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EXISTENCE OF MANY SIGN-CHANGING NONRADIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS ON THIN ANNULI

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Abstract. We study the existence of many nonradial sign-changing solutions of a superlinear Dirichlet boundary value problem in an annulus in $\mathbb{R}^N$. We use Nehari-type variational method and group invariance techniques to prove that the critical points of an action functional on some spaces of invariant functions in $H_0^{1,2}(\Omega_\varepsilon)$, where Ω_ε is an annulus in $\mathbb{R}^N$ of width ε, are weak solutions (which in our case are also classical solutions) to our problem. Our result generalizes an earlier result of Castro et al. (See [4])

1. Introduction

In this article we discuss the existence of many sign-changing nonradial solutions of semilinear elliptic equations on an annulus in $\mathbb{R}^N, N \geq 2$:

$$\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 - \varepsilon < |x| < \varepsilon\},$$

where $\varepsilon > 0$.

We consider the Dirichlet boundary value problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.1}$$

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where the non-linearity f is of class $C^1(\mathbb{R})$ and satisfies the following assumptions:

(A1) $f(0) = 0$ and $f'(0) < \lambda_1$, where λ_1 is the smallest eigenvalue of $-\Delta$ with zero Dirichlet boundary condition in Ω_ε.

(A2) $f'(u) > f(u)/u$ for all $u \neq 0$.

(A3) (Superlinearity)

$$\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty.$$

(A4) (Subcritical growth) There exist constants $p \in (1, (N+2)/(N-2))$ and $C > 0$ such that

$$|f'(u)| \leq C(|u|^{p-1} + 1) \quad \text{for all } u \in \mathbb{R}.$$

(A5) There exist constants $m \in (0,1)$ and ρ such that

$$uf(u) \geq \frac{2}{m} F(u) > 0,$$

where $|u| > \rho$ and $F(u) = \int_0^u f(s)\,ds$.

If $N = 2$, then $p \in (1, \infty)$. A typical nonlinearity is the function $f(t) = t^3$, although our results are not restricted to an odd nonlinearity.

We note that the condition $f'(0) < \lambda_1$ is necessary for the existence of sign-changing solutions (see [2]).

In [11], Wang proved that, over a smooth bounded domain, problem (1.1) has a positive solution, a negative solution, and a third solution with no information about its sign. In [2], Castro et al. proved the existence of a third solution that changes sign exactly once. Later in [4], they established the existence of a nonradial sign-changing solution when the underlying domain is an annulus in $\mathbb{R}^N$. Furthermore, if the annulus is two dimensional they proved that (1.1) has many sign-changing nonradial solutions. The purpose of this paper is to extend their result to higher dimensions.

Our main result is the following

Theorem 1.1. *Assume f satisfies* (A1)–(A5). *Then for each positive integer k there exists $\varepsilon_1(k) > 0$ such that if $0 < \varepsilon < \varepsilon_1(k)$ then* (1.1) *has k sign-changing nonradial solutions.*

In our context, by a solution to (1.1) we mean a function $u \in H_0^{1,2}(\Omega_\varepsilon)$ that satisfies

$$\int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v - vf(u))\,dx = 0, \tag{1.2}$$

for all $v \in C_0^\infty(\Omega_\varepsilon)$, where $H_0^{1,2}(\Omega_\varepsilon)$ is the Sobolev space with inner product $\langle u, v\rangle = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v\,dx$ (see [1]). Note that (1.2) is obtained by multiplying the equation in (1.1) by v and integrating by parts. So classical solutions of (1.1) (that is, the ones which are in $C^2(\Omega_\varepsilon)\cap C(\overline{\Omega_\varepsilon})$) are also weak solutions. By the assumptions on f and the regularity theory for elliptic boundary value problems (see [7]), a weak solution of (1.1) is also a classical solution.

The left-hand side of (1.2) is just the Fréchet derivative of the functional

$$J(u)=\int_{\Omega_\varepsilon}\left\{\frac{1}{2}|\nabla u|^2-F(u)\right\}dx$$

defined on $H_0^{1,2}(\Omega_\varepsilon)$. Note that $J\in C^2(H_0^{1,2}(\Omega_\varepsilon),\mathbb{R})$ (see [10]). Moreover, u is a solution to (1.1) if and only if u is a critical point of J.

Instead of looking for sign-changing critical points of the functional J on $H_0^{1,2}(\Omega_\varepsilon)$, we look for them on a subset of a submanifold of invariant functions in $H_0^{1,2}(\Omega_\varepsilon)$.

Our main tools for proving existence and multiplicity results consist of an idea in [8] and [9] and critical point theory, i.e., we consider the functional J defined above and the functional

$$\gamma(u)=\int_{\Omega_\varepsilon}\left(|\nabla u|^2-uf(u)\right)\,dx.$$

For a positive integer k, we define

$$\begin{aligned}H(\varepsilon,k):=&\operatorname{Fix}(G(k))\\ =&\{v\in H_0^{1,2}(\Omega_\varepsilon):v(gx,Ty)=v(x,y),\ \text{for all } (g,T)\in G(k)\}\\ =&\{v\in H_0^{1,2}(\Omega_\varepsilon):v(x,y)=u(x,|y|),\ \text{for some } u\\ &\quad\text{which satisfies } u(gx,|y|)=u(x,|y|) \text{ for all } g\in G_k\},\end{aligned}$$

where $G(k)=G_k\times\mathbf{O}(N-2)$, $\mathbf{O}(j)$ denotes the group of $j\times j$ orthogonal matrices, and

$$\begin{aligned}G_k:=\Big\{&g\in\mathbf{O}(2):\\ &g(x_1,x_2)=\left(x_1\cos\frac{2\pi l}{k}+x_2\sin\frac{2\pi l}{k},-x_1\sin\frac{2\pi l}{k}+x_2\cos\frac{2\pi l}{k}\right),\\ &(x_1,x_2)\in\mathbb{R}^2,\ l\in\mathbb{Z}\Big\}.\end{aligned}$$

Note that $H(\varepsilon,k)$ can be regarded as the class of functions that are periodic of period $2\pi/k$ in the θ variable, where (r,θ) are the polar coordinate of $x=(x_1,x_2)$, and that depend on $|y|$, where $y=(x_3,\dots,x_N)$.

Also, we consider the Nehari manifold

$$S(\varepsilon,k)=\{v\in H(\varepsilon,k)\setminus\{0\}:\gamma(v)=0\}.$$

Of particular interest is the subset of $S(\varepsilon, k)$ given by

$$S^1(\varepsilon, k) = \{v \in S(\varepsilon, k) : v_+, v_- \in S(\varepsilon, k)\},$$

where $v_+(x) = \max\{v(x), 0\}$ and $v_-(x) = \min\{v(x), 0\}$ are the positive and negative parts of v respectively.

Similarly, we define

$$\begin{aligned} H(\varepsilon, \infty) :&= \{v \in H^1_0(\Omega_\varepsilon) : v(gx, Ty) = v(x, y), \\ &\qquad \text{for all } (g, T) \in \mathbf{O}(2) \times \mathbf{O}(N-2)\} \\ &= \{v \in H^{1,2}_0(\Omega_\varepsilon) : v(x, y) = u(|x|, |y|), \text{ for some } u\}, \end{aligned}$$

the manifold

$$S(\varepsilon, \infty) = \{v \in H(\varepsilon, \infty) \setminus \{0\} : \gamma(v) = 0\},$$

and the set

$$S^1(\varepsilon, \infty) = \{v \in S(\varepsilon, \infty) : v_+, v_- \in S(\varepsilon, \infty)\}.$$

Note that if $u \in H(\varepsilon, \infty)$ then u is θ-independent.

We consider the following numbers associated with the above sets

$$j^\varepsilon_k = \inf_{v \in S^1(\varepsilon, k)} J(v), \qquad j^\varepsilon_\infty = \inf_{v \in S^1(\varepsilon, \infty)} J(v).$$

We will obtain many sign-changing nonradial solutions to (1.1) by establishing the following properties:

(i) j^ε_k is achieved by some $u_{\varepsilon,k} \in S^1(\varepsilon, k)$ and $u_{\varepsilon,k}$ is a critical point of J on $H(\varepsilon, k)$.

(ii) $u_{\varepsilon,k}$ is a critical point of J on $H^{1,2}_0(\Omega_\varepsilon)$.

(iii) $j^\varepsilon_k < j^\varepsilon_\infty$ for $k \geq 1$ and $0 < \varepsilon < \varepsilon_1(k)$.

(iv) $j^\varepsilon_k < j^\varepsilon_{kn}$ whenever $j^\varepsilon_{kn} < j^\varepsilon_\infty$.

Note that assertion (ii) is related to the symmetric criticality principle: if $u_{\varepsilon,k}$ is a critical point of J on $H(\varepsilon, k)$, then $u_{\varepsilon,k}$ is a critical point of J on $H^{1,2}_0(\Omega_\varepsilon)$ (see [12]).

The paper is organized as follows: in Section 2, we discuss assertions (i), (iii), and (iv). In Section 3, we prove Theorem 1.1.

2. Existence results

Assertion (i) of the previous paragraph is a direct consequence of the following theorem

Theorem 2.1. *For each positive integer $k = 1, 2\ldots$ and $\varepsilon > 0$ there exists a minimizer $u_{\varepsilon,k}$ of j^ε_k which changes sign. Moreover, $u_{\varepsilon,k}$ is a critical point of J on $H(\varepsilon, k)$.*

Proof. This follows from a recent result of Castro, Cossio, and Neuberger [2]. □

As for assertion (iii) we have

Theorem 2.2. *For a positive integer k, there exists $\varepsilon_1(k) > 0$ such that if $0 < \varepsilon < \varepsilon_1(k)$ then $j_k^\varepsilon < j_\infty^\varepsilon$. Thus, $u_{\varepsilon,k}$ is θ-dependent.*

Proof. A proof of this theorem can be found in [6]. □

The following lemma, which establishes assertion (iv), shows that if k divides n and $j_n^\varepsilon < j_\infty^\varepsilon$ then $j_k^\varepsilon < j_n^\varepsilon$.

Lemma 2.3. *Let f satisfies* (A1)–(A5). *For $n = 2, 3, \ldots$, $k = 1, 2, \ldots$. if $j_{kn}^\varepsilon < j_\infty^\varepsilon$ then $j_k^\varepsilon < j_{kn}^\varepsilon$.*

Proof. Fix k and n. For $\varepsilon > 0$, Theorem 2.1 guarantees the existence of a sign-changing minimizer u of J on $S^1(\varepsilon, kn)$. According to Theorem 2.1 and assertion (ii), u is a solution to (1.1). Furthermore, invoking Theorem 2.2 with $0 < \varepsilon < \varepsilon_1(k)$, we know that u is θ-dependent. Now, by the regularity theory of elliptic equations we know that u is a C^2 function. Let $x = (r, \theta)$ be the polar coordinate of $x \in \mathbb{R}^2$ and write $u = u(r, \theta, |y|)$. Then

$$\int_{\Omega_\varepsilon} |\nabla u|^2\,dx\,dy = \int_{(r,|y|)} \int_0^{2\pi} (u_r^2 + \frac{1}{r^2} u_\theta^2 + |\nabla_y u|^2) r\,dr\,d\theta\,dy$$

and

$$\int_{\Omega_\varepsilon} F(u)\,dx\,dy = \int_{(r,|y|)} \int_0^{2\pi} F(u) r\,dr\,d\theta\,dy.$$

Define the function

$$v(r, \theta, |y|) = u(r, \theta/n, |y|), \quad 0 \le \theta \le 2\pi.$$

Since u is θ-dependent and changes sign so does v. Also,

$$v_\pm(r, \theta + 2\pi/k, |y|) = v_\pm(r, \theta, |y|).$$

It follows that $v_\pm \in H(\varepsilon, k)$.

An easy calculation yields the following equalities

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla v_\pm|^2\,dx\,dy = \int_{(r,|y|)} \int_0^{2\pi} &((u_\pm)_r^2(r, \theta, |y|) + \frac{1}{r^2 n^4} (u_\pm)_\theta^2(r, \theta, |y|) \\ &+ |\nabla_y u_\pm(r, \theta, |y|)|^2) r\,dr\,d\theta\,dy\end{aligned}$$

and

$$\int_{\Omega_\varepsilon} F(v_\pm)\,dx\,dy = \int_{(r,|y|)} \int_0^{2\pi} F(u_\pm(r, \theta, |y|)) r\,dr\,d\theta\,dy.$$

Since u does not belong to $S^1(\varepsilon, \infty)$ we have

$$\int_{(r,|y|)} \int_0^{2\pi} (u_\pm)_\theta^2(r, \theta, |y|) r\,dr\,d\theta\,dy > 0.$$

This implies that $\gamma(v_\pm) < 0$. That is

$$\int_{\Omega_\varepsilon} |\nabla v_\pm|^2 \, dx \, dy < \int_{\Omega_\varepsilon} v_\pm f(v_\pm) \, dx \, dy. \tag{2.1}$$

Now, by Lemma 2.2 of [2] we can find $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha v_+ \in S(\varepsilon, k)$ and $\beta v_- \in S(\varepsilon, k)$. Let $w = \alpha v_+ + \beta v_- \in S^1(\varepsilon, k)$. Using the fact that $P_v(\lambda) = \lambda v f(\lambda v)/2 - F(\lambda v)$ is monotonically increasing for $\lambda > 0$ and the definition of j_k^ε we have

$$j_k^\varepsilon \le P_{v_+}(\alpha) + P_{v_-}(\beta) < P_{v_+}(1) + P_{v_-}(1) = J(u) = j_{kn}^\varepsilon.$$

Putting together all the arguments above we conclude a proof of the lemma. □

3. Proof of Theorem 1.1

Let $k \ge 1$ be an integer. According to Theorem 2.2 there exists $\varepsilon_1(2^k)$ such that if $0 < \varepsilon < \varepsilon_1(2^k)$ then $j_{2^k}^\varepsilon < j_\infty^\varepsilon$. Applying Lemma 2.3 to obtain

$$j_2^\varepsilon < j_{2^2}^\varepsilon < \ldots < j_{2^k}^\varepsilon < j_\infty^\varepsilon. \tag{3.1}$$

According to Theorem 2.1 there exists $u_i \in S^1(\varepsilon, 2^i)$, $i = 1, \ldots, k$, such that $j_{2^i}^\varepsilon = J(u_i)$. Moreover, u_i is a solution of (1.1). Also, according to Theorem 2.2, u_i is θ-dependent. Finally, by (3.1), $\{u_i\}_{i=1}^k$ are distinct. The proof of Theorem 1.1 is now complete. □

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