

**BIFURCATION OF PERIODIC SOLUTIONS
OF THE NAVIER–STOKES EQUATIONS
IN A THIN DOMAIN**

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ABSTRACT. Aim of this paper is to provide conditions in order to guarantee that the periodic solutions in time and in the space variables of the Navier–Stokes equations bifurcate. Specifically, we study this problem when the considered state domain has one dimension which is small with respect to the others which we let to tend to zero. The thinness of the domain represents the bifurcation parameter in our situation.

1. Introduction

The aim of this paper is to study the bifurcation of the periodic solutions of the incompressible Navier–Stokes equations in a thin three-dimensional domain $Q_\varepsilon = \Omega \times (0, \varepsilon)$. Here $\Omega \subset \mathbb{R}^2$ is a rectangle, and $\varepsilon > 0$ is a small parameter, which represents the parameter of the bifurcation problem. The Navier–Stokes equations have the form:

$$(1) \quad \begin{cases} \frac{\partial U}{\partial t} = \nu \Delta U - \nabla P - (U \cdot \nabla)U + F, \\ \nabla \cdot U = 0, \end{cases}$$

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for $t > 0$ and $(x_1, x_2, x_3) \in Q_\varepsilon$. As usual $U = (U_1, U_2, U_3)$ is a 3-dimensional vector function of (t, x_1, x_2, x_3) which represents the velocity of a fluid element at time t and at position (x_1, x_2, x_3) . The coefficient ν is the kinematic viscosity. The scalar quantity $P = P(t, x_1, x_2, x_3)$ is the pressure, and $F = F(t, x_1, x_2, x_3)$ is an external force, T -periodic in time.

We impose periodic boundary conditions on (1); thus we are in effect studying the Navier–Stokes equations in a three-dimensional torus which is thin in one direction. It turns out that, with these boundary conditions, problem (1) has a natural limit as $\varepsilon \rightarrow 0$; this limit is defined by the reduced Navier–Stokes equations in Ω . The passage to the limit is effectuated using properties of the Green’s function of a certain ordinary differential operator (see [2]). The fact that the Green’s function behaves regularly as $\varepsilon \rightarrow 0$ is of basic significance and seems to be noted explicitly for the first time in [2].

We will assume that the external force F satisfies

$$\int_{Q_\varepsilon} F(t, x_1, x_2, x_3) dx_1 dx_2 dx_3 = 0 \quad \text{for all } t \geq 0 \text{ and for small } \varepsilon > 0.$$

(If F is independent of t , this condition is actually necessary for the existence of non trivial periodic solutions of (1).) We will also assume that the reduced Navier–Stokes equations admit a T -periodic solution u_0 ; for simplicity we will assume that

$$\int_{\Omega} u_0(0, x_1, x_2) dx_1 dx_2 = 0.$$

We first solve the problem obtained from (1) by applying the Leray projector \mathbf{P}_ε on the divergence-free subspace of $L_2(Q_\varepsilon)$. The corresponding solutions of (1) are recovered by standard methods, see [1]. If the external force F is bounded then using the approach presented in [9] one can prove that there is always a T -periodic solution of (1).

As we showed in [2] the divergence-free part of these solutions converge to a T -periodic solution of the reduced equations as $\varepsilon \rightarrow 0$. Therefore there exists a continuous branch of T -periodic solutions parametrized by $\varepsilon > 0$.

The paper is organized as follows. In Section 2, we discuss certain basic facts concerning the formulation of problem (1). We introduce the reduced Navier–Stokes equations in Ω , and we formulate an abstract Krasnosel’skiĭ type bifurcation theorem (see e.g. [4]), which is the theoretical tool for solving our problem. In Section 3 we pose our assumptions and we state and prove our main Theorem on the existence of a second branch of T -periodic solutions of (1). Its proof is based on the results proved in [2].

2. Statement of the problem and preliminary results

Let us write again the problem which we will study:

$$(2) \quad \begin{cases} \frac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla)U + \nabla P = F(t, x_1, x_2, x_3), \\ \nabla \cdot U = 0, \end{cases}$$

where $t > 0$ and $(x_1, x_2, x_3) \in Q_\varepsilon$. Here $Q_\varepsilon = \Omega \times (0, \varepsilon)$, where $\Omega \subset \mathbb{R}^2$ is the rectangle $[0, \ell_1] \times [0, \ell_2]$, and ε is a small positive parameter. The problem (1) is to be viewed as an evolution equation for the velocity $U = U(t, x_1, x_2, x_3)$ and the pressure $P = P(t, x_1, x_2, x_3)$. We will study (1) in the presence of periodic boundary conditions:

- (2a) $U(t, 0, x_2, x_3) = U(t, \ell_1, x_2, x_3),$
- (2b) $U_{x_1}(t, 0, x_2, x_3) = U_{x_1}(t, \ell_1, x_2, x_3),$
- (2c) $U(t, x_1, 0, x_3) = U(t, x_1, \ell_2, x_3),$
- (2d) $U_{x_2}(t, x_1, 0, x_3) = U_{x_2}(t, x_1, \ell_2, x_3),$
- (2e) $U(t, x_1, x_2, 0) = U(t, x_1, x_2, \varepsilon),$
- (2f) $U_{x_3}(t, x_1, x_2, 0) = U_{x_3}(t, x_1, x_2, \varepsilon).$

With our methods we can treat various types of external force field F ; we indicate two possibilities. It is useful to think of F as a function depending parametrically on ε : we write $F = F_\varepsilon(t, x_1, x_2, x_3)$.

(i) Suppose that there exists $\varepsilon_0 > 0$ and a continuous function $F : [0, \infty) \times [0, \ell_1] \times [0, \ell_2] \times [0, \varepsilon_0] \rightarrow \mathbb{R}^3$, such that $F_\varepsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, x_3)$. In this case, we require that F be T -periodic in t for a fixed $T > 0$, and that

$$\int_{Q_\varepsilon} F(t, x_1, x_2, x_3) dx_1 dx_2 dx_3 = 0 \quad \text{for all } t \geq 0 \text{ and } 0 \leq \varepsilon \leq \varepsilon_0.$$

This last condition is satisfied if, for instance,

$$F(t, x_1, x_2, x_3) = \sum_{n=0}^{\infty} F_n(t, x_1, x_2) x_3^n,$$

where F_n is T -periodic with $\int_{\Omega} F_n(t, x_1, x_2) dx_1 dx_2 = 0$ ($n \geq 0, t \geq 0$).

(ii) Write $y = x_3/\varepsilon$ so that $0 \leq y \leq 1$, and let F be a continuous T -periodic mapping from $[0, \infty)$ to $L_2(Q)$ where $Q = [0, \ell_1] \times [0, \ell_2] \times [0, 1]$. Let $F_\varepsilon(t, x_1, x_2, x_3) = F(t, x_1, x_2, x_3/\varepsilon)$ for $0 \leq \varepsilon \leq \varepsilon_0$. We can write $F(t, x_1, x_2, y) = \sum_{n=-\infty}^{\infty} F_n(t, x_1, x_2) e^{2\pi i n y}$, we impose the sole condition that

$$\int_{\Omega} F_0(t, x_1, x_2) dx_1 dx_2 = 0 \quad \text{for all } t \geq 0.$$

We see that, as $\varepsilon \rightarrow 0$, F_ε converges weakly in $L_2(Q)$ to F_0 for each $t \geq 0$; it will turn out that this weak convergence is sufficient for the validity of our results.

We also suppose that the initial value $U_0(x_1, x_2, x_3) = U(0, x_1, x_2, x_3)$ satisfies $\int_{Q_\varepsilon} U_0(x_1, x_2, x_3) dx_1 dx_2 dx_3 = 0$. As remarked in [7], this implies that $\int_{Q_\varepsilon} U(t, x_1, x_2, x_3) dx_1 dx_2 dx_3 = 0$ for all $t \geq 0$.

We introduce the change of variables

$$\begin{aligned} x_1 &= x_1, & x_2 &= x_2, & x_3 &= \varepsilon y, \\ u(t, x_1, x_2, y) &= U(t, x_1, x_2, \varepsilon y), \\ p(t, x_1, x_2, y) &= P(t, x_1, x_2, \varepsilon y). \end{aligned}$$

Then equations (1) take the form

$$(1_\varepsilon) \quad \frac{\partial u}{\partial t} - \nu \Delta_\varepsilon u + (u \cdot \nabla_\varepsilon)u + \nabla_\varepsilon p = F(t, x_1, x_2, \varepsilon y),$$

where

$$\nabla_\varepsilon = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{1}{\varepsilon} \frac{\partial}{\partial y} \right) \quad \text{and} \quad \Delta_\varepsilon = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial y^2}$$

are singular differential operators. These differential operators act on functions defined on the fixed domain

$$Q = (0, \ell_1) \times (0, \ell_2) \times (0, 1).$$

The boundary conditions (2a)–(2f) become

$$\begin{aligned} (2a_\varepsilon) \quad & u(t, 0, x_2, y) = u(t, \ell_1, x_2, y), \\ (2b_\varepsilon) \quad & u_{x_1}(t, 0, x_2, y) = u_{x_1}(t, \ell_1, x_2, y), \\ (2c_\varepsilon) \quad & u(t, x_1, 0, y) = u(t, x_1, \ell_2, y), \\ (2d_\varepsilon) \quad & u_{x_2}(t, x_1, 0, y) = u_{x_2}(t, x_1, \ell_2, y), \\ (2e_\varepsilon) \quad & u(t, x_1, x_2, 0) = u(t, x_1, x_2, 1), \\ (2f_\varepsilon) \quad & u_y(t, x_1, x_2, 0) = u_y(t, x_1, x_2, 1). \end{aligned}$$

Let $L_2(Q)$ be the set of \mathbb{R}^3 -valued vector functions u on Q with finite norm $\|u\|_2 = (\int_Q |u(x_1, x_2, y)|^2 dx_1 dx_2 dy)^{1/2}$; here $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^3 . We emphasize that $L_2(Q)$ denotes a space of *vector* functions. The differential expression Δ_ε together with the conditions (2_ε) can be taken to define a self-adjoint operator on $L_2(Q)$, whose domain can be identified with the closure in the Sobolev space $H^2(Q)$ (of \mathbb{R}^3 -valued functions) of those vector fields defined on \overline{Q} which are C^∞ -smooth when extended periodically to \mathbb{R}^3 . We also introduce the spaces $L_p(Q)$ of \mathbb{R}^3 -valued functions on Q with finite norm $\|u\|_p = (\int_Q |u(x_1, x_2, y)|^p dx_1 dx_2 dy)^{1/p}$; Δ_ε and conditions (2_ε) define a closed unbounded linear operator on $L_p(Q)$ whose domain can be identified with the closure of the smooth periodic vector fields in the Sobolev space $W^{2,p}(Q)$.

Having said all this, we recall that, using a well-known method (e.g., [1], [6], [8]), one can reduce the study of problem (1_ε) , (2_ε) to that of an abstract evolution equation on the space of divergence-free vector fields on Q . In fact, let

$$H_\varepsilon = \text{cls} \left\{ u \in H^1_{\text{per}}(Q) \mid \nabla_\varepsilon \cdot u = 0 \text{ and } \int_Q u \, dx_1 \, dx_2 \, dx_3 = 0 \right\} \subset L_2(Q),$$

where the set $H^1_{\text{per}}(Q)$ of periodic vector fields in the Sobolev space $H^1(Q)$ is identified with a subset of $L_2(Q)$, and the closure (cls) is in $L_2(Q)$. Let $\mathbf{P}_\varepsilon : L_2(Q) \rightarrow H_\varepsilon$ be the orthogonal projection (the *Leray projector*). Define

$$\tilde{L}_\varepsilon = -\mathbf{P}_\varepsilon \Delta_\varepsilon,$$

where as stated Δ_ε is defined using the periodic boundary conditions (2_ε) . Further, define the bilinear operator B_ε on $H_\varepsilon \times H_\varepsilon$ by

$$B_\varepsilon(v, w) = \mathbf{P}_\varepsilon(v \cdot \nabla_\varepsilon)w.$$

Then problems (1_ε) , (2_ε) are equivalent to the problem

$$(3_\varepsilon) \quad \frac{\partial u}{\partial t} + \nu \tilde{L}_\varepsilon u + B_\varepsilon(u, u) = \mathbf{P}_\varepsilon F.$$

The equivalence is to be understood in the following sense. If $u = u(t, x_1, x_2, y)$ is a solution of (3_ε) , then there is a function $p = p(t, x_1, x_2, y)$, which defines an element of $H^1(Q)$ for each $t \in \mathbb{R}$, such that (u, p) is a solution of (1_ε) , (2_ε) . Moreover, the pressure p is unique up to an additive constant. (See [1], Proposition 1.6).

REMARKS 1. (a) Note that, in the presence of the periodic boundary conditions (2_ε) , we have

$$\mathbf{P}_\varepsilon \nabla_\varepsilon = \nabla_\varepsilon \mathbf{P}_\varepsilon \quad \text{and} \quad \mathbf{P}_\varepsilon \Delta_\varepsilon = \Delta_\varepsilon \mathbf{P}_\varepsilon,$$

where all operators act on $L_2(Q)$. This can be proved by an elementary Fourier series argument; see [1, p. 43] and [7].

(b) Note that 0 is an eigenvalue of \tilde{L}_ε , and that the rest of the spectrum of \tilde{L}_ε lies in the positive real half-line and is discrete. In fact, for each $\gamma > 0$, $(\gamma I + \tilde{L}_\varepsilon)^{-1}$ is a compact self-adjoint operator. The same remarks apply to the operator Δ_ε on $L_2(Q)$.

The equation (3_ε) is of parabolic type, and can be written in the abstract form

$$(4_\varepsilon) \quad u' + \nu \tilde{L}_\varepsilon u = \tilde{f}(t, u, \varepsilon),$$

where $\tilde{f}(t, u, \varepsilon) = -B_\varepsilon(u, u) + \mathbf{P}_\varepsilon F$, ($u \in H_\varepsilon$). It is convenient to introduce a positive constant γ and add a term γI to both sides of (4_ε) . Writing

$$L_\varepsilon = \nu \tilde{L}_\varepsilon + \gamma I, \quad f = \tilde{f} + \gamma I,$$

we transform (4_ε) into

$$(5_\varepsilon) \quad u' + L_\varepsilon u = f(t, u, \varepsilon), \quad u \in H_\varepsilon.$$

The positive constant γ will be held fixed for the rest of the paper.

Roughly speaking, we propose to solve (5_ε) by “multiplying (5_ε) by $e^{L_\varepsilon t}$ and integrating”. This will “work” because of two facts (among others). First, L_ε is positive and self-adjoint, hence $e^{-L_\varepsilon t}$ is an analytic semigroup. Second, the nonlinear term in (5_ε) is subordinate to a fractional power of L_ε . This basic observation has been exploited in [9], [5].

We pause for a brief review of the basic facts about analytic semigroups and fractional powers of operators which we will need. As a general reference for these results we give [5].

Let A be a closed linear operator in a Banach space E . Suppose that, for all complex numbers λ satisfying $\operatorname{Re} \lambda \geq \sigma$, the resolvent $(\lambda I + A)^{-1}$ exists and satisfies the following inequality:

$$(6) \quad \|(\lambda I + A)^{-1}\| \leq \frac{c}{1 + |\lambda|}.$$

Then A generates an analytic semigroup e^{-At} , and moreover the fractional powers A^z are defined for $\operatorname{Re} z \neq 0$ (see [5]). If A is unbounded, then A^z is unbounded for $\operatorname{Re} z > 0$.

If $w \in \mathbf{C}$ and $\tau \in (-\pi, \pi)$, define the ray

$$\Gamma(\tau, w) = \{\lambda \in \mathbf{C} \mid \lambda = w + \rho e^{i\tau}, \rho \in [0, \infty)\}.$$

Let $\sigma \in \mathbb{R}$ and c be as above, and fix $\beta \in (\pi/2, \pi/2 + \arcsin(1/c))$. Set

$$\Gamma_1 = \Gamma(-\beta, \sigma), \quad \Gamma_2 = \Gamma(\beta, \sigma),$$

and write $\Gamma_1 \cup \Gamma_2$ for the curve obtained by joining Γ_1 and Γ_2 at the vertex σ . We traverse this curve from bottom to top, i.e., so that the spectrum of $-A$ is to the left of $\Gamma_1 \cup \Gamma_2$. Then the following formula is valid:

$$(7) \quad e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} e^{\lambda t} (\lambda I + A)^{-1} d\lambda.$$

One has further that, for $t > 0$ and $\alpha \geq 0$,

$$(8) \quad A^\alpha e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} \lambda^\alpha e^{\lambda t} (\lambda I + A)^{-1} d\lambda.$$

For the negative fractional powers $-1 < -\alpha < 0$, one has

$$(9) \quad A^{-\alpha} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty t^{-\alpha} (tI + A)^{-1} dt.$$

We note finally the following estimate:

$$(10) \quad \|A^\alpha e^{-At}\| \leq \frac{C(\alpha, c, \sigma)}{t^\alpha} \quad (t > 0, \alpha \geq 0),$$

where C depends on the quantities indicated but not on t .

Observe that for each fixed $\varepsilon > 0$ the operator L_ε satisfies (6) for $\text{Re}\lambda \geq \sigma = 0$. As is shown in [2] the constant $c(\varepsilon)$ in (6) can be fixed independently of ε . Later, we will discuss the proof of this basic fact. As in [2], by a T -periodic solution of problem (5_ε) we mean a function $u = u(\cdot) \in C_T(H_\varepsilon)$ which satisfies the equation

$$u(t) = e^{-L_\varepsilon t}(I - e^{-L_\varepsilon T})^{-1} \int_0^T L_\varepsilon^\alpha e^{-L_\varepsilon(T-s)} f(s, L_\varepsilon^{-\alpha} u(s), \varepsilon) ds + \int_0^t L_\varepsilon^\alpha e^{-L_\varepsilon(t-s)} f(s, L_\varepsilon^{-\alpha} u(s), \varepsilon) ds.$$

Here $C_T(H_\varepsilon)$ with the usual sup-norm is the Banach space of T -periodic, continuous mappings from \mathbb{R} to H_ε and $e^{-L_\varepsilon t}$ is the analytic semigroup on H_ε defined by the operator L_ε , and α is an appropriate positive number. In our considerations $\alpha = 3/4$.

Let us denote by $\tilde{\Phi}_\varepsilon(u)(t)$ the right hand side of the above equation. It is convenient to extend the domain of definition of $\tilde{\Phi}_\varepsilon$ from $C_T(H_\varepsilon)$ to $C_T(L_2(Q))$. For this, let Δ_ε be defined as above by the differential expression

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial y^2}$$

together with the periodic boundary conditions (2_ε) . Let

$$D_\varepsilon = \gamma I - \nu \Delta_\varepsilon,$$

so that D_ε is a positive definite, self-adjoint operator on $L_2(Q)$. Define $\Phi_\varepsilon : C_T(L_2(Q)) \rightarrow C_T(L_2(Q))$ by

$$(11) \quad \Phi_\varepsilon(u)(t) = e^{-D_\varepsilon t}(I - e^{-D_\varepsilon T})^{-1} \int_0^T D_\varepsilon^\alpha e^{-D_\varepsilon(T-s)} f(s, D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon u(s), \varepsilon) ds + \int_0^t D_\varepsilon^\alpha e^{-D_\varepsilon(t-s)} f(s, D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon u(s), \varepsilon) ds.$$

It turns out that if $\alpha = 3/4$ and $\varepsilon > 0$, the map $\Phi_\varepsilon : C_T(L_2(Q)) \rightarrow C_T(L_2(Q))$ is well-defined and completely continuous, see ([2], Lemma 2.2). Observe that the image of Φ_ε actually lies in H_ε . This is because $\mathbf{P}_\varepsilon D_\varepsilon = D_\varepsilon \mathbf{P}_\varepsilon$. Moreover, $\Phi_\varepsilon(u) = \tilde{\Phi}_\varepsilon(u)$ whenever $u = u(\cdot)$ is in $C_T(H_\varepsilon)$. So the fixed points of Φ_ε coincide with those of $\tilde{\Phi}_\varepsilon$. From now on, we will study almost exclusively the map Φ_ε .

We recall that in [2] we gave general sufficient conditions for the existence of a continuous branch $\{u^\varepsilon\}$ of functions in $L_2(Q)$ so that, for each sufficiently

small $\varepsilon > 0$, w^ε is a T -periodic solution of problem (5 $_\varepsilon$). By a continuous branch we mean a continuum which can be parametrized continuously by the single real variable ε .

To solve the fixed point problem for the operator defined in (11) is equivalent, see [5, p. 501], to solving the fixed point problem for the quasi-translation operator which, in our case, is defined as follows: consider the solution w^ε of the operator equation

$$(12) \quad w(t) = e^{-D_\varepsilon t} w_0 + \int_0^t D_\varepsilon^\alpha e^{-D_\varepsilon(t-s)} f(s, D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon w(s), \varepsilon) ds,$$

in the space $C([0, T], L_2(Q))$. Here we consider only the initial conditions $w_0 \in L_2(Q)$ for which the solution of the equation (12) exists on all of $[0, T]$. Since $f(s, D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon w, \varepsilon)$ satisfies a local Lipschitz condition, the solution w^ε of the equation (12) is unique. The quasi-translation operator W is then defined by the formula

$$(13) \quad W(\varepsilon, w_0) = w^\varepsilon(T).$$

Next we introduce the “reduced” Navier–Stokes equations in Ω . These are obtained by passing to the limit as $\varepsilon \rightarrow 0$ in (5 $_\varepsilon$). Write \mathbf{P}_2 for the Leray projector in $L_2(\Omega)$; thus \mathbf{P}_2 is the orthogonal projection onto the subspace

$$\left\{ v \in H^1_{\text{per}}(\Omega) \mid \nabla_2 \cdot v = 0 \text{ and } \int_\Omega v(x_1, x_2) dx_1 dx_2 = 0 \right\}.$$

Here ∇_2 is the two-dimensional divergence. We have written $L_2(\Omega)$, resp. $H^1(\Omega)$ to indicate the appropriate spaces of \mathbb{R}^2 -valued functions.

Define $M : L_2(Q) \rightarrow L_2(Q)$ to be the projection obtained by integrating with respect to the y -variable:

$$(Mu)(x_1, x_2) = \int_0^1 u(x_1, x_2, y) dy.$$

Thus we agree to identify the function Mu with the element \tilde{u} of $L_2(Q)$ defined by $\tilde{u}(x_1, x_2, y) = Mu(x_1, x_2)$. We recall the following fact, proved in [7]: let $g \in L_2(Q)$ be a vector field which depends only on the variables $(x_1, x_2) \in \Omega$ and such that $\int_\Omega g_3(x_1, x_2) dx_1 dx_2 = 0$. Then one has

$$(14) \quad \mathbf{P}_\varepsilon \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ g_3 \end{pmatrix}.$$

Following [7, p. 513], we write $\bar{u} = Mu$ and $w = (I - M)u$. Applying M and $I - M$ to (3 $_{\varepsilon}$), we get

$$\begin{aligned}
 (15) \quad & \frac{\partial \bar{u}}{\partial t} + \nu \tilde{L}_{\varepsilon} \bar{u} + B_{\varepsilon}(\bar{u}, \bar{u}) = M \mathbf{P}_{\varepsilon} F \\
 & \quad - M[B_{\varepsilon}(\bar{u}, w) + B_{\varepsilon}(w, \bar{u}) + B_{\varepsilon}(w, w)] \\
 & \frac{\partial w}{\partial t} + \nu \tilde{L}_{\varepsilon} w = (I - M) \mathbf{P}_{\varepsilon} F \\
 & \quad - (I - M)[B_{\varepsilon}(\bar{u}, w) + B_{\varepsilon}(w, \bar{u}) + B_{\varepsilon}(w, w)].
 \end{aligned}$$

If $\mathbf{P}_{\varepsilon} F$ depends only on (t, x_1, x_2) , and if the initial value $w(0) = 0$, then $w(t) = 0$ for all $t > 0$, and the first equation in (15) becomes an equation for \bar{u} alone. To obtain the reduced Navier-Stokes equations, we replace the forcing function F_{ε} by an appropriate limiting function F_0 . Let us consider the two cases discussed earlier.

- (i) If $F_{\varepsilon}(t, x_1, x_2, x_3) = F(t, x_1, x_2, x_3)$ for a fixed, T -periodic continuous function defined on $[0, \infty) \times [0, \ell_1] \times [0, \ell_2] \times [0, \varepsilon_0]$, we set $F_0(t, x_1, x_2) = F(t, x_1, x_2, 0)$.
- (ii) If $F_{\varepsilon}(t, x_1, x_2, x_3) = \sum_{n=-\infty}^{\infty} F_n(t, x_1, x_2) e^{2\pi i n y}$, where $y = x_3/\varepsilon$ and where the series on the right defines a continuous, T -periodic function of $[0, \infty)$ into $L_2(Q)$, then we let $F_0(t, x_1, x_2)$ be the 0th Fourier coefficient in the expansion of F_{ε} .

In case (i), $\mathbf{P}_{\varepsilon} F_{\varepsilon} \rightarrow F_0$ strongly in $L_2(Q)$. In case (ii), $\mathbf{P}_{\varepsilon} F_{\varepsilon} \rightarrow F_0$ weakly in $L_2(Q)$ as $\varepsilon \rightarrow 0$. Our results will require only the weak convergence of $\mathbf{P}_{\varepsilon} F_{\varepsilon}$ to F_0 as $\varepsilon \rightarrow 0$.

The reduced Navier-Stokes equations are the equations for $\bar{u} = Mu$ in (15) when F_0 is substituted for F . We can rewrite the reduced Navier-Stokes equations in the following form. Write $\bar{u} = Mu = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$. Put $\bar{v} = (\bar{u}_1, \bar{u}_2)$. Then the equations for \bar{u} in (15) are

$$\begin{aligned}
 (16) \quad & \frac{\partial \bar{v}}{\partial t} - \nu \mathbf{P}_2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \bar{v} + \mathbf{P}_2(\bar{v} \cdot \nabla_2) \bar{v} = \mathbf{P}_2 \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}, \\
 & \frac{\partial \bar{u}_3}{\partial t} - \nu \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \bar{u}_3 + \left(\bar{u}_1 \frac{\partial}{\partial x_1} + \bar{u}_2 \frac{\partial}{\partial x_2} \right) \bar{u}_3 = F_{03}.
 \end{aligned}$$

These equations are supplemented by the periodic boundary conditions

- (17a) $\bar{u}(t, 0, x_2) = \bar{u}(t, \ell_1, x_2),$
- (17b) $\bar{u}_{x_1}(t, 0, x_2) = \bar{u}_{x_1}(t, \ell_1, x_2),$
- (17c) $\bar{u}(t, x_1, 0) = \bar{u}(t, x_1, \ell_2),$
- (17d) $\bar{u}_{x_2}(t, x_1, 0) = \bar{u}_{x_2}(t, x_1, \ell_2).$

We also impose the mean value condition

$$(17e) \quad \int_{\Omega} \bar{u}(0, x_1, x_2) dx_1 dx_2 = 0.$$

This last condition implies that $\int_{\Omega} \bar{u}(t, x_1, x_2) dx_1 dx_2 = 0$ for all $t \geq 0$.

Now if the equation (5 $_{\varepsilon}$) has a T -periodic solution $u^{\varepsilon} \in H_{\varepsilon}$ then we can localize the Navier–Stokes equations around it. In fact if $u \in H_{\varepsilon}$ write $u = w + u^{\varepsilon}$ in (5 $_{\varepsilon}$) to obtain

$$(18) \quad w' + L_{\varepsilon}w - \gamma w + B_{\varepsilon}(u^{\varepsilon}, w) + B_{\varepsilon}(w, u^{\varepsilon}) + B_{\varepsilon}(w, w) = 0.$$

We will refer to its linear part

$$(19) \quad w' + L_{\varepsilon}w - \gamma w + B_{\varepsilon}(u^{\varepsilon}, w) + B_{\varepsilon}(w, u^{\varepsilon}) = 0$$

as the linearized equation.

If u^{ε} , $\varepsilon \geq 0$, are uniformly bounded in $L_2(Q)$ then by using the approach of [2] one can prove that for every sequence $\varepsilon_n \rightarrow 0$ the sequence $\{u^{\varepsilon_n}\}$ has a limit point which represents a T -periodic solution u^0 of the reduced equation (16). Let us define

$$H_0 = \left\{ g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in L_2(Q) \mid g_i = g_i(x_1, x_2), 1 \leq i \leq 3, \right. \\ \left. \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \mathbf{P}_2 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \text{ and } \int_{\Omega} g(x_1, x_2) dx_1 dx_2 = 0 \right\}.$$

Then $u^0 \in H_0$. We can now proceed as before by letting $u = v + u^0$ in (16), where $u \in H_0$, we obtain

$$(20) \quad v' + L_0v - \gamma v + B_0(u^0, v) + B_0(v, u^0) + B_0(v, v) = 0,$$

here

$$L_0 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\nu \mathbf{P}_2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ -\nu \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) v_3 \end{pmatrix} + \gamma \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

together the boundary conditions (17a)–(17d). The operator B_0 is given by

$$B_0(w, v) = \begin{pmatrix} \mathbf{P}_2 \begin{pmatrix} w_1 \frac{\partial v_1}{\partial x_1} + w_2 \frac{\partial v_1}{\partial x_2} \\ w_1 \frac{\partial v_2}{\partial x_1} + w_2 \frac{\partial v_2}{\partial x_2} \\ w_1 \frac{\partial v_3}{\partial x_1} + w_2 \frac{\partial v_3}{\partial x_2} \end{pmatrix} \end{pmatrix}$$

with

$$\begin{aligned} v(x_1, x_2) &= (v_1(x_1, x_2), v_2(x_1, x_2), v_3(x_1, x_2)), \\ w(x_1, x_2) &= (w_1(x_1, x_2), w_2(x_1, x_2), w_3(x_1, x_2)), \end{aligned}$$

and \mathbf{P}_2 is the Leray projector in $L_2(\Omega)$.

The linearized equation corresponding to (20) has the following form

$$(21) \quad v' + L_0v - \gamma v + B_0(u^0, v) + B_0(v, u^0) = 0.$$

It is convenient to rewrite the quasi-translation operator $W(\varepsilon, w_0)$ for the equation (18). For this it is sufficient to replace in (12) the term $f(s, \phi, \varepsilon)$ by $h(\phi, \varepsilon) = \gamma\phi - B_\varepsilon(u^\varepsilon, \phi) - B_\varepsilon(\phi, u^\varepsilon) - B_\varepsilon(\phi, \phi)$, where $\phi = D_\varepsilon^{-\alpha}\mathbf{P}_\varepsilon w$. We obtain

$$w(t) = e^{-D_\varepsilon t}w_0 + \int_0^t D_\varepsilon^\alpha e^{-D_\varepsilon(t-s)}h(D_\varepsilon^{-\alpha}\mathbf{P}_\varepsilon w(s), \varepsilon) ds.$$

Observe that, in this case, the unique solution corresponding to $w_0 = 0$ is the zero solution.

We also need the quasi-translation operator corresponding to (20). Taking (14) and (16) into account, we see that the quasi-translation operator V for (20) defined on $ML_2(Q)$ is

$$w(t) = e^{-D_0 t}Mw_0 + \int_0^t D_0^\alpha e^{-D_0(t-s)}h_0(D_0^{-\alpha}\mathbf{P}_\varepsilon Mw(s), \varepsilon) ds,$$

where $w_0 \in L_2(Q)$, $h_0(v) = \gamma v - B_0(u^0, v) - B_0(v, u^0) - B_0(v, v)$ and D_0 is the operator generated by $(\gamma I - \nu\mathbf{P}_\varepsilon\Delta_2)$ and the boundary conditions (17a)–(17d). Here $\Delta_2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. As will see Proposition 1 in the sequel will permit to show, (Proposition 2), that $W(\varepsilon, w_0) \rightarrow VMw_0$ as $\varepsilon \rightarrow 0$, uniformly with respect to w_0 in a bounded set of $L_2(Q)$. We note in particular that the said uniform convergence of $W(\varepsilon, w_0) \rightarrow VMw_0$ as $\varepsilon \rightarrow 0$ requires only the weak convergence of $\mathbf{P}_\varepsilon F_\varepsilon$ to F_0 in $L_2(Q)$, because this implies the strong convergence of u^ε to u^0 .

The main tool for proving our bifurcation result is the following well-known theorem which we formulate here in an abstract form suitable for our purposes. Here $\mathcal{B}_E(0, d)$ denotes the ball in the Banach space E centered at 0 with radius $d > 0$.

THEOREM 1. *Let $W : [0, \varepsilon_0) \times \mathcal{B}_E(0, d) \rightarrow E$ be a compact operator. Suppose that*

$$(22) \quad W(\varepsilon, w) = W'(\varepsilon, 0)w + H(\varepsilon, w, w),$$

where $W'(\varepsilon, 0)$ denotes the derivative of $W(\varepsilon, \cdot)$ with respect to the second variable calculated at 0, and $H(\varepsilon, \cdot, \cdot)$ is a bilinear form. Furthermore, assume that $\|H(\varepsilon, h, h)\|$ is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$, $h \in \partial\mathcal{B}_E(0, 1)$ and

that $W'(\varepsilon, 0)$ is continuous in the operator norm with respect to $\varepsilon > 0$. Finally, assume that the following conditions are satisfied:

- (C1) 1 is a simple eigenvalue of $W'(0, 0)$,
- (C2) $1 \notin \sigma(W'(\varepsilon, 0))$ for $\varepsilon \in (0, \varepsilon_0)$,
- (C3) $\zeta_0 = \langle H(0, e_0, e_0), g_0^* \rangle \neq 0$ where $e_0 \in E$, $\|e_0\| = 1$, $W'(0, 0)e_0 = e_0$, $g_0^* \in E^*$, $\|g_0^*\| = 1$, $W'^*(0, 0)g_0^* = g_0^*$, $\langle e_0, g_0^* \rangle = 1$.

Then there exists $d_0 > 0$ such that for $\varepsilon > 0$ sufficiently small the equation

$$w = W(\varepsilon, w)$$

has a unique non zero solution $w^\varepsilon \in \mathcal{B}_E(0, d_0)$, which depends continuously on ε and $w^\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$.

We will need also the following crucial result stated in [2], which we report below in some detail for the reader's convenience. Here $\partial_i = \partial/\partial x_i$ ($i = 1, 2$), and $\partial_3 = (1/\varepsilon)(\partial/\partial y)$.

PROPOSITION 1. As $\varepsilon \rightarrow 0$, we have for each $\alpha \in (0, 1)$ and $1 < p < \infty$:

$$D_\varepsilon^{-\alpha} \rightarrow D_0^{-\alpha}M, \quad e^{-D_\varepsilon t} \rightarrow e^{-D_0 t}M, \quad D_\varepsilon^\alpha e^{-D_\varepsilon t} \rightarrow D_0^\alpha e^{-D_0 t}M \quad (t > 0).$$

The convergences are in the operator norm on the space of bounded linear operators from $L_p(Q)$ to $L_p(Q)$. Moreover,

$$\|D_\varepsilon^\alpha e^{-D_\varepsilon t}\|_{L_p(Q) \rightarrow L_p(Q)} \quad \text{and} \quad \|D_0^\alpha e^{-D_0 t}M\|_{L_p(Q) \rightarrow L_p(Q)} \leq C(\alpha)/t^\alpha$$

where the constant $C(\alpha)$ is independent of $t > 0$ and of ε . If $\alpha = 3/4$, we also have

$$D_\varepsilon^{-3/4} \rightarrow D_0^{-3/4}M, \quad \partial_i D_\varepsilon^{-3/4} \rightarrow \partial_i D_0^{-3/4}M \quad (i = 1, 2), \quad \partial_3 D_\varepsilon^{-3/4} \rightarrow 0.$$

Here the convergences are with respect to the operator norm on the space of bounded linear transformations from $L_2(Q)$ to $L_q(Q)$, $2 \leq q < 3$.

PROOF. We discuss how this result can be proved. Let $1 < p < \infty$, and view D_ε as a bounded linear operator on $L_p(Q)$. For simplicity we normalize and set the kinematic viscosity $\nu = 1$.

The first step is to prove the basic estimates:

$$(23) \quad \|(\lambda I + D_\varepsilon)^{-1}\| \leq \frac{c}{1 + |\lambda|} \quad (\text{Re } \lambda \geq 0),$$

$$(24) \quad \|\partial_i(\lambda I + D_\varepsilon)^{-1}\| \leq \frac{\hat{c}}{\sqrt{1 + |\lambda|}} \quad (\text{Re } \lambda \geq 0, \quad i = 1, 2, 3),$$

where the constants c and \hat{c} do not depend on ε . To this end, we introduce the formula of Grisvard. Let A_1 and A_2 be two closed linear operators in a Banach space E whose domains have dense intersection D . Suppose that A_1 and A_2

commute on a dense subset of D . Suppose that the resolvents exist and satisfy

$$\|(\lambda I + A_i)^{-1}\| \leq \frac{c}{1 + |\lambda|} \quad \text{for } \operatorname{Re} \lambda \geq 0, \quad i = 1, 2.$$

Let $\Gamma_1 = \Gamma(-\beta, 0)$ and $\Gamma_2 = \Gamma(\beta, 0)$ be the curves introduced earlier, and let $\Gamma = \Gamma_1 \cup \Gamma_2$ be traversed from bottom to top. Then

$$(25) \quad (A_1 + A_2)^{-1} = \frac{1}{2\pi i} \int_{\Gamma} (A_1 - z)^{-1} (A_2 + z)^{-1} dz.$$

We apply the formula (25) with

$$A_1 = \frac{\lambda}{2} + \frac{\gamma}{2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \quad \text{and} \quad A_2^\varepsilon = \frac{\lambda}{2} + \frac{\gamma}{2} - \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial y^2}.$$

Here the operators are defined using periodic boundary conditions in L_p -spaces of the appropriate number of variables. The resolvent of A_2^ε can be studied in detail by writing down the Green's function of the ordinary differential operator $A_2^\varepsilon + z$ with periodic boundary conditions. In fact, write $G(y, \eta; \omega, \varepsilon)$ for the Green's function of the non-homogeneous problem

$$(26) \quad -\frac{1}{\varepsilon^2} v'' + \omega v = \varphi(y), \quad v(0) = v(1), \quad v'(0) = v'(1),$$

where the prime ' denotes d/dy and

$$\omega = z + \frac{\lambda}{2} + \frac{\gamma}{2}.$$

It turns out that, for all values z and λ of interest, the quantity ω lies in a closed sector in \mathbf{C} which is disjoint from the negative real axis. Choose $\sqrt{\omega}$ to be the square root of ω with positive real part. In this case there is a positive constant d such that

$$\frac{1}{d} \operatorname{Re} \sqrt{\omega} \leq |\sqrt{\omega}| \leq d \operatorname{Re} \sqrt{\omega}$$

for all relevant values of ω . One finds that

$$(27) \quad G(y, \eta; \omega, \varepsilon) = \varepsilon \frac{e^{-\varepsilon\sqrt{\omega}|y-\eta|} + e^{-\varepsilon\sqrt{\omega}(1-|y-\eta|)}}{2\sqrt{\omega}(1 - e^{-\varepsilon\sqrt{\omega}})}.$$

Moreover, one can verify that

$$(28) \quad \partial_3 G(y, \eta; \varepsilon, \omega) = \begin{cases} \varepsilon \frac{e^{-\varepsilon\sqrt{\omega}(1-y+\eta)} - e^{-\varepsilon\sqrt{\omega}(y-\eta)}}{2(1 - e^{-\varepsilon\sqrt{\omega}})} & y > \eta, \\ \varepsilon \frac{e^{-\varepsilon\sqrt{\omega}(y-\eta)} - e^{-\varepsilon\sqrt{\omega}(1-y+\eta)}}{2(1 - e^{-\varepsilon\sqrt{\omega}})} & y < \eta. \end{cases}$$

□

The following Lemma shows how to obtain explicit estimates for G and $\partial_3 G$ which do not depend on ε . These estimates together with the Grisvard formula permit to prove (23) and (24), see ([2], Proposition 3.1) for the details.

LEMMA 1. *There is a positive constant c with the following properties:*

(a) *if $\varphi \in L_p[0, 1]$, then*

$$\left\| y \mapsto \int_0^1 G(y, \eta; \omega, \varepsilon) \varphi(\eta) \, d\eta \right\|_p \leq \frac{c}{|\omega|} \|\varphi\|_p,$$

(b) *if $\varphi \in L_p[0, 1]$, then*

$$\left\| y \mapsto \int_0^1 \partial_3 G(y, \eta; \omega, \varepsilon) \varphi(\eta) \, d\eta \right\|_p \leq \frac{c}{|\sqrt{\omega}|} \|\varphi\|_p,$$

for any $p \geq 1$.

PROOF. Observe first that G and $\partial_3 G$ can be estimated as follows:

$$(29) \quad |G(y, \eta; \omega, \varepsilon)| \leq \begin{cases} \frac{c}{|\omega|} & \text{for } |\varepsilon\sqrt{\omega}| \leq 1, \\ c\varepsilon(e^{-\varepsilon\operatorname{Re}\sqrt{\omega}|y-\eta|} + e^{-\varepsilon\operatorname{Re}\sqrt{\omega}(1-|y-\eta|)})/|\sqrt{\omega}| & \text{for } |\varepsilon\sqrt{\omega}| \geq 1, \end{cases}$$

$$(30) \quad |\partial_3 G(y, \eta; \omega, \varepsilon)| \leq \begin{cases} \frac{c}{|\sqrt{\omega}|} & \text{for } |\varepsilon\sqrt{\omega}| \leq 1, \\ c\varepsilon(e^{-\varepsilon\operatorname{Re}\sqrt{\omega}|y-\eta|} + e^{-\varepsilon\operatorname{Re}\sqrt{\omega}(1-|y-\eta|)}) & \text{for } |\varepsilon\sqrt{\omega}| \geq 1, \end{cases}$$

where c is a constant independent of ω and ε .

Suppose that $|\varepsilon\sqrt{\omega}| \leq 1$. We have

$$\begin{aligned} & \left(\int_0^1 \left| \int_0^1 G(y, \eta; \omega, \varepsilon) \varphi(\eta) \, d\eta \right|^p dy \right)^{1/p} \\ & \leq \left[\int_0^1 \left(\int_0^1 \frac{c}{|\omega|} |\varphi(\eta)| \, d\eta \right)^p dy \right]^{1/p} \\ & \leq \frac{c}{|\omega|} \left[\int_0^1 \int_0^1 |\varphi(\eta)|^p \, d\eta dy \right]^{1/p} = \frac{c}{|\omega|} \|\varphi\|_p. \end{aligned}$$

On the other hand, if $|\varepsilon\sqrt{\omega}| \geq 1$, we have

$$\begin{aligned} & \left(\int_0^1 \left| \int_0^1 G(y, \eta; \omega, \varepsilon) \varphi(\eta) \, d\eta \right|^p dy \right)^{1/p} \\ & \leq c \left[\left(\int_0^1 \left| \int_0^1 \frac{\varepsilon e^{-\operatorname{Re}\sqrt{\omega}|y-\eta|}}{|\sqrt{\omega}|} |\varphi(\eta)| \, d\eta \right|^p dy \right)^{1/p} \right. \\ & \quad \left. + \left(\int_0^1 \left| \int_0^1 \frac{\varepsilon e^{-\operatorname{Re}\sqrt{\omega}(1-|y-\eta|)}}{|\sqrt{\omega}|} |\varphi(\eta)| \, d\eta \right|^p dy \right)^{1/p} \right] = c[I_1(\omega) + I_2(\omega)]. \end{aligned}$$

We will estimate $I_1(\omega)$ and $I_2(\omega)$ separately. First of all, we have

$$I_1(\omega) = \frac{1}{|\sqrt{\omega}|} \left(\int_0^1 \left| \int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon\operatorname{Re}\sqrt{\omega}|y-\eta|} \tilde{\varphi}(\eta) \, d\eta \right|^p dy \right)^{1/p},$$

where $\tilde{\varphi}$ is the extension of φ to \mathbb{R} obtained by setting $\tilde{\varphi}(\eta) = 0$ for $\eta \notin [0, 1]$, $\tilde{\varphi}(\eta) = \varphi(\eta)$ for $\eta \in [0, 1]$. Setting $y - \eta = \tau$ and using the generalized Minkowski inequality of [5, p. 46], we see that

$$\begin{aligned} I_1(\omega) &= \frac{1}{|\sqrt{\omega}|} \left(\int_0^1 \left| \int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} |\tau|} |\tilde{\varphi}(y - \tau)|^p d\tau \right|^p dy \right)^{1/p} \\ &\leq \frac{1}{|\sqrt{\omega}|} \left\{ \int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} |\tau|} \left[\int_0^1 |\tilde{\varphi}(y - \tau)|^p dy \right]^{1/p} d\tau \right\}^{p(1/p)} \\ &\leq \frac{1}{|\sqrt{\omega}|} \frac{2}{\operatorname{Re} \sqrt{\omega}} \|\varphi\|_p \leq \frac{c}{|\omega|} \|\varphi\|_p \end{aligned}$$

where the constant c is independent of ε , ω and p .

Turning to $I_2(\omega)$, we have

$$\begin{aligned} I_2(\omega) &= \frac{1}{|\sqrt{\omega}|} \left(\int_0^1 \left| \int_0^y \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} (1 - |y - \eta|)} \varphi(\eta) d\eta \right. \right. \\ &\quad \left. \left. + \int_y^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} (1 - |y - \eta|)} \varphi(\eta) d\eta \right|^p dy \right)^{1/p} \\ &\leq \frac{1}{\sqrt{\omega}} \left[\left(\int_0^1 \left| \int_0^y \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} (1 - y + \eta)} \varphi(\eta) d\eta \right|^p dy \right)^{1/p} \right. \\ &\quad \left. + \left(\int_0^1 \left| \int_y^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} (1 - \eta + y)} \varphi(\eta) d\eta \right|^p dy \right)^{1/p} \right] \\ &= \frac{1}{|\sqrt{\omega}|} \left[\left(\int_0^1 \left| \int_{1-y}^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} \tau} \varphi(y + \tau - 1) d\tau \right|^p dy \right)^{1/p} \right. \\ &\quad \left. + \left(\int_0^1 \left| \int_y^1 \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} \tau} \varphi(y - \tau + 1) d\tau \right|^p dy \right)^{1/p} \right] \\ &\leq \frac{1}{|\sqrt{\omega}|} \left[\int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} |\tau|} \left(\int_0^1 |\tilde{\varphi}(y + \tau - 1)|^p dy \right)^{1/p} d\tau \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \varepsilon e^{-\varepsilon \operatorname{Re} \sqrt{\omega} |\tau|} \left(\int_0^1 |\tilde{\varphi}(y - \tau + 1)|^p dy \right)^{1/p} d\tau \right]. \end{aligned}$$

Here $\tilde{\varphi}$ is the extension of φ to \mathbb{R} introduced previously, and we have used the generalized Minkowski inequality again. So we have

$$I_2(\omega) \leq \frac{c}{|\omega|} \|\varphi\|_p,$$

where c is independent of ω , ε and p . Putting together the above estimates, we obtain (a) of the lemma for a constant c which is independent of ω , ε and p .

The estimate (30) for $|\partial_3 G|$ differs from the estimate (29) for $|G|$ only by a factor of $|\sqrt{\omega}|$. So going through the above calculations again, one has

$$\left(\int_0^1 \left| \int_0^1 \partial_3 G(y, \eta; \omega, \varepsilon) \varphi(\eta) d\eta \right|^p dy \right)^{1/p} \leq \frac{c}{|\sqrt{\omega}|} \|\varphi\|_p$$

for a constant c independent of ω , ε and p . This complete the proof of Lemma 1. \square

We return now to the proof of Proposition 1. All the statements for $\alpha \in (0, 1)$ and $1 < p < \infty$ which regard convergence and estimates in the space of bounded linear operators on $L_p(Q)$ follow fairly quickly from (23) and (24) together with (7)–(10). For the details we refer to ([2], Proposition 3.3).

Consider now the case when $\alpha = 3/4$, to prove the assertion regarding convergence of $D_\varepsilon^{-3/4}$ to $D_0^{-3/4}M$ in the space of bounded linear transformation from $L_2(Q)$ to $L_q(Q)$, we use the formula (9):

$$D_\varepsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(tI + D_\varepsilon)^{-1} dt.$$

This formula is valid if $0 < \alpha < 1$ and if D_ε is viewed as a bounded linear operator on $L_p(Q)$ for fixed $p \in (1, \infty)$. Let us write

$$A_1 = \frac{t}{2} + \frac{\gamma}{2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \quad \text{and} \quad A_2^\varepsilon = \frac{t}{2} + \frac{\gamma}{2} - \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial y^2}.$$

Consider the resolvent of A_2^ε , proceeding as in Lemma 1, we write down the Green’s function $G(y, \eta; \omega, \varepsilon)$ of (26) where now $\omega = z + t/2 + \gamma/2$, and we find that, if $\varphi \in L_1[0, 1]$, then

$$\left\| y \mapsto \int_0^1 G(y, \eta; \omega, \varepsilon) \varphi(\eta) d\eta \right\|_\infty \leq \frac{c}{|\sqrt{\omega}|} \|\varphi\|_1.$$

Using this fact together with the Grisvard formula and the Riesz interpolation theory [5], one can find that, if $0 < p \leq q < \infty$, then

$$(31) \quad \|(tI + D_\varepsilon)^{-1}\|_{L_p(Q) \rightarrow L_q(Q)} \leq c(1+t)^{-1+(3/2)(1/p-1/q)},$$

where the constant c does not depend on ε . Observe that this inequality reduces to (23) if $p = q$. Therefore, if

$$(32) \quad \frac{1}{q} > \frac{1}{p} - \frac{2\alpha}{3},$$

then the right-hand side of the formula (9) for $D_\varepsilon^{-\alpha}$ converges to $D_0^{-\alpha}M$ as $\varepsilon \rightarrow 0$. Since (32) holds if $\alpha = 3/4$, $p = 2$, and $2 \leq q < \infty$, we have proved that $D_\varepsilon^{-3/4} \rightarrow D_0^{-3/4}M$ as bounded linear transformation from $L_2(Q)$ to $L_q(Q)$ if $2 \leq q < \infty$.

To prove the remaining assertions of Proposition 1 for $\alpha = 3/4$, one uses estimates on $\partial_3 G$ of Lemma 1(b), together with the formula

$$D_\varepsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi(1-\alpha)} \int_0^\infty t^{1-\alpha}(tI + D_\varepsilon)^{-2} dt,$$

valid for $0 < \alpha < 1$ (see [5, p. 281]). In fact, we obtain

$$(33) \quad \partial_i D_\varepsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi(1-\alpha)} \int_0^\infty t^{1-\alpha} \partial_i (tI + D_\varepsilon)^{-1} (tI + D_\varepsilon)^{-1} dt$$

where the integrand on the right is at first viewed as an operator from $L_p(Q)$ to $L_q(Q)$, ($1 < p \leq q < \infty$). We estimate the integrand to obtain

$$\begin{aligned} & \|\partial_i(tI + D_\varepsilon)^{-1}(tI + D_\varepsilon)^{-1}\|_{L_p(Q) \rightarrow L_q(Q)} \\ & \leq \|\partial_i(tI + D_\varepsilon)^{-1}\|_{L_q(Q) \rightarrow L_q(Q)} \|(tI + D_\varepsilon)^{-1}\|_{L_p(Q) \rightarrow L_q(Q)} \\ & \leq \frac{c}{(1+t)^{1/2}}(1+t)^{-1+(3/2)(1/p-1/q)}. \end{aligned}$$

Here we have used (24) and (31). Once again, if $p \geq 2$ and

$$(34) \quad \frac{1}{q} > \frac{1}{p} - \frac{2\alpha - 1}{3},$$

then the integral in (33) converges and defines a bounded linear operator from $L_p(Q)$ to $L_q(Q)$.

Now set $\alpha = 3/4$, $p = 2$, $2 \leq q < 3$. Then (34) is valid, and so, as $\varepsilon \rightarrow 0$, we get:

$$\partial_i D_\varepsilon^{-3/4} \rightarrow \partial_i D_0^{-3/4} M \quad (i = 1, 2), \quad \partial_3 D_\varepsilon^{-3/4} \rightarrow 0.$$

The convergences are in the operator norm on bounded linear operators from $L_p(Q)$ to $L_q(Q)$. This completes the proof of Proposition 1. □

3. The bifurcation result

We now assume conditions on our specific problem which will guarantee that all the hypotheses of Theorem 1 are satisfied.

- (A1) The linearized reduced equation (21) has a nontrivial T -periodic solution z^0 and it does not admit other T -periodic solutions linearly independent of z^0 . Furthermore (21) does not possess solutions of the form $v(t, x_1, x_2) + (t/T)z^0(t, x_1, x_2)$ where v is a T -periodic function.

This condition means that the derivative $V'(0)$ of the quasi-translation operator V for the equation (20) has 1 as a simple eigenvalue. Obviously, the same applies to the operator $\mathcal{J}V'(0)M$, where \mathcal{J} is the natural embedding of $L_2(\Omega)$ into $L_2(Q)$. In the sequel we will omit \mathcal{J} when this does not give rise to ambiguity and by $\|\cdot\|$ will denote both the norm in $L_2(Q)$ and in $L_2(\Omega)$. Without loss of generality we can assume that $\|z^0(0)\| = 1$. Let us denote by g_0^* the normalized eigenvector of $(V'(0)M)^*$ corresponding to the eigenvalue 1 such that $\langle z^0(0), g_0^* \rangle = 1$.

In what follows we will show that $W'(\varepsilon, 0) \rightarrow V'(0)M$ as $\varepsilon \rightarrow 0$ in the operator norm. Therefore by the perturbation theory of [3] there exists a unique continuous branch $\mu(\varepsilon)$ of eigenvalues of $W'(\varepsilon, 0)$ such that $\mu(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$.

We assume that

- (A2) $\mu(\varepsilon) \neq 1$ for $\varepsilon \in (0, \varepsilon_0)$.

This implies that equation (19) for $\varepsilon \in (0, \varepsilon_0)$ does not possess nontrivial T -periodic solutions.

We finally assume that

$$(A3) \quad \zeta_0 = \int_0^T \langle U(T, s)B_0(z^0(0), z^0(0)), g_0^* \rangle ds \neq 0, \text{ where } U(t, s) \text{ is the evolution operator defined by the equation (21).}$$

We are now in a position to formulate the main result.

THEOREM 2. *Assume (A1)–(A3). Then for sufficiently small $\varepsilon > 0$ there exists a continuous branch of T -periodic solutions \bar{u}^ε of (5 $_\varepsilon$) such that $\bar{u}^\varepsilon \neq u^\varepsilon$ and $\bar{u}^\varepsilon \rightarrow u^0$ as $\varepsilon \rightarrow 0$.*

The proof is based on the following preliminary result.

PROPOSITION 2. *There exist $\varepsilon_0 > 0$ and $d > 0$ such that the operators W and VM are well defined on $(0, \varepsilon_0) \times \mathcal{B}_{L_2(Q)}(0, d)$ and $\mathcal{B}_{L_2(Q)}(0, d)$ respectively. Moreover,*

$$(35) \quad W(\varepsilon, w_0) \rightarrow VMw_0$$

as $\varepsilon \rightarrow 0$ uniformly with respect to $w_0 \in \mathcal{B}_{L_2(Q)}(0, d)$.

PROOF. First let us prove that the operator W is well defined on $(0, \varepsilon_0) \times \mathcal{B}_{L_2(Q)}(0, d)$ for some $\varepsilon_0 > 0$ and $d > 0$. Assume to the contrary that there exist $r_0 > 0$, $\varepsilon_n \rightarrow 0$, $w_0^n \rightarrow 0$, $t_n \in [0, T]$ such that

$$(36) \quad \|w^n(t_n)\| = r_0$$

and $\|w^n(t)\| < r_0$ for $0 \leq t < t_n$. Here w^n is the solution of the equation

$$(37) \quad w(t) = e^{-D\varepsilon_n t} w_0^n + \int_0^t D_{\varepsilon_n}^\alpha e^{-D\varepsilon_n(t-s)} h(D_{\varepsilon_n}^{-\alpha} \mathbf{P}_{\varepsilon_n} w(s), \varepsilon_n) ds$$

where

$$(38) \quad h(\phi, \varepsilon) = \gamma\phi - B_\varepsilon(u^\varepsilon, \phi) - B_\varepsilon(\phi, u^\varepsilon) - B_\varepsilon(\phi, \phi).$$

This is equivalent to proving that for any $r > 0$ there exist $\varepsilon_0, d > 0$ such that for any $w_0 \in \mathcal{B}_{L_2(Q)}(0, d)$ the corresponding solution $w(t)$ of (37) is such that $\|w(t)\| < r$ for any $t \in [0, T]$. Note that $\liminf_{n \rightarrow \infty} t_n = \beta > 0$ since by Proposition 1 we have that $\|w^n(t)\| \rightarrow 0$ as $t \rightarrow 0$ uniformly with respect to n . Without loss of generality we will suppose that $\lim_{n \rightarrow \infty} t_n = \beta$. Moreover, for any $n \in \mathbb{N}$ there exists $\tau_n \in (0, t_n)$ such that $\|w^n(\tau_n)\| = r_0/2$ and $\|w^n(t)\| < r_0/2$ for any $t \in [0, \tau_n)$, otherwise (36) would be contradicted. We want to prove that $\tau_n \rightarrow \beta$. We argue by contradiction; thus, by passing to a subsequence if necessary, we assume that $\tau_n \rightarrow \tau_0 < \beta$. Therefore there exists $\delta > 0$ such that $\|w^n(t)\| < r_0$ for any $t \in [0, \tau_0 + \delta] \subset [0, \beta]$ and for n sufficiently large, hence $\{w^n\} \subset C([0, \tau_0 + \delta], L_2(Q))$ is bounded and so it has a limit point which

represents a solution w^0 of the reduced equation (20) with initial condition zero, thus w^0 is identically zero. This contradicts the fact that $\|w^n(t)\| \geq r_0/3$, for sufficiently large $n \in \mathbb{N}$, in an interval $(\tau_0 - \delta_1, \tau_0 + \delta_1)$ for some $0 < \delta_1 < \delta$. In conclusion $\tau_n \rightarrow \beta$ as $n \rightarrow \infty$ with $\tau_n < t_n$ for every $n \in \mathbb{N}$. Evaluate now

$$(39) \quad w^n(t_n) - w^n(\tau_n) = (e^{-D_{\varepsilon_n}(t_n - \tau_n)} - I) \left[(e^{-D_{\varepsilon_n}\tau_n} w_0^n) + \int_0^{\tau_n} D_{\varepsilon_n}^\alpha e^{-D_{\varepsilon_n}(\tau_n - s)} h(D_{\varepsilon_n}^{-\alpha} \mathbf{P}_{\varepsilon_n} w^n(s), \varepsilon_n) ds \right] + \int_{\tau_n}^{t_n} D_{\varepsilon_n}^\alpha e^{-D_{\varepsilon_n}(t_n - s)} h(D_{\varepsilon_n}^{-\alpha} \mathbf{P}_{\varepsilon_n} w^n(s), \varepsilon_n) ds.$$

Observe that in (39) the operator $(e^{-D_{\varepsilon_n}(t_n - \tau_n)} - I)$ is applied to elements of a compact set, and so for n sufficiently large we have that the norm of the first term of the right hand side of (39) is less than $r_0/8$. On the other hand by using the inequalities of Proposition 1, the term

$$\int_{\tau_n}^{t_n} D_{\varepsilon_n}^\alpha e^{-D_{\varepsilon_n}(t_n - s)} h(D_{\varepsilon_n}^{-\alpha} \mathbf{P}_{\varepsilon_n} w^n(s), \varepsilon_n) ds$$

can be estimated in norm from above by $c(t_n - \tau_n)^{1-\alpha}$, where c is a positive constant, thus for n sufficiently large it is less than $r_0/8$. In conclusion, the norm of the right hand side of (39) for n sufficiently large is less than $r_0/4$. Thus

$$r_0/2 = \|w^n(t_n)\| - \|w^n(\tau_n)\| \leq \|w^n(t_n) - w^n(\tau_n)\| < r_0/4,$$

which is a contradiction. Therefore there exist $d_0 > 0$ and $\varepsilon_0 > 0$ such that $W(\varepsilon, w_0)$ is defined for any $\varepsilon \in (0, \varepsilon_0)$ and any $w_0 \in \mathcal{B}_{L_2(Q)}(0, d_0)$. Repeating the same arguments for VM one can show that there exists $d_1 > 0$ such that the operator VM is well defined on $\mathcal{B}_{L_2(Q)}(0, d_1)$. Taking $d = \min\{d_0, d_1\}$ we have the assertion.

Finally, the convergence (35) is a consequence of the Proposition 1, in fact

$$w^\varepsilon(T) = e^{-D_\varepsilon T} w_0 + \int_0^T D_\varepsilon^\alpha e^{-D_\varepsilon(T-s)} h(D_\varepsilon^{-\alpha} \mathbf{P}_\varepsilon w^\varepsilon(s), \varepsilon) ds$$

and we have already proved that w^ε is bounded in $C([0, T], L_2(Q))$ and hence relatively compact. □

As is shown in [5] the operator $W'(\varepsilon, 0)$ is the quasi-translation operator of the linearized equation (19); also $V'(0)$ is the quasi translation operator of (21). Thus as a direct consequence of Proposition 2 we have the following result.

PROPOSITION 3. $W'(\varepsilon, 0) \rightarrow V'(0)M$ as $\varepsilon \rightarrow 0$ in the operator norm.

PROOF OF THEOREM 2. It is enough to show that the operator W satisfies all the conditions of Theorem 1. Indeed, it is compact due to the fact that the

operator $D_\varepsilon^\delta W(\varepsilon, \cdot)$, with $\delta < \alpha$, is bounded, hence $W(\varepsilon, w_0) \in \bigcup_{\varepsilon \in (0, \varepsilon_0)} D_\varepsilon^{-\delta} \mathcal{B}$, where \mathcal{B} is a ball centered at the origin with sufficiently large radius. On the other hand, by Proposition 1, $D_\varepsilon^{-\delta} w$ is compact with respect to both the variables ε, w . The representation (22) follows from (38), the analogous formula for f_0 and [5, Theorem 23.14, p. 494]. Finally, assumption (A1) implies condition (C1), condition (A2) implies (C2), and (C3) is obtained from (A3) and [5, Theorem 23.14, p. 494]. \square

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