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NONLINEAR EIGENVALUE PROBLEMS ADMITTING EIGENFUNCTIONS WITH KNOWN GEOMETRIC PROPERTIES

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Abstract. We consider nonlinear eigenvalue problems of the form

$$(*) \qquad A_0 y + B(y)y = \lambda y$$

in a real Hilbert space $\mathcal{H}$, where A_0 is a semi-bounded self-adjoint operator and, for every y from a certain dense subspace X of $\mathcal{H}, B(y)$ is a bounded symmetric linear operator. The left hand side is assumed to be the gradient of a functional $\psi \in C^1(x)$, and the associated linear problems

$$(**) \qquad A_0 v + B(y)v = \mu v$$

are supposed to have discrete spectrum ($y \in X$). We present a new topological method which permits, under appropriate assumptions, to construct solutions of $(*)$ on a sphere $S_R := \{y \in X \mid \|y\|_{\mathcal{H}} = R\}$ whose ψ-value is the nth Ljusternik–Schnirelman level of $\psi|_{S_R}$ and whose corresponding eigenvalue is the nth eigenvalue of the associated linear problem $(**)$, where $R > 0$ and $n \in \mathbb{N}$ are given. In applications, the eigenfunctions thus found share any geometric property enjoyed by an n-th eigenfunction of a linear problem of the form $(**)$. We discuss applications to nonlinear Sturm–Liouville problems, to the nonlinear Hill's equation, to periodic solutions of second-order systems, and to elliptic partial differential equations with radial symmetry.

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1. Introduction

To motivate the present work, let us first consider a nonlinear Sturm–Liouville problem of the form

$$-(p(t)u')' + q(t)u + g(t,u) = \lambda u \quad \text{for } a \le t \le b \tag{1.1}$$

with Dirichlet boundary conditions

$$u(a) = u(b) = 0 \tag{1.2}$$

and with an isoperimetric constraint of the form

$$\int_a^b u^2\,dt = R^2, \tag{1.3}$$

where $R > 0$ is given. One then asks for *eigensolutions*, i.e. pairs $(\lambda, u) \in \mathbb{R} \times C^2[a,b]$ which solve (1.1)–(1.3). Solutions of the Dirichlet problem (1.1)–(1.2) may be viewed as critical points of an appropriate functional ψ defined on the Sobolev space $X := W_0^{1,2}([a,b])$, and hence the full problem can be treated by applying critical point theory to the restriction of ψ to the surface

$$S_R := \{u \in X \mid \|u\|_{L^2([a,b])} = R\}.$$

If the nonlinearity is odd (i.e. $g(t,-u) = -g(t,u)$ for all $(t,u) \in [a,b] \times \mathbb{R}$), the existence of infinitely many eigensolutions (λ, u) can be established using Ljusternik–Schnirelman theory on general level sets as developed, for instance, by Zeidler [25], and these can be arranged in a sequence $(\lambda_n, \pm u_n)_{n\ge1}$ in such a way that the critical value $c_n := \psi(u_n)$ satisfies an inf-sup-characterization analogous to the classical inf-sup-description of eigenvalues of linear Sturm–Liouville problems. In fact, for $g \equiv 0$ one has $c_n = \lambda_n R^2/2$, where λ_n denotes the nth eigenvalue, and the Ljusternik–Schnirelman characterization of c_n reduces to the classical Courant–Fischer principle for λ_n. Therefore the Ljusternik–Schnirelman levels c_n should perhaps be considered as the proper generalization of eigenvalues to semi-linear problems, a viewpoint which has been stressed especially by E. Zeidler [24], [25] and which has led, for instance in the work of Chiappinelli [4], to interesting extensions of deep results about the eigenvalues of linear problems to the eigenvalues of appropriately constrained nonlinear problems. The question now arises whether the classical result on the number of zeroes of the nth eigenfunction of a linear Sturm–Liouville problem can be extended to the nonlinear case. An affirmative answer to this question has been given by Heinz [12] for problem (1.1)–(1.3) as well as for a certain type of singular nonlinear Sturm–Liouville problem, and Shibata [20] has extended the results and methods of [12] to certain problems in which the right-hand side and the constraint are more complicated. Also, similar connections between nodal structure and variational characterizations have been established for *unconstrained* problems by Coffman [6], [7] and Heinz [13].

However, all these results involve the assumption that the function $u \mapsto g(t,u)/u$ is *strictly increasing* on $]0,\infty[$ for every $t \in [a,b]$, and hence they do not include the linear case $g \equiv 0$. This very unfortunate flaw can now be remedied as a consequence of a much more general theory which is applicable to a broad range of boundary value problems for differential and integro-differential equations and which pertains not only to the nodal structure of solutions, but makes it possible to extend any known relationship between the number n and the geometric properties of nth eigenfunctions of a class of linear problems to a corresponding class of nonlinear problems. We now proceed to explain this in detail, giving an abstract setting which encompasses all the problems we have in mind.

Let $\mathcal{H}$ denote a real Hilbert space with scalar product $(\,\cdot\,|\,\cdot\,)$ and norm $\|\cdot\|$, and let $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ be an (unbounded) linear operator in $\mathcal{H}$ which is self-adjoint and bounded from below. Let X be its form domain, equipped with its natural norm, and let X^* be the topological dual of X. We shall consider equations of the form

$$A_0 y + B(y)y = \lambda y \tag{1.4}$$

together with a constraint

$$\|y\| = R \tag{1.5}$$

with $R > 0$ given. Here $(B(y))_{y\in C}$ is a continuous family of symmetric linear perturbations of A_0, parametrized on a space C such that $X \subseteq C \subseteq \mathcal{H}$. Thus, for every $y \in C$ we have an *associated linear eigenvalue problem*

$$A_0 v + B(y)v = \mu v. \tag*{$(1.6)_y$}$$

Our assumptions (which will be stated precisely in Section 3) imply that the self-adjoint operators

$$A(y) := A_0 + B(y)$$

have compact resolvent, and hence, for every $y \in C$, we have the unbounded increasing sequence of eigenvalues

$$\mu_1(y) \le \mu_2(y) \le \dots$$

of $(1.6)_y$ (counted with multiplicity, as usual). Moreover, we require that the map

$$N : X \to X^*, \quad y \mapsto B(y)y$$

is a potential operator, i.e. that it is the differential of a functional $\varphi \in C^1(X)$. By definition of X, the operator A_0 has a canonical extension to a map A_0 :

$X \to X^*$, and hence equation (1.4) may be considered as an equation in X^*. We put

$$\psi(y) := \frac{1}{2}\langle A_0 y, y\rangle + \varphi(y), \quad \text{for } y \in X,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the dual pairing between X^* and X. Then the solutions of (1.4), (1.5) are precisely the critical points of ψ restricted to the surface

$$S_R := \{y \in X \mid \|y\| = R\},$$

and in case the nonlinearity is odd (i.e. $B(-y) = B(y)$ for all y) one can again construct infinitely many solution pairs $(\lambda_n, \pm y_n)$ $(n = 1, 2, \dots)$ by means of Ljusternik–Schnirelman theory. The critical levels $c_n = \psi(y_n)$ enjoy the variational characterization

$$c_n = c_n(R) = \inf_{M \in \Sigma_n(R)} \sup_{u \in M} \psi(u), \tag{1.7}$$

where $\Sigma_n(R)$ denotes the family of all closed symmetric subsets of S_R whose Krasnosel'skiĭ genus $\gamma(M)$ (taken with respect to the topology of X) is not less than n. The eigenvalues λ_n appear as Lagrange multipliers, and the theory does not furnish any additional information about them or about the y_n.

To improve on this situation, consider an $n \in \mathbb{N}$ such that $\mu_n(y) < \mu_{n+1}(y)$ for $\|y\| \le R$. For all such y we may consider the orthogonal projector $P(y)$ onto the span of the first n eigenvectors of $A(y)$, and we define the *fixed point set* $K = K(R, n)$ by

$$K := \{y \in S_R \mid P(y)y = y\}.$$

The main point of the present paper is to establish reasonable sufficient conditions for the following crucial property to hold:

(CP) K is compact and nonempty, $\gamma(K) = n, c_n = \max_{u \in K} \psi(u)$, and every $y \in K \cap \psi^{-1}(c_n)$ is a solution of equation (1.4) with eigenvalue $\lambda = \mu_n(y)$.

Thus, if (CP) holds, we obtain a solution (λ_n, y_n) for which $\psi(y_n) = c_n$ and $\lambda_n = \mu_n(y_n)$, so that y_n shares any property enjoyed by an nth eigenfunction of a problem of type $(1.6)_y$. Conversely, if (λ, y) is a solution of (1.4), (1.5) such that $\lambda = \mu_m$ with $m \le n$ (which might, for instance, be read off the geometric properties of y), then $y \in K$ and hence $\psi(y) \le c_n$ by (CP). However, it is not clear whether $\lambda = \mu_n(y)$ implies that $\psi(y) = c_n$, and we suspect that this does not always hold.

After developing some topological preliminaries in Section 2, we prove (CP) under suitable hypotheses in Section 3. To avoid technical difficulties, we do not do this in full generality, but rather limit ourselves to the case where the perturbations $B(y)$ are *bounded* linear operators in $\mathcal{H}$. This suffices for the applications worked out in this paper, and a more general version is contained in [9]. Our hypotheses involve a certain "comparison condition" which, for the case of problem (1.1), amounts to requiring that the map $u \mapsto g(t,u)/u$ is nondecreasing on $]0,\infty[$ for every $t \in [a,b]$. We shall exhibit a broad and natural class of nonlinearities for which the comparison condition holds. Moreover, we have to impose a "global boundedness" condition on the family $(B(y))_{y\in S_R}$, and this assumption is very restrictive in terms of applications to differential equations. In each of our applications, however, we will be able to force this condition on the problem by means of a cut-off procedure based on appropriate a priori estimates.

In Section 4 we shall apply the abstract theory to periodic solutions of a nonlinear Hill's equation. The Dirichlet problem (1.1)–(1.3) which has been used above as a motivation will not be treated in detail because this could be done by a straightforward adaption of the material of Section 4 to a slightly simpler situation (cf. Remark (b) in Section 4). Thus, we shall consider an equation of the form (1.1), but with *periodic* data p, q, $g(\,\cdot\,,u)$ defined on all of $\mathbb{R}$, and (1.2) will be replaced by periodic boundary conditions. The integral in (1.3) is, of course, taken over a period interval. What renders this problem more interesting than the Dirichlet problem is the fact that multiple eigenvalues may occur in the associated linear problems, but on the other hand one always has $\mu_n(y) < \mu_{n+1}(y)$ whenever n is *odd*, as is well known from the theory of Hill's equation (cf. [8], for instance). Thus, under the assumption that $g(t,-u) = -g(t,u)$ and that $g(t,u)/u$ is monotonically nondecreasing for $u \in]0,\infty[$ and every fixed t, the theory of Section 3 is applicable after a suitable cut-off procedure, and one obtains the existence of a solution (λ_n, u_n) having $n-1$ simple zeroes in the period interval whenever $R > 0$ is arbitrary and $n \in \mathbb{N}$ is odd. Moreover, the eigenfunction u_n corresponds to the critical value c_n. Note that because of the occurrence of double eigenvalues topological degree methods and, in particular, the classical results of Rabinowitz [18] on nonlinear Sturm–Liouville problems are not applicable here, so that the present theory yields a new result on the existence of solutions with prescribed number of zeroes even when the connection with Ljusternik–Schnirelman critical values is ignored.

In Section 5 we try to generalize the material of Section 4 to the vector-valued case. Thus we consider semilinear second-order systems of ordinary differential equations with periodic data, and we assume that the nonlinearity satisfies a "comparison condition" generalizing the monotonicity condition required in the scalar case. We seek periodic solutions satisfying an appropriate L^2-constraint. In order to deal with the problem of multiple eigenvalues, we pick a number n such that

$$\mu_n(0) < \mu_{n+1}(0), \tag{1.8}$$

and we restrict attention to *small* $R > 0$. This situation involves some additional technical difficulties because the L^2-norm is too weak to enforce persistence of

$\mu_n(y) < \mu_{n+1}(y)$ even for small $\|y\|$, and moreover it is not clear how to perform the necessary cut-off in such a way that the comparison condition remains valid. Nevertheless we shall prove that for every $n \in \mathbb{N}$ satisfying (1.8) there is $R_0 > 0$ such that for every $0 < R < R_0$ there exists a solution $(\lambda_{n,R}, u_{n,R})$ of the desired type and that these solutions bifurcate from $(\mu_n(0), 0)$. For a geometric interpretation, the number of zeroes has to be replaced by the Morse index of the associated linear problem. Most of the results of Sections 4 and 5 were announced in [10].

Finally, we shall briefly sketch a further application in Section 6. As a multidimensional variant of (1.1)–(1.3) we consider the semilinear elliptic equation

$$-\Delta u + q(x)u + g(x, u) = \lambda u \tag{1.9}$$

with radially symmetric data q, $g(\,\cdot\,, u)$ on the unit ball $B := \{x \in \mathbb{R}^N \mid \|x\| \leq 1\}$ together with Dirichlet boundary conditions as well as the constraint

$$\int_B u(x)^2\, dx = R^2,$$

and we look for radial solutions having a prescribed number of nodal surfaces. This is not a special case of (1.1)–(1.3) since the introduction of polar coordinates leads to a singularity at the origin, and, in fact, this singularity causes various technical difficulties as well as some interesting regularity questions which are treated in [9]. In particular, the more general version of the abstract theory given in [9] is needed to treat equation (1.9). Therefore we only give a brief account of the results and refer the reader to [9] for details.

Many other applications are conceivable, and we hope to be able to report on some others in the near future. For instance, for problems with symmetry the nodal structure could be replaced by the transformation law under the action of the symmetry group. Furthermore, multiple solutions to nonlinear boundary value problems have been obtained by Ljusternik–Schnirelman-theory on other level sets besides S_R and by other minimax principles (e.g. various kinds of symmetric mountain pass theorems), and in each case a sequence $(c_n)_{n\geq 1}$ of critical values enjoying a variational characterization is furnished by the respective minimax method (see e.g. [3], [13], [15], [19], [20], [22]–[25]). However, it seems that the only known connections between these variational characterizations and the geometry of the corresponding solutions refer to the ground state, i.e. the lowest critical level, where for many situations it is known that the corresponding solutions do not change sign. To what extent and under what circumstances results such as those of the present paper can be extended to these other variational principles is an open question.

2. A topological lemma

In this section we shall establish a basic topological lemma which is the point of departure for all further developments. We begin by introducing some notation.

Let $(X, \|\cdot\|)$ be a real Hilbert space, and let $\mathcal{L}(X)$ be the space of bounded linear operators in X, endowed with the usual operator norm. More generally, for two normed spaces E, F the normed space of bounded linear operators $E \to F$ will be denoted by $\mathcal{L}(E, F)$. For $T \in \mathcal{L}(E, F)$, we shall write $\mathcal{N}(T)$ (resp. $\mathcal{R}(T)$) for the kernel (resp. the range) of the linear map T. For $\delta > 0$, the δ-neighbourhood of a set $A \subseteq X$ is written $U_\delta(A)$, i.e. we have

$$U_\delta(A) := \{x \in X \mid \operatorname{dist}(x, A) < \delta\},$$

and similarly for subsets of any other metric space. For a singleton $A = \{a\}$ we also write $B_\delta(a)$ in place of $U_\delta(\{a\})$. A subset $A \subseteq X$ will be called *symmetric* if it is invariant with respect to the action of the group $\mathbb{Z}_2$ given by reflection at the origin, i.e. if

$$\forall x: \quad x \in A \Rightarrow -x \in A.$$

A map $h : A \to B$, where $A, B \subseteq X$ are symmetric subsets, is called *odd* if it is equivariant with respect to this $\mathbb{Z}_2$-action, i.e. if

$$h(-x) = -h(x) \quad \text{for all } x \in A.$$

We denote by Σ the family of all closed symmetric subsets of $X \setminus \{0\}$, and for every $A \in \Sigma$, $\gamma(A)$ will denote the Krasnosel'skiĭ genus of A. For definition and properties of the Krasnosel'skiĭ genus, see any book on critical point theory, e.g. [3], [15], [19], [22], [23], or [24]. We shall also need some very basic notions and results about vector bundles, for which we refer the reader to the books by Atiyah [1] and Husemoller [14].

The basic result of this section now reads as follows:

Proposition 2.1. *Let $S \in \Sigma$ be bounded and such that for every finite-dimensional subspace U of X there is an odd homeomorphism h of $S \cap U$ onto the unit sphere in U. Fix a number $n \in \mathbb{N}$, and consider a continuous map $H : [0, 1] \times S \to \mathcal{L}(X)$ having the following properties:*

(i) *$H(t, -y) = H(t, y)$ for $0 \le t \le 1$ and every $y \in S$,*
(ii) *$H(0, \cdot\,)$ is constant on S,*
(iii) *for every $(t, y) \in [0, 1] \times S$, $H(t, y)$ is an orthogonal projection of rank n,*
(iv) *the range $H([0, 1] \times S)$ is a relatively compact subset of $\mathcal{L}(X)$.*

Put $P := H(1, \cdot\,)$ and $K := \{y \in S \mid P(y)y = y\}$. Then $K \in \Sigma$, K is compact, and $\gamma(K) = n$. In particular, $K \neq \emptyset$.

We prove this first for the finite-dimensional case. Thus, assume that $\dim X = N+1 < \infty$, $N+1 \geq n$. By assumption there is an odd homeomorphism h of S onto the unit sphere S^N in X. For $x \in X \setminus \{0\}$, we denote by $\langle x \rangle$ the span of x, considered as a point of real projective N-space $\mathbb{RP}^N$. Because of assumption (i) the map H factors in the form

$$H : [0,1] \times S \xrightarrow{\alpha} [0,1] \times \mathbb{RP}^N \xrightarrow{\widetilde{H}} \mathcal{L}(X),$$

where $\alpha(t,y) := (t, \langle h(y) \rangle)$, and where $\widetilde{H}$ is continuous. Now, let $\mathbf{G}_n(X)$ be the Grassmannian manifold of n-planes (i.e. n-dimensional linear subspaces) of X, let γ_n be the universal n-plane bundle over $\mathbf{G}_n(X)$, and let $\gamma_n^\perp$ be its orthogonal complement in the trivial bundle $\mathbf{G}_n(X) \times X$. Thus, the total space of $\gamma_n^\perp$ consists of the pairs $(V,v) \in \mathbf{G}_n(X) \times X$ such that $v \in V^\perp$, and the projection is induced by the canonical projection $\mathbf{G}_n(X) \times X \to \mathbf{G}_n(X)$. We define a map

$$\eta : [0,1] \times \mathbb{RP}^N \to \mathbf{G}_n(X)$$

by

$$\eta(t,p) := \mathcal{R}(\widetilde{H}(t,p)).$$

This map is continuous since the topology of $\mathbf{G}_n(X)$ is induced by the metric

$$d(V,W) := \|Q_V - Q_W\|,$$

where $Q_V, Q_W \in \mathcal{L}(X)$ are the orthogonal projections with ranges V, W, respectively. Hence, by assumption (ii), the map

$$\eta_1 := \eta(1,\,\cdot\,) : \mathbb{RP}^N \to \mathbf{G}_n(X)$$

is *nullhomotopic*, and this implies that the pull-back $\xi := \eta_1^* \gamma_n^\perp$ is trivializable. Now ξ can clearly be identified with the bundle whose total space is

$$\begin{aligned} E &= \{(p,v) \in \mathbb{RP}^N \times X \mid v \in \eta_1(p)^\perp\} \\ &= \{(p,v) \in \mathbb{RP}^N \mid v \in \mathcal{N}(\widetilde{H}(1,p))\} \\ &= \{(\langle h(y) \rangle, v) \mid y \in S, v \in \mathcal{N}(P(y))\} \end{aligned}$$

and whose projection is induced by the canonical projection $\mathbb{RP}^N \times X \to \mathbb{RP}^N$. Let $\tau : E \to \mathbb{RP}^N \times \mathbb{R}^{N+1-n}$ be a trivialization of ξ, and define a map $\varphi : S \to \mathbb{R}^{N+1-n}$ as the composition

$$\varphi : S \xrightarrow{\sigma} E \xrightarrow{\tau} \mathbb{RP}^N \times \mathbb{R}^{N+1-n} \longrightarrow \mathbb{R}^{N+1-n},$$

where σ is given by

$$\sigma(y) := (\langle h(y) \rangle, y - P(y)y)$$

and where the last arrow is canonical projection. Since τ is linear on fibers, h is odd, and $P(-y) = P(y)$ by assumption, it clearly follows that φ is *odd*.

Now consider $B \in \Sigma$ such that $B \subseteq S$ and $B \cap K = \emptyset$. Then, for every $y \in B$ we have $y - P(y)y \neq 0$ and hence $\varphi(y) \neq 0$. Thus the restriction of φ to B is an odd map $B \to \mathbb{R}^{N+1-n} \setminus \{0\}$ and hence we find

$$\gamma(B) \leq N + 1 - n. \tag{2.1}$$

From the definitions together with assumption (i) it is clear that $K \in \Sigma$ and that K is compact. As is well known, it follows that $\gamma(K)$ is finite, and that there exists $\delta > 0$ such that

$$\gamma(\overline{U_\delta(K)}) = \gamma(K).$$

We choose such a $\delta > 0$ and take $B := S \setminus U_\delta(K)$ (in case $K = \emptyset$ we take $B = S$). Then $B \in \Sigma$ and $B \cap K = \emptyset$, so we have (2.1). Moreover, $S \subseteq B \cup \overline{U_\delta(K)}$ and $\gamma(S) = \gamma(S^N) = N + 1$ because of the odd homeomorphism h. Thus we obtain

$$N + 1 = \gamma(S) \leq \gamma(B) + \gamma(\overline{U_\delta(K)}) = \gamma(B) + \gamma(K) \leq N + 1 - n + \gamma(K),$$

and hence $\gamma(K) \geq n$. To prove $\gamma(K) \leq n$, we consider the bundle $\zeta := \eta_1^*(\gamma_n)$ which is again trivializable. Clearly it can be identified with the sub-bundle of $\mathbb{RP}^N \times X$ whose total space is

$$F = \{(p, v) \in \mathbb{RP}^N \times X \mid v \in \mathcal{R}(\widetilde{H}(1, p))\} = \{(\langle h(y)\rangle, v) \mid y \in S,\ v \in \mathcal{R}(P(y))\}.$$

Now, using a trivialization $\tau_1 : F \to \mathbb{RP}^N \times \mathbb{R}^n$ of ζ, we define an odd continuous map $\phi_1 : S \to \mathbb{R}^n$ as the composition

$$\phi_1 : S \xrightarrow{\sigma_1} F \xrightarrow{\tau_1} \mathbb{RP}^N \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

where $\sigma_1(y) := (\langle h(y)\rangle, P(y)y)$ and where again the last arrow is canonical projection. By definition of K the restriction of ϕ_1 to K has no zero, and hence $\gamma(K) \leq n$, as desired. Thus we have proved our result in the finite-dimensional case.

To treat the infinite-dimensional case, we need the following lemma:

Lemma 2.2. *Let M be a topological space and let $n \in \mathbb{N}$ be fixed. Moreover, let $P : M \to \mathcal{L}(X)$ be a continuous mapping whose range is a relatively compact subset of $\mathcal{L}(X)$ consisting of orthogonal projections of rank n. Then, for each $\varepsilon > 0$ there exists a finite-dimensional subspace U of X and a continuous mapping $P_\varepsilon : M \to \mathcal{L}(X)$ whose values are orthogonal projections onto n-dimensional subspaces of U and for which we have*

$$\|P(x) - P_\varepsilon(x)\| < \varepsilon \quad \text{for all } x \in M.$$

Moreover, if $P \circ j = P$ for some map $j : M \to M$, then also $P_\varepsilon \circ j = P_\varepsilon$, and if P is constant on some subset M_0 of M, then so is P_ε.

Proof. Let $\varepsilon > 0$, and choose $\delta > 0$ such that

$$\delta < \min(1/4, \varepsilon^2/49).$$

Since $P(M)$ is relatively compact, there are points $a_1, \dots, a_m \in M$ such that

$$P(M) \subseteq \bigcup_{i=1}^{m} B_\delta(P(a_i)),$$

and we define U to be the span of $\bigcup_{i=1}^{m} \mathcal{R}(P(a_i))$, which is evidently a finite-dimensional subspace of X. Let Q be the orthogonal projection of X onto U. Then, for any $x \in M$, there is $i \in \{1, \dots, m\}$ such that $P(x) \in B_\delta(P(a_i))$ and hence $\|P(x) - QP(x)\| \leq \|P(x) - P(a_i)\| + \|QP(a_i) - QP(x)\| \leq (1 + \|Q\|)\|P(x) - P(a_i)\| < 2\delta$. Thus we have

$$\|P(x) - QP(x)\| < 2\delta \quad \text{for all } x \in M. \tag{2.2}$$

In particular, for every $v \in \mathcal{R}(P(x))$ we have $\|v - Qv\| = \|P(x)v - QP(x)v\| < 2\delta\|v\|$ and hence

$$\|Qv\| \geq \|v\| - \|Qv - v\| \geq (1 - 2\delta)\|v\| > \|v\|/2.$$

This shows that the restriction of Q to $\mathcal{R}(P(x))$ is an isomorphism, $\mathcal{R}(P(x)) \to \mathcal{R}(QP(x))$, and that for the inverse isomorphism $R(x)$ we have

$$\|R(x)\| \leq 2. \tag{2.3}$$

In particular, $\dim \mathcal{R}(QP(x)) = \dim \mathcal{R}(P(x)) = n$ for all $x \in M$. We take for $P_\varepsilon(x)$ the orthogonal projection $X \to \mathcal{R}(QP(x))$. Then certainly $P_\varepsilon(x)$ is an orthogonal projection onto an n-dimensional subspace of U, and the map $P_\varepsilon : M \to \mathcal{L}(X)$ has the additional properties mentioned at the end of the lemma. Fix $x \in M$. In order to estimate $\|P_\varepsilon(x) - P(x)\|$ we consider an arbitrary $u \in X$ with $\|u\| = 1$, and we put

$$D := \|u - P_\varepsilon(x)u\| = \operatorname{dist}(u, \mathcal{R}(QP(x))),$$

$$v := R(x)P_\varepsilon(x)u.$$

Then $P_\varepsilon(x)u = QP(x)v$ and $\|v\| \leq \|R(x)\| \cdot \|P_\varepsilon(x)\| \cdot \|u\| \leq 2$ by (2.3). Using (2.2) repeatedly, we therefore obtain

$$\begin{aligned}\|u - QP(x)u\| &\leq 2\delta + \|u - P(x)u\| = 2\delta + \operatorname{dist}(u, \mathcal{R}(P(x))) \\ &\leq 2\delta + \|u - P(x)v\| \leq 2\delta + \|u - QP(x)v\| + 2\delta\|v\| \leq D + 6\delta.\end{aligned}$$

Since $P_\varepsilon(x)u - QP(x)u \in \mathcal{R}(QP(x))$, this vector is orthogonal to $u - P_\varepsilon(x)u$, and hence

$$\begin{aligned}\|P_\varepsilon(x)u - QP(x)u\|^2 &= \|u - QP(x)u\|^2 - \|u - P_\varepsilon(x)u\|^2 \\ &\leq (D + 6\delta)^2 - D^2 = 12D\delta + 36\delta^2.\end{aligned}$$

Noting that $D \le \|u\| + \|P_\varepsilon(x)u\| \le 2$ and remembering that $\delta < 1/4$ we infer

$$\|P_\varepsilon(x)u - QP(x)u\| \le \sqrt{33\delta}$$

and hence, using (2.2) again,

$$\|P_\varepsilon(x)u - P(x)u\| \le \sqrt{33\delta} + 2\delta < (\sqrt{33}+1)\sqrt{\delta} < 7\sqrt{\delta}.$$

Since u is an arbitrary vector of norm 1, this yields the desired result by the choice of δ.

It remains to show that the map $P_\varepsilon : M \to \mathcal{L}(X)$ is continuous. This could be done by direct estimation, but we prefer to give a shorter proof using some general results on vector bundles, e.g. those from Section 1.3 of [1], which carry over to the present situation without any change. Thus, let us consider the trivial Hilbert space bundle $M \times X \to X$. Since P is continuous, the map $M \times X \to M \times X$, $(x, v) \mapsto (x, QP(x)v)$ is a bundle homomorphism of constant finite rank n. The image of this homomorphism is therefore a subbundle ξ of $M \times X$, and by construction the orthogonal projection homomorphism $M \times X \to \xi$ is nothing but the map $(x, v) \mapsto (x, P_\varepsilon(x)v)$. This proves the continuity of P_ε, and so the proof of Lemma 2.2 is complete. □

Now choose a null sequence $(\varepsilon_j)_{j\ge 1} \subseteq \,]0,\infty[$. We can apply Lemma 2.2 to the given map $H : [0,1]\times S \to \mathcal{L}(X)$, taking $\varepsilon = \varepsilon_j$, $M = [0,1]\times S$, $M_0 = \{0\}\times S$ and $j : M \to M$ as the map given by $j(t,y) := (t,-y)$. This yields a sequence $(U_j)_j$ of finite-dimensional subspaces of X and a sequence $(H_j)_j$ of continuous maps $H_j : [0,1]\times S \to \mathcal{L}(X, U_j)$ such that for every j we have

$$\|H(t,y) - H_j(t,y)\| < \varepsilon_j \quad \text{for all } t \in [0,1],\ y \in S \tag{2.4}$$

and such that the maps $\rho_j \circ H_j|_{[0,1]\times(S\cap U_j)} : [0,1]\times(S\cap U_j) \to \mathcal{L}(U_j)$, where $\rho_j : \mathcal{L}(X, U_j) \to \mathcal{L}(U_j)$ denotes restriction, satisfy the assumptions of Proposition 2.1 with X replaced by U_j and S replaced by $S \cap U_j$. According to the finite-dimensional version of Proposition 2.1 which has already been established, we therefore know that

$$\gamma(K_j) = n \tag{2.5}$$

for the compact symmetric sets

$$K_j := \{y \in S\cap U_j \mid P_j(y)y = y\},$$

where $P_j := H_j(1,\,\cdot\,)$ for $j = 1, 2, \ldots$. We shall complete the proof by means of a compactness argument based on the following lemma, in which we use the maps $\beta : S \to X$, $\beta_j : S\cap U_j \to U_j$ defined by

$$\beta(y) := P(y)y, \quad \beta_j(y) := P_j(y)y$$

for $j \in \mathbb{N}$.

Lemma 2.3. *The sets $\beta(S)$ and*

$$C := \bigcup_{j=1}^{\infty} \beta_j(S \cap U_j)$$

are relatively compact in X.

Proof. (i) To prove the assertion for $\beta(S)$, consider an arbitrary sequence $(y_k)_k \subseteq S$ and put $z_k := \beta(y_k)$. Moreover, put

$$b := \sup_{y \in S} \|y\|,$$

which is finite by assumption. By hypothesis (iv), we may assume, after passing to a suitable subsequence, that the limit

$$P_\infty := \lim_{k \to \infty} P(y_k)$$

exists in $\mathcal{L}(X)$. Now it is well known (and easy to see) that the set of orthogonal projections is strongly closed and that the rank is a lower semicontinuous function on that set. Hence P_∞ has finite rank, so (after passing to a subsequence again) we can assume that the bounded sequence $(P_\infty y_k)_k$ has a limit z in X. But

$$||P(y_k)y_k - P_\infty y_k\| \leq b\|P(y_k) - P_\infty\| \quad \text{for all } k,$$

hence

$$\|z - z_k\| \leq \|z - P_\infty y_k\| + b\|P_\infty - P(y_k)\| \to 0 \quad \text{as } k \to \infty.$$

Thus we have found a convergent subsequence of $(z_k)_k$, as desired.

(ii) It is clear from (2.4) that

$$\|\beta(y) - \beta_j(y)\| < \varepsilon_j b \quad \text{for all } y \in S \cap U_j,\ j \in \mathbb{N}.$$

Hence, for every $\varepsilon > 0$, we have $\beta_j(S \cap U_j) \subseteq U_\varepsilon(\beta(S))$ for all but finitely many j. Therefore, if we put

$$C_m := \bigcup_{j=1}^{m} \beta_j(S \cap U_j)$$

for $m \in \mathbb{N}$, we find that for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$C \subseteq C_m \cup U_\varepsilon(\beta(S)).$$

Observe that every $\beta_j(S \cap U_j)$ is relatively compact, being a bounded subset of the finite-dimensional space U_j, and hence every C_m is relatively compact. From this and the result of part (i) the relative compactness of C easily follows. $\square$

End of the proof of Proposition 2.1. $K = \{y \in S \mid \beta(y) = y\}$ is compact since it is a closed subset of the compact space $\overline{\beta(S)}$, and $K \in \Sigma$ is clear from assumption (i). The estimate $\gamma(K) \leq n$ follows because we can construct an odd continuous map $\phi_1 : K \to \mathbb{R}^n \setminus \{0\}$ in the same way as in the finite-dimensional case. We only have to replace $\mathbb{RP}^N$ by the compact space $\widehat{K}$ which arises from K by identifying y and $-y$, and we have to take $[y] = \{y, -y\} \in \widehat{K}$ instead of $\langle h(y) \rangle$ throughout.

It remains to show that $\gamma(K) \geq n$. Let $\mathcal{C}$ be the metric space of all non-empty closed subsets of the compact space $S \cap \overline{C}$, equipped with the Hausdorff distance. As is well known (see e.g. [16]), this space is again compact. Thus, after passing to a suitable subsequence we may assume that we have a limit

$$K_\infty = \lim_{j\to\infty} K_j$$

with respect to the Hausdorff distance. Since reflection at the origin induces a homeomorphism $\mathcal{C} \to \mathcal{C}$ and $\Sigma \cap \mathcal{C}$ is the fixed point set of that homeomorphism, $\Sigma \cap \mathcal{C}$ is closed in $\mathcal{C}$, and, in particular, $K_\infty \in \Sigma$. Moreover, since the Hausdorff distance can be described in the form

$$D(A, B) = \inf\{\delta > 0 \mid A \subseteq U_\delta(B) \text{ and } B \subseteq U_\delta(A)\},$$

it easily follows from (2.5) and the properties of the Krasnosel'skiĭ genus that

$$\gamma(K_\infty) \geq n.$$

Thus the desired result follows from

$$K \supseteq K_\infty. \tag{2.6}$$

To see this, consider an arbitrary $y \in K_\infty$ and note that by definition of the Hausdorff distance there must be a subsequence $(K_{j(m)})_m$ such that for every $m \in \mathbb{N}$ there exists $y_m \in K_{j(m)}$ satisfying

$$\|y_m - y\| < 1/m.$$

Then $y = \lim_{m\to\infty} y_m$ as well as $\beta(y) = \lim_{m\to\infty} \beta(y_m)$ by continuity of β. However, from $y_m = \beta_{j(m)}(y_m)$ we get

$$\|\beta(y_m) - y_m\| = \|P(y_m)y_m - P_{j(m)}(y_m)\| < \varepsilon_{j(m)}\|y_m\| \leq \varepsilon_{j(m)} b \to 0$$

as $m \to \infty$, and hence $y = \lim_{m\to\infty} y_m = \lim_{m\to\infty} \beta(y_m) = \beta(y)$, which implies $y \in K$. Thus we have established (2.6), and the proof is complete. $\square$

COROLLARY 2.4. *Let $S_1 := \{v \in X \mid \|v\| = 1\}$ be the unit sphere, and let $\rho : X \setminus \{0\} \to S_1$, $v \mapsto v/\|v\|$ be the radial projection. The assertions of Proposition 2.1 remain true when S is replaced by a closed subset S_R of $X \setminus \{0\}$ such that ρ restricts to a homeomorphism $S_R \to S_1$.*

PROOF. Let $h : S_R \to S_1$ be the odd homeomorphism obtained by restricting ρ to S_R. Given a continuous $H : [0, 1] \times S_R \to \mathcal{L}(X)$ satisfying conditions (i)–(iv) from Proposition 2.1, we define $\widetilde{H} : [0,1] \times S_1 \to \mathcal{L}(X)$ by

$$\widetilde{H}(t,z) := H(t, h^{-1}(z)).$$

Then obviously $S = S_1$ and $\widetilde{H}$ satisfy all the assumptions of Proposition 2.1, and hence the set

$$\widetilde{K} := \{z \in S_1 \mid \widetilde{H}(1,z)z = z\}$$

has the desired properties. But the equations

$$H(1,y)y = y, \quad y \in S_R$$

and

$$\widetilde{H}(1,z)z = z, \quad z \in S_1$$

are evidently equivalent via the substitution $z = h(y)$. This means that $K = h^{-1}(\widetilde{K})$, whence the result. □

3. Abstract nonlinear eigenvalue problems

In this section we discuss Problem (1.4), (1.5) from Section 1, and we retain all the notations introduced there. Thus, $\|\cdot\|$ respectively $(\,\cdot\,|\,\cdot\,)$ will denote the norm respectively the scalar product in the real Hilbert space $\mathcal{H}$, while other norms will be identified by suitable subscripts if there is danger of confusion. For spaces of bounded linear operators we will use the notations introduced in Section 2. In particular, for any bounded linear operator T, $\|T\|$ denotes the operator norm of T. Moreover, the self-adjoint operator A_0 in $\mathcal{H}$ is assumed to be bounded below, and its greatest lower bound will be denoted by $-\nu$, so that we have

$$(A_0 u|u) + \nu\|u\|^2 \geq 0 \quad \text{for all } u \in \mathcal{D}(A_0). \tag{3.1}$$

The domain $\mathcal{D}(A_0)$ is endowed with the graph norm, so that it is a Hilbert space in its own right. The form domain X is, by definition, the domain of the positive self-adjoint operator $(A_0 + \nu I)^{1/2}$, equipped with the corresponding graph norm, i.e. we have

$$\|y\|_X^2 := \|(A_0 + \nu I)^{1/2} y\|^2 + \|y\|^2 \tag{3.2}$$

for $y \in X$, which reduces to

$$\|y\|_X^2 = (A_o y \,|\, y) + (\nu + 1)\|y\|^2$$

in case $y \in \mathcal{D}(A_0)$. The canonical pairing between X and its topological dual X^* will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$, and we shall identify $\mathcal{H}$ with a subspace of X^* by means of the natural injection which assigns to every $u \in \mathcal{H}$ the continuous linear form $(u|\,\cdot\,)$ on X. Given a Banach space C with $X \subseteq C \subseteq \mathcal{H}$ and a continuous map $B : C \to \mathcal{L}(\mathcal{H})$ as in Section 1, we can then assign to every $y \in C$ an operator $A(y) \in \mathcal{L}(X, X^*)$ which is the unique continuous extension of $A_0 + B(y) : \mathcal{D}(A_0) \to X^*$ to X. Thus $A(y)$ is the unique bounded linear operator $X \to X^*$ which agrees with the left-hand side of $(1.6)_y$ for every $v \in \mathcal{D}(A_0)$.

We need the following fundamental hypotheses about these data:

(H1) The embedding $X \hookrightarrow C$ is compact, and the embedding $C \hookrightarrow \mathcal{H}$ is continuous.

(H2) For every $y \in C$, $B(y)$ is a bounded self-adjoint operator in $\mathcal{H}$.

(H3) $B(-y) = B(y)$, for all $y \in C$.

(H4) The map

$$X \to X^*, \quad y \mapsto B(y)y$$

is the differential of a C^1-functional $\phi : X \to \mathbb{R}$.

It follows from (H1) that the embedding $\mathcal{D}(A_0) \hookrightarrow \mathcal{H}$ is compact, and because of (H2) the graph norm of $A_0 + B(y)$ is equivalent to that of A_0 for every $y \in C$. Hence every $A_0 + B(y)$ has pure point spectrum, and we can write the spectrum of $A_0 + B(y)$ in the form of an increasing sequence

$$\mu_1(y) \le \mu_2(y) \le \dots ,$$

where the eigenvalues $\mu_k(y)$ are counted with multiplicity. In case

$$\mu_n(y) < \mu_{n+1}(y) \tag{3.3}$$

we introduce the space $V_n(y)$ which is defined to be the span of the first n eigenvectors of $A_0 + B(y)$, i.e.

$$V_n(y) := \Bigg\{ \sum_{k=1}^{n} \alpha_k u_k \,\Bigg|\, \alpha_k \in \mathbb{R},\ u_k \in \mathcal{D}(A_0), \\ A_0 u_k + B(y) u_k = \mu_k(y) u_k,\ k = 1, \dots , n \Bigg\}.$$

Hypothesis (H4) embodies the requirement that equation (1.4) be of variational type. As usual for such equations, we normalize ϕ so as to have

$$\phi(0) = 0. \tag{3.4}$$

Moreover, none of the assumptions are altered if we replace A_0 by $A_0 + B(0)$ and $B(y)$ by $B(y) - B(0)$. Therefore we may assume without loss of generality that

$$B(0) = 0. \tag{3.5}$$

It follows from (H4) that

$$A(y)y = d\psi(y) \quad \text{for all } y \in X,$$

where $\psi \in C^1(X)$ is defined by

$$\psi(y) := \frac{1}{2}\langle A_0 y, y\rangle + \phi(y) \quad \text{for } y \in X.$$

(By abuse of notation, we write A_0 to denote $A(0) \in \mathcal{L}(X, X^*)$.) By (H3), the functional ψ is *even*, and it makes sense to consider the Ljusternik–Schnirelman levels $c_n = c_n(R)$ for given $R > 0$ as defined by (1.7).

Apart from hypotheses (H1)–(H4), which provide a general framework, we shall make use of the following two additional conditions, in which we use the notation

$$T_R := \{y \in X \mid \|y\| \le R\}.$$

(GB) ("global boundedness") There is $\eta > 0$ such that

$$\|B(y)\| \le \eta \quad \text{for all } y \in T_R.$$

(CC) ("comparison condition") For arbitrary vectors $y, v \in X$ we have

$$2(\phi(v) - \phi(y)) \ge \langle B(y)v, v\rangle - \langle B(y)y, y\rangle.$$

Remarks. (a) The auxiliary space C has been introduced essentially for reasons of convenience. We could take $C = X$ throughout if we replaced (H1) by the requirement that the embedding $X \hookrightarrow \mathcal{H}$ be compact. However, the setting described above is closer to what actually happens in the applications.

(b) Condition (CC) will be elucidated by the example below, which also provides a rich and natural class of functionals satisfying (CC). However, let us note some simple consequences of (CC): Taking $v = 0$ and using (3.4) we see

$$\phi(y) \le \frac{1}{2}\langle B(y)y, y\rangle \quad \text{for all } y \in X. \tag{3.6}$$

Furthermore, if we assume (3.5) to hold and take $y = 0$ in (CC), it follows that

$$\phi(v) \ge 0 \quad \text{for all } v \in X. \tag{3.7}$$

This together with (3.2) yields the estimate

$$\|v\|_X^2 \le 2\psi(v) + (\nu + 1)\|v\|^2, \tag{3.7a}$$

which will be useful for applications.

Example 3.1. Consider functionals of the form

$$\phi = \Phi \circ q, \tag{3.8}$$

where $\Phi \in C^1(X)$ and where q is an X-valued quadratic form on X, i.e.

$$q(y) = b(y, y) \quad \text{for } y \in X$$

for a unique symmetric continuous bilinear map $b : X \times X \to X$. Thus, $q \in C^\infty(X, X)$, and its first derivative is given by

$$q'(y)v = 2b(y, v) = 2b(v, y) \quad \text{for } y, v \in X,$$

hence the chain rule yields $\phi \in C^1$ and

$$d\phi(y) = 2d\Phi(q(y)) \circ b(y, \,\cdot\,) \quad \text{for all } y \in X.$$

Thus, if we define $B : X \to \mathcal{L}(X, X^*)$ by

$$\langle B(y)v, w\rangle := 2\langle d\Phi(q(y)), b(v, w)\rangle = 2\langle d\Phi(q(y)), b(w, v)\rangle$$

for $v, w, y \in X$, we indeed have $d\phi(y) = B(y)y$ for all $y \in X$. In particular, $\langle B(y)v, v\rangle = 2\langle d\Phi(q(y)), q(v)\rangle$, and hence condition (CC) is equivalent to

$$\Phi(q(v)) - \Phi(q(y)) \geq \langle d\Phi(q(y)), q(v) - q(y)\rangle \quad \text{for all } y, v \in X. \tag{3.9}$$

Now suppose that Φ is *convex*. Then

$$\Phi(w) - \Phi(z) \geq \langle d\Phi(z), w - z\rangle \quad \text{for all } w, z \in X.$$

Hence, taking $w = q(v)$, $z = q(y)$ we see that (3.9) is satisfied. Thus, a class of nonlinearities satisfying conditions (H4) and (CC) is given by the differentials of the functionals ϕ of the form (3.8) with convex $\Phi \in C^1$. Note that these nonlinearities also satisfy (H2) and (H3) (at least when $\mathcal{R}(B(y)) \subseteq \mathcal{H}$ and $B(y)$ can be extended to an operator in $\mathcal{L}(\mathcal{H})$).

The importance of condition (CC) stems from the fact that it enables us to compare Ljusternik–Schnirelman levels of $\psi|_{S_R}$ and eigenvalues of an associated linear problem. This is expressed by the following proposition, which is related to Lemma 4.3 of [6]. It is one of the main results of the present section.

Proposition 3.2. *Suppose hypotheses* (H1)–(H4) *and* (CC) *are satisfied, and consider $n \in \mathbb{N}$ and $y \in S_R$ such that* (3.3) *also holds. If $y \in V_n(y)$, then we have*

(a) *$\psi(y) \leq c_n(R)$, and moreover*
(b) *if $\psi(y) = c_n(R)$, then (λ, y) is a solution of problem* (1.4), (1.5) *for $\lambda = \mu_n(y)$.*

Proof. Fix $y \in S_R$ satisfying the assumptions of the proposition, and define the Rayleigh quotient $\rho : X \setminus \{0\} \to \mathbb{R}$ by

$$\rho(u) := \frac{\langle A(y)u, u\rangle}{\|u\|^2}.$$

Since $y \in S_R$, (CC) implies, for every $v \in S_R$,

$$\psi(v) - \psi(y) \geq \frac{R^2}{2}(\rho(v) - \rho(y)).$$

Choose eigenvectors $u_1, \ldots, u_{n-1}$ corresponding to $\mu_1(y), \ldots, \mu_{n-1}(y)$, and let $W \subseteq V_n(y)$ be the span of $u_1, \ldots, u_{n-1}$. Then it is clear from spectral theory that

$$\mu_n(y) = \inf_{v \in S_R \cap W^\perp} \rho(v).$$

Moreover, $y \in V_n(y)$ implies $\rho(y) \leq \mu_n(y)$. Under the assumptions of part (a) we therefore obtain

$$\inf_{v \in S_R \cap W^\perp} \psi(v) - \psi(y) \geq \frac{R^2}{2}(\mu_n(y) - \rho(y)) \geq 0.$$

Now let $\Sigma(R)$ denote the family of all closed symmetric subsets of S_R as in Section 2, and define

$$\Sigma_m^*(R) := \{C \in \Sigma(R) \mid \gamma(C') \leq m \text{ for every } C' \in \Sigma(R) \text{ such that } C \cap C' = \emptyset\}$$

for $m \in \mathbb{N} \cup \{0\}$. With this notation, the Ljusternik–Schnirelman level $c_n = c_n(R)$ can be described as

$$c_n = \sup_{C \in \Sigma_{n-1}^*(R)} \inf_{v \in C} \psi(v).$$

This was stated in [11] (see also [7]), and it follows easily from the observation that

$$\begin{aligned} c_n &= \inf\{c \in \mathbb{R} \mid \gamma(S_R \cap \psi^{-1}(]-\infty, c])) \geq n\} \\ &= \sup\{c \in \mathbb{R} \mid \gamma(S_R \cap \psi^{-1}(]-\infty, c])) < n\}. \end{aligned}$$

Since $X \cap W^\perp$ is a closed linear subspace of X of codimension $n-1$, orthogonal projection (with respect to the scalar product of X) shows at once that $S_R \cap W^\perp \in \Sigma_{n-1}^*(R)$, and hence

$$\inf_{v \in S_R \cap W^\perp} \psi(v) \leq c_n.$$

Thus we find that $y \in V_n(y)$ implies

$$c_n - \psi(y) \geq \frac{R^2}{2}(\mu_n(y) - \rho(y)) \geq 0,$$

and the assertion of part (a) follows. If moreover $\psi(y) = c_n$, we must have $\rho(y) = \mu_n(y)$, and this is possible only if y is an eigenvector of $A_0 + B(y)$ with eigenvalue $\mu_n(y)$, which proves part (b). □

Next, we shall construct a family of projections to which the results of Section 2 can be applied. For this the variational structure of the problem is immaterial, and we shall not require (H4). However, condition (GB) will play an important role.

In the next proposition and its proof it will be important to distinguish carefully between orthogonality with respect to the scalar product of $\mathcal{H}$ and that w.r.t. the scalar product of X, which will be denoted by $(\cdot\,|\,\cdot)_X$. We shall therefore talk about $\mathcal{H}$- orthogonal vectors versus X-orthogonal vectors etc. For instance, eigenvectors of $A_0 + B(y)$ corresponding to different eigenvalues are $\mathcal{H}$-orthogonal, but, in general, not X-orthogonal.

Proposition 3.3. *Suppose* (H1), (H2) *are satisfied, and consider* $n \in \mathbb{N}$, $R > 0$ *such that* (3.3) *holds for every* $y \in T_R$. *For each* $y \in T_R$, *let* $P_n(y)$ *be the* X*-orthogonal projector of* X *onto* $V_n(y)$. *Then we have*

(a) *the map* $P_n : T_R \to \mathcal{L}(X)$ *is continuous,*
(b) *if, in addition,* (GB) *is satisfied, then the range* $P_n(T_R)$ *is relatively compact in* $\mathcal{L}(X)$.

Proof. (a) As a first step, let us consider the $\mathcal{H}$-orthogonal projector $Q_n(y)$ of $\mathcal{H}$ onto $V_n(y)$ for $y \in T_R$, and let us prove that the map

$$Q_n : T_R \to \mathcal{L}(\mathcal{H}, \mathcal{D}(A_0))$$

is continuous. To this end, fix $y_0 \in T_R$ and choose a closed Jordan curve Γ in the complex plane such that $\mu_1(y_0), \dots, \mu_n(y_0)$ lie in the interior of Γ, while all other $\mu_k(y_0)$ lie in the exterior. Then we have

$$Q_n(y_0) = \frac{1}{2\pi i} \int_\Gamma (\lambda I - A_0 - B(y_0))^{-1}\, d\lambda. \tag{3.10}$$

(Strictly speaking, this is true for the complexifications of the operators in question, but we shall not distinguish between operators in the real space $\mathcal{H}$ and their complexifications in the notation.) Now, by the assumptions, $B : C \to \mathcal{L}(\mathcal{H})$ restricts to a continuous map $B : X \to \mathcal{L}(\mathcal{H})$, and hence $\mu_k(y)$ depends *continuously* on $y \in X$, as can easily be seen (e.g. from the Courant–Fischer principle). More precisely, we have

$$|\mu_k(y_1) - \mu_k(y_2)| \le \|B(y_1) - B(y_2)\| \tag{3.11}$$

for every $k \in \mathbb{N}$ and arbitrary $y_1, y_2 \in X$. Thus there is $\delta > 0$ such that $\mu_1(y), \dots, \mu_n(y)$ are in the interior, all other $\mu_k(y)$ in the exterior of Γ provided $y \in B_\delta(y_0)$. In particular, (3.10) still holds when y_0 is replaced by $y \in B_\delta(y_0)$. Elementary estimates from perturbation theory show that the set

$$\Lambda := \{T \in \mathcal{L}(\mathcal{H}) \mid A_0 + T \text{ has a bounded inverse}\}$$

is open in $\mathcal{L}(\mathcal{H})$, that $(A_0+T)^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{D}(A_0))$ for all $T \in \Lambda$, and that the map

$$\Lambda \to \mathcal{L}(\mathcal{H}, \mathcal{D}(A_0)), \quad T \mapsto (A_0 + T)^{-1}$$

is continuous. Applying this remark to $T = B(y) - \lambda I$, where $y \in B_\delta(y_0)$ and $\lambda \in \Gamma$, shows that the map

$$\Gamma \times B_\delta(y_0) \to \mathcal{L}(\mathcal{H}, \mathcal{D}(A_0)), \quad (\lambda, y) \mapsto (\lambda I - A_0 - B(y))^{-1}$$

is continuous. Hence (3.10), with y_0 replaced by y yields the desired continuity result in $B_\delta(y_0)$. However, this is enough because y_0 was arbitrary.

Next, observe that we have continuous embeddings $J_0 : X \hookrightarrow \mathcal{H}$ and $J_1 : \mathcal{D}(A_0) \hookrightarrow X$. They obviously induce a continuous embedding

$$J : \mathcal{L}(\mathcal{H}, \mathcal{D}(A_0)) \hookrightarrow \mathcal{L}(X), \quad T \mapsto J_1 \circ T \circ J_0,$$

and hence we have the continuous map $\widetilde{Q}_n := J \circ Q_n : T_R \to \mathcal{L}(X)$. Since it has the constant rank n, the set

$$\{(y, v) \in T_R \times X \mid v \in \mathcal{R}(\widetilde{Q}(y))\}$$

is a sub-bundle of the trivial bundle $T_R \times X$, and (as in the proof of Proposition 2.1) it follows that $P_n : T_R \to \mathcal{L}(X)$ is continuous, as claimed.

(b) We shall prove the relative compactness of $P_n(T_R)$ in $\mathcal{L}(X)$ by exhibiting a compact space Y and a continuous map α such that $P_n(T_R) \subseteq \alpha(Y)$. To do this, note first that by (GB) and (3.11) we have

$$|\mu_k(y)| \leq |\mu_k(0)| + 2\eta$$

for every $y \in T_R$, $k \in \mathbb{N}$. Now consider an eigenvector $u \in \mathcal{N}(\mu_k(y)I - A_0 - B(y))$ with $\|u\| = 1$, where $y \in T_R$, $k \in \mathbb{N}$. Then

$$A_0 u = \mu_k(y)u - B(y)u,$$

hence $\|A_0 u\| \leq (|\mu_k(y)| + \eta)\|u\| \leq (|\mu_k(0)| + 3\eta)\|u\|$, and hence we obtain

$$\|A_0 u\| \leq |\mu_k(0)| + 3\eta.$$

This means that for every $k \in \mathbb{N}$ the set

$$E_k := \{u \in \mathcal{D}(A_0) \mid \|u\| = 1 \text{ and } u \in \mathcal{N}(\mu_k(y)I - A_0 - B(y)) \text{ for some } y \in T_R\}$$

is bounded in $\mathcal{D}(A_0)$. But the compact embedding $X \hookrightarrow \mathcal{H}$ leads to a compact embedding $\mathcal{D}(A_0) \hookrightarrow X$, as is easily seen from the definitions of the norms and the fact that $(A_0 + \nu I)^{1/2}$ is a *closed* operator in $\mathcal{H}$. Hence $\overline{E_k}$ (the closure being taken in X) is a compact space for every k.

Now let us put

$$Y := \{(u_1, \ldots, u_n) \in \overline{E_1} \times \overline{E_2} \times \ldots \times \overline{E_n} \mid (u_j | u_k) = \delta_{jk} \text{ for } j, k = 1, \ldots, n\}.$$

Since the scalar product of $\mathcal{H}$ is continuous on $X \times X$, Y is closed in $\overline{E_1} \times \ldots \times \overline{E_n}$, and hence Y is *compact*.

To construct α, we first consider the map $X \times X \to \mathcal{L}(X)$, $(u, v) \mapsto Z_{u,v}$, where $Z_{u,v}$ is given by

$$Z_{u,v} x := (x \mid u)_X v \quad \text{for } x \in X.$$

Since $\|Z_{u,v}x\|_X \le \|u\|_X\|v\|_X\|x\|_X$ for all u, v, x, this bilinear map is continuous. Therefore a continuous map $\Pi : X^n \to \mathcal{L}(X)$ is given by

$$\Pi(v_1, \dots , v_n) := \sum_{j=1}^{n} Z_{v_j,v_j}.$$

Now put

$$\mathcal{B}_n(X) := \{(u_1, \dots , u_n) \in X^n \mid u_1, \dots , u_n \text{ are linearly independent}\}.$$

Clearly $Y \subseteq \mathcal{B}_n(X)$, and Gram–Schmidt orthonormalization with respect to $(\,\cdot\,|\,\cdot\,)_X$ provides a continuous map $\omega : \mathcal{B}_n(X) \to \mathcal{B}_n(X)$. Hence

$$\alpha := \Pi \circ \omega$$

is a continuous map $\mathcal{B}_n(X) \to \mathcal{L}(X)$. To show that $P_n(T_R) \subseteq \alpha(Y)$, consider an arbitrary $y \in T_R$ and choose an $\mathcal{H}$-orthonormal basis $\{u_1, \dots, u_n\}$ of $V_n(y)$ such that $u_k \in \mathcal{N}(\mu_k(y)I - A_0 - B(y))$ for $k = 1, \dots , n$. By definition, $(u_1, \dots , u_n) \in Y$, and if we put $(v_1, \dots , v_n) := \omega(u_1, \dots , u_n)$, then the vectors $v_1, \dots , v_n$ form an X-orthonormal basis of $V_n(y)$. Hence the X-orthogonal projection $P_n(y)$ is given by

$$P_n(y)x = \sum_{j=1}^{n}(x \mid v_j)_X v_j \quad \text{for } x \in X,$$

which means that $P_n(y) = \Pi(v_1, \dots , v_n) = \alpha(u_1, \dots , u_n) \in \alpha(Y)$, and the proof is complete. □

Corollary 3.4. *Suppose that* (H1)–(H3) *are satisfied and that* (GB) *holds for suitable* $R > 0$. *Consider* $n \in \mathbb{N}$ *such that* (3.3) *holds for all* $y \in T_R$ *and put*

$$K := \{y \in S_R \mid y \in V_n(y)\}.$$

Then K *is compact and symmetric, and* $\gamma(K) = n$.

Proof. Let $P_n : T_R \to \mathcal{L}(X)$ be the map from Proposition 3.3. Then

$$P_n(-y) = P_n(y) \quad \text{for all } y \in T_R$$

by (H3). Hence it follows from Proposition 3.3 that the homotopy $H : [0, 1] \times S_R \to \mathcal{L}(X)$ given by

$$H(t, y) := P_n(ty)$$

has the properties (i)–(iv) from Proposition 2.1. Moreover, radial projection is evidently an odd homeomorphism of S_R onto the unit sphere in X. The assertion now follows from Corollary 2.4. □

In our applications the last corollary will be combined with Proposition 3.2 to derive property (CP) from Section 1. The next theorem is a result of this type, but its assumptions are too restrictive for most applications. Still, it is useful as a first step:

Theorem 3.5. *Suppose* (H1)–(H4), (CC) *and* (GB) *are all satisfied for suitable* $R > 0$, *and consider* $n \in \mathbb{N}$ *such that* (3.3) *also holds on* T_R. *Then condition* (CP) *from Section* 1 *holds true for these values of* R *and* n.

Proof. Let K be the compact symmetric subset of S_R considered in Corollary 3.4, and let $b_n := \max_{u\in K} \psi(u)$. Then $b_n \geq c_n$ by Corollary 3.4 and the definition of c_n. On the other hand, we have $b_n \leq c_n$ by Proposition 3.2(a), hence $b_n = c_n$. The rest of the assertion now follows from Proposition 3.2(b). □

Remark. The material of the present section can be generalized to situations where the $B(y)$ are *unbounded* symmetric operators. Condition (GB) is then replaced by the assumption that there exists an operator T which is relatively compact with respect to A_0 and which satisfies

$$\|B(y)v\| \leq \|Tv\| \quad \text{for all } y \in T_R,\ v \in \mathcal{D}(A_0).$$

As we shall see in Section 6, this version is important for applications to partial differential equations. For details cf. [9].

4. Periodic solutions of a nonlinear Hill's equation

In this section we consider equations of the form (1.1) with periodic data. Without loss of generality the period is taken equal to 1. Moreover, since the nonlinearity g is supposed to be odd, we write it in the form

$$g(x, y) = f(x, y^2)y.$$

Thus our equation reads

$$-(p(x)y')' + q(x)y + f(x, y^2)y = \lambda y, \tag{4.1}$$

where $p : \mathbb{R} \to \,]0, \infty[\,$, $q : \mathbb{R} \to \mathbb{R}$ are given 1-periodic continuous functions, $p \in C^1$, and where $f : \mathbb{R} \times [0, \infty[\, \to \mathbb{R}$ is continuous and 1-periodic in the x-variable. Equation (4.1) is considered together with periodic boundary conditions

$$y(0) = y(1), \quad y'(0) = y'(1) \tag{4.2}$$

as well as the isoperimetric constraint

$$\int_0^1 y(x)^2\, dx = R^2. \tag{4.3}$$

As a matter of convenience we also assume that

$$f(x, 0) \equiv 0 \quad \text{on } \mathbb{R}, \tag{4.4}$$

which can always be arranged by taking q appropriately.

Roughly speaking, the main result of this section says that property (CP) from Section 1 holds for the above problem for arbitrary $R > 0$ and every *odd* integer n provided f satisfies

(M) For every $x \in \mathbb{R}$, $f(x, \cdot\,)$ is nondecreasing on $[0, \infty[$.

To be more precise, let X be the Sobolev space of real 1-periodic $W^{1,2}$-functions, i.e.

$$X := \{v \in W^{1,2}[0,1] \mid v(0) = v(1)\}$$

with its standard norm, and define the functionals ϕ, ψ on X, by

$$\phi(v) := \frac{1}{2} \int_0^1 F(x, v(x)^2)\, dx,$$

where

$$F(x,s) := \int_0^s f(x,t)\, dt \quad \text{for } x \in \mathbb{R},\ s \geq 0,$$

and

$$\psi(v) := \frac{1}{2} \int_0^1 (p(x)v'(x)^2 + q(x)v(x)^2)\, dx + \varphi(v).$$

It is well known that $\phi, \psi \in C^1(X)$, that the solutions of (4.1)–(4.3) correspond to the critical points of $\psi|_{S_R}$ and that every weak $W^{1,2}$-solution of (4.1) is, in fact, a classical solution. We take the auxiliary space C to be the space of real continuous 1-periodic functions on $\mathbb{R}$ equipped with the sup-norm $\|\cdot\|_\infty$. Then for every $y \in C$ we have an associated linear problem which reads

$$(4.5)_y \qquad \begin{cases} -(p(x)v')' + q(x)v + f(x, y(x)^2)v = \mu v, \\ v(0) = v(1), \\ v'(0) = v'(1). \end{cases}$$

This is the periodic boundary value problem for the classical Hill equation, and hence, as is well known (see e.g. [8]), the problem has an unbounded sequence of eigenvalues

$$\mu_1(y) < \mu_2(y) \leq \mu_3(y) < \mu_4(y) \leq \dots,$$

where $\mu_{2m}(y) > \mu_{2m-1}(y)$ and where every eigenfunction corresponding to $\mu_{2m-1}(y)$ has exactly $2m-2$ zeroes in $[0,1[$ $(m = 1, 2, \dots)$. Thus, for every odd $n \in \mathbb{N}$ the fixed point set K can be formed, and we have the following basic result:

Theorem 4.1. *Suppose condition* (M) *is satisfied, and consider an arbitrary* $R > 0$ *and an odd integer* $n \in \mathbb{N}$. *Then problem* (4.1)–(4.3) *has property* (CP) *for these values of* R *and* n. *In particular, there is a solution* (y, λ) *of* (4.1)–(4.3) *such that* $\psi(y) = c_n$ *and* y *has exactly* $n - 1$ (*simple*) *zeroes in* $[0, 1[$.

To prove this theorem, let us begin by casting our problem into the abstract framework of Section 3. For convenience, we shall consider all relevant function spaces as spaces of functions on the period interval $I := [0,1]$, which is no loss of generality, because for our spaces of periodic functions restriction to a period interval is always an isometric isomorphism. Thus, we take $\mathcal{H} := L^2(I)$ with the standard norm $\|\cdot\|$ and scalar product $(\cdot\,|\,\cdot)$, and we define the self-adjoint operator A_0 in $\mathcal{H}$ by

$$\mathcal{D}(A_0) := \{v \in W^{2,2}(I) \mid v(0) = v(1),\ v'(0) = v'(1)\},$$

$$A_0 v := -(pv')' + qv.$$

The form domain of A_0 is then the space X defined above, its norm being equivalent to the norm defined by (3.2). For $y \in C$ we define $B(y) \in \mathcal{L}(\mathcal{H})$ to be multiplication by the continuous function

$$x \mapsto f(x, y(x)^2).$$

With these choices, equations (1.4), (1.5) are clearly equivalent to (4.1)–(4.3), and hypotheses (H1)–(H4) are obviously satisfied. Moreover, (3.3)–(3.5) hold because of (4.4), the definition of ϕ and the assumption that n is odd. In the next lemma we see that condition (CC) is also satisfied, but (GB) is not, in general. To surmount this difficulty we shall introduce a *truncated problem* below.

Lemma 4.2.

(a) *If f satisfies condition* (M), *then ϕ satisfies condition* (CC).
(b) *If f is bounded, then B satisfies condition* (GB) *for any $R > 0$.*

Proof. (a) From (M) it follows that $F(x,\cdot\,)$ is convex for every $x \in I$, hence the functional Φ given by

$$\Phi(u) := \int_0^1 F(x, u(x))\,dx$$

is convex. Obviously $\phi = \Phi \circ Q$, where $Q(u) := u^2$ is an X-valued quadratic form on X. Hence the result follows from Example 3.1. (It is, however, not difficult to check condition (CC) directly.)

(b) Suppose

$$\eta := \sup_{\substack{x \in I \\ s \geq 0}} |f(x,s)| < \infty.$$

Then, for every $y \in C$, $B(y) \in \mathcal{L}(\mathcal{H})$ is multiplication by the bounded function

$$g(x) := f(x, y(x)^2) \quad \text{for } x \in I,$$

and hence $\|B(y)\| \leq \eta$, as desired. □

Proof of Theorem 4.1. Fix $R > 0$ and an odd integer $n \geq 1$. Since (CC) and (3.5) are valid, we also have (3.7a). The continuous embedding $X \hookrightarrow C$

therefore yields

$$\|y\|_\infty^2 \le b_1\psi(y) + b_0 R^2 \tag{4.6}$$

for every $y \in S_R$, where the positive constants b_0, b_1 depend on A_0, but not on y or $B(y)$. Choose

$$\tau > b_1 c_n(R) + b_0 R^2$$

and define $\widehat{f} : \mathbb{R} \times [0, \infty[\to \mathbb{R}$ by

$$\widehat{f}(x,s) := \begin{cases} f(x,s) & \text{if } 0 \le s \le \tau, \\ f(x,\tau) & \text{if } s \ge \tau. \end{cases}$$

The truncated problem now consists of combining the modified equation

$$-(p(x)y')' + q(x)y + \widehat{f}(x,y^2)y = \lambda y \tag{4.1$'$}$$

with the side conditions (4.2), (4.3). Clearly this problem satisfies all the assumptions of Theorem 4.1, and, in addition, $\widehat{f}$ is bounded on $I \times [0,\infty[$, so that (GB) is also satisfied. Hence Corollary 3.4 and Theorem 3.5 can be applied to problem (4.1')–(4.3). Denoting by $\widehat{\psi}$, $\widehat{B}(y)$, $\widehat{c}_n$, $\widehat{V}_n(y)$, $\widehat{K}$ etc. the objects and quantities analogous to ψ, $B(y)$, c_n, $V_n(y)$, K, respectively, but referring to the truncated problem, we therefore know that $\gamma(\widehat{K}) \ge n$ and that $\widehat{c}_n = \max_{y\in\widehat{K}} \widehat{\psi}(y)$. The assertion now follows from Proposition 3.2(b) and the next lemma. □

LEMMA 4.3. *$\widehat{c}_n = c_n$, $\widehat{K} = K$, and for every $y \in K$ we have $\widehat{\psi}(y) = \psi(y)$.*

PROOF. It is clear from the definitions that $\widehat{\psi} \le \psi$ and hence $\widehat{c}_n \le c_n$. Also, the definition of $\widehat{f}$ implies that for every $y \in X$ with $\|y\|_\infty < \sqrt{\tau}$ we have $\widehat{f}(x,y(x)^2) = f(x,y(x)^2)$ and $\widehat{F}(x,y(x)^2) = F(x,y(x)^2)$ on I and hence $\widehat{\psi}(y) = \psi(y)$ as well as $\widehat{B}(y) = B(y)$. Now consider $y \in K$. Then $\psi(y) \le c_n$ by Proposition 3.2(a), hence $\|y\|_\infty < \sqrt{\tau}$ by (4.6). But then $B(y) = \widehat{B}(y)$, hence $y \in \widehat{K}$. Since A_0 is not altered by passing to the truncated problem, we also have (4.6) with ψ replaced by $\widehat{\psi}$. Therefore we can repeat the preceding argument with the roles of K and $\widehat{K}$ interchanged. Hence we obtain

$$K = \widehat{K} \subseteq \{y \in S_R \mid \|y\|_\infty \le \sqrt{\tau}\},$$

so that $\gamma(K) = \gamma(\widehat{K}) = n$ and $\psi = \widehat{\psi}$ on K. Hence

$$c_n \le \max_{y\in K} \psi(y) = \max_{y\in\widehat{K}} \widehat{\psi}(y) = \widehat{c}_n \le c_n,$$

which proves $c_n = \widehat{c}_n$. □

REMARKS. (a) Consider a solution $(\lambda, y) \in \mathbb{R} \times C^2[0,1]$ of (4.1)–(4.3) and an arbitrary $m \in \mathbb{N}$ such that $\psi(y) > c_m$. Periodicity implies that the number of zeroes of y in $[0,1[$ is even, say $2k$. Then the theory of Hill's equation says that $\lambda = \mu_{2k}(y)$ or $\lambda = \mu_{2k+1}(y)$. In either case, $y \in V_{2k+1}(y)$ and hence

$$c_m < \psi(y) \le c_{2k+1}$$

by Proposition 3.2. It follows that $m \le 2k$, i.e. the number of zeroes of y is at least m if $\psi(y) > c_m$. (In fact, it is at least $m+1$ if m is odd.) Following an idea of Coffman [7], it can be shown that the multiplicity of Ljusternik–Schnirelman levels is at most 2, i.e. we have

$$c_m < c_{m+2} \quad \text{for all } m \ge 1 \tag{4.7}$$

(cf. [9]). If $n \ge 3$ is odd, we can therefore take $m = n-2$ in the above argument, and we find that *every* solution $y \in S_R \cap \psi^{-1}(c_n(R))$ has at least $n-1$ zeroes in $[0,1[$.

(b) If we replace (4.2) by Dirichlet boundary conditions and drop the periodicity requirement on the data, we can evidently repeat all the considerations of this section for *arbitrary* $n \in \mathbb{N}$, because for Dirichlet problems all eigenvalues are simple. Also, it can be shown that in this case the Ljusternik–Schnirelman levels are simple in the sense that we have

$$c_n < c_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

A proof for this was sketched in [7] and worked out in detail by Heid [9]. It follows that every solution $y \in S_R \cap \psi^{-1}(c_n)$ must have at least $n-1$ interior zeroes, and that there exists one having exactly $n-1$ interior zeroes. Thus we recover all the results of [12] under the weaker hypothesis (M).

5. Periodic solutions of periodic second-order systems

This section is devoted to a vector-valued variant of the periodic problem treated in the preceding section. To avoid unnecessary technicalities, we only consider a model problem which exhibits the typical difficulties. Thus, on the period interval $I := [0,1]$ we consider the second-order system

$$-y'' + \mathbf{A}(x)y + \mathbf{B}(x,y)y = \lambda y \tag{5.1}$$

together with the periodic boundary conditions

$$y(0) = y(1), \quad y'(0) = y'(1) \tag{5.2}$$

and the constraint

$$\int_0^1 |y(x)|^2\,dx = R^2. \tag{5.3}$$

Here y is an $\mathbb{R}^N$-valued function, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. $\mathbf{A}(x)$ and $\mathbf{B}(x,y)$ are $N\times N$-matrices, and all such matrices will be identified with operators from $\mathcal{L}(\mathbb{R}^N)$ via matrix multiplication from the left. Our assumptions are as follows:

(A) $\mathbf{A} : I \to \mathcal{L}(\mathbb{R}^N)$ is continuous, $\mathbf{A}(0) = \mathbf{A}(1)$, and for every $x \in I$, $\mathbf{A}(x)$ is symmetric.

(B) $\mathbf{B} : I \times \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^N)$ is continuous, $\mathbf{B}(0, y) = \mathbf{B}(1, y)$ for all $y \in \mathbb{R}^N$, $\mathbf{B}(x, 0) = 0$ for all $x \in I$, and $B(x, y)$ is symmetric for all $x \in I$, $y \in \mathbb{R}^N$.

(S) ($\mathbb{Z}_2$-*symmetry*) $\mathbf{B}(x, -y) = \mathbf{B}(x, y)$ for all $x \in I$, $y \in \mathbb{R}^N$.

(VS) (*variational structure*) There exists $G \in C^1(I \times \mathbb{R}^N)$ such that

$$\mathbf{B}(x, y)y = \frac{\partial G}{\partial y}(x, y) \quad \text{for all } x \in I,\ y \in \mathbb{R}^N.$$

(QM) (*quasi-monotonicity*) For all $x \in I$ and $y, v \in \mathbb{R}^N$ we have

$$2(G(x, v) - G(x, y)) \geq \mathbf{B}(x, y)v \cdot v - \mathbf{B}(x, y)y \cdot y,$$

(where the dot denotes the scalar product in $\mathbb{R}^N$).

Remarks. (a) The special form of the nonlinearity is only a very weak restriction on G. To see this, note that every $G \in C^2(I \times \mathbb{R}^N)$ with $\dfrac{\partial G}{\partial y}(x, 0) = 0$ satisfies

$$\frac{\partial G}{\partial y}(x, y) = \mathbf{B}(x, y)y,$$

where

$$\mathbf{B}(x, y) := \int_0^1 \frac{\partial^2 G}{\partial y^2}(x, ty)\, dt$$

is a symmetric matrix depending continuously on x, y.

(b) Clearly, condition (QM) just means that for every fixed x, $G(x, \cdot\,)$ satisfies the comparison condition in $X = \mathbb{R}^N$. Hence we see from Example 3.1 that functions of the form

$$G(x, y) = g(x, q_1(x, y), \ldots, q_N(x, y))$$

satisfy (QM) provided $g \in C^1(I \times \mathbb{R}^N)$ is such that for every $x \in I$, $g(x, \cdot\,)$ is convex and the $q_k : \mathbb{R}^N \to \mathbb{R}$ are quadratic forms on $\mathbb{R}^N$. In particular, for $N = 1$ and $G(x, y) = (F(x, y^2))/2$ as in Section 4, (QM) reduces to (M).

It is clear how to cast this problem into our abstract setting. We take $\mathcal{H} = L^2(I, \mathbb{R}^N)$, $\mathcal{D}(A_0) = \{v \in W^{2,2}(I, \mathbb{R}^N) \mid v(0) = v(1), v'(0) = v'(1)\}$, $A_0 v := -v'' + \mathbf{A}v$, so that the form domain X is given by

$$X = \{v \in W^{1,2}(I, \mathbb{R}^N) \mid v(0) = v(1)\},$$

and we take $C := \{y \in C^0(I, \mathbb{R}^N) \mid y(0) = y(1)\}$. For any $y \in C$ the perturbation $B(y) \in \mathcal{L}(\mathcal{H})$ is then given by

$$[B(y)v](x) = \mathbf{B}(x, y(x))v(x) \quad \text{for } x \in I,$$

and the functionals $\phi, \psi \in C^1(X)$ are given by

$$\phi(y) := \int_0^1 G(x, y(x))\, dx,$$
$$\psi(y) := \frac{1}{2}\int_0^1 (|y'(x)|^2 + \mathbf{A}(x)y(x)\cdot y(x))\, dx + \varphi(y).$$

It clearly follows from standard results that conditions (H1)–(H4) as well as (3.4), (3.5) and (CC) are satisfied. In particular, for every $y \in C$ we have the increasing sequence $(\mu_k(y))_{k\geq 1}$ of eigenvalues of the associated linear problem, and every eigenvalue has multiplicity not greater than $2N$. However, it is no longer clear where to expect the gaps $\mu_n(y) < \mu_{n+1}(y)$ or how they depend on y. Therefore we pick a number $n \in \mathbb{N}$ such that

$$\mu_n(0) < \mu_{n+1}(0), \tag{5.4}$$

and we look at the situation for *small* $R > 0$ only. Our main result reads as follows:

Theorem 5.1. *Suppose hypotheses* (A), (B), (S), (VS) *and* (QM) *are satisfied, and consider $n \in \mathbb{N}$ such that* (5.4) *holds. For $R > 0$ define*

$$K(R) := \{y \in S_R \mid \mu_n(y) < \mu_{n+1}(y) \text{ and } y \in V_n(y)\}.$$

Then there exists $R_0 > 0$ such that for $0 < R < R_0$ we have

- (i) *$K(R)$ is compact and symmetric, and $\gamma(K(R)) = n$. In particular, $K(R) \neq \emptyset$.*
- (ii) $c_n(R) = \max_{y\in K(R)} \psi(y)$,
- (iii) *Every $y \in K(R)$ such that $\psi(y) = c_n(R)$ is an eigenfunction of* (5.1)–(5.3) *with eigenvalue $\lambda = \mu_n(y)$.*

Proof. Choose $0 < \eta < (\mu_{n+1}(0) - \mu_n(0))/2$. By (B) and the continuity of $\mathbf{B}$ there exists $\tau > 0$ such that

$$|y| \leq \tau \Rightarrow |\mathbf{B}(x, y)| \leq \eta \quad \text{for all } x \in I.$$

Put $\rho(y) := \tau y/|y|$ for $y \in \mathbb{R}^N \setminus \{0\}$ and define $\widetilde{\mathbf{B}} : I \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$\widetilde{\mathbf{B}}(x, y) := \begin{cases} \mathbf{B}(x, y) & \text{if } |y| \leq \tau, \\ \mathbf{B}(x, \rho(y)) & \text{if } |y| \geq \tau. \end{cases}$$

We now introduce a *truncated problem* which consists of the equation

$$-y'' + \mathbf{A}(x)y + \widetilde{\mathbf{B}}(x, y)y = \lambda y \tag{5.1'}$$

together with the side conditions (5.2), (5.3). This problem can evidently be cast into the abstract framework using the same spaces $\mathcal{H}$, X, C and the same operator A_0 as before, and it satisfies (H1)–(H3) as well as (3.5). It also satisfies (GB), for by the choice of τ we have

$$\|\widetilde{B}(y)\| = \max_{x\in I} |\widetilde{\mathbf{B}}(x, y(x))| \le \eta \tag{5.5}$$

for all $y \in C$. (Here and in the sequel we use the tilde to denote objects and quantities referring to the truncated problem.) Furthermore, it follows from (3.5), (5.5) and (3.11) that

$$|\widetilde{\mu}_k(y) - \widetilde{\mu}_k(0)| \le \eta \tag{5.5a}$$

for arbitrary $k \in \mathbb{N}$, $y \in C$, and since clearly $\widetilde{\mu}_k(0) = \mu_k(0)$, we see from the choice of η that

$$\widetilde{\mu}_n(y) < \widetilde{\mu}_{n+1}(y) \quad \text{for all } y.$$

In particular, (3.3) is valid for the truncated problem for every $R > 0$. Thus Corollary 3.4 can be applied to the truncated problem, and we obtain

$$\gamma(\widetilde{K}(R)) = n$$

for all $R > 0$. We shall prove below that there exists $R_0 > 0$ such that

$$K(R) = \widetilde{K}(R) \quad \text{for } 0 < R < R_0 \tag{5.6}$$

For such R it then follows that assertion (i) of the theorem is valid. In particular, $K(R) \in \Sigma_n(R)$ and hence $c_n(R) \le \max_{y\in K(R)} \psi(y)$ by the definition of c_n. Assertions (ii) and (iii) now follow from Proposition 3.2.

It remains to prove (5.6). First consider $y \in \widetilde{K}(R)$. By (3.2) and (5.5a) we have

$$\|y\|_X^2 \le (\widetilde{\mu}_n(y) + \nu)R^2 \le (\mu(0) + \eta + \nu)R^2,$$

and hence we see from the continuous embedding $X \hookrightarrow C$ that there exists $R_1 > 0$ such that for $0 < R < R_1$ we have $\|y\|_\infty < \tau$. But then $\widetilde{B}(y) = B(y)$, hence $\widetilde{\mu}_k(y) = \mu_k(y)$ for all k, in particular $\mu_n(y) < \mu_{n+1}(y)$, and moreover $\widetilde{V}_n(y) = V_n(y)$, whence $y \in K(R)$. This shows that

$$\widetilde{K}(R) \subseteq K(R)$$

for $0 < R < R_1$. To prove the converse, note first that (4.6) is available in the present context, and moreover

$$\lim_{R\to 0+} c_n(R) = 0. \tag{5.7}$$

This is a standard result in Ljusternik–Schnirelman theory and has been proved for many situations. In our case it immediately follows from the observations that on one hand

$$c_n(R) \ge \inf_{u\in S_R} \psi(u) \ge -\frac{\nu}{2}R^2$$

by (3.1), (3.7) and the fact that $\mathcal{D}(A_0)$ is dense in X, and that on the other hand, if we pick a fixed subspace $W \subseteq X$ of finite dimension $m \geq n$, then

$$c_n(R) \leq \max_{u \in W \cap S_R} \psi(u) \underset{R \to 0+}{\longrightarrow} 0$$

by the continuity of ψ at the origin. Now it follows from (4.6) and Proposition 3.2 that $y \in K(R)$ implies

$$\|y\|_\infty^2 \leq b_1 c_n(R) + b_0 R^2,$$

and hence by (5.7) there exists $R_2 > 0$ such that $\|y\|_\infty < \tau$ whenever $0 < R < R_2$. But then $B(y) = \widetilde{B}(y)$, and hence $y \in \widetilde{K}(R)$ follows as before. Thus we see that

$$K(R) \subseteq \widetilde{K}(R)$$

whenever $0 < R < R_2$. Choosing $R_0 := \min(R_1, R_2)$, we therefore obtain (5.6), and the proof is complete. □

Remarks. (a) The fact that $\lambda = \mu_n(y)$ for the solutions y exhibited by Theorem 5.1 can be interpreted geometrically in terms of the Morse index of y as a solution of the associated linear problem

$$\begin{gathered} -v'' + \mathbf{A}(x)v + \mathbf{B}(x, y(x))v = \mu v, \\ v(0) = v(1), \quad v'(0) = v'(1). \end{gathered}$$

Since the multiplicity of the eigenvalue $\mu_n(y)$ is at most $2N$, the Morse index is $\geq n - 2N$.

(b) No special features of the periodic boundary conditions were used here, and, in fact, there are analogous results for any self-adjoint boundary conditions. The special importance of the periodic case arises from the fact that our solutions correspond to periodic orbits of the dynamical system associated to equation (5.1) when it is considered as a differential equation with periodic data.

(c) If $(R_j)_j$ is a null sequence and y_j is a solution on S_{R_j} as given by Theorem 5.1, then we have

$$\lim_{j \to \infty} \|y_j\|_\infty = 0,$$

as is clear from the arguments used to prove (5.6). This implies that $\mu_n(y_j) \to \mu_n(0)$ as $j \to \infty$, and hence we see that the solutions given by Theorem 5.1 *bifurcate* from $(\mu_n(0), 0)$ in the topology of $\mathbb{R} \times C$.

6. Further applications

In this final section we discuss some applications to elliptic partial differential equations with radial symmetry. For the sake of brevity we only give an informal treatment, referring the reader to [9] for details. Let

$$D := \{x \in \mathbb{R}^N \mid |x| < 1\}$$

be the unit ball in Euclidean N-space ($N \geq 2$), and consider the radially symmetric semilinear elliptic Dirichlet problem

$$-\Delta u + q(|x|)u + f(|x|, u^2)u = \lambda u \quad \text{for } x \in D, \tag{6.1}$$

$$u|_{\partial D} = 0 \tag{6.2}$$

together with the isoperimetric constraint

$$\int_D u(x)^2 \, dx = R^2. \tag{6.3}$$

Here $R > 0$ is given, and the data functions $q : [0,1] \to \mathbb{R}$, $f : [0,1] \times [0, \infty[\to \mathbb{R}$ satisfy the following assumptions:

(A1) q is continuous,
(A2) f is continuous, $f(r, 0) \equiv 0$,
(A3) There are constants $r_0 \in]0,1]$, $c > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ such that

$$|f(r,s)| \leq c r^\alpha s^\beta \quad \text{for } 0 < r \leq r_0$$

and $\alpha - (N-2)\beta > 0$,
(M) For every $r \in [0,1]$, $f(r, \cdot)$ is monotonically nondecreasing on $[0, \infty[$.

Here the spaces $\mathcal{H}$, X, $\mathcal{D}(A_0)$ consist of the radially symmetric functions in $L^2(D)$, $W_0^{1,2}(D)$, $W^{2,2}(D) \cap W_0^{1,2}(D)$, respectively, and the self-adjoint operator A_0 is given by

$$A_0 u := -\Delta u + qu$$

in the sense of distributions. An important new aspect is the fact that the functions in X may have a singularity at $x = 0$, so that C can no longer be chosen as a space of bounded functions. Instead, for $\theta > (N-2)/2$ suitably chosen, we take C to be the space of continuous radially symmetric functions u on $D \setminus \{0\}$ for which the norm

$$\|u\|_C := \sup_{0 < |x| \leq 1} |x|^{-\theta} |u(x)|$$

is finite. Using (A3) as well as certain ramifications of the Strauss lemma, it can then be shown that

(i) there is a compact embedding $X \hookrightarrow C$
(ii) for every $y \in C$, multiplication by the function $x \mapsto f(|x|, y(x)^2)$ defines a symmetric linear operator $B_0(y) \in \mathcal{L}(\mathcal{D}(A_0), \mathcal{H})$, and
(iii) the map $B_0 : C \to \mathcal{L}(\mathcal{D}(A_0), \mathcal{H})$ thus defined is continuous.

It follows that $B_0(y)$ is always A_0-compact, and hence, for every $y \in C$ we have the sequence $(\mu_k(y))_{k \geq 1}$ of eigenvalues of the associated linear problem. In polar coordinates the associated linear problem can be rewritten as a (singular) boundary value problem for a second-order ordinary differential equation, and hence all its eigenvalues are simple, and the eigenfunctions corresponding to

$\mu_k(y)$ have precisely $k-1$ nodal surfaces in $D\setminus\{0\}$. In particular, (3.3) is satisfied globally. Moreover, it follows from (A2), (A3) that a functional $\phi \in C^1(X)$ is given by

$$\phi(y) := \frac{1}{2}\int_D F(|x|, y(x)^2)\,dx$$

with $F(r,s) := \int\limits_0^s f(r,t)\,dt$, and that the derivative $d\phi(y)$ is just $B(y)y$, where $B(y)$ denotes the canonical extension of $B_0(y)$ to an element of $\mathcal{L}(X, X^*)$ for $y \in X$. The problem therefore has variational structure, with the functional ψ given by

$$\psi(y) := \frac{1}{2}\int_D (|\nabla y(x)|^2 + q(|x|)y(x)^2)\,dx + \phi(y),$$

and we can introduce the Ljusternik–Schnirelman levels $c_n(R)$, the fixed point sets $K = K(R,n)$ etc.

As before, (CC) is satisfied because of (M). Condition (GB) has to be replaced by a condition of "uniform A_0-compactness" as described at the end of Section 3. But again this condition can only be satisfied after a cut-off procedure. This time the truncated nonlinearity is (for fixed $R > 0$, $n \in \mathbb{N}$) given by

$$\widehat{f}(r,s) := \begin{cases} f(r,s) & \text{if } s \le \tau r^{-2\theta}, \\ f(r, \tau r^{-2\theta}) & \text{if } s \ge \tau r^{-2\theta}, \end{cases}$$

where $\tau > 0$ is chosen large enough to ensure

$$y \in S_R \cap \psi^{-1}(]-\infty, c_n(R)]) \Rightarrow \|y\|_C^2 < \tau.$$

One can now proceed essentially as in Section 4 to obtain property (CP). Moreover, some regularity theory shows that the nodal solutions thus exhibited are, in fact, classical C^2-solutions on D, and finally, the Ljusternik–Schnirelman levels turn out to be simple, as can be shown by an appropriate adaption of the ideas of Coffman [7], using the strong maximum principle. Thus, the main result concerning problem (6.1)–(6.3) is

Theorem 6.1. *Suppose assumptions* (A1)–(A3) *and* (M) *are satisfied, and consider arbitrary $R > 0$, $n \in \mathbb{N}$. Put*

$$K := \{y \in S_R \mid y \in V_n(y)\}.$$

Then we have

(a) *K is compact and symmetric, and $\gamma(K) = n$. In particular, $K \neq \emptyset$.*
(b) $c_n = \max\limits_{y\in K} \psi(y)$.
(c) *Every $y \in K \cap \psi^{-1}(c_n)$ is a classical solution of* (6.1)–(6.3) *having precisely $n-1$ nodal surfaces in $D \setminus \{0\}$.*
(d) $c_n < c_{n+1}$.

REMARKS. (a) Note that assumption (A3) admits both subcritical and supercritical growth of the nonlinearity for $s \to \infty$, depending on the asymptotics as $r \to 0$. Actually, property (CP) has been established for problem (6.1)–(6.3) in [9] under somewhat weaker assumptions. The function f need only be defined and continuous on $]0,1[\times[0,\infty[$, and for the exponents α, β appearing in (A3) we only need to require

$$\alpha-(N-2)\beta>\begin{cases}-2 & \text{if } N\geq 4,\\ -3/2 & \text{if } N=3,\\ -1 & \text{if } N=2.\end{cases}$$

Thus, for $N=2$ any polynomial growth is admissible, and for $N\geq 4$ supercritical cases are included whenever $\alpha>0$, while for the autonomous case ($\alpha=0$) any subcritical growth is allowed (note that the critical Sobolev exponent corresponds to $\beta=2/(N-2)$). Only for $N=3$ our assumptions are more restrictive than those under which classical Ljusternik–Schnirelman theory would work. This has to do with the behaviour of the multiplication operators $B_0(y)$ on $\mathcal{D}(A_0)$, and probably this shortcoming could be remedied by invoking some more sophisticated perturbation theory.

(b) Condition (CC) introduces a definite sign into the nonlinearity (exhibited, for instance, by (3.7)). In this sense all the problems treated here are *sublinear*. Superlinear problems, i.e. equations of the type

$$\Delta u+c(x)u+f(x,u^2)u=0,$$

where $f(x,0)\equiv 0$ and f satisfies (M), have been considered by many authors, and solutions whose number of zeroes (resp. nodal surfaces) is known have been constructed in many situations. We mention Z. Nehari's classical paper [17], the work by Coffman [5] and Struwe [21], and the recent paper by Bartsch and Willem [2]. Usually in this type of work an appropriate functional is minimized (or maximized) under the side condition that a nodal configuration is prescribed, yielding a function which solves the given equation outside the nodal set. Then the parameters determining the prescribed nodal configuration are varied until an "optimal" configuration is reached for which the corresponding piecewise solution actually solves the equation everywhere. For a class of problems satisfying a strict monotonicity assumption, Coffman [5] identified Nehari's characteristic numbers (which are obtained in the fashion just described) with the Ljusternik–Schnirelman levels of a certain functional on the Nehari manifold. Under his assumptions the Nehari manifold is spherelike as required for S_R in our Corollary 2.4, and it is tempting to try to adapt the present method to such situations. The explicit reference to nodal configurations could then be replaced by a general way to construct solutions y for which the Lagrange multiplier $\lambda=0$, which is generated by the Nehari constraint, is known to be the n-th eigenvalue of the associated linear problem. However, such an adaption is far from obvious.

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