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$\mathcal{F}$-EPI MAPS

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Abstract. The concept of 0-epi maps is a known homotopic analogue to maps with nonzero degree. There exist various related notions on unbounded sets and for multivalued maps. We introduce a concept which unifies these definitions. We also compare the various concepts. In particular, we prove that proper 0-epi maps are also 0-multiepi.

1. Introduction

The probably simplest homotopic concept for equations is the concept of 0-epi maps which has been introduced in [5]: If X, Y are Banach spaces and $\Omega \subseteq X$ is bounded, then a continuous map $F: \overline{\Omega} \to Y$ with $0 \notin F(\partial\Omega)$ is called 0-*epi* if the equation $F(x) = \varphi(x)$ has a solution for any compact map $\varphi: \overline{\Omega} \to Y$ with $\varphi|_{\partial\Omega} = 0$. Roughly speaking, a map is 0-epi if it has a homotopically stable 0. In particular, it turns out that the above definition is independent under perturbations by compact homotopies.

It is a consequence of Hopf's theorems on the connection of homology and homotopy theory that if $\Omega \subseteq X = Y$ is open and bounded, and if F has the form $F = \mathrm{id} - C$ where C is compact (or just so-called *strictly condensing*), then F is 0-epi if and only if $\deg(F, \Omega_0, 0) \neq 0$ for some component Ω_0 of Ω (see [6]).

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However, the above definition is also useful if no degree for F is defined, in particular if $X \neq Y$.

In view of this observation, the notion of 0-epi maps has already found numerous applications which go beyond the realms of degree theory. For an overview, we refer to the monograph [8].

A corresponding multivalued concept where φ is replaced by a multivalued function has been introduced in [14]. However, in [14], emphasis was put on the extension to *noncompact* functions. The question was left open whether each 0-epi map is also 0-epi in the multivalued sense. In Section 4, we give a positive answer for proper maps.

If the set Ω above is an *unbounded* subset of a topological vector space, one may repeat the above definition but require additionally that φ have bounded support as was done in [5]. However, this is a very restrictive requirement on φ. It appears more natural to modify the definition in this case by requiring that $F(x) = \varphi(x)$ have a solution for any compact map φ with bounded range with $\varphi|_{\partial\Omega} = 0$. (Here, *compact* means as usual that φ maps bounded sets into precompact sets). However, also the requirement that φ have bounded range is too restrictive for many applications. In [4], this condition was replaced in case $\Omega = X$ by the sublinear growth condition

$$\limsup_{\|x\|\to\infty} \frac{\|\varphi(x)\|}{\|x\|} = 0.$$

The corresponding maps F have been called *stably-solvable* in [4].

We shall introduce a concept in Sections 2 and 3 which contains all of the above notions (both, in the single-valued and in the multi-valued case). Our concept also easily covers generalized growth conditions like

$$\limsup_{\|x\|\to\infty} \frac{\|\varphi(x)\|}{q(\|x\|)} = 0$$

(with $q(t) \uparrow \infty$ as $t \uparrow \infty$). The relation between the single-valued and the multivalued case in our general unifying concept is discussed in Section 4.

It will be convenient (although mathematically not quite precise) to identify single-valued maps with multi-valued maps, i.e.occassionally we do not notationally distinguish the map $\varphi: A \to B$ from the map $\Phi: A \to 2^B$ defined by $\Phi(x) = \{\varphi(x)\}$. In particular, we will use certain multi-valued concepts also for single-valued maps without further remarks. We adopt the usual notation for the image of a set $\Phi(C) = \bigcup_{x\in C} \Phi(x)$.

Throughout this paper, we assume the *axiom of dependent choices* DC which allows countably many recursive or nonrecursive choices ([9]) and which usually suffices for real analysis, in particular for applications in physics. If we use the (general) axiom of choice, we mention this explicitly.

2. $\mathcal{F}$-epi maps

We introduce now the general concept mentioned in the introduction. In view of later applications, we study not only the case that we have maps between Banach spaces but only between topological spaces which is a more natural setting for a homotopical concept. Only for the image space Y we also require a vector structure for simplicity, but it suffices to consider maps which attain their values in a certain subset $K \subseteq Y$. This is of some interest for applications, since this enables us e.g. to look for *positive* solutions (if K is the cone of positive functions).

Throughout this paper, let D be a normal space, $P \subseteq D$ closed, and $\Omega \subseteq D$. Let Y be a topological vector space, and $K \subseteq Y$. We will be interested in solutions of the equation $F(x) = \varphi(x)$ where $F\colon D \to Y$ and $\varphi\colon D \to K$. In the situation of the introduction, one should think of $D = \overline{\Omega}$ and $P = \partial\Omega$. We call this the "canonical" situation.

However, we have an important reason why we do not restrict ourselves to this case: For the homotopical definition of a coincidence index (the fixed point index of morphisms [7], [11], see also [17]), one studies instead of solutions of the equation $F(x) = \varphi(x)$ the fixed points of the multivalued map $\varphi \circ F^{-1}$. In particular, this index is defined in terms of subsets of K. In this connection, it is more natural to put e.g. $D = F^{-1}(\overline{O})$ and $P = F^{-1}(\partial O)$ where $O \subseteq K$ is open in K and the boundary (and closure) are understood with respect to K.

Some general remark: Roughly speaking, one might consider the theory presented in this paper as a "purely homotopical" approach to the solution of coincidence equations. It is somewhat surprising that this approach is formulated most naturally in terms of subsets of D (or in other words: in terms of the multivalued map $F^{-1} \circ \varphi$) while the corresponding homological approach (via coincidence index) is formulated more naturally in terms of subsets of K (or in terms of the multivalued map $\varphi \circ F^{-1}$). In the classical "fixed point setting" (i.e. if $F(x) = x$), the Hopf extension theorem implies that the two approaches in fact coincide in some sense [6], [18]. However, in the general situation the above described difference makes it even hard to just formulate such a connection (although a generalization of the Hopf theorem for the coincidence index is known [10]).

One should think of K as a *cone*, i.e. a closed and convex subset of Y with $0 \in K + K \subseteq K$. In this connection, we emphasize that in the canonical situation even if Ω is a subset of a normed space X, it might be convenient to consider Ω as a subset of some cone D in X and to understand the boundary $P = \partial\Omega$ with respect to this cone (many cones have no interior points). This allows e.g. to look for positive solutions of certain problems.

We point out that another possible choice for subsets Ω of a normed space X is to consider Ω as a subset of $D = X \cup \{\infty\}$ (endowed with the natural topology), and to understand the boundary $P = \partial\Omega$ with respect to D. In this case the condition $\varphi|_P = 0$ means that φ has bounded support and $\varphi|_{\partial_X\Omega} = 0$ where $\partial_X\Omega$ is the boundary of Ω with respect to X. This choice will lead to the original concept of 0-epi maps from [5], [8] mentioned in the beginning.

Throughout, we assume that we have given a family $\mathfrak{B}$ of subsets of D. One should think of $\mathfrak{B}$ as the "bounded" subsets of D. We require compactness conditions only on the "bounded" sets:

Definition 2.1. We call a map $\Phi: D \to 2^Y$ $(\mathfrak{B}, K)$-*compact*, if $\overline{\mathrm{conv}}(\Phi(B))$ is compact and contained in K for any $B \in \mathfrak{B}$. Similarly, we call a homotopy $H: [0,1] \times D \to Y$ $(\mathfrak{B}, K)$-*compact*, if $\overline{\mathrm{conv}}(H([0,1] \times B))$ is compact and contained in K for any $B \in \mathfrak{B}$.

Example 2.1. Let Y be a Fréchet space, and $K \subseteq Y$ be closed and convex. If D is a subset of some topological space X and $\mathfrak{B}$ denotes the system of all bounded subsets of D, then a map $\Phi: \overline{\Omega} \to 2^Y$ is $(\mathfrak{B}, K)$-compact if and only if it maps bounded sets into precompact sets and if its range is contained in K.

If Y is for example an uncomplete normed space, then the requirement that $\overline{\mathrm{conv}}(\Phi(B))$ be compact is in general more restrictive than the apparently more natural requirement that $\overline{\Phi(B)}$ be compact. However, the former is more convenient and useful in many proofs (it is also needed in the results from [14]).

We assume throughout that we have given two sets of functions $\mathcal{G}$ and $\mathcal{G}_-$ such that $(\mathcal{G}, \mathcal{G}_-)$ represents a growth condition in the following sense:

Definition 2.2. Let $\mathcal{G}$ be a set of multi-valued functions $H: [0,1] \times D \to 2^Y$. If $\Phi: D \to 2^Y$ has the form $\Phi = H(0, \cdot\,)$ with some $H \in \mathcal{G}$, we also write $\Phi \in \mathcal{G}$. Similarly for single-valued functions. Let $\mathcal{G}_-$ be a set of (single-valued) functions $F: D \to Y$. We say that $(\mathcal{G}, \mathcal{G}_-)$ defines a $(\mathfrak{B}, K)$-*growth condition* if the following properties are satisfied:

(1) If $H \in \mathcal{G}$ and $\lambda: X \to [0,1]$ is continuous, then also $(t,x) \mapsto H(\lambda(t), x)$ belongs to $\mathcal{G}$.
(2) If $\varphi: D \to Y$ belongs to $\mathcal{G}$, then also $\lambda\varphi \in \mathcal{G}$ for any $\lambda \in \mathbb{R}$.
(3) If $\varphi: D \to Y$ belongs to $\mathcal{G}$, then $h(t,x) = t\varphi(x)$ $(0 \le t \le 1)$ belongs to $\mathcal{G}$.
(4) If $\Phi: D \to 2^Y$ and $\varphi: D \to Y$ belong to $\mathcal{G}$, then also $\Phi + \varphi \in \mathcal{G}$.
(5) If $\Phi, \Psi: D \to 2^Y$ belong to $\mathcal{G}$, and if $\Omega_0 \subseteq \Omega$ is such that the piecewise defined function

$$\chi(x) = \begin{cases} \Phi(x) & \text{if } x \in \Omega_0, \\ \Psi(x) & \text{if } x \notin \Omega_0, \end{cases}$$

is upper semicontinuous, then $\chi \in \mathcal{G}$.

(6) For each $(\mathfrak{B}, K)$-compact $H \in \mathcal{G}$ and each $F \in \mathcal{G}_-$ there is a neighbourhood $V \subseteq Y$ of 0 such that for each compact $C \subseteq V$ there is some $B \in \mathfrak{B}$ with $\{x \in D : F(x) \in H([0,1] \times \{x\}) + C\} \subseteq B$.

The first properties mean, roughly speaking, that $\mathcal{G}$ indeed consists of all functions which satisfy a certain type of growth condition (the first condition means that the growth condition for $H(t, \cdot\,)$ is uniform with respect to t). However, the last requirement in Definition 2.2 is the most essential one. This condition implies in particular that we have an a priori estimate for the solutions of the inclusion $F(x) \in \Phi(x)$ for $F \in \mathcal{G}$ and $\Phi \in \mathcal{G}_-$. In general, the larger the class $\mathcal{G}$, the smaller must be $\mathcal{G}_-$ to satisfy this requirement. Roughly speaking: The less restrictive the "growth condition" on Φ, the "faster" must F grow.

We note that not every property of Definition 2.2 is needed for every result. In particular, the somewhat technical property (5) is only needed for Corollary 3.2 (restriction property for f-multiepi maps). The reader who has no usage for the restriction property thus may eliminate the requirement (5) from the above definition.

We are mainly interested in the following three special cases which correspond to the earlier mentioned definition of 0-epi maps on bounded sets, on unbounded sets, and of stably-solvable maps, respectively:

Example 2.2. If $D \in \mathfrak{B}$, then we may let $\mathcal{G}$ and $\mathcal{G}_-$ consist of all (single- or multi-valued) functions from D (resp. $[0,1] \times D$) into Y. In other words, if Ω is bounded in the situation of Example 2.1 (or if we are only interested in functions Φ with compact $\overline{\text{conv}}\Phi(D)$), there is no need to consider any growth condition.

Example 2.3. Consider the situation of Example 2.1. Let $\mathcal{G}$ denote the system of all functions with bounded range, and $\mathcal{G}_-$ the system of all functions with the property that preimages of bounded sets are bounded. Then $(\mathcal{G}, \mathcal{G}_-)$ defines a $(\mathfrak{B}, K)$-growth condition (and one may even choose $V = Y$). Indeed, since compact sets are bounded, it follows that for any $\Phi \in \mathcal{G}$ the set $\Phi(D) + C$ is bounded, and so even $F^{-1}(\Phi(D) + C) \in \mathfrak{B}$.

Example 2.4. Let X and Y be normed spaces, $D \subseteq X$, and let $\mathfrak{B}$ denote the system of all bounded subsets of D. Assume that we have given monotone increasing maps $p, q\colon [0,\infty) \to [0,\infty)$. We assume that p satisfies a Δ_2-*condition*, i.e. there are constants $t_0, c \in [0,\infty)$ with

$$p(2t) \leq c \cdot p(t) \quad (t \geq t_0).$$

A typical example is $p(t) = t^\alpha$ $(0 < \alpha < \infty)$. Moreover, we assume that $q(t) \to \infty$ as $t \to \infty$. Given a map $H\colon [0,1] \times D \to 2^Y$, we put

$$[H]^{p,q} := \limsup_{\substack{\|x\|\to\infty \\ x\in D}} \sup \left\{ \frac{p(\|y\|)}{q(\|x\|)} : y \in H([0,1]\times\{x\}) \right\}$$

and

$$[H]_{p,q} := \liminf_{\substack{\|x\|\to\infty \\ x\in D}} \inf \left\{ \frac{p(\|y\|)}{q(\|x\|)} : y \in H([0,1]\times\{x\}) \right\}$$

where we put, to avoid special cases, $\sup\emptyset := 0$ and $\inf\emptyset := \infty$. We also put $[H]^{p,q} := 0$ and $[H]_{p,q} := \infty$ if D is bounded. If $\Phi\colon D \to 2^Y$, we put $[\Phi]^{p,q} := [H]^{p,q}$ and $[\Phi]_{p,q} := [H]_{p,q}$ where $H(t,\,\cdot\,) := \Phi$; analogously for single-valued maps. Then the pair $(\mathcal{G},\mathcal{G}_-)$ satisfies a $(\mathfrak{B},K)$-growth condition when $\mathcal{G}$ denotes the class of all functions Φ with $[\Phi]^{p,q} = 0$ and $\mathcal{G}_-$ denotes the class of all functions F with $[F]_{p,q} > 0$.

Indeed, observe first that if a function Φ is "dominated" by a function $\Psi \in \mathcal{G}$ (in the sense that for each $x \in D$ and each $y \in \Phi(x)$ there is some $z \in \Psi(x)$ with $\|y\| \le \|z\|$), then $\Phi \in \mathcal{G}$. Moreover, if $\Phi, \Psi \in \mathcal{G}$, then also the "union function" $(\Phi\cup\Psi)(x) := \Phi(x)\cup\Psi(x)$ belongs to $\mathcal{G}$. This already implies the property (5) of Definition 2.2. The Δ_2-condition implies that for any $\Phi \in \mathcal{G}$ we have $2\Phi \in \mathcal{G}$ and thus even $2^n\Phi \in \mathcal{G}$ for any n which now implies $\lambda\Phi \in \mathcal{G}$ for any $\lambda \in \mathbb{R}$. Moreover, if also $\varphi \in \mathcal{G}$, then $2(\Phi\cup\varphi)$ dominates $\Phi+\varphi$ and belongs to $\mathcal{G}$; hence $\Phi+\varphi \in \mathcal{G}$.

Finally, if $F \in \mathcal{G}_-$, $H \in \mathcal{G}$, then for any bounded $C \subseteq Y$ the set $S = \{x \in D : F(x) \in H([0,1]\times\{x\}) + C\}$ is bounded. Indeed, suppose that C is bounded by $m < \infty$, without loss of generality $m \ge t_0$. We find $\varepsilon > 0$ and $r > 0$ such that for any $x \in D$ with $\|x\| > r$ the estimates

$$p(\|F(x)\|) > \varepsilon q(\|x\|) \ge c\cdot p(m)$$

and

$$p(\|y\|) \le \varepsilon q(\|x\|)/c \quad (y \in H([0,1]\times\{x\}))$$

hold. For each such x, each $y \in H([0,1]\times\{x\})$, and each $z \in C$, we thus find $p(\|y+z\|) \le p(2\max\{\|y\|, m\}) \le \max\{cp(\|y\|), cp(m)\} < p(\|F(x)\|)$ which implies $x \notin S$. Hence, S is bounded by r.

The reader familiar with the paper [4] might consider Example 2.4 as the deeper reason why the conditions $[\varphi]^{\mathrm{id},\mathrm{id}} = 0$ and $[F]_{\mathrm{id},\mathrm{id}} > 0$ play such a crucial role for stably-solvable maps. There is usually no difficulty to replace them by the more general conditions $[\varphi]^{p,q} = 0$ and $[F]_{p,q} > 0$. (However, be aware that for some applications in [4] it is required that $F = \mathrm{id}$ satisfies $[F]_{p,q} > 0$ which excludes many choices for p and q).

Now we come to our main definitions:

Definition 2.3. We call a family $\mathcal{F}$ of maps $\Phi\colon D \to 2^Y$ *appropriate* (with respect to $(\Omega, P, K, Y, \mathcal{G})$) if the following holds:

(1) Any $\Phi \in \mathcal{F}$ is upper semicontinuous, and all values $\Phi(x)$ are closed.
(2) For any $\Phi \in \mathcal{F}$ and any $(\mathfrak{B}, K)$-compact continuous map $\varphi\colon D \to Y$ which satisfies $\varphi \in \mathcal{G}$ and $\varphi|_P = 0$, the sum $\Phi + \varphi$ belongs to $\mathcal{F}$.

In the following results we always (tacitly) assume that $\mathcal{F}$ is appropriate. However, although we will prove our results only under this assumption, it will be sometimes convenient to use the following essential notion even if $\mathcal{F}$ fails to be appropriate.

Definition 2.4. We call a map $F\colon D \to Y$

(1) *$\mathcal{F}$-epi* (on Ω), if for any $\Phi \in \mathcal{F}$ the inclusion $F(x) \in \Phi(x)$ has a solution $x \in \Omega$.
(2) *$\mathcal{F}$-admissible* (on P), if for any $\Phi \in \mathcal{F}$ the inclusion $F(x) \in \Phi(x)$ has no solution on P.

The crucial property of $\mathcal{F}$-epi maps is that this class is stable under homotopic perturbations (if $\mathcal{F}$ is appropriate):

Poposition 2.1 (Homotopy invariance for $\mathcal{F}$-epi maps). *Let $F\colon D \to Y$ be continuous, and $H\colon [0,1] \times D \to Y$ be continuous, $(\mathfrak{B}, K)$-compact, and belong to $\mathcal{G}$. Assume that $H(0, \cdot\,) = 0$ and that each of the functions $F_\lambda(x) = F(x) - H(\lambda, x)$ is $\mathcal{F}$-admissible. If F_0 is $\mathcal{F}$-epi, then all F_λ are $\mathcal{F}$-epi.*

Proof. We have to prove that F_{λ_1} is $\mathcal{F}$-epi for any $\lambda_1 \in [0,1]$. Replacing H by $\tilde{H}(\lambda, x) = H(\lambda\lambda_1, x)$ if necessary, it suffices to consider the case $\lambda_1 = 1$. Let $\Phi \in \mathcal{F}$ be given.

The set $S = \{x \in D : F_\lambda(x) \in \Phi(x) \text{ for some } \lambda \in [0,1]\}$ is closed. Indeed, given $x_0 \notin S$, let Λ denote the system of all open sets $L \subseteq [0,1]$ with the property that $F_\lambda(x) \notin \Phi(x)$ for all x in some neighbourhood U of x_0 for any $\lambda \in L$. Then Λ is an open covering of $[0,1]$: For any $\lambda_0 \in [0,1]$ the closed set $\Phi(x_0)$ is disjoint from the compact set $\{F_{\lambda_0}(x_0)\}$, since $x_0 \notin S$. By [13, Theorem 1.10] there are disjoint open sets $V_1, V_2 \subseteq Y$ which contain $\Phi(x_0)$ and $F_{\lambda_0}(x_0)$, respectively. By the (upper semi-)continuity, we find neighbourhoods $U \subseteq D$ of x_0 and $L \subseteq [0,1]$ of λ_0 with $\Phi(x) \subseteq V_1$ and $F_\lambda(x) \in V_2$ for $(x, \lambda) \in U \times L$, in particular $F_\lambda(x) \notin \Phi(x)$. The compact set $[0,1]$ is covered by finitely many sets $L_1, \ldots, L_k \in \Lambda$; let $U_1, \ldots, U_k$ denote corresponding neighbourhoods of x_0 with $F_\lambda(x) \notin \Phi(x)$ for $(x, \lambda) \in U_i \times L_i$. The intersection $U_0 = U_1 \cap \cdots \cap U_k$ is a neighbourhood of x_0 which thus satisfies $U_0 \cap S = \emptyset$. Hence, $x_0 \notin \overline{S}$, and so S is closed.

Since each of the functions F_λ is $\mathcal{F}$-admissible, it follows that $S \cap P = \emptyset$. By Urysohn's lemma, we thus find a continuous function $\lambda\colon D \to [0,1]$ with $\lambda|_P = 0$

and $\lambda|_S = 1$. Put $\Psi(x) = \Phi(x) + H(\lambda(x), x)$. We have $\Psi \in \mathcal{F}$, because $\mathcal{F}$ is admissible. Since F is $\mathcal{F}$-epi, the inclusion $F(x) \in \Psi(x)$ has a solution $x \in \Omega$. This means $F_{\lambda(x)}(x) \in \Phi(x)$, and so $x \in S$ which in turn implies $\lambda(x) = 1$, i.e. $F_1(x) \in \Phi(x)$. Hence, F_1 is $\mathcal{F}$-epi. □

The following observation is not an immediate consequence of Mazur's lemma, since we do not assume that Y is a Fréchet space:

Lemma 2.1. *If $A, B \subseteq Y$ are convex and compact, then $\overline{\mathrm{conv}}(A \cup B) = \mathrm{conv}(A \cup B)$ and $\overline{\mathrm{conv}}(A + B) = A + B$ are compact.*

Proof. Consider the map $f: A \times B \times [0,1] \to Y$, defined by $f(a,b,\lambda) = \lambda a + (1-\lambda)b$. Since A and B are convex, the range of the continuous function f contains (and thus is equal to) $\mathrm{conv}(A \cup B)$. We point out that the compactness of a finite product $A \times B \times [0,1]$ of compact spaces can be proved without appealing to the axiom of choice [15] (although the proof is much more cumbersome with this restriction than other proofs of Tychonoff's theorem). The compactness of $A + B$ follows analogously by the continuity of $g: A \times B \to Y$, $g(a,b) = a + b$, and a straightforward calculation shows that $A + B$ is convex if A and B are convex. □

Proposition 2.2 (Rouché stability for $\mathcal{F}$-epi maps). *Let $K \subseteq Y$ be convex with $0 \in K$, and $F, G: D \to Y$ be continuous such that $F - G$ belongs to $\mathcal{G}$ and is $(\mathfrak{B}, K)$-compact. If the boundary condition*

$$F(x) + \lambda(G(x) - F(x)) \notin \Phi(x) \quad (\Phi \in \mathcal{F},\ x \in P,\ 0 \leq \lambda \leq 1)$$

holds and F is $\mathcal{F}$-epi, then G is $\mathcal{F}$-epi.

Proof. Lemma 2.1 implies that $\overline{\mathrm{conv}}((F-G)(B) \cup \{0\})$ is a compact subset of K for any $B \in \mathfrak{B}$, and so $H(\lambda, x) = \lambda(F(x) - G(x))$ is $(\mathfrak{B}, K)$-compact. The statement now follows from Proposition 2.1. □

Corollary 2.1 (Boundary dependence). *Let $K \subseteq Y$ be convex with $0 \in K$, and $F, G: D \to Y$ be continuous with $F|_P = G|_P$. Assume that $F - G$ belongs to $\mathcal{G}$ and is $(\mathfrak{B}, K)$-compact. If F is $\mathcal{F}$-admissible and $\mathcal{F}$-epi then G is $\mathcal{F}$-admissible and $\mathcal{F}$-epi.*

3. $\mathcal{F}_f^\circ$-epi maps

Definition 2.4 is somewhat too general for our purpose: We are mainly interested in a particular type of systems $\mathcal{F}$.

Definition 3.1. Given some appropriate family $\mathcal{F}$, we denote by $\mathcal{F}^\circ$ the subset of all $\Phi \in \mathcal{F}$ which belong to $\mathcal{G}$ and are $(\mathfrak{B}, K)$-compact. Similarly, we denote by $\mathcal{F}_f$ the subset of all $\Phi \in \mathcal{F}$ which satisfy $\Phi|_P = f$ (put $\mathcal{F}_f = \mathcal{F}$ if $P = \emptyset$). We also write $\mathcal{F}_f^\circ := (\mathcal{F}^\circ)_f = (\mathcal{F}_f)^\circ$.

Note that $\mathcal{F}^\circ$ implicitly depends on $\mathcal{G}$, $\mathfrak{B}$, and K.

Proposition 3.1. *If $K + K \subseteq K$, then $\mathcal{F}^\circ$, $\mathcal{F}_f$, and $\mathcal{F}_f^\circ$ are appropriate.*

Proof. Let $\Phi \in \mathcal{F}^\circ$ and some $(\mathfrak{B}, K)$-compact continuous map $\varphi\colon D \to Y$ with $\varphi \in \mathcal{G}$ and $\varphi|_P = 0$ be given. Then $\Phi + \varphi \in \mathcal{G}$. Moreover, for any $B \in \mathfrak{B}$, the sets $A_1 = \overline{\operatorname{conv}}(\Phi(B))$ and $A_2 = \overline{\operatorname{conv}}(\psi(B))$ are compact subsets of K, and so in view of Lemma 2.1, also $\overline{\operatorname{conv}}((\Phi + \psi)(B)) \subseteq \overline{\operatorname{conv}}(A_1 + A_2) = A_1 + A_2$ is a compact subset of K (here, we use that $K + K \subseteq K$). Hence, $\Phi + \psi$ is $(\mathfrak{B}, K)$-compact, and so $\Phi + \psi \in \mathcal{F}^\circ$. This proves that $\mathcal{F}^\circ$ is appropriate. Since trivially $\mathcal{F}_f$ is appropriate for appropriate $\mathcal{F}$, the statement follows. $\square$

If $\mathcal{F}_f \neq \emptyset$, then $F\colon D \to Y$ is $\mathcal{F}_f$-admissible if and only if $F(x) \notin f(x)$ for all $x \in \partial\Omega$. To denote the latter, we simply say that F is *f-admissible.* From the homotopy invariance of $\mathcal{F}$-epi maps, we get immediately:

Corollary 3.1 (Homotopy invariance for $\mathcal{F}_f$-epi maps). *Let $K + K \subseteq K$, and $F\colon D \to Y$ be continuous. Let the $(\mathfrak{B}, K)$-compact homotopy $H\colon [0,1] \times D \to K$ belong to $\mathcal{G}$, satisfy $H(0, \cdot\,) = 0$, and be such that each of the functions $F_\lambda(x) = F(x) - H(\lambda, x)$ is f-admissible. If F_0 is $\mathcal{F}_f$-epi then each of the functions F_λ is $\mathcal{F}_f$-epi.*

Of course, also an anologue of the Rouché theorem holds. We leave the formulation to the reader, but mention instead a more powerful result under an additional assumption for F:

Recall that a map is called *proper* if preimages of compact sets are compact. For maps in topological vector spaces, a more natural and less restrictive requirement is that the map be proper on closed bounded subsets. This is the case, for example, if $F = \mathrm{id} - C$ with a continuous compact (linear or nonlinear) map C. The analogue to that definition in our abstract situation is:

Definition 3.2. We call a map $F\colon D \to Y$ $(\mathfrak{B}, K)$-*proper*, if for any compact $C \subseteq K$ and any $B \in \mathfrak{B}$ the set $\overline{F^{-1}(C) \cap B}$ is compact.

Proposition 3.2 (Uniform Rouché stability for proper maps). *Let $K \subseteq Y$ be convex with $0 \in K + K \subseteq K$. Let $F\colon D \to Y$ be continuous, $(\mathfrak{B}, K)$-proper, and belong to $\mathcal{G}_-$. If F is f-admissible, then there is a balanced neighbourhood $V \subseteq Y$ of 0 such that $(F(x) + V) \cap f(x) = \emptyset$ for any $x \in P$. Moreover, for any such V the following holds: If F is $\mathcal{F}_f^\circ$-epi, and $G\colon D \to Y$ is continuous with $G(x) \in F(x) + V$ for $x \in P$ and such that $F - G$ belongs to $\mathcal{G}$ and is $(\mathfrak{B}, K)$-compact, then G is $\mathcal{F}_f^\circ$-epi.*

Proof. Assume by contradiction that for any balanced neighbourhood $V \subseteq V_0$ of 0, we find some $x_V \in P$ and some $y_V \in V$ with $F(x_V) - y_V \in f(x_V)$. If we partially order the index set of all V by inclusion, we may assume by the axiom of choice that $(x_V)_V$ and $(y_V)_V$ are nets.

Recall that the system of all balanced neighbourhoods of 0 forms a base, see e.g. [13, Theorem 1.14]. In particular, $y_V \to 0$, and so the set $C = \{y_V : V\}$ is compact. Choose some $\Phi \in \mathcal{F}_f$. Passing to a subnet if necessary, it is no loss of generality to assume that C is contained in the neighbourhood of Definition 2.2, and so the set $A = \{x_V : V\}$ is contained in some $B \in \mathfrak{B}$. Since Φ is $(\mathfrak{B}, K)$-compact, it follows that $f(A) \subseteq \overline{\operatorname{conv}}(\Phi(B))$ is compact, and so also $F(A)$ is contained in the compact set $f(A) + C$. Since $A \subseteq B$ and F is $(\mathfrak{B}, K)$-proper, it follows that $\overline{A}$ is compact, i.e. $x_V \in P$ contains a subnet which converges to some $x \in P$. By the continuity, and since $f(x)$ is closed, we find $F(x) \in f(x)$ which contradicts the assumption that F be f-admissible. The second statement now follows from Proposition 2.2. Indeed, since V is balanced, we have for any $\lambda \in [0,1]$ and any $x \in P$ that $F(x) + \lambda(G(x) - F(x)) \in F(x) + V$ which is disjoint from $f(x)$. □

If Y is not metrizable, our proof of Proposition 3.2 requires the axiom of choice. We do not know whether it is possible to prove the proposition without that axiom in this case (for metrizable Y one just has to pass to a countable base of balanced neighbourhoods $V \subseteq Y$ of 0 in the proof).

If $\mathcal{F} = C(D, K)$ is the system of all single-valued continuous maps, then we call the $\mathcal{F}_f^\circ$-epi maps simply *f-epi* (note that this definition implicitly depends on Ω, B, K, Y, $\mathfrak{B}$, and $\mathcal{G}$). If additionally $f \equiv 0$, we call these maps *0-epi*. If K is a subspace of Y, it suffices to study 0-epi maps:

Proposition 3.3. *Let $0 \in K + K \subseteq K$ and $-K \subseteq K$, and $f\colon B \to Y$ be the restriction of a $(\mathfrak{B}, K)$-compact continuous map $f_0 \in \mathcal{G}$. Then a map $F\colon D \to K$ is f-epi if and only if $F - f_0$ is 0-epi.*

Proof. For $\mathcal{F} = C(D, K)$, we have $\Phi \in \mathcal{F}_f^\circ$ if and only if $\Phi + f_0 \in \mathcal{F}_0^\circ$. Indeed, if $\Phi \in \mathcal{F}^\circ$ then a similar argument as in Proposition 3.1 shows that $\Phi + f_0 \in \mathcal{F}^\circ$. Since $-K \subseteq K$, also $-f_0$ is $(\mathfrak{B}, K)$-compact, and so also conversely $\Psi = \Phi + f_0 \in \mathcal{F}^\circ$ implies $\Phi = \Psi + (-f_0) \in \mathcal{F}^\circ$. □

In the situation of Example 2.3 (and if we are in the canonical situation in normed spaces), we get the earlier mentioned concept of 0-epi maps from [5]. Moreover, if we define $\mathcal{G}$ as in Example 2.4, our 0-epi maps are precisely the stably-solvable maps from [4]. In both situations, the homotopy invariance (Corollary 3.1) has been proved separately in the corresponding papers.

If K is a cone, and $\mathcal{F} = \mathcal{K} = \mathcal{K}(D, K)$ is the system of all upper semi-continuous maps with nonempty, closed and convex values in K, then we call $\mathcal{F}_f^\circ$-epi maps *f-multiepi*. If $\mathfrak{B}$ is as in Example 2.2 (and if M is metric), this definition was introduced in [14]. For this particular situation, the homotopy invariance (Corollary 3.1) has already been obtained in [14] for the class of f-epi resp. f-multiepi maps separately.

As some substitute for the additivity of the degree, we have for $\mathcal{F}_f$-epi maps a restriction property which states roughly speaking that if a map F is $\mathcal{F}_f$-epi on some Ω, then it has this property also on subsets $\Omega_0 \subseteq \Omega$ if it is admissible on the difference $\Omega \setminus \Omega_0$. Sadly, in our general situation, the precise formulation of this idea is rather technical:

Definition 3.3. If $\Omega_0 \subseteq \Omega$ and $f_0: P \cup (\Omega \setminus \Omega_0) \to 2^Y$, we denote by $\mathcal{F}_{f_0}$ the set of all functions $\Phi \in \mathcal{F}$ with $\Phi|_{P \cup (\Omega \setminus \Omega_0)} = f_0$. Moreover, if $D_0 \subseteq D$, we denote by $\mathcal{F}|_{D_0}$ the set of all restrictions $\Phi|_{D_0}$ with $\Phi \in \mathcal{F}$.

In particular, if $f_0|_P = f$, then $\mathcal{F}_{f_0} \subseteq \mathcal{F}_f$.

The definition does not conflict with our previous notation, for if $P \cup (\Omega \setminus \Omega_0) = P$, then $f_0 = f$.

Proposition 3.4 (Restriction property). *Let $F: D \to Y$ be $\mathcal{F}_f$-epi (on Ω). Let $\Omega_0 \subseteq \Omega$, $f_0: P \cup (\Omega \setminus \Omega_0) \to 2^Y$, and suppose that $F(x) \notin f_0(x)$ for $x \in \Omega \setminus \Omega_0$. Then F is $\mathcal{F}_{f_0}$-epi on Ω_0. In particular, $F|_{D_0}$ is $\mathcal{F}_{f_0}|_{D_0}$-epi on Ω_0 when $\Omega_0 \subseteq D_0 \subseteq D$.*

Proof. Given some $\Psi \in \mathcal{F}_{f_0}$, we find by definition some $\Phi \in \mathcal{F}_f$ with $\Phi|_{\Omega \setminus \Omega_0} \equiv f_0$ and $\Psi = \Phi|_{D_0}$. Since F is $\mathcal{F}_f$-epi, the inclusion $F(x) \in \Phi(x)$ has a solution $x \in \Omega$. We must have $x \in \Omega_0$, since otherwise $F(x) \notin f_0(x) = \Phi(x)$, by assumption. Hence, $F(x) \in \Phi(x) = \Psi(x)$. $\square$

In the following corollary, we understand that the growth condition $\mathcal{G}_0$ on the "smaller" set D_0 is defined by restriction of the corresponding growth condition $\mathcal{G}$ on D (i.e. $\mathcal{G}_0 = \{\Phi|_{D_0} : \Phi \in \mathcal{G}\}$).

Corollary 3.2 (Restriction property for f-multiepi maps). *Let $\Omega_0 \subseteq \Omega$, and $\overline{\Omega}_0 \subseteq D_0 \subseteq D$. Suppose $P_0 \subseteq D_0$ is closed in D_0, and $\partial\Omega_0 \cup (P \cap \Omega) \subseteq P_0$ (the boundary is understood with respect to D). Let f_0 be the restriction of some function $\Phi_0 \in \mathcal{K}(D,K)^\circ_f$ to P_0. If $F: D \to Y$ is f-multiepi on Ω and $F(x) \notin \Phi_0(x)$ for all $x \in \Omega \setminus \Omega_0$, then $F|_{D_0}$ is f_0-multiepi on Ω_0.*

Proof. Put $\mathcal{F} = \mathcal{K}(D,K)^\circ$. Any map $\Psi \in \mathcal{K}(D_0,K)^\circ_{f_0}$ belongs to $\mathcal{F}_{\Phi_0}|_{D_0}$. Indeed, we have $\Psi = \Phi|_{D_0}$ for

$$\Phi(x) = \begin{cases} \Psi(x) & \text{if } x \in \Omega_0, \\ \Phi_0(x) & \text{if } x \in D \setminus \Omega_0. \end{cases}$$

Since we have on $\partial\Omega_0$ that $\Psi(x) = f_0(x) = \Phi_0(x)$, it follows that Φ is upper semicontinuous, and so $\Phi \in \mathcal{K}(D,K)$. Since $\Psi \in \mathcal{G}_0$ and $\Psi_0 \in \mathcal{G}$, we have $\Phi \in \mathcal{G}$.

Moreover, for any $B \in \mathfrak{B}$ the sets $A_1 = \overline{\mathrm{conv}}(\Psi(B))$ and $A_2 = \overline{\mathrm{conv}}(f_0(B))$ are compact, and so also $\overline{\mathrm{conv}}(\Phi(B)) \subseteq \overline{\mathrm{conv}}(A_1 \cup A_2)$ is compact by Lemma 2.1.$\square$

Replacing $\mathcal{K}$ by C in the previous proof, we find:

Corollary 3.3 (Restriction property for f-epi maps). *Let $\Omega_0 \subseteq \Omega$, and $\overline{\Omega}_0 \subseteq D_0 \subseteq D$. Suppose $P_0 \subseteq D_0$ is closed in D_0, and $\partial\Omega_0 \cup (P \cap \Omega) \subseteq P_0$ (the boundary is understood with respect to D). Let f_0 be the restriction of some function $\varphi_0 \in C(D,K)^\circ_f$ to P_0. If $F: D \to Y$ is f-epi on Ω and $F(x) \neq \varphi_0(x)$ for all $x \in \Omega \setminus \Omega_0$, then $F|_{D_0}$ is f_0-epi on Ω_0.*

In [5] it has been proved that in the situation of Examples 2.2 and 2.3 the identity is 0-epi (with respect to $P = \partial\Omega$) if (and only if) $0 \in \Omega$. This result may be considered as an analogue to the normalization property of the degree. However, since the main advantage of 0-epi maps over the degree occurs if $X \neq Y$, we intend to prove a normalization result also in this situation, i.e. when we replace the identity by some "almost homeomorphism". In the proof we will exhibit that the normalization property follows from the restriction property and the Tychonoff fixed point theorem. In the multivalued case, one needs of course the Ky Fan fixed point theorem:

Proposition 3.5 (Normalization property for f-multiepi maps on bounded sets). *Let $D \in \mathfrak{B}$, and let $\mathcal{G}_-$ and $\mathcal{G}$ contain all maps* (*recall Example* 2.2). *Assume that Y is locally convex and D is metrizable. Let $K \subseteq Y$ be a cone, and $F: D_0 \to K$ be the restriction of a homeomorphism $G: D_1 \to K$ where $D_1 \subseteq D$. Let $f: P \to 2^Y$ where $P \subseteq D_0$ is closed in D_0. Let $\Omega \subseteq D_0$ be such that the boundary of Ω with respect to D_1 is contained in P. If $G^{-1}(\overline{\mathrm{conv}}(f(P))) \subseteq \Omega$ then F is f-multiepi.*

Proof. Without loss of generality, we assume $D = D_1$ and that $C = \overline{\mathrm{conv}}(f(P))$ is a compact subset of K.

By Ma's generalization of Dugundji's extension theorem [12, Theorem 2.1] we may extend f to a map $\Phi_0 \in \mathcal{K}(D,C)$; in case $C = \emptyset$, put $\Phi_0(x) \equiv \{c\}$ where $c \in G(\Omega)$. Put $\Omega_1 = D$ and $P_1 = \emptyset$, and let $f_1 = \Phi_0|_{P_1}$ be the "empty" function. We claim that G is f_1-multiepi on Ω_1 (with the growth condition from Example 2.2).

Indeed, if $\Phi \in \mathcal{K}(D,K)^\circ_{f_1} = \mathcal{K}(D,K)^\circ$ is given, consider the map $G_0 = \Phi \circ G^{-1}$. Since $\overline{\mathrm{conv}}(G_0(C)) \subseteq C$ is compact, the Ky Fan fixed point theorem implies that G_0 has a fixed point $y \in C$. For $x = G^{-1}(y) \in \Omega_1$, we have $\Phi(x) = G_0(y) \ni y = G(x)$, and so G is f_1-multiepi on Ω_1, as claimed.

Since G is one-to-one and $G^{-1}(C) \subseteq \Omega$, we have $G(x) \notin C$ for $x \in D \setminus \Omega$, and so $G(x) \notin \Phi_0(x)$ for $x \in \Omega_1 \setminus \Omega$ (the latter holds also if $C = \emptyset$, since then

$G(x) \neq c \in G(\Omega)$ for $x \in \Omega_1 \setminus \Omega$). Corollary 3.2 thus implies that $F = G|_{D_0}$ is f-multiepi on Ω. □

If Y is metrizable and D is separable, one can also apply the extension result from [15] to find the extension Φ_0 of f in the previous proof. Without this assumption, our previous proof required the axiom of choice for this step. However, usually the existence of such an extension is trivial (in particular, if $f \equiv 0$).

If we are only interested in the single-valued case and if Y is metrizable, we do not need the axiom of choice to find the extension. Moreover, in this case we may even drop the requirement that D be metrizable:

Proposition 3.6 (Normalization property for f-epi maps on bounded sets). *Let $D \in \mathfrak{B}$, and let $\mathcal{G}_-$ and $\mathcal{G}$ contain all maps. Assume that Y is locally convex and metrizable. Let $K \subseteq Y$ be a cone, and $F: D_0 \to K$ be the restriction of a homeomorphism $G: D_1 \to K$ where $D_1 \subseteq D$. Let $f: P \to Y$ where $P \subseteq D_0$ is closed in D_0. Let $\Omega \subseteq D_0$ be such that the boundary of Ω with respect to D_1 is contained in P. If $G^{-1}(\overline{\operatorname{conv}}(f(P))) \subseteq \Omega$, then F is f-epi.*

Proof. The proof proceeds analogous to Proposition 3.5 (using Tychonoff's fixed point theorem and Corollary 3.3) if we can prove that f has an extension to a function from $C(D, C)$ (if $C \neq \emptyset$). But C is a compact metric ANR and thus isomorphic to a neighbourhood retract of the Hilbert cube H. Identifying C with its image under the corresponding isomorphism, we may extend f to a continuous map $f: D \to H$ by the Tietze–Urysohn extension theorem. Since C is even an absolute metric retract, there is a retraction ρ of H onto C; the composition $\rho \circ f$ is the desired extension of f. □

On unbounded sets, we need an additional requirement for the homeomorphism G.

Proposition 3.7 (Normalization property for f-(multi)epi maps on vector spaces). *Consider the situation of Example* 2.3. *Assume that Y is locally convex and D is metrizable. Let $K \subseteq Y$ be a cone, and $F: D_0 \to K$ be the restriction of a homeomorphism $G: D_1 \to K$ where $D_1 \subseteq D$ such that G^{-1} maps bounded sets into bounded sets* (*i.e.* $G \in \mathcal{G}_-$). *Let $f: P \to 2^Y$ where $P \subseteq D_0$ is closed in D_0. Let $\Omega \subseteq D_0$ be such that the boundary of Ω with respect to D_1 is contained in P. If $G^{-1}(\overline{\operatorname{conv}}(f(P))) \subseteq \Omega$ then F is f-multiepi* (*and thus f-epi if f is single-valued*).

Proof. Put $C = \overline{\operatorname{conv}}(f(P))$. By Ma's extension theorem, we may extend f to a function $\Phi_0 \in \mathcal{K}(D_1, C)$ (if $C \neq \emptyset$). As in the proof of Proposition 3.5, we put $\Omega_1 = D_1$, $P_1 = \emptyset$, $f_1 = \Phi_0|_{\partial\Omega_1}$, and prove that G is f_1-multiepi on Ω_1 (but now with the growth condition from Example 2.3): If $\Phi \in \mathcal{K}(D_1, K)^\circ_{f_1} = \mathcal{K}(D_1, K)^\circ$

is given, the set $R = \overline{\mathrm{conv}}(\Phi(D_1))$ is bounded, and thus also $B = G^{-1}(R)$ is bounded. Consequently, the set $K_0 = \overline{\mathrm{conv}}(\Phi(B))$ is a compact subset of K. The map $G_0 = \Phi \circ G^{-1}$ maps K_0 into itself and thus has a fixed point by the Ky Fan fixed point theorem. As in the proof of Proposition 3.5 this implies that $G(x) \in \Phi(x)$ for some $x \in G^{-1}(K_0) \subseteq B$.

Since G is one-to-one and $G^{-1}(C) \subseteq \Omega$, we find as in the proof of Proposition 3.5 that $G(x) \notin \Phi_0(x)$ for $x \in D_1 \setminus \Omega$, and may conclude the proof with an application of Corollary 3.3. □

Note that also Proposition 3.7 required the axiom of choice to find the extension Φ_0 for f (which nevertheless is trivial in many cases). The axiom of choice is not required if Y is metrizable, $C = \overline{\mathrm{conv}}(f(\partial\Omega))$ is complete, and D is separable.

We point out that the proof of the normalization property involved the fixed point theorems for the map $\Phi \circ G^{-1}$ which corresponds to the "homologic" approach to coincidence points, as mentioned in the beginning. This is not surprising, since the Tychonoff/Ky Fan fixed point theorem is of a "homologic nature".

If one applies deeper homology theory instead of these fixed point theorems, one may weaken the assumption that G be a homeomorphism. Recall that a continuous surjection $G\colon D_1 \to K$ is called a *Vietoris map*, if it is proper, and if for any $y \in K$ the fibre $G^{-1}(\{y\})$ is acyclic with respect to the Čech cohomlogy with rational coefficients. Of course, each homeomorphism is a Vietoris map.

Theorem 3.1 (Vietoris normalization for f-(multi)epi maps). *Assume that either $D \in \mathfrak{B}$, and $\mathcal{G}_-$ and $\mathcal{G}$ contain all maps, or that we are in the situation of Example* 2.3. *Assume that Y is locally convex, and D and Y are metrizable. Let $K \subseteq Y$ be a cone, and $F\colon D_0 \to K$ be the restriction of a (surjective) Vietoris map $G\colon D_1 \to K$ with $G \in \mathcal{G}$ where $D_1 \subseteq D$. Let $f\colon P \to 2^Y$ where $P \subseteq D_0$ is closed in D_0. Let $\Omega \subseteq D_0$ be such that the boundary of Ω with respect to D_1 is contained in P. Assume that the set $G(D_1 \setminus \Omega)$ is disjoint from $\overline{\mathrm{conv}}(f(P))$* (*and also $\neq K$ if $P = \emptyset$*)*. Then F is f-multiepi* (*and thus f-epi if f is single-valued*)*.*

Proof. The proof is similar to the previous proofs. The main difference is that one has to prove that if $K_0 \subseteq K$ is closed and convex and $\Phi \in \mathcal{K}(D_1, K_0)^\circ$, then the inclusion $G(x) \in \Phi(x)$ has a solution $x \in B_0 = G^{-1}(K_0)$. Since $G\colon B_0 \to K_0$ is a Vietoris map, this follows e.g. from the last corollary in [17]. □

As before, we applied the axiom of choice for the proof.

4. Relation between f-epi and f-multiepi maps

We study the question whether f-epi maps and f-multiepi maps actually are the same. This question of course only makes sense if f is single-valued.

Actually, we are not only interested in convex-valued maps Φ but in a larger class of maps.

Definition 4.1. Let $\Phi: D \to 2^Y$. Given some neighbourhood $\mathfrak{U} \subseteq D \times Y$ of the graph of Φ, we call a single-valued map $\varphi: D \to Y$ a $\mathfrak{U}$-*approximation*, if φ is continuous and its graph is contained in $\mathfrak{U}$. We call Φ

(1) *approximable* if for each neighbourhood $\mathfrak{U}$ of the graph of Φ there is a $\mathfrak{U}$-approximation.

(2) $\mathfrak{B}$-*approximable*, if for each $B \in \mathfrak{B}$ there is some $B_0 \in \mathfrak{B}$ with the following property: For any neighbourhood $\mathfrak{U} \subseteq D \times Y$ of the graph of Φ there is some $\mathfrak{U}$-approximation φ with

$$\varphi(B) \subseteq \overline{\mathrm{conv}}(\Phi(B_0))$$

and such that for each $B_1 \in \mathfrak{B}$ there is some $B_2 \in \mathfrak{B}$ with

$$\varphi(B_1) \subseteq \overline{\mathrm{conv}}(\Phi(B_2)).$$

(Note that B_0 may only depend on B, but B_2 may also depend on $\mathfrak{U}$ and φ).

We denote the system of all upper semicontinuous and $\mathfrak{B}$-approximable maps $\Phi: D \to 2^K$ with closed values by $\mathcal{A} = \mathcal{A}(D, K)$.

Proposition 4.1. *Each $\mathfrak{B}$-approximable map Φ is approximable. The converse holds if Y is locally convex and there are $B_1, B_2, \ldots \in \mathfrak{B}$ with the following properties:*

(1) $\bigcup \mathfrak{B} = D$, *and each $B \in \mathfrak{B}$ is contained in some B_n.*

(2) *The set $\overline{B}_n$ is contained in the interior (with respect to D) of B_{n+1}.*

(3) $C_n = \overline{\mathrm{conv}}(\Phi(B_n))$ *are neighbourhood retracts in Y.*

Proof. Without loss of generality, we may assume that $C_n \neq \emptyset$ for each n. Since C_n are neighbourhood retracts, we find open sets $V_n \subseteq Y$ with $V_n \supseteq C_n$ and retractions ρ_n of V_n onto C_n. Let I_n denote the interior of B_n. Then $\overline{I}_n \subseteq \overline{B}_n \subseteq I_{n+1}$, and for each $x \in D$ there is some n with $x \in B_n \subseteq I_{n+1}$.

Let a neighbourhood $\mathfrak{U}$ of the graph of Φ be given. Given some pair (x, y) of the graph of Φ, we find some smallest n with $x \in I_n$. Moreover, we find some open $P \subseteq I_n$ with $x \in P$ and some convex neighbourhood $V \subseteq Y$ of 0 such that $P \times (y + V) \subseteq \mathfrak{U}$ (here, we used that Y is locally convex). Since $y \in C_n \subseteq C_{n+1}$, we have $\rho_n(y) = \rho_{n+1}(y) = y$, and so there is some open $W \subseteq y + V$ with $y \in W \subseteq V_n \cap V_{n+1}$ and $\rho_n(W) \cup \rho_{n+1}(W) \subseteq y + V$. Let $\mathfrak{U}_0$ denote the system of all sets of the form $P \times W$ which can be obtained in this way (i.e. for any choice of x, y, V, P and W which satisfy the above requirements). Then $\mathfrak{U}_0$ is open and contains the graph of Φ by construction.

Since Φ is approximable, there is some $\mathfrak{U}_0$-approximation ψ. Since D is normal, we find by Urysohn's lemma continuous functions $\lambda_n: \overline{I}_{n+1} \to [0,1]$ with $\lambda_n|_{I_n} = 1$ and $\lambda_n|_{\partial I_{n+1}} = 0$. For $x \in I_1$, put $\varphi(x) = \rho_1(\psi(x))$, and for $x \in I_{n+1} \setminus I_n$ put

$$\varphi(x) = \lambda_n(x)\rho_n(\psi(x)) + (1 - \lambda_n(x))\rho_{n+1}(\psi(x)).$$

Then φ defines an $\mathfrak{U}$-approximation. Indeed, given $x \in D$, let n be the smallest number with $x \in I_n$. Since ψ is an $\mathfrak{U}_0$-approximation, we find by definition of $\mathfrak{U}_0$ some $y \in \Phi(x)$, some open $P \subseteq I_n$, and some convex neighbourhood $V \subseteq Y$ of 0 with $P \times (y + V) \subseteq \mathfrak{U}$ such that $w = \psi(x) \in V_n \cap V_{n+1}$ and $\rho_n(w), \rho_{n+1}(w) \in y + V$. In particular, $\varphi(x)$ is defined and belongs to $\operatorname{conv}(y + V) = y + V$, i.e. $(x, \varphi(x)) \in \mathfrak{U}$. Our choice of λ_n implies that φ is continuous on ∂I_n.

Finally, we have by induction that $\varphi(I_n) \subseteq C_n$: This is trivial for $n = 1$, and if this is true for n, it is also true for $n+1$, becaues $\varphi(I_{n+1} \setminus I_n) \subseteq \operatorname{conv}(C_n \cup C_{n+1}) = C_{n+1}$. For each $B \in \mathfrak{B}$ there is some smallest n with $B \subseteq B_n \subseteq I_{n+1}$, and so $\varphi(B) \subseteq C_{n+1} = \overline{\operatorname{conv}}(\Phi(B_{n+1}))$. Note that our choice of n only depends from B and $\mathfrak{B}$. □

In view of Proposition 4.1, we recall that if Y is a metrizable locally convex space, then any closed convex set is even a retract of Y by Dugundji's extension theorem ([3]). The latter requires the axiom of choice, in general, but there is a constructive proof for compact convex subsets [16] which is the only case of interest for us in the following. (In the extension result from [16] it is required that Y be a Fréchet space, but since compact sets are complete, one may just consider the completion of Y; see [17] for details). In particular, we have:

Corollary 4.1. *Let Y be a metrizable locally convex space, M be a subset of a normed space X, and $\mathfrak{B}$ be the system of all bounded subsets of D. If $\Phi: D \to 2^Y$ is approximable and $(\mathfrak{B}, K)$-compact, then Φ is $\mathfrak{B}$-approximable.*

Proof. Put $B_n = \{x \in D : \|x\| \leq n\}$ in Proposition 4.1. □

Maps with convex values are $\mathfrak{B}$-approximable:

Proposition 4.2 ($\mathcal{K}^\circ \subseteq \mathcal{A}^\circ$ for bounded Ω). *Let Y be a locally convex space, and $D \in \mathfrak{B}$ be metrizable. Then any upper semicontinuous map $\Phi: D \to 2^Y$ with nonempty, convex and compact values is $\mathfrak{B}$-approximable, in particular, $\mathcal{K}(D, K)^\circ \subseteq \mathcal{A}(D, K)^\circ$.*

Proof. Let a neighbourhood $\mathfrak{U}$ of the graph of Φ be given. Given $x \in D$, we find some $\varepsilon_x > 0$ and some open $V_x \supseteq \Phi(x)$ such that $K(\varepsilon_x, x) \times V_x \subseteq \mathfrak{U}$ where $K(\varepsilon_x, x)$ denotes the ball with center x and radius ε_x. Since the compact set $\Phi(x)$ is disjoint from the closed complement of V_x, we find by [13, Theorem 1.10] a neighbourhood $U_x \subseteq Y$ of 0 with $\Phi(x) + U_x \subseteq V_x$. Since Y is locally convex, we may assume that U_x is convex. Since $\Phi(x)$ is convex, it follows that $\Phi(x)+U_x$ is a convex neighbourhood of $\Phi(x)$.

By shrinking V_x and ε_x if necessary, we may thus assume that V_x is convex and, since Φ is upper semicontinuous, that $\Phi(K(\varepsilon_x,x)) \subseteq V_x$. By Stone's theorem, D is paracompact, and so we find a partition of unity $(\psi_x)_x$ which is subordinate to $(K(\varepsilon_x/3,x))_x$. Choose an arbitrary single-valued selection $\varphi_0\colon D\to Y$, i.e. $\varphi_0(x)\in\Phi(x)$, and put

$$\varphi(z)=\sum_{x\in D}\varphi_0(x)\psi_x(z)\quad (z\in D).$$

Since $(\psi_x)_x$ is a partition of unity and φ_0 attains its values in $\Phi(D)$, the function φ is continuous and attains its values in $\operatorname{conv}(\Phi(D))$. Given $z\in D$, consider the finite set $I=\{x:\varphi_x(z)\neq 0\}$, and let $x_0\in I$ be such that ε_{x_0} becomes maximal. Then we have for any $x\in I$ that $d(x,x_0)\le d(x,z)+d(z,x_0)\le 2\varepsilon_{x_0}/3$, and so $K(\varepsilon_x/3,x)\subseteq K(\varepsilon_{x_0},x_0)$ which implies $\psi_x(z)\in\Phi(\varepsilon_{x_0},x_0)\in V_{x_0}$. Since V_{x_0} is convex, we thus have $(z,\varphi(z))\in K(\varepsilon_{x_0},x_0)\times V_{x_0}\subseteq\mathfrak{U}$, as desired. $\square$

If Y is normed, Proposition 4.2 has been proved in [1]. Our proof follows [2, Theorem 24.2].

We point out that if D is not compact, Proposition 4.2 makes essential use of the axiom of choice. The same holds for the following analogous result:

Proposition 4.3 ($\mathcal{K}^\circ\subseteq\mathcal{A}^\circ$ in normed spaces). *Let Y be a locally convex space, and M be a subset of a normed space X. Let $\mathfrak{B}$ denote the system of all bounded subsets of $D\subseteq M$. Then any upper semicontinuous map $\Phi\colon D\to 2^Y$ with nonempty, convex and compact values is $\mathfrak{B}$-approximable, in particular, $\mathcal{K}(D,K)^\circ\subseteq\mathcal{A}(D,K)^\circ$.*

Proof. We may not apply Corollary 4.1 for the proof, since if Y is not metrizable, it is not clear whether closed convex subsets are neighbourhood retracts. However, we may just repeat the construction of Proposition 4.2: It is no loss of generality in this construction that we always have $\varepsilon_x\le 1$. It follows that if $B\subseteq D$ is bounded by some smallest constant $C\ge 0$, we have for any $z\in B$ that $\psi_x(z)=0$ for any x which does not belong to the set $B_0:=\{x\in D:\|x\|\le C+1\}$. The definition of φ then implies $\varphi(B)\subseteq\operatorname{conv}(\Phi(B_0))$. Note that $I(B)$ actually depends only from B. $\square$

For applications, it is important to note that even if X and Y are Banach spaces, and $M\subseteq X$, the set $\mathcal{A}(D,K)$ usually contains much more maps than $\mathcal{K}(D,K)$ in view of Corollary 4.1.

For example, if D is an ANR, then all upper semicontinuous maps $\Phi\colon D\to 2^K$ are approximable by [11, Corollary 1.36] if the images $\Phi(x)$ are nonempty and compact and so-called UV^ω sets; in particular, R_δ sets (i.e. the intersection of a decreasing sequence of nonempty compact contractible sets) have this property. Moreover, the composition of an approximable maps with a continuous (single-valued) map is approximable. This observation is of interest, since the evolution operator of differential inclusions usually is the composition of an R_δ-valued map with a continuous map. For more results on approximable maps, we refer to [11].

Lemma 4.1 (Rouché for the graph). *Let $K \subseteq Y$ be convex with $0 \in K + K \subseteq K$, and $\mathcal{F} = C(D, K)$. Assume that $F\colon D \to Y$ is f-admissible, f-epi, and belongs to $\mathcal{G}_-$. Then for each upper semicontinuous map $\Phi\colon D \to 2^Y$ with $\Phi|_P = f|_P$ the following holds:*

- *There is a neighbourhood $\mathfrak{U} \subseteq D \times Y$ of the graph of Φ such that, for any $\mathfrak{U}$-approximation φ, we have: if there is some $\psi \in \mathcal{F}_f$ such that $\varphi - \psi$ is $(\mathfrak{B}, K)$-compact and belongs to $\mathcal{G}$, then the equation $F(x) = \varphi(x)$ has a solution $x \in \Omega$.*

Proof. For any $x \in P$, the compact set $\{F(x)\}$ is disjoint from the closed set $\{f(x)\}$, by assumption. By [13, Theorem 1.14], we thus find a balanced open neighbourhood $V \subseteq Y$ of 0 such that the sets $F(x) + V$ and $f(x) + V$ are disjoint. By the continuity, we find an open neighbourhood U of x such that $F(U) \subseteq F(x) + V$ and $\Phi(U) \subseteq f(x) + V$. Let $\mathfrak{U}_0$ be the union of all sets of the form $U \times (f(x) + V)$ obtained in this way.

Note that in the previous construction $x \in U$, and so the projection of $\mathfrak{U}_0$ onto the first component contains a neighbourhood of P. Since D is normal, we find a closed neighbourhood of P which is contained in this projection. Let O be the complement of this neighbourhood, and $\mathfrak{U} = \mathfrak{U}_0 \cup (O \times Y)$.

Then $\mathfrak{U}$ is a neighbourhood of the graph of Φ with the following property: If $(x, y) \in \mathfrak{U}$ and $x \in P$, then there is a balanced neighbourhood $V \subseteq Y$ of 0 with $y \in f(x) + V$ and $F(x) \notin f(x) + V$.

Hence, if φ is an $\mathfrak{U}$-approximation, we have for any $x \in P$ and $0 \leq \lambda \leq 1$ that $F(x) \neq f(x) + \lambda(\varphi(x) - f(x))$.

Choose some $\psi \in \mathcal{F}_f$ such that $\varphi - \psi$ is $(\mathfrak{B}, K)$-compact, and put $G = F - (\varphi - \psi)$. Then $F - G = \varphi - \psi$, and for any $x \in P$ and $0 \leq \lambda \leq 1$ we have $F(x) + \lambda(G(x) - F(x)) = F(x) - \lambda(\varphi(x) - f(x)) \neq f(x)$.

Proposition 2.2 thus implies that G is f-epi, and in particular, the equation $G(x) = \psi(x)$ has a solution $x \in \Omega$. But this equation means $F(x) = \varphi(x)$. □

To prove that proper 0-epi maps are 0-multiepi, we need a slightly more restrictive condition:

Definition 4.2. Let $\Phi \in \mathcal{A}(D, K)$ and $F \in \mathcal{G}_-$. We say that (F, Φ) is *graph-admissible* if there is some neighbourhood $\mathfrak{U} \subseteq D \times Y$ of the graph of Φ such that we have for some $B \in \mathfrak{B}$: Any $\mathfrak{U}$-approximation φ with the additional property that for each $B_1 \in \mathfrak{B}$ there is some $B_2 \in \mathfrak{B}$ with

$$\varphi(B_1) \subseteq \overline{\operatorname{conv}}(\Phi(B_2)) \tag{4.1}$$

satisfies:

(1) $\varphi \in \mathcal{G}$,
(2) $\{x \in \overline{\Omega} : F(x) = \varphi(x)\} \subseteq B$.

We say that an f-admissible map $F \in \mathcal{G}_-$ is *f-graph-admissible*, if for any $\Phi \in \mathcal{A}_f^\circ \cap \mathcal{G}$ the pair (F, Φ) is graph-admissible.

Example 4.1. If $(\mathcal{G}, \mathcal{G}_-)$ is a growth condition as in some of our previous examples, then any f-admissible $F \in \mathcal{G}_-$ is f-graph-admissible.

This is non-trivial only for Example 4.1 if Y is not normed, since in this case neighbourhoods need not be bounded. However, for $\Phi \in \mathcal{G}$, the set $B_0 = \overline{\operatorname{conv}}(\Phi(D))$ is bounded. Since (4.1) implies $\varphi(D) \subseteq B_0$, it follows that the set $\{x \in D : F(x) = \varphi(x)\}$ is contained in the bounded set $B := F^{-1}(B_0)$.

Now we are in a position to formulate our main theorem on multiepi maps:

Theorem 4.1 (Proper f-epi maps are $\mathcal{A}_f^\circ$-epi). *Let Y be a locally convex space, and $K \subseteq Y$ be convex with $0 \in K + K \subseteq K$. Let the continuous map $F\colon D \to Y$ be f-admissible, f-epi, $(\mathfrak{B}, K)$-proper, and belong to $\mathcal{G}_-$. Let $\Phi \in \mathcal{A}(D, K)_f^\circ$ be such that (F, Φ) is graph-admissible, and assume:*

- *For each $B \in \mathfrak{B}$ there is some $\psi \in C(D, K)_f^\circ$ such that $\Phi(B) - \psi(B)$ is contained in K.*

Then the inclusion $F(x) \in \Phi(x)$ has a solution $x \in \overline{\Omega} \setminus P$.

Proof. Choose a neighbourhood $\mathfrak{U}_0$ of the graph of Φ with the properties described in Lemma 4.1. We may assume that $\mathfrak{U}_0$ also has the properties described in Definition 4.2; let $B \in \mathfrak{B}$ denote the corresponding set, and let B_0 be the corresponding set from Definition 4.1.

Then for any neighbourhood $\mathfrak{U} \subseteq \mathfrak{U}_0$, we find an $\mathfrak{U}$-approximation $\varphi_{\mathfrak{U}}$ with the following properties:

(1) $\varphi_{\mathfrak{U}} \in \mathcal{G}$,
(2) $\varphi_{\mathfrak{U}}$ is $(\mathfrak{B}, K)$-compact (because for any $B_1 \in \mathfrak{B}$ there is some $B_2 \in \mathfrak{B}$ such that $\overline{\operatorname{conv}}(\varphi_{\mathfrak{U}}(B_1)) \subseteq \overline{\operatorname{conv}}(\Phi(B_2))$ is a compact subset of K),
(3) the coincidence set $A_{\mathfrak{U}} = \{x \in \overline{\Omega} : F(x) = \varphi_{\mathfrak{U}}(x)\}$ is contained in B, and so $F(A_{\mathfrak{U}}) = \varphi(A_{\mathfrak{U}}) \subseteq \varphi(B) \subseteq \overline{\operatorname{conv}}(\Phi(B_0)) =: C$.

Since F is f-epi, we find some $x_{\mathfrak{U}} \in A_{\mathfrak{U}}$. By the axiom of choice, we may assume that $(x_{\mathfrak{U}})_{\mathfrak{U}}$ is a net. Since $C \subseteq K$ is compact, $A_{\mathfrak{U}} \subseteq F^{-1}(C) \cap B$, and F is $(\mathfrak{B}, K)$-proper, it follows that $x_{\mathfrak{U}}$ contains a subnet which converges to some point $x \in \overline{\Omega}$.

We claim that $F(x) \in \Phi(x)$. Otherwise, we find disjoint open sets V_F, V_Φ with $F(x) \in V_F$ and $\Phi(x) \subseteq V_\Phi$. By the (upper semi-)continuity, we find a neighbourhood $U_1 \subseteq D$ of x with $F(U_1) \subseteq V_F$ and $\Phi(U_1) \subseteq V_\Phi$. Since D is normal, we find a neighbourhood U_2 of x with $\overline{U}_2 \subseteq U_1$. Then $(U_1 \times V_\Phi) \cup ((D \setminus \overline{U}_2) \times Y)$ is a neighbourhood of the graph of Φ and thus contains some $\mathfrak{U}$ for which $x_\mathfrak{U} \in U_2$. Then $F(x_\mathfrak{U}) = \varphi_\mathfrak{U}(x_\mathfrak{U}) \in V_\Phi$ and $F(x_\mathfrak{U}) \in V_F$, a contradiction.

This contradiction proves $F(x) \in \Phi(x)$, and since F is f-admissible, we may conclude that $x \notin P$. □

If we assume the axiom of choice, we thus have proved the following:

Corollary 4.2 (Proper f-epi maps are f-multiepi). *Let Y be a Fréchet space, $K = Y$, and $P \supseteq \overline{\Omega} \setminus \Omega$. Assume that one of the following conditions is satisfied:*

(1) *$D \in \mathfrak{B}$ is metrizable, and $\mathcal{G}$ and $\mathcal{G}_-$ consist of all maps* (*recall Example* 2.2),
(2) *D is a subset of a normed space X, and $\mathfrak{B}$ is the system of bounded subsets of $D \subseteq X$.*

Let $f: P \to Y$ be such that $C(D,Y)^\circ_f \neq \emptyset$. Then for any continuous map $F: D \to Y$ which is f-graph-admissible and proper on each $\overline{B}$ $(B \in \mathfrak{B})$ the following statements are equivalent:

(1) *F is f-epi,*
(2) *F is f-multiepi,*
(3) *F is $\mathcal{A}^\circ_f$-epi,*
(4) *F is $\mathcal{F}^\circ_f$-epi where $\mathcal{F}$ denotes the system of all approximable upper semi-continuous maps $\Phi: D \to 2^Y$ with closed values.*

Proof. If F is f-epi, Theorem 4.1 implies that F is $\mathcal{A}^\circ_f$-epi. Moroever, Proposition 4.1 implies that $\mathcal{A}^\circ_f = \mathcal{F}^\circ_f$ (recall the remarks following Proposition 4.1 concerning the existence of a retraction), and Proposition 4.2, resp. Proposition 4.3 implies that if F is $\mathcal{A}^\circ_f$-epi, then F is $\mathcal{K}^\circ_f$-epi, i.e. f-multiepi. Finally, since single-valued continuous maps belong to $\mathcal{K}$, it follows that any f-multiepi map is f-epi. □

In connection with Corollary 4.2, we point out once more that in all our previous examples any f-admissible map $F \in \mathcal{G}_-$ is f-graph-admissible.

We do not know whether Corollary 4.2 holds without the artificial assumption that F be proper. Note, however, that in all definitions of a degree or index for a map that we found in literature, properness of the corresponding map F is assumed. So in this sense, our result is not "worse" than the corresponding analogous results for the degree.

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