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REMARKS ON THE EQUIVARIANT DEGREE THEORY

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Abstract. We present the computations of the secondary obstruction groups for the first stem of stable equivariant homotopy groups, used in the setting for the equivariant degree introduced by Ize *et al.*, in the case of the same action of a compact Lie group on the domain and co-domain.

1. Introduction

The equivariant degree $\deg_G(f,\Omega)$ on a bounded invariant open set $\Omega\subset W$ for an equivariant map $f:W\to V$ between two representations of a compact Lie group G ($\dim V\geq\dim W$) was introduced by Ize *et al.* (cf. [7-10]) as an element of the equivariant homotopy groups of spheres $\Pi^G_{S^V}(S^W)$, where S^V denotes the one-point compactification of V. It was proved (cf. [7]) that this equivariant degree has all the standard properties expected from a "degree theory". From the applications point of view, the most interesting case is $W=\mathbb{R}^n\oplus V$ (we assume that G acts trivially on $\mathbb{R}^n$). In this case, by applying the regular normal approximation theorem (cf. [13]) or general position results (cf. [14]), the map f can be deformed on Ω to a map $\widetilde{f}$ such that the set of zeros of $\widetilde{f}$ in Ω is a disjoint union of compact closed G-submanifolds M_α, indexed by the orbit types α in Ω with $(G_x)=\alpha$ for all $x\in M_\alpha$. As it is well known, the equivariant degree

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expresses topological obstructions for the existence of equivariant extensions over Ω^H without zeros (cf. [7]). These obstructions are called *primary* if $\dim W(H) = n$, and are called *secondary* if $\dim W(H) < n$, where $W(H)$ denotes the Weyl group of H. We denote these obstruction groups by $\Pi(H)$. For the computations of the primary obstructions we refer to [3], [6], [9], [10], [14], [16].

It was proved in [9] that under reasonable conditions, in the case of the so called *first stem* (i.e. corresponding to the orbit types (H) with $\dim W(H) = n-1$), the secondary obstruction groups for abelian actions are finite. It was conjectured by J. Ize that similar results should hold for an arbitrary compact Lie group.

The objective of this paper is to compute the secondary obstruction groups for an arbitrary compact Lie group for the first stem in the case of the same action on the domain and co-domain (cf. Theorem 4.3). In particular, under the additional assumption that the related space of principal orbits is one-connected, we obtain that $\mathbb{Z}_2$ is a subgroup of $\Pi(H)$ and $\Pi(H)/\mathbb{Z}_2 \simeq W(H)/[W(H), W(H)]$, where $[W(H), W(H)]$ denotes the commutator subgroup of $W(H)$, and the finiteness of $\Pi(H)$ follows.

The rest of the paper is organized as follows. Section 2 contains preliminaries on stable equivariant homotopy groups and a slightly modified definition of the equivariant degree, originally introduced by Ize *et al.* (cf. [7]). This modified equivariant degree is defined only for maps $f : \mathbb{R}^n \times V \to V$, i.e. under the assumption that the same representation V appears in domain and co-domain, and is "stabilized" after several suspensions, so the additivity and suspension properties are satisfied without additional assumptions. These simplifications may cause the loss of the universality property, but otherwise, the obtained in this way equivariant degree is not different from the original definition given in [7] (for this particular case). Section 3 is devoted to the bordism theory and in Section 4 we present and prove the main result of this paper – Theorem 4.3 on the first stem secondary obstruction groups.

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2. Equivariant homotopy groups and equivariant degree

For an Euclidean space U we denote by $B(U)$ the unit ball in U, by $S(U)$ the unit sphere in U, and by S^U the one point compactification of U. Via the standard stereographic projection, S^U can be identified with the unit sphere $S(\mathbb{R} \times U)$.

Let V be an orthogonal representation of a compact Lie group G. Let $n \geq 0$ be a fixed integer and $W := \mathbb{R}^n \oplus V$, where G is assumed to act trivially on $\mathbb{R}^n$. Let H be a (closed) subgroup of G and X a subset of W. We denote $X^H = \{x \in X;\ \text{for all } g \in H,\ gx = x\}$ and $X_H = \{x \in X;\ G_x = H\}$, where G_x is the isotropy group of x. We also put $X_{(H)} = GX_H$.

Let $N \geq 1$ be an integer. It is clear that by Tietze–Gleason Lemma (cf. [11]), the set of the equivariant homotopy classes $[S^{\mathbb{R}^{N-1}\oplus W}; S^{\mathbb{R}^{N-1}\oplus V}]^G$ of equivariant maps $\varphi : S^{\mathbb{R}^{N-1}\oplus W} \to S^{\mathbb{R}^{N-1}\oplus V}$, can be identified with the set of equivariant homotopy classes

$$\Pi_N := [\overline{B}(\mathbb{R}^{N+n} \times V), S(\mathbb{R}^{N+n} \times V); \mathbb{R}^N \times V, (\mathbb{R}^N \times V) \setminus \{0\}]^G.$$

It is well known (see [7]) that for $N > 1$ the set Π_N has a natural structure of an abelian group. We denote by $\xi_m : \Pi_N \to \Pi_{N+m}$ the standard m-th suspension homomorphism.

The following result is a particular case of an equivariant version of the Freudenthal Suspension Theorem (cf. [15]):

Theorem 2.1. *Let $x^H = N - 1 + n + \dim V^H$ and $y_H = N - 2 + \dim V^H$, where $H \subset G$. If for every isotropy subgroup H in V we have $x^H \leq 2y_H$, $y_H \geq 1$, then the suspension homomorphism $\xi_m : \Pi_N \to \Pi_{N+m}$ is an isomorphism for all $m \geq 1$. In particular, ξ_m is an isomorphism for all $N \geq n + 3$.*

Put $\Pi := \Pi_N$, for $N \geq n+3$, i.e. Π is the "limit" of the groups Π_N.

We are now in a position to introduce a slightly modified definition of the equivariant degree. The original definition, introduced by Ize *et al.*, was established for the general case of equivariant maps between two arbitrary representations of the group G. In this paper, we consider only the case of equivariant maps $f : \mathbb{R}^n \times V \to V$.

Let $\Omega \subset \mathbb{R}^n \times V$ be a bounded invariant open subset. We will call an equivariant map $f : \mathbb{R}^n \times V \to V$ *Ω-admissible* if $f(x) \neq 0$ for all $x \in \partial\Omega$. If f is Ω-admissible, then there exists an invariant neighbourhood $\mathcal{N}$ of $\partial\Omega$ such that $f(x) \neq 0$ for all $x \in \mathcal{N}$. We put $\Omega_{\mathcal{N}} := \Omega \cup \mathcal{N}$. Let $R > 0$ be a real number such that $\overline{\Omega_{\mathcal{N}}} \subset B_R(0) := \{x \in \mathbb{R}^n \times V;\ \|x\| < R\}$. Let $\eta : \overline{B_R(0)} \to \mathbb{R}$ be an invariant Urysohn function such that

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \notin \Omega_{\mathcal{N}}. \end{cases}$$

We define $F : ([-1,1] \times \overline{B_R(0)}, \partial([-1,1] \times B_R(0)) \to (\mathbb{R} \times V, (\mathbb{R} \times V) \setminus \{0\})$ by

$$F(t,x) = (t + 2\eta(x), f(x)), \quad (t,x) \in [-1,1] \times \overline{B_R(0)}.$$

Since $[-1,1]\times\overline{B_R(0)}$ is equivariantly homotopic to $B(\mathbb{R}\times\mathbb{R}^n\times V)$, we obtain that the map F defines an equivariant homotopy class $[F]$ in Π_1. Set

$$\deg_G(f,\Omega) := \xi_{n+3}[F]\in\Pi,$$

and we call it the *G-equivariant degree* of f in Ω.

We have the following result due to Ize *et al.* (cf. [7]):

Theorem 2.2. *The G-equivariant degree* $\deg_G(f,\Omega)$ *has the following properties:*

(P1) (Existence) *If* $\deg_G(f,\Omega)\neq 0$ *then there exists* $x\in\Omega$ *such that* $f(x)=0$*.*
(P2) (Additivity) *If* $f^{-1}(0)\cap\Omega\subset\Omega_1\cup\Omega_2$*, where* Ω_1 *and* Ω_2 *are two disjoint open invariant subsets of* Ω*, then*

$$\deg_G(f,\Omega)=\deg_G(f,\Omega_1)+\deg_G(f,\Omega_2).$$

(P3) (Homotopy) *If* $f_t:[0,1]\times V\to W$ *is an equivariant homotopy of* Ω*-admissible maps then* $\deg_G(f_t,\Omega)=$ *constant.*
(P4) (Suspension) $\deg_G(Id\times f,(-1,1)\times\Omega)=\deg_G(f,\Omega)$.

As our objective is to compute the equivariant homotopy group Π, we will decompose Π into a direct sum of "simpler" subgroups. To this end, we will approximate $F:\mathbb{R}^{N+n}\times V\to\mathbb{R}^N\times V$ by G-equivariant "regular" maps transverse to zero on the fixed point spaces. By applying the additivity property of the equivariant degree to "regular" representatives of elements in Π, we are able to separate their zeros according to the orbit types.

Definition 2.3.

(i) An equivariant map $F:\mathbb{R}^{N+n}\times V\to\mathbb{R}^N\times V$ is called *normal* in $\Omega\subset\mathbb{R}^{N+n}\times V$ if

$$\forall x\in F^{-1}(0)\cap\Omega\ \exists\delta>0\ \forall v\perp T_x\Omega_{(G_x)}\quad \|v\|<\delta\Rightarrow F(x+v)=v,$$

where $T_x\Omega_{(G_x)}$ denotes the tangent space to $\Omega_{(G_x)}$ at x.
(ii) A normal in Ω map F is called *regular normal* in Ω if it is of class C^1 in Ω and for every orbit type (H) in Ω the map $F_{|\Omega_H}:\Omega_H\to W^H$ has zero as a regular value.
(iii) An equivariant homotopy $h:[0,1]\times\mathbb{R}^{N+n}\times V\to\mathbb{R}^N\times V$ is called a *normal homotopy* in Ω if it is a normal map on $[0,1]\times\Omega$.

We have the following approximation result:

THEOREM 2.4 (cf. [13], see also [14]). *Let $F : \mathbb{R}^{N+n} \times V \to \mathbb{R}^N \times V$ be an Ω-admissible map. Then for every $\varepsilon > 0$ there exists an equivariant map $\widetilde{F}$ such that*

(i) *$\widetilde{F}$ is Ω-admissible,*
(ii) *$\widetilde{F}$ is regular normal in Ω,*
(iii) $\sup_{x\in\Omega} \|F(x) - \widetilde{F}(x)\| < \varepsilon$.

Similar statement is also valid for normal homotopies.

REMARK 2.5. It follows directly from the definition that a suspension of a regular normal in Ω map F is regular normal in $(-1,1) \times \Omega$. By Theorem 2.4, every element α in Π has a regular normal representative F. The set of zeros $F^{-1}(0)$ can be decomposed into a union of *compact* disjoint G-invariant submanifolds indexed by their orbit types.

DEFINITION 2.6. For every orbit type (H) in V we define the subset $\Pi(H)$ of Π which consists of all the elements $a \in \Pi$ such that there exists a regular normal in $B(\mathbb{R}^{N+n} \times V)$ map $F : \mathbb{R}^{N+n} \times V \to \mathbb{R}^N \times V$ with the following properties:

(i) F is $B(\mathbb{R}^{N+n} \times V))$-admissible,
(ii) $F^{-1}(0) \cap B(\mathbb{R}^{N+n} \times V)) = (F^{-1}(0) \cap B(\mathbb{R}^{N+n} \times V)))_{(H)}$, i.e. the set $F^{-1}(0)$ is of orbit type (H),
(iii) $\deg_G(F, B(\mathbb{R}^{N+n} \times V)) = a$.

PROPOSITION 2.7. *For every orbit type $\alpha = (H)$ in V such that $\dim W(H) \leq n$, the set $\Pi(H)$ is a subgroup of Π and, in addition, we have*

$$\Pi = \bigoplus_{\dim W(H) \leq n} \Pi(H).$$

PROOF. It follows from the definition of the group operation in Π (cf. [7]), that if $a \in \Pi(H)$ then $-a \in \Pi(H)$, and if $a, b \in \Pi(H)$ then $a + b \in \Pi(H)$. Suppose that $a \in \Pi(H) \cap \Pi(K)$, where K is another orbit type in V. Let Ω denote the unit ball in $\mathbb{R}^{N+n} \times V$. By Theorem 2.4, there exist regular normal representatives F_0 and F_1 of a, such that $F_0^{-1}(0)$ contains only zeros of the orbit type (H) and $F_1^{-1}(0)$ contains only zeros of the orbit type (K). By assumption, F_0 and F_1 are equivariantly homotopic. Take a normal equivariant homotopy $h(t,x)$ between F_0 and F_1 provided by Theorem 2.4. Choose an invariant Urysohn function $\eta : [0,1] \times \overline{\Omega} \to \mathbb{R}$ such that for every point $(t,x) \in [0,1] \times \Omega$ with $h(t,x) = 0$ we have $\eta(x,t) = 1$ for $(G_x) = (H)$ and $\eta(x,t) = 0$ for $(G_x) \neq (H)$. Define the equivariant homotopy $g(t,x)$ between F_0 and $F(x) := h(\eta(x),x)$ by $g(t,x) := h(t\eta(x),x)$. It is clear that if $h(\eta(x),x) = 0$ then either $(G_x) = (H)$, so $\eta(x) = 1$ and $h(\eta(x),x) = F_1(x) = 0$, but this is impossible because F_1 has no zeros of the orbit type (H), or $(G_x) \neq (H)$, so $\eta(x) = 0$ and $h(\eta(x), x) = F_0(x) = 0$, but this is also impossible, because F_0 has no zeros of the orbit type different from (H). Consequently, $F^{-1}(0) = \emptyset$ and $a = \deg_G(F, \Omega)$, so $\Pi(H) \cap \Pi(K) = \{0\}$ for $(H) \neq (K)$.

Let $a \in \Pi$ be an arbitrary element and F a regular normal representative of a. Then $F^{-1}(0) = \bigcup_{(H)} M_{(H)}$, where $M_{(H)}$ has the orbit type (H). Let $\Omega_{(H)}$ be an isolating neighbourhood of $M_{(H)}$. Then, by the additivity property,

$$a = \deg_G(F, \Omega) = \sum_{(H)} \deg_G(F, \Omega_{(H)}) = \sum_{(H)} a_{(H)},$$

where $a_{(H)} = \deg_G(F, \Omega_{(H)})$. Let $\eta_{(H)} : \Omega \to \mathbb{R}$ be an invariant differentiable Urysohn function such that $\eta_{(H)}(x) = 0$ for $x \in M_{(H)}$ and $\eta_{(H)}(x) = 1$ for $x \notin \Omega_{(H)}$. Then $F_{(H)}(t, x) = (t + 2\eta_{(H)}(x), F(x))$ is regular normal in $(-1, 1) \times \Omega$, and by the suspension property, $a_{(H)} = \deg_G(F, \Omega_{(H)}) = \deg_G(F_{(H)}, (-1, 1) \times \Omega)$, so $a_{(H)} \in \Pi(H)$. Therefore, this representation is unique and the statement follows.

Finally, if $\widetilde{F}$ is an Ω-admissible regular normal map and $W(H) > n$, then by the transversality condition for a regular normal map, $\widetilde{F}^{-1}(0) \cap \Omega_{(H)} = \emptyset$. $\square$

In what follows, we will denote by $a_{(H)}$ the $\Pi(H)$-component of $a \in \Pi$. Let $f : \mathbb{R}^n \times V \to V$ be an Ω-admissible equivariant map. We will write

$$\deg_G(f, \Omega) = \sum_{(H)} a_{(H)} \in \Pi = \bigoplus_{\dim W(H) \leq n} \Pi(H). \tag{2.1}$$

As an immediate consequence we obtain the following result, which can be considered as a refinement of the existence property:

Proposition 2.8. *If $\deg_G(f, \omega) = \sum_{(H)} a_{(H)} \neq 0$, i.e. there exists $a_{(H)} \neq 0$, then there exists $x \in \Omega^H$ such that $f(x) = 0$, i.e. there exists a solution in Ω of the equation $f(x) = 0$ with symmetries at least H. Moreover, if f is normal in Ω and $a_{(H)} \neq 0$, then $f^{-1}(0) \cap \Omega_H \neq \emptyset$.*

3. Equivariant framed bordism

Let $f : \mathbb{R}^n \times V \to V$ be a G-equivariant Ω-admissible map. It is well known that for a subgroup H such that $\dim W(H) = n$, the group $\Pi(H)$ is either $\mathbb{Z}$ or $\mathbb{Z}_2$ (see [3], [6], [9], [10], [14], [16] for more details), and consequently the coefficient $a_{(H)}$ of $\deg_G(f, \Omega)$ (given in (2.1)) is an integer or integer modulo 2.

The main objective of this and the next sections is the computation of $\Pi(H)$ corresponding to secondary obstructions with $W(H)$ finite. In the case of an abelian action, the groups $\Pi(H)$ with $W(H)$ finite, were computed in [9], [10] using the fundamental domain method combined with the geometric obstruction theory (cf. [14]). Our method, which can be equally applied to abelian and non-abelian actions, is based on the use of the fundamental domain method and the classical bordism theory. In this paper we present only the computations in the case of the first stem, i.e. for $n = 1$, but we believe that this method should also work for $n > 1$.

Let us begin with the definition of the equivariant bordism relation.

Let $\Gamma := W(H)$ and U denote the Γ-representation $\mathbb{R}^{N+n} \times V^H$, $N \geq n+3$, in particular $\dim U^\Gamma \geq 3$. We denote by $A \subset U$ the set of all non-principal orbits and put $U_0 = \mathbb{R}^N \times V^H$. We fix an orientation in U_0 which naturally induces an orientation in U. Notice that Γ acts freely on $U \setminus A$.

Let M be a compact n-dimensional Γ-submanifold (possibly with boundary) of $U \setminus A$. We denote by $\nu(M)$ the normal bundle to M in U, and by $\nu_x(M)$ the fiber of $\nu(M)$ at $x \in M$. By a Γ-*framing* on M we mean a Γ-trivialization of $\nu(M)$, i.e. a Γ-vector bundle isomorphism:

$$\varepsilon : \nu(M) \to M \times U_0, \quad \varepsilon(x, v) = (x, \varepsilon_x v),$$

where $\varepsilon_x : \nu_x(M) \to U_0$ is an isomorphism such that $g\varepsilon_x(v) = \varepsilon_{gx}(gv)$ for all $x \in M$, $v \in \nu_x(M)$ and $g \in \Gamma$. The pair (M, ε) is called a Γ-framed n-submanifold of U. It is clear that the manifold M has a natural orientation induced by the framing ε.

Definition 3.1. Two closed Γ-framed n-submanifolds (M_0, ε_0) and (M_1, ε_1) are called Γ-*bordant* if there exists a compact $n+1$-dimensional Γ-submanifold (with boundary) $W \subset [0,1] \times (U \setminus A)$ with a Γ-framing $\eta : \nu(W) \to W \times U_0$ such that:

(i) $\partial W = \{0\} \times M_0 \cup \{1\} \times M_1$,
(ii) $\eta(0, \cdot\,) = \varepsilon_0$, $\eta(1, \cdot\,) = \varepsilon_1$,

and we will write

$$(M_0, \varepsilon_0) \sim (M_1, \varepsilon_1).$$

The relation $\sim$ is an equivalence relation and we will denote by $[M, \varepsilon]$ the class of (M, ε). We will also say that (M, ε) is *null-bordant* if (M, ε) is bordant to an empty set and we will write $(M, \varepsilon) \sim 0$. We denote by $\Omega^\Gamma(U \setminus A)$ the set of all equivalence classes of the relation $\sim$.

The following standard fact can be proved by using the (equivariant) Pontrjagin–Thom construction for the framed bordism (cf. [2], [19]), and the one-to-one correspondence between G-equivariant homotopy classes of G-equivariant maps and G-equivariant extensions of $W(H)$-equivariant maps (cf. [3], p. 122).

Proposition 3.2. *Under the above assumptions, the set $\Omega^\Gamma(U \setminus A)$ has a natural structure of an abelian group which is isomorphic to the group $\Pi(H)$.*

Remark 3.3. Let $f : \mathbb{R}^n \times V \to V$ be an equivariant Ω-admissible regular normal map. Then $f_H^{-1}(0)$, where $f_H := f_{|\Omega_H} : \Omega_H \to V^H$, admits a $W(H)$-framing in $\mathbb{R}^N \times \Omega_H \subset U \setminus A$, $N \geq 3+n$, with the framing ε_f induced by the gradients of the Nth suspension of f_H. Then, for H, such that $\dim W(H) \leq n$, the coefficient $a_{(H)}$ of $\deg_G(f,\Omega)$ is equal to the $W(H)$-equivariant bordism class $[f_H^{-1}(0), \varepsilon_f]$ in Π, i.e. we have the following "analytic formula" for the equivariant degree of f:

$$\deg_G(f,\Omega) = \sum_{(H)} a_{(H)}, \quad a_{(H)} = [f_H^{-1}(0), \varepsilon_f].$$

Let $\mathcal{R} := (U \setminus A)/\Gamma$. Since Γ acts freely on $U \setminus A$, $\mathcal{R}$ is a manifold. It is well known (cf. [11], Theorem 4.27), that the orbit space $\mathcal{R}$ is connected. Let $\pi : U \setminus A \to \mathcal{R}$ be the natural projection. In what follows we will denote by $\Omega_1(\mathcal{R})$ the usual oriented (non-framed) singular bordism group defined for 1-dimensional closed singular submanifolds of $\mathcal{R}$ (cf. [2], [19]). In the case $N \geq 4$, by the classical Whitney theorem, the group $\Omega_1(\mathcal{R})$ is naturally isomorphic to bordism group of *smooth* oriented one-dimensional submanifolds of $\mathcal{R}$.

As it is well known, any compact closed oriented one-dimensional (respectively, two-dimensional) manifold is a boundary of a compact oriented two-dimensional (respectively, three-dimensional) manifold. Therefore, we have:

Proposition 3.4. *The group $\Omega_1(\mathcal{R})$ is isomorphic to the first singular homology group $H_1(\mathcal{R})$.*

For a formal proof of Proposition 3.4, we refer to [19], Chapter IX, or [17, Proposition II.4.5, Chapter IV, §7] and Chapter VI in [19].

Corollary 3.5. $\Omega_1(\mathcal{R}) \simeq \pi_1(\mathcal{R})/[\pi_1(\mathcal{R}), \pi_1(\mathcal{R})]$*, where* $[\pi_1(\mathcal{R}), \pi_1(\mathcal{R})]$ *denotes the commutator group of* $\pi_1(\mathcal{R})$*. In particular, if* $U \setminus A$ *is simply connected, then* $\Omega_1(\mathcal{R}) \simeq (\Gamma/\Gamma_0)/[\Gamma/\Gamma_0, \Gamma/\Gamma_0]$*, where* Γ_0 *denotes the connected component of* Γ *containing the unity.*

Proof. By Hurewicz Homomorphism Theorem (cf. [5, §14, Theorem 3], or [18, Chapter 7, §4]), $H_1(\mathcal{R}) \simeq \pi_1(\mathcal{R})/[\pi_1(\mathcal{R}), \pi_1(\mathcal{R})]$. The natural projection $\pi : (U \setminus A)/\Gamma_0 \to \mathcal{R}$ is a covering with the fiber Γ/Γ_0. If $U \setminus A$ is simply connected, then $(U \setminus A)/\Gamma_0$ is also simply connected (cf. [1]), and $\pi : (U \setminus A)/\Gamma_0 \to \mathcal{R}$ is a universal covering of $\mathcal{R}$. Therefore, $\pi_1(\mathcal{R}) = \Gamma/\Gamma_0$.

Remark 3.6. Since $\mathcal{R}$ is a connected manifold, the set $[S^1, \mathcal{R}]$ of all free homotopy classes of maps from S^1 to $\mathcal{R}$ is in one-to-one correspondence with the conjugacy classes of elements in $\pi_1(\mathcal{R}, x_0)$ (see [4, Chapter 4, §17, Theorem 4]). Therefore, by Corollary 3.5, we have that $\Omega_1(\mathcal{R})$ is isomorphic to $[S^1, \mathcal{R}]$.

Let us recall that a compact Lie group Γ is called *bi-orientable* if it admits an orientation invariant with respect to left and right translations. It is clear that finite, connected or abelian compact Lie groups are bi-orientable (cf. [6], [16]). Since $\mathcal{R}$ does not need to be an orientable manifold, we need the following:

LEMMA 3.7. *Assume that* Γ *is bi-orientable and* $\dim\Gamma = n-1$. *Let* (W,η) *be a compact* $n+1$*-dimensional* Γ*-framed submanifold* (*with boundary*) *of* $\mathbb{R}\times(U\setminus A)$, *satisfying the conditions* (i) *and* (ii) *of Definition* 3.1. *Then* W/Γ *is an oriented* 2*-submanifold* (*with boundary*) *of* $\mathbb{R}\times\mathcal{R}$.

PROOF. It is clear that the Γ-framing on W induces an orientation on W in such a way that the orientation of the tangent space $\tau_x(W)$ to W at the point x, followed by the orientation of $\nu_x(W)$, given by the Γ-framing η, coincides with the fixed orientation of $\mathbb{R}\times U$. Notice that every $g\in\Gamma$ changes the orientation of $\mathbb{R}\times U = \tau_x(W)\oplus\nu_x(W)$ if and only if it changes the orientation of U_0. Hence, from the fact that the Γ-framing $\eta:\nu(W)\to W\times U_0$ is Γ-equivariant it follows that $g_{|\nu_x(W)}:\nu_x(W)\to\nu_{gx}(W)$ changes the orientations if and only if $g:\mathbb{R}\times U\to\mathbb{R}\times U$ changes the orientations. Consequently, $g_{\tau_x(W)}:\tau_x(W)\to\tau_{gx}(W)$ always preserves the orientations. Therefore, W/Γ is an oriented 2-submanifold of $\mathbb{R}\times\mathcal{R}$ (cf., for instance, [6], [16]). $\square$

As it follows from the proof, Lemma 3.7 is valid in more general setting, namely in the case of two different *concordant* (cf. [14]) actions of Γ on U and U_0.

It is well known that in the case of classical (non-equivariant) framed bordism theory, every bordism class can be represented by a connected submanifold (cf. [4, §23, Theorem 2]). We cannot expect so much in the case of Γ-framed bordism. However, we have:

LEMMA 3.8. *Assume that* $n=1$ *and* Γ *is finite. For every* Γ*-framed closed* 1*-submanifold* (M,ε) *of* $U\setminus A$ *such that* M/Γ *is not connected, there exists a* Γ*-framed submanifold* $(\widetilde{M},\widetilde{\varepsilon})$ *of* $U\setminus A$ *such that* $\widetilde{M}/\Gamma$ *is connected and* $[M,\varepsilon]=[\widetilde{M},\widetilde{\varepsilon}]$.

PROOF. Assume for simplicity that M/Γ is composed of two connected components N_1 and N_2, and let $x_1\in N_1$ and $x_2\in N_2$. Since $\mathcal{R}$ is connected, there exists a smooth path $\xi:[0,1]\to\mathcal{R}$ joining x_1 to x_2 (we can assume that ξ is also an embedding and $\xi\big((0,1)\big)\subset\mathcal{R}\setminus M/\Gamma$). Define $\widetilde{\sigma}:[0,1]\to\mathcal{R}$ by

$$\widetilde{\sigma}(t)=\begin{cases}\sigma_{N_1}(4t) & \text{for } t\in[0,1/4],\\ \xi(4t-1) & \text{for } t\in[1/4,1/2],\\ \sigma_{N_2}(4t-2) & \text{for } t\in[1/2,3/4],\\ \xi(4-4t) & \text{for } t\in[3/4,1].\end{cases}$$

By the Whitney theorem (since $\dim\mathcal{R}\ge 4$), using a small perturbation, $\widetilde{\sigma}$ can be deformed into a smooth embedding σ. Put $N=\sigma([0,1])$ and let $\widetilde{M}=\pi^{-1}(N)$. Since π is a covering, there exist smooth liftings $\widehat{\sigma}:[0,1]\to U\setminus A$ of $\widetilde{\sigma}$ and

$\bar{\sigma} : [0,1] \to U \setminus A$ of σ. Notice that the set $D := \overline{\sigma}([0,1])$ is a fundamental domain for $\widetilde{M}$. The set $T := \widehat{\sigma}([1/4,1/2]))$ is contractible, hence there exists a "framing" on T which coincides at $\widehat{\sigma}(1/4)$ and $\widehat{\sigma}(1/2)$ with the Γ-framing ε on M. Since there is an element $g \in \Gamma$ such that $g\widehat{\sigma}([1/4,1/2]) = \widehat{\sigma}(3/4,1])$, we can extend this "framing", by using g, on the set $\widehat{\sigma}([3/4,1])$. By equivariance, we extend the obtained "framing" over the set $M \cup \widetilde{T}$, where $\widetilde{T} = \Gamma(T)$. The set $M \cup \widetilde{T}$ can be made arbitrarily closed to $\widetilde{M}$. Therefore, by using the standard argument (cf. [4, §23, Theorem 2]), there exists a (non-equivariant) framing on D "transferred" from the framing on $M \cup \widetilde{T}$. We can extend it equivariantly to a Γ-framing on $\widetilde{M}$. It can be verified, by using the standard argument, that $[M,\varepsilon] = [\widetilde{M},\widetilde{\varepsilon}]$ □

4. Computations of $\Pi(H)$ for the first stem

In this section we will assume that $n = 1$ and $\Gamma = W(H)$ is a finite group. In order to compute $\Omega^\Gamma(U \setminus A)$ we will study its relation to $\Omega_1(\mathcal{R})$. We have

Lemma 4.1. *The map $\Phi : \Omega^\Gamma(U) \to \Omega_1(\mathcal{R})$ given by $\Phi([M,\varepsilon]) = [M/\Gamma]$, where $[M,\varepsilon] \in \Omega^\Gamma(U)$ and $[M/\Gamma]$ denotes the oriented bordism class of the 1-manifold M/Γ, is a well defined epimorphism of abelian groups.*

Proof. We consider the following natural projection

$$U \setminus A \xrightarrow{\pi} \mathcal{R}.$$

In order to check that Φ is a well defined homomorphism we need to show that if (M_0,ε_0) and (M_1,ε_1) are two Γ-bordant 1-dimensional Γ-framed submanifolds of $U \setminus A$, then $\pi(M_0)$ is bordant to $\pi(M_1)$ (with respect to oriented bordism). Assume that there exists a Γ-framed 2-submanifold (W,η) in $[0,1] \times (U \setminus A)$ satisfying the conditions of Definition 3.1. Then, by Lemma 3.7, W/Γ is an oriented submanifold of $[0,1] \times \mathcal{R}$ with the boundary composed exactly of $\pi(M_0)$ and $\pi(M_1)$, therefore $[\pi(M_0)] = [\pi(M_1)]$ in $\Omega_1(\mathcal{R})$. Hence Φ is well defined. Since π maps a union of disjoint Γ-framed submanifolds in $U \setminus A$ onto a union of disjoint submanifolds in $\mathcal{R}$, clearly, Φ is a homomorphism of abelian groups.

To show the surjectivity of Φ, consider a 1-dimensional oriented connected submanifold N of $\mathcal{R}$. It is clear that we can identify N with an embedding $\sigma : [0,1] \to \mathcal{R}$ such that $\sigma(0) = \sigma(1)$, $\sigma([0,1]) = N$ with σ preserving the orientations of $[0,1]$ and N. Since $\pi : U \setminus A \to \mathcal{R}$ is a covering, there exists a lifting $\widetilde{\sigma} : [0,1] \to (U \setminus A)$ and an element $g \in \Gamma$ such that $\widetilde{\sigma}(1) = g\widetilde{\sigma}(0)$. Put $M := \pi^{-1}(N)$. It is clear that $D := \widetilde{\sigma}([0,1])$ is a fundamental domain for M. Since D is contractible, the normal vector bundle $\nu(D)$ is trivial and there exists a framing $\widetilde{\varepsilon}$ on D such that $\widetilde{\varepsilon}_{\widetilde{\sigma}(0)} = g^{-1} \circ \widetilde{\varepsilon}_{\widetilde{\sigma}(1)} \circ g$, where $g\widetilde{\sigma}(0) = \widetilde{\sigma}(1)$. The extension of $\widetilde{\varepsilon}$ to an equivariant trivialization $\varepsilon : \nu(M) \to M \times U_0$ defines a

Γ-framing on M. By construction, $\Phi([M,\varepsilon]) = [M/\Gamma] = [N]$, therefore Φ is an epimorphism. □

Let N be an oriented connected 1-submanifold of $\mathcal{R}$. We can associate with N an embedding, denoted by $\sigma_N : S^1 \to \mathcal{R}$, such that $\sigma_N(S^1) = N$ and σ_N preserves the orientations of S^1 and N.

Lemma 4.2. *Let $[M,\varepsilon] \in \operatorname{Ker}\Phi$. Then $\sigma_{M/\Gamma}$ is (freely) homotopic to a constant map.*

Proof. Suppose that $\Phi([M,\varepsilon]) = 0$. By Lemma 3.8, we can assume that M/Γ is connected. Then, by Remark 3.6, $[\sigma_{M/\Gamma}]$ is the zero element in the quotient group $\pi_1(\mathcal{R})/[\pi_1(\mathcal{R}),\pi_1(\mathcal{R})] = [S^1,\mathcal{R}]$, so $\sigma_{M/\Gamma}$ is freely homotopic to a constant map. □

Theorem 4.3. *Assume that $n = 1$ and Γ is finite. Then we have the following short exact sequence of abelian groups*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Omega^\Gamma(U\setminus A) \xrightarrow{\Phi} (\pi_1(\mathcal{R})/[\pi_1(\mathcal{R}),\pi_1(\mathcal{R})] \longrightarrow 0.$$

Proof. Let $\alpha \in \operatorname{Ker}\Phi$ and (M,ε) be a representative of α such that M/Γ is connected (Lemma 3.8) and $\sigma_{M/\Gamma} : S^1 \to \mathcal{R}$ is homotopic to a constant map (Lemma 4.2). Consequently, there exists a lifting $\widehat{\sigma} : S^1 \to U\setminus A$ of $\sigma_{M/\Gamma}$. Let $j := \widehat{\sigma}\circ\sigma_{M/\Gamma}^{-1} : M/\Gamma \to M$. The map j is a left inverse of $\pi_{|M}$. We denote by $j^*(\nu(M))$ the pull-back of $\nu(M)$ by j. Clearly, $j^*(\nu(M))$ is isomorphic to $\nu(M/\Gamma)$. Therefore, we have the following diagram of vector bundles

$$\begin{array}{ccccc}
j^*(\nu(M)) & \underset{\simeq}{\longrightarrow} & \nu(M)_{j(M/\Gamma)} & \underset{\simeq}{\xrightarrow{\varepsilon}} & j(M/\Gamma)\times U_0 \\
\simeq\big\downarrow j & & & & \simeq\big\downarrow j^{-1} \\
\nu(M/\Gamma) & & \xrightarrow[\varepsilon_0]{\hspace{6em}} & & M/\Gamma\times U_0
\end{array}$$

The submanifold M/Γ has a framing ε_0 in $\mathcal{R}$ induced from (M,ε).

On the other hand, since $\widehat{\sigma}(S^1) \subset U\setminus A$ is a fundamental domain for M, a trivialization $\varepsilon_0 : \nu(M/\Gamma) \to M/\Gamma\times U_0$ induces a Γ-trivialization $\varepsilon : \nu(M) \to M\times U_0$, and consequently, $\operatorname{Ker}\Phi$ is in one-to-one correspondence with the framed bordism classes of connected and null homotopic 1-submanifolds (N,ε_0) in $\mathcal{R}$. It is clear that there are only two equivalence classes for such framed bordism relation, i.e. $\operatorname{Ker}\Phi \simeq \mathbb{Z}_2$. The result follows from Lemma 4.1. □

Corollary 4.4. *Under the assumptions of Theorem 4.3 we have the following:*

(i) $\Omega^\Gamma(U\setminus A)/\mathbb{Z}_2 \simeq \pi_1(\mathcal{R})/[\pi_1(\mathcal{R}),\pi_1(\mathcal{R})]$,

(ii) *if $U\setminus A$ is simply connected then* $\Omega^\Gamma(U\setminus A)/\mathbb{Z}_2 \simeq \Gamma/[\Gamma,\Gamma]$,

(ii) *if $U\setminus A$ is simply connected and $\Gamma/[\Gamma,\Gamma]$ is of odd order, then* $\Omega^\Gamma(U\setminus A) \simeq \mathbb{Z}_2\oplus\Gamma/[\Gamma,\Gamma]$.

Proof. Since $\pi : U \setminus A \to \mathcal{R}$ is a universal covering, $\pi_1(\mathcal{R}) = \Gamma$, (ii) follows from Theorem 4.3. If $\Gamma/[\Gamma,\Gamma]$ is of odd order, then (iii) follows from the classification theorem for abelian groups. $\square$

Examples 4.5. (a) Let $\Gamma = \mathbb{Z}_n$, where $n \geq 3$ is an odd integer. If $U \setminus A$ is simply connected, then $\Omega^\Gamma(U \setminus A) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_n \simeq \mathbb{Z}_{2n}$. More general, if $\Gamma = \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_r}$, and p_i, p_j are relatively prime and odd for all $i \neq j$, $i, j \in \{1, 2, \ldots, r\}$, then $\Omega^\Gamma(U \setminus A) = \mathbb{Z}_2 \times \Gamma$ (compare with [9, Theorem 8.5]).

(b) Let $\Gamma = H(\mathbb{Z}_3)$ be the Heisenberg group of the matrices

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix},$$

with $a, b, c \in \mathbb{Z}_3$. Obviously, $[\Gamma, \Gamma] = \mathbb{Z}_3$, thus $\Omega^\Gamma(U \setminus A) = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[2] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer-Verlag, 1964.

[3] T. tom Dieck, *Transformation Groups*, de Gruyter, Berlin, 1987.

[4] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry – Methods and Applications* Part II, Springer, New York, 1992.

[5] A. T. Fomenko and D. B. Fuks, *Курс Гомотопической Топологии* (*The Course of Homotopic Topology*), Nauka, Moscow, 1989.

[6] K. Gęba, W. Krawcewicz and J. Wu, *An equivariant degree with applications to symmetric bifurcation problems I: Construction of the degree*, Bull. London. Math. Soc. **69** (1994), 377–398.

[7] J. Ize, I. Massabó and A. Vignoli, *Degree theory for equivariant maps, I*, Trans. Amer. Math. Soc. **315** (1989), 433–510.

[8] ———, *Degree theory for equivariant maps, the general S^1-action*, Memoirs Amer. Math. Soc. **481** (1992).

[9] J. Ize and A. Vignoli, *Equivariant degree for abelian actions, Part I; Equivariant homotopy groups*, Topol. Methods Nonlinear Anal. **2** (1993), 367–413.

[10] ———, *Equivariant degree for abelian actions, Part II; Index Computations*, Topol. Methods Nonlinear Anal. **7** (1996), 369–430.

[11] K. Kawakubo, *The Theory of Transformation Groups*, Oxford University Press, 1991.

[12] W. Krawcewicz and P. Vivi, *Normal bifurcation of equivariant maps*, preprint (1995).

[13] W. Krawcewicz and H. Xia, *Analytic constructon of an equivariant degree*, Izv. Vyssh. Uchebn. Zaved. Mat. **6** (1996), 37–53 (Russian); Russian Math. (Iz. VUZ) **40** (1996), 34–49.

[14] A. Kushkuley and Z. Balanov, *Geometric Methods in Degree Theory for Equivariant Maps*, Springer LNM 1632, Berlin, 1996.

[15] U. Namboodiri, *Equivariant vector fields on spheres*, Trans. Amer. Math. Soc. **278** (1983), 431–460.

[16] G. Peschke, *Degree of certain equivariant maps into a representation sphere*, Topology Appl. **59** (1994), 137–156.

[17] Y. Rudyak, *On Thom Spectra, Orientability and Cobordism*, Springer, New York, 1998.

[18] E. H. Spanier, *Algebraic Topology*, Springer, New York, 1966.

[19] R. E. Stong, *Notes on Cobordism Theory*, Princeton University Press, New Jersey, 1968.

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