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SIGN CHANGING SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We are interested in solutions $u \in H^1(\mathbb{R}^N)$ of the linear Schrödinger equation $-\delta u + b_\lambda(x)u = f(x,u)$. The nonlinearity f grows superlinearly and subcritically as $|u| \to \infty$. The potential b_λ is positive, bounded away from 0, and has a potential well. The parameter λ controls the steepness of the well. In an earlier paper we found a positive and a negative solution. In this paper we find third solution. We also prove that this third solution changes sign and that it is concentrated in the potential well if $\lambda \to \infty$. No symmetry conditions are assumed.

1. Introduction

We consider the problem

$$\text{(S)} \qquad \begin{cases} -\Delta u + b(x)u = f(x,u) & \text{for } x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

If b and f are independent of $x \in \mathbb{R}^N$ or depend radially on x one can try to find radial solutions with a prescribed number of nodes. In this way one can obtain multiple existence of solutions, in particular solutions which change sign arbitrarily often, depending on b and f, of course. This has been done by many authors using various techniques; see e.g. [10], [7], [5] and the references therein.

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In [4] one can also find the existence of nonradial solutions which change sign in an explicit way breaking the radial symmetry of the equation.

In the general case where b and f depend on x nonradially Rabinowitz [12] proved the existence of a positive and a negative solution of (S) provided $b(x) \to \infty$ as $|x| \to \infty$, and f is superlinear and subcritical. In [2], among other things we weakened the conditions on the potential b and still obtained a positive and a negative solution. Under similar hypotheses as in [2] we shall prove here the existence of a third solution for (S) which changes sign.

On a bounded domain Ω instead of $\mathbb{R}^N$ the existence of a third solution $u_1 \in H_0^1(\Omega)$ of the Dirichlet problem

$$\text{(D)} \qquad \begin{cases} -\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for a superlinear and subcritical nonlinearity g has first been proved in [13]. The existence of a sign changing solution of (D) on an arbitrary smooth bounded domain has been established only quite recently in [8], [9], [6] and [3]. The results from our earlier paper [3] suggest that most critical point theorems which are based on an at least two-dimensional linking argument yield in fact a sign changing solution of (D). In particular, the three solutions theorem from [13] follows as a corollary from the results in [3] with the third solution u_1 being sign changing. Moreover, if g satisfies a certain convexity condition then this third solution has precisely two nodal domains; see [6] and [1] for more results in this direction.

The existence of a third solution on (S) under the hypotheses as in [2] is new. We are not aware of any result which guarantees that there is a solution of (S) which changes sign, except when symmetry in the x-variable can be used. Unfortunately, the techniques from our earlier papers [3], [1] cannot be translated directly to problems on $\mathbb{R}^N$. There it was essential that the positive cone $P = \{u \in \mathcal{C}_0^1(\Omega) : u \geq 0\}$ has nonempty interior in the $\mathcal{C}_0^1$-topology. This is not the case if Ω is unbounded. The idea of this paper is to approximate the equation on $\mathbb{R}^N$ by the Dirichlet problem on the balls $B_R(0)$ and let $R \to \infty$. We have to make sure that the limit still changes sign. It would be interesting to find an approach to the existence of sign changing solutions of (S), or of related problems on unbounded domains, which does not depend on the cone P having nonempty interior.

2. Statement of results

We assume the following hypotheses on the nonlinearity.

(f_1) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies $f(x,u) = o(|u|)$ as $u \to 0$, unifomly in x.

(f_2) There are constants $a_1, a_2 > 0$ and $s > 1$ with $s < (N+2)/(N-2)$ if $N \geq 3$, such that

$$|f(x,u)| \leq a_1 + a_2|u|^s \quad \text{for every } x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

(f_3) There exists $q > 2$ such that

$$0 < qF(x,u) \equiv q \int_0^u f(x,t)\,dt \leq uf(x,u)$$

for every $x \in \mathbb{R}^N$, $u \in \mathbb{R} \setminus \{0\}$.

(f_4) For any $R > 0$ there exists $c_R > 0$ such that

$$\frac{f(x,t) - f(x,s)}{t - s} > -c_R \quad \text{for } x \in \mathbb{R}^N,\ -R < s < t < R.$$

Concerning the potential we consider two situations. For our first result we assume:

(b_1) $b \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ satisfies $b_0 := \inf_{x \in \mathbb{R}^N} b(x) > 0$.

(b_2) For every $M > 0$

$$\mu(\{x \in \mathbb{R}^N : b(x) \leq M\}) < \infty,$$

where μ denotes the Lebesque measure in $\mathbb{R}^N$.

In [2] we obtained under these conditions the existence of a positive and a negative solution of (S).

Theorem 2.1. *If* (b_1), (b_2), *and* (f_1)–(f_3) *hold then* (S) *has a sign changing solution* u_1.

Next we consider a parametrized version of (S):

$$(\mathrm{S}_\lambda) \qquad \begin{cases} -\Delta u + (1 + \lambda a(x))u = f(x,u) & \text{for } x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Here we assume:

(a_1) $a \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ satisfies $a \geq 0$ and $a^{-1}(0)$ has nonempty interior.

(a_2) There exists $M_0 > 0$ such that

$$\mu(\{x \in \mathbb{R}^N : a(x) \leq M_0\}) < \infty.$$

Theorem 2.2. *If* (a_1), (a_2), *and* (f_1)–(f_3) *hold then* (S_λ) *has a sign changing solution* u_λ *for every* λ *large. If in addition* $\mu(\partial a^{-1}(0)) = 0$ *then any sequence* $\lambda_n \to \infty$ *has a subsequence such that* u_{λ_n} *converges along this subsequence towards a solution* $u \in H^1_0(\Omega)$ *of the equation* $-\Delta u + u = f(x,u)$ *on* $\Omega := \operatorname{int} a^{-1}(0)$.

As in the case of (S), the solution u_λ is a third nontrivial solution. The existence of a positive and a negative solution of (S_λ) for λ large has been proved in [2]. The potential $b_\lambda(x) = 1 + \lambda a(x)$ satisfies (b_1) but not (b_2). The infimum of b_λ is normalized to 1. Observe that in both theorems it is allowed that $\liminf_{|x|\to\infty} b(x) = \inf_{x\in\mathbb{R}^N} b(x)$. In particular, the potentials need not have a bounded well.

3. Proofs

Let

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} b(x)u^2\,dx < \infty \right\}$$

be equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + b(x)u^2)\,dx.$$

Similarly, we write E_λ and $\|\cdot\|_\lambda$ when working with $b_\lambda = 1 + \lambda a$ instead of b. Clearly, for $\lambda > 0$ the space

$$\begin{aligned} E_\lambda = & \left\{ u \in H^1(\mathbb{R}^N) : \|u\|_\lambda^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + b_\lambda(x)u^2)\,dx < \infty \right\} \\ = & \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2\,dx < \infty \right\} \end{aligned}$$

is independent of λ and all norms $\|\cdot\|_\lambda$ are equivalent. We also write $|u|_p$ for the L^p-norm. We consider first the existence of sign changing solutions of the problem

$$(\mathrm{D}_k) \qquad \begin{cases} -\Delta u + b(x)u = f(x,u) & \text{in } B_k(0), \\ u = 0 & \text{on } \partial B_k(0), \end{cases}$$

and similarly of ($D_{\lambda,k}$) where b is replaced by b_λ.

Theorem 3.1.

(a) *Under the assumptions of Theorem* 2.1 *problem* (D_k) *has a sign changing solution* $w_k \neq 0$ *such that* $\|w_k\| \le \beta_0$ *for some* β_0 *independent of* $k \in \mathbb{N}$.

(b) *Under the assumptions of Theorem* 2.2 *problem* ($D_{\lambda,k}$) *has a sign changing solution* $w_{\lambda,k} \neq 0$ *such that* $\|w_{\lambda,k}\|_\lambda \le \beta_0$ *for some* $\beta_0 > 0$ *independent of* $\lambda \ge 0$ *and* $k \in \mathbb{N}$.

Proof. (a) The existence of a sign changing solution has essentially been proved in [3] and [1] except that there it is assumed that f is $\mathcal{C}^1$. Moreover, in [3] f is independent of x and f' has to be bounded away from 0. This is not necessary, however, arguing as follows. For $n \in \mathbb{N}$ we replace f by f_n which coincides with f if $|u| \le R_n$, is increasing in u for $|u| > R_n$ and grows superlinearly and subcritically, where $R_n \to \infty$ as $n \to \infty$. We let $F_n(x,u) := \int_0^u f_n(x,t)\,dt$ be the primitive of f_n as usual. By (f_4) there exists $c_n > 0$ such that $f_n(x,u)+c_nu$ is strictly increasing in u. This has the effect that the gradient vector field of the $\mathcal{C}^1$-functional

$$H_0^1(B_k(0)) \to \mathbb{R}, \quad u \mapsto \int_{B_k(0)} \left(F_n(x,u) + \frac{1}{2}c_n u^2\right) dx$$

with respect to the scalar product

$$\langle u, v\rangle_n := \int_{B_k(0)} (\nabla u \cdot \nabla v + b(x)uv + c_n uv)\,dx$$

on $H_0^1(B_k(0))$ is order preserving. Moreover, it induces a strongly order preserving vector field on $\mathcal{C}_0^1(B_k(0)) := \mathcal{C}^1(B_k(0)) \cap H_0^1(B_k(0)$. Using the convexity of th positive and the negative cone it is easy to construct a pseudo-gradient vector field for the functional

$$\begin{aligned}\Phi_{k,n}(u) =& \frac{1}{2}\int_{B_k(0)} (|\nabla u|^2 + b(x)u^2)\,dx - \int_{B_k(0)} F_n(x,u)\,dx \\ =& \frac{1}{2}\|u\|_n^2 - \int_{B_k(0)} (F_n(x,u) + \frac{1}{2}c_n u^2)\,dx\end{aligned}$$

on $\mathcal{C}_0^1(B_k(0))$ such that the associated flow leaves the positive and negative cone invariant. It is this property which is used in [3] and [1] and which yields a sign changing critical point $w_{k,n}$ of $\Phi_{k,n}$. An inspection of the proofs in [3] and [1] shows that the critical point comes from a linking which also yields a bound for the critical value. More precisely, let $e_1 \in H_0^1(B_\rho(0))$ be the positive eigenfunction of $-\Delta$ and let $e_2 \in H_0^1(B_\rho(0))$ be orthogonal to e_1, some $\rho > 0$ fixed. Then an upper bound for the critical value is given by the supremum of $\Phi_{k,n}$ on the span of e_1, e_2 which is independent of k and n. It follows that $\|w_{k,n}\|$ is bounded independent of k and n. Then it follows from elliptic theory that for n large enough the critical point satisfies $|w_{k,n}|_\infty < R_n$, hence $w_k := w_{k,n}$ solves (D_k).

(b) We may proceed as in [3] with the same changes as sketched in (a). Also we may assume that $B_\rho(0) \subset \text{ int } a^{-1}(0)$ for $\rho > 0$ small. This has the effect that the associated functional restricted to the span of e_1, e_2 is independent of λ yielding bounds independent of λ. □

Lemma 3.1. *There exists $\alpha_0 > 0$ independent of $\lambda \geq 0$ and $k \in \mathbb{N}$ such that $|w^{\pm}|_{s+1} \geq \alpha_0$ for any sign changing solution w of (D_k) or of $(D_{\lambda,k})$. Here $w^+ = \max\{0, w\}$ and $w^- = \min\{0, w\}$.*

Proof. We consider the parameter dependent problem $(\mathrm{D}_{\lambda,k})$. The proof for (D_k) is the same, one simply drops all λ's appearing below. Multiplying the equation by $w^{\pm}$ we get

$$\|w^{\pm}\|_{\lambda}^2 = \int_{B_k(0)} (|\nabla w^{\pm}|^2 + b_{\lambda}(x)|w^{\pm}|^2)\, dx = \int_{B_k(0)} f(x, w^{\pm}) w^{\pm}\, dx.$$

By (f_1) and (f_2) there exists $A > 0$ such that

$$f(x,t) \cdot t \leq \frac{1}{2}|t|^2 + A|t|^{s+1} \quad \text{for } x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

This implies

$$\int_{B_k(0)} f(x, w^{\pm}) w^{\pm} \leq \frac{1}{2}\|w^{\pm}\|_{\lambda}^2 + A|w^{\pm}|_{s+1}^{s+1}$$

and therefore

$$\|w^{\pm}\|_{\lambda}^2 \leq 2A|w^{\pm}|_{s+1}^{s+1}.$$

By the embedding theorem there exists $c_0 > 0$ such that

$$|w^{\pm}|_{s+1} \leq c_0\|w^{\pm}\|_{H^1(\mathbb{R}^N)} \leq c_0\|w^{\pm}\|_{\lambda}.$$

Consequently we obtain

$$|w^{\pm}|_{s+1}^{s-1} \geq 1/2c_0^2 A. \qquad \square$$

Lemma 3.2.

(a) *Suppose* (b_1) *and* (b_2) *hold. Then given $\beta_0 > 0$ and $\varepsilon > 0$ there exists $R > 0$ such that $\|u\| \leq \beta_0$ implies $\left|u|_{B_R^c}\right|_{s+1} \leq \varepsilon$, for any $u \in E$. Here $B_R^c := \{x \in \mathbb{R}^N : |x| > R\}$.*

(b) *Suppose* (a_1) *and* (a_2) *hold. Then given $\beta_0 > 0$ and $\varepsilon > 0$ there exists $\lambda_0 > 0$ and $R > 0$ such that $\|u\|_{\lambda} \leq \beta_0$ implies $\left|u|_{B_R^c}\right|_{s+1} \leq \varepsilon$ for any $u \in E_{\lambda}$, any $\lambda \geq \lambda_0$.*

Proof. The proof is similar to that of Lemma 5.2 in [2]. $\square$

Now we can prove Theorem 2.1 and Theorem 2.2. We choose $0 < \varepsilon_0 < \alpha_0/2$ where α_0 is from Lemma 3.1. For this ε_0 and for β_0 as in Theorem 3.1 we choose $R > 0$ and $\lambda_0 > 0$ as in Lemma 3.2. In order to simplify notations we fix $\lambda \geq \lambda_0$ and write $w_k \in H_0^1(B_k(0))$ both for the solution of (D_k) from Theorem 3.1(a) and for the solution of $(\mathrm{D}_{\lambda,k})$ from Theorem 3.1(b). Since $\|w_k\|_{\lambda} \leq \beta_0$ we may assume that up to a subsequence $w_k \rightharpoonup w$ in E_{λ} and $w_k \to w$ in $L_{\mathrm{loc}}^{\theta}(\mathbb{R}^N)$ for $2 \leq \theta < 2^*$. It is easy to see that w is a solution of (S) or (S_{λ}) respectively. We want to show that w is a sign changing solution. Without loss of generality, we may also assume that $w_k^{\pm} \rightharpoonup w^{\pm}$ in E_{λ} and $w_k^{\pm} \to w^{\pm}$ in $L_{\mathrm{loc}}^{s+1}(\mathbb{R}^N)$. Now

Lemma 3.1 yields $\left|w_k^{\pm}\right|_{s+1} \geq \alpha_0$ and Lemma 3.2 yields $\left|w_k^{\pm}|_{B_R^c}\right|_{s+1} \leq \varepsilon_0$. Since $w_k^{\pm} \to w^{\pm}$ in $L^{s+1}(B_R(0))$ we obtain $w^{\pm}|_{B_R(0)} \neq 0$, hence $w^{\pm} \neq 0$.

It remains to prove the last statement in Theorem 2.2. Consider a sequence of solutions $u_n := u_{\lambda_n}$ of (S_{λ_n}) with $\lambda_n \to \infty$. By our above argument we know that $\Phi_{\lambda_n}(u_n)$ and $\|u_n\|_{H^1}$ are bounded. Here

$$\Phi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + b_\lambda(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx$$

is the functional associated to (S_λ). Since (u_n) is bounded in $H^1(\mathbb{R}^N)$ we may assume $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and

$$u_n \to u \quad \text{in } L^\theta_{\mathrm{loc}}(\mathbb{R}^N),\ \text{for } 2 \leq \theta < 2^*. \tag{3.1}$$

We claim $u|_{\Omega^c} = 0$ where $\Omega = \operatorname{int} a^{-1}(0)$ and $\Omega^c = \{x \in \mathbb{R}^N : x \notin \Omega\}$. If $u|_{\Omega^c} \neq 0$ then there exists a compact subset F with $\operatorname{dist}(F, a^{-1}(0)) > 0$ and $\mu(F) > 0$ such that $u|_F \neq 0$. Here we used $\mu(\partial a^{-1}(0)) = 0$. Then by (3.1)

$$\int_F u_n^2\,dx \to \int_F u^2\,dx > 0.$$

Setting $a_F := \min a|_F > 0$ it follows that

$$\Phi_{\lambda_n}(u_n) \geq \lambda_n \int_F a(x)u_n^2\,dx \geq \lambda_n a_F \int_F u_n^2\,dx \to \infty \quad \text{as } n \to \infty,$$

a contradiction. Since u_n solves $-\Delta u_n + (\lambda_n a + 1)u_n = f(x,u_n)$ we have that $u \in H_0^1(\Omega)$ is a solution of $-\Delta u + u = f(x,u)$ in Ω.

Next we claim that

$$u_n \to u \quad \text{in } L^\theta(\mathbb{R}^N),\ \text{for } 2 < \theta < 2^*. \tag{3.2}$$

If not, then by the concentration compactness principle of P. L. Lions (Lemma I.1 in [11]) there exist $r > 0$ and a sequence $x_n \in \mathbb{R}^N$ with $|x_n| \to \infty$ such that

$$\int_{B_r(x_n)} u_n^2\,dx \geq \delta > 0.$$

Now an argument similar to showing $u|_{\Omega^c} = 0$ gives a contradiction.

Finally, in order to see $u_n \to u$ in $H^1(\mathbb{R}^N)$ observe that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2\,dx &+ \int_{\mathbb{R}^N} (\lambda_n a + 1)(u_n - u)^2\,dx \\
&= \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} (\lambda_n a + 1)u_n^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx \\
&\quad - \int_{\mathbb{R}^N} (\lambda_n a + 1)u^2\,dx + o(1) \\
&= \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx - \int_{\mathbb{R}^N} f(x,u)u\,dx + o(1) = o(1)
\end{aligned}$$

where we used (3.2).

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