has two fixed points $x_+$ in $D^k_+$ and $x_-$ in $D^k_-$ of index respectively $+1$ and $-1$, and that is taut over $X^G$. We can now extend, in a unique way, $h$ to an equivariant map $Gh : GX^{H_1} = X^1_s \to X^1_s$ defined on the singular set of $X$. The free locus of the action, $X_{(1)} = X \setminus X^1_s \subset X$, is the union of six open $k+1$-balls which have the boundary in $X^1_s$. Let us pick up one of them, and call it $D^{k+1}$. We can extend $Gh$ to $D^{k+1}$ in a way such that its image is contained in $X \setminus D^{k+1}$ and it is taut on $\partial D^{k+1}$. Then there is a unique equivariant map on $X$ which has these values on $D^{k+1}$ and let us call it $f : X \to X$.

Let us compute $\mathcal{I}_G(f) = \mathcal{I}(f^G|X_G) + \mathcal{I}(f^{H_1}|X_{H_1}) + \mathcal{I}(f|X_{(1)})$. Because by definition $f^G = a_{k-1}$, its fixed point index $\mathcal{I}(f^G|X_G) = 0$ is zero; on the other hand the fixed point index $\mathcal{I}(f^{H_1}|X_{H_1}) = \mathcal{I}(h|D^k_+) + \mathcal{I}(h|D^k_-) = 1 + z_{-1}$, and because there are no fixed point in $D^{k+1}$, the last fixed point index $\mathcal{I}(f|X_{(1)}) = 0$ is zero. Therefore $\mathcal{I}_G(f) = 1 + z_{-1}$ which is non-zero.

If we look at the restrictions $f$, $f^{H_1}$ and $f^G$, we see that they are homotopic to the antipodal maps $a_{k-1}$, $a_k$ and $a_{k+1}$, and hence $N(f) = N(f^{H_1}) = N(f^G) = 0$. Moreover, looking at the degrees, the maps $a_{k+1}$ and $f$ have the same set of degrees $\deg(a^H_{k+1}) = \deg(f^H)$ for every isotropy group $H \subset G$, while they are not $G$-homotopic. This is a well-known fact in equivariant homotopy of spheres; however, we included the example to show how the equivariant fixed point index might help in computations.

Remark 5.4. Looking at the examples above and at the definition of $\mathcal{I}_G(f)$ in Section 4, it is easy to see how to extend it to more general settings, like self-maps of stratified spaces, orbifolds or collared pairs. All the properties are carried out in the same way.

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